Boundary control problem and optimality conditions for the Cahn–Hilliard equation with dynamic boundary conditions

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Abstract

This paper is concerned with a boundary control problem for the Cahn–Hilliard equation coupled with dynamic boundary conditions. In order to handle the control problem, we restrict our analysis to the case of regular potentials defined on the whole real line, assuming the boundary potential to be dominant. The existence of optimal control, the Fréchet differentiability of the control-to-state operator between appropriate Banach spaces, and the first-order necessary conditions for optimality are addressed. In particular, the necessary condition for optimality is characterized by a variational inequality involving the adjoint variables.

Key words: Cahn–Hilliard equation, dynamic boundary conditions, phase separation, double-well potentials, optimal control, optimality conditions, adjoint problem.

AMS (MOS) Subject Classification: 35K61, 49J20, 49K20, 49J50.

1 Introduction

The Cahn–Hilliard equation plays a fundamental role in material science (see, e.g., the review paper [34] and the vast literature therein). Such an equation was historically proposed for the study of phase segregation in cooling binary alloys (see [2]). On the other hand, from then onward, it has been shown how versatile this equation can be for several applications in very different fields such as engineering, biology, tumor growth, image inpainting, population dynamics, bacterial films, and many others. The huge efforts by the mathematical community have made the classical Cahn–Hilliard equation well-understood from a mathematical point of view, at least as far as the existence, uniqueness and regularity of solutions are concerned. Here, we address a boundary optimal control problem for the Cahn–Hilliard equation coupled with some non-standard boundary conditions, the so-called dynamic ones.

For a fixed finite final time $T > 0$, the Cahn–Hilliard equation reads as follows

$$\partial_t y - \Delta w = 0 \quad \text{in } Q := \Omega \times (0, T), \tag{1.1}$$

$$w = -\Delta y + f'(y) \quad \text{in } Q, \tag{1.2}$$

where Ω represents the space domain in which the evolution takes place, and the occurring variables y and w stand for the order parameter and the corresponding chemical potential, respectively. Moreover, f' denotes the derivative of a nonlinearity that possesses a double-well behavior. For this latter, the prototype is the regular double-well potential f_{reg}, defined by

$$f_{reg}(r) = \frac{1}{4}(r^2 - 1)^2, \quad \text{whence} \quad f'_{reg}(r) = r^3 - r, \quad \text{for } r \in \mathbb{R}. \tag{1.3}$$

Besides, we endow the above system with an initial condition of the form $y(0) = y_0$, and suitable boundary conditions. As for boundary conditions, the widespread types in literature are the no-flux conditions for both the variables y and w. It is worth noting that, from a phenomenological point of view, the no-flux condition for w is quite natural since it ensures the mass conservation during the evolution process: this can be easily checked by testing the equation (1.1) by 1 and integrating by parts over Ω. In fact, denoting by $(v)^\Omega$ the mean value of the function $v : \Omega \to \mathbb{R}$, we realize that

$$(\partial_t y(t))^\Omega = 0 \quad \text{for a.a. } t \in (0, T), \quad \text{and} \quad (y(t))^\Omega = m_0 \quad \text{for every } t \in [0, T], \\ \text{where } m_0 := (y_0)^\Omega \text{ is the mean value of } y_0. \tag{1.4}$$

In this contribution, we also deal with the no-flux condition for the chemical potential, whereas a dynamic boundary condition for the order parameter is prescribed. These boundary conditions are quite new and were recently proposed in order to take into account the dynamics between the walls. In this regard, let us address to [19], where both the viscous and the non-viscous Cahn–Hilliard equations, combined with these kinds of boundary conditions, have been investigated by assuming the boundary potential to be dominant on the bulk one. Furthermore, we have to mention [4, 9, 13, 16, 23, 25, 33, 36–38, 42], where other problems related to the Cahn–Hilliard equation combined with dynamic boundary conditions have been analyzed, and [3, 7, 8, 11, 20, 29, 35] for the coupling of dynamic boundary conditions with different phase field models such as the Allen-Cahn or the Penrose-Fife model. So, according to [19] we supply the above system (1.1)–(1.2) with

$$\partial_n w = 0 \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.5}$$
$$\partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + f'_\Gamma(y_\Gamma) = u_\Gamma \quad \text{on } \Sigma, \tag{1.6}$$

where Γ is the boundary of Ω, y_Γ denotes the trace of y, Δ_Γ stands for the Laplace–Beltrami operator on the boundary, and ∂_n represents the outward normal derivative. Furthermore, the term f'_Γ is a nonlinearity similar to the previous f', but operating on the values on the boundary instead of on the bulk, whereas u_Γ is the so-called control variable which can be interpreted as a boundary source term.

Summing up, the system we are going to deal with reads

$$\partial_t y - \Delta w = 0 \quad \text{in } Q, \tag{1.7}$$
$$w = -\Delta y + f'(y) \quad \text{in } Q, \tag{1.8}$$
$$\partial_n w = 0 \quad \text{on } \Sigma, \tag{1.9}$$
$$y_\Gamma = y_{|\Gamma} \quad \text{and} \quad \partial_t y_\Gamma + \partial_n y - \Delta_\Gamma y_\Gamma + f'_\Gamma(y_\Gamma) = u_\Gamma \quad \text{on } \Sigma, \tag{1.10}$$
$$y(0) = y_0 \quad \text{in } \Omega. \tag{1.11}$$

Once that the state system (1.7)–(1.11) has been described, we can address the corresponding control problem. Among several possibilities, we consider the following tracking-type cost functional

$$\begin{aligned} \mathcal{J}(y, y_\Gamma, u_\Gamma) := {} & \frac{b_Q}{2} \|y - z_Q\|^2_{L^2(Q)} + \frac{b_\Sigma}{2} \|y_\Gamma - z_\Sigma\|^2_{L^2(\Sigma)} + \frac{b_\Omega}{2} \|y(T) - z_\Omega\|^2_{L^2(\Omega)} \\ & + \frac{b_\Gamma}{2} \|y_\Gamma(T) - z_\Gamma\|^2_{L^2(\Gamma)} + \frac{b_0}{2} \|u_\Gamma\|^2_{L^2(\Sigma)}, \end{aligned} \tag{1.12}$$

where the symbols $b_Q, b_\Sigma, b_\Omega, b_\Gamma, b_0$ and $z_Q, z_\Sigma, z_\Omega, z_\Gamma$ denote nonnegative constants and some target functions, respectively. Moreover, we require the control variable u_Γ to belong to the non-empty control-box $\mathcal{U}_{\text{ad}}$ which is defined by

$$\begin{aligned} \mathcal{U}_{\text{ad}} := \big\{ & u_\Gamma \in H^1(0,T;L^2(\Gamma)) \cap L^\infty(\Sigma) : \\ & u_{\Gamma,\min} \le u_\Gamma \le u_{\Gamma,\max} \ \text{ a.e. on } \Sigma, \ \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \le M_0 \big\}, \end{aligned} \tag{1.13}$$

for suitable functions $u_{\Gamma,\min}, u_{\Gamma,\max} \in L^\infty(\Sigma)$, and for a positive constant M_0. Note that, owing to the weak lower semicontinuity of norms, $\mathcal{U}_{\text{ad}}$ is a closed convex subset of $L^2(\Sigma)$. Therefore, our minimization problem consists in seeking an admissible control variable u_Γ such that, along with its corresponding solution to system (1.7)–(1.11), minimizes the cost functional (1.12).

Concerning the interpretation of the optimal control problem, let us point out that, since the target functions $z_Q, z_\Sigma, z_\Omega, z_\Gamma$ provide some particular configuration, we are looking for an admissible control variable u_Γ which forces its corresponding solution to (1.7)–(1.11) to be as close as possible to the prescribed configuration. Conversely, the last term of (1.12) penalizes the large values of the L^2-norm of the control so that it can be seen as the cost we have to pay in order to follow that strategy.

As for previous contributions on optimal control problems for Cahn–Hilliard systems possibly involving dynamic boundary conditions, let us mention the papers [5, 6, 10, 12, 14, 15, 17, 18, 21, 22, 26–28, 39, 43, 44]. In particular, we focus our attention on [17], where the optimal control problem for the viscous Cahn–Hilliard equation endowed with dynamic boundary conditions is investigated by exploiting the well-posedness of the state system discussed in [19]. Moreover, we also point out [18], where the optimal control problem has been extended to the non-viscous case. In fact, by employing suitable asymptotic arguments and letting the viscosity parameter tend to zero, in [18] it is shown how the optimal control results for the viscous case allow to recover other results for the pure setting. It is worth underlining that the optimal control problem is exactly the one we are going to address here, but in this contribution we follow a direct approach and are able to obtain better results.

Indeed, it occurs that in the limit procedure of [18] some information on the limiting terms turns out to be lost and especially the results concerning the first-order conditions for optimality and the adjoint system are somehow unsatisfactory since they hold in a very weak sense. Moreover, the adjoint system at the limit has not an explicit structure and the uniqueness for its solution is not at all clear. Namely, the related existence result states the existence of proper elements in dual spaces that satisfy some properties and are the (weak star) limits of some terms or groups of terms of the adjoint system for the viscous case (see [18, Thm. 2.7, p. 318] for a precise statement). On the other hand, it may appear that the optimality condition there obtained is very similar to the one we will point out here since they formally consist in the same variational inequality (see [18, eq. (2.54)] and compare with (2.48)). However, the results are substantially different and the difference is hidden in the two adjoint systems. Lastly, let us point out that the cost functional of [18] is less general than ours since, as a consequence of the results of [17], the constant b_Ω and b_Γ are taken identically zero.

In the present contribution, provided we restrict the analysis on everywhere defined potentials like (1.3), we show that also for the non-viscous case the optimality condition can be completely characterized. As a matter of fact, the existence, uniqueness and also further regularity for the adjoint system will be proved (cf. Theorem 2.7). Moreover, since from a technical viewpoint the strategies are very different, we have to perform the proofs ex novo without relying on the results proved in [17].

After showing the existence of optimal controls, we characterize the first-order necessary conditions that every optimal control has to satisfy through a variational inequality. In this direction, a key point will be showing the Fréchet differentiability of the control-to-state operator. Then, as usual for optimal control problems (see, e.g., [31,41]), in order to simplify the obtained optimality conditions, a new system, called adjoint, has to be introduced and solved in order to reformulate the necessary condition in a more convenient way. The adjoint system turns out to be a backward-in-time boundary value problem of the following form

$$\begin{aligned}
&q = -\Delta p \quad \text{in } Q,\\
&-\partial_t p - \Delta q + \lambda q = \varphi_Q \quad \text{in } Q,\\
&\partial_n p = 0 \quad \text{on } \Sigma,\\
&-\partial_t q_\Gamma + \partial_n q - \Delta_\Gamma q_\Gamma + \lambda_\Gamma q_\Gamma = \varphi_\Sigma \quad \text{on } \Sigma,
\end{aligned}$$

where q and p are the adjoint variables, q_Γ stands for the trace of q, and the functions λ, λ_Γ, φ_Q and φ_Σ are somehow related to $z_Q, z_\Sigma, z_\Omega, z_\Gamma$ and to the constants $b_Q, b_\Sigma, b_\Omega, b_\Gamma, b_0$ appearing in (1.12), as well as to the optimal state $(\overline{y}, \overline{y}_\Gamma)$, which is the state associated to the optimal control $\overline{u}_\Gamma$. Furthermore, the above system will be coupled with suitable final conditions.

The plan of the paper is as follows. In Section 2 we specify the mathematical setting and recollect the results we have established. From the third section on, we begin with the corresponding proofs. Section 3 is devoted to the existence of optimal controls. Furthermore, Section 4 is the place in which the main novelties appear: there, we discuss the properties of the control-to-state operator $\mathcal{S}$ proving its Lipschitz continuity and the Fréchet differentiability in suitable Banach spaces. Finally, the well-posedness of the adjoint system and the first-order necessary conditions for optimality are discussed in Section 5.

2 Statement of the problem and results

In this section, we set the notation and present in detail the established results. We start by pointing out that Ω represents the body where the evolution takes place and we assume $\Omega \subset \mathbb{R}^3$ to be open, connected, bounded and smooth, with Lebesgue measure denoted by $|\Omega|$. Moreover, let us fix once for all that the symbols Γ, ∂_n, ∇_Γ and Δ_Γ stand for the boundary of Ω, the outward normal derivative, the surface gradient, and the Laplace–Beltrami operator, respectively. Given a finite final time $T > 0$, we set for convenience

$$Q_t := \Omega \times (0,t), \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T], \tag{2.1}$$
$$Q := Q_T\,, \quad \text{and} \quad \Sigma := \Sigma_T\,. \tag{2.2}$$

Before diving into the mathematical setting, let us emphasize a typical issue of control problems. Although some of the results we need hold under rather weak conditions, we will require quite strong hypotheses for the involved potentials and for the initial data in order to handle the corresponding control problem. As a consequence, the following requirements surely comply with the framework of [19].

On the potentials f and f_Γ we make the following structural assumptions

$$f,\ f_\Gamma : \mathbb{R} \to [0,+\infty) \text{ are } C^4 \text{ functions.} \tag{2.3}$$
$$f'(0) = f'_\Gamma(0) = 0, \text{ and } f'' \text{ and } f''_\Gamma \text{ are bounded from below.} \tag{2.4}$$
$$|f'(r)| \le \eta\,|f'_\Gamma(r)| + C \quad \text{for some } \eta,\ C > 0 \quad \text{and every } r \in \mathbb{R}. \tag{2.5}$$
$$\lim_{r\searrow\,-\infty} f'(r) = \lim_{r\searrow-\infty} f'_\Gamma(r) = -\infty, \text{ and } \lim_{r\nearrow+\infty} f'(r) = \lim_{r\nearrow+\infty} f'_\Gamma(r) = +\infty\,. \tag{2.6}$$

Remark 2.1. The above conditions imply the possibility of splitting f' as $f' = \beta + \pi$, where β is a monotone function, which diverges as its argument goes to $-\infty$ or to $+\infty$, while π is a regular perturbation with bounded derivative. Likewise, it goes for the boundary contribution f'_Γ that can be possibly written as $f'_\Gamma = \beta_\Gamma + \pi_\Gamma$, for suitable functions satisfying the same properties as β and π.

It is worth emphasizing that in our treatment, owing to (2.3)–(2.6), the case of (1.3) is allowed, while other significant cases like, e.g., the logarithmic potential

$$f_{log}(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r) - kr^2, \quad r \in (-1,1), \tag{2.7}$$

(with $k > 1$ to ensure non-convexity) are not. On the other hand, the above setting (2.3)–(2.6) perfectly fits the framework of [19] since the assumption (2.5) postulates the domination of the boundary potential on the bulk one. For the converse case, namely the one in which the bulk potential is the leading one between the two, we refer to the contributions [24, 25]. Now, let us introduce some functional spaces that will be useful later on by defining

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad V_\Gamma := H^1(\Gamma), \quad H_\Gamma := L^2(\Gamma), \tag{2.8}$$
$$\mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : \ v_\Gamma = v_{|\Gamma}\}, \quad \text{and} \quad \mathcal{G} := V^* \times H_\Gamma, \tag{2.9}$$

and we endow them with their natural norms to get some Banach spaces. Besides, for an arbitrary Banach space X, we agree to use $\|\cdot\|_X$ to denote its norm, the standard symbol X^* for its topological dual, and ${}_{X^*}\langle\cdot,\cdot\rangle_X$ for the corresponding duality product between X^* and X. Meanwhile, we will use $\|\cdot\|_p$ for the usual norm in L^p spaces. In the following, we understood that H is embedded in V^* in the usual way, i.e. $V \subset H \cong H^* \subset V^*$. This constitutes a Hilbert triplet, namely we have the following identification

$$\langle u, v\rangle = (u, v) \quad \text{for every } u \in H \text{ and } v \in V, \tag{2.10}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in H.

In addition, whenever $u \in V^*$ and $\underline{u} \in L^1(0, T; V^*)$, we define their generalized mean values $u^\Omega \in \mathbb{R}$ and $\underline{u}^\Omega \in L^1(0, T)$ by

$$u^\Omega := \frac{1}{|\Omega|}\langle u, 1\rangle, \quad \text{and} \quad \underline{u}^\Omega(t) := \big(\underline{u}(t)\big)^\Omega \quad \text{for a.a. } t \in (0, T), \tag{2.11}$$

where (2.11) reduces to the usual mean values when it is applied to elements of H or $L^1(0, T; H)$.

Next, since in the last two sections we are going to use test functions with zero mean value, it is convenient to set

$$\mathcal{G}_\Omega := \{(v, v_\Gamma) \in \mathcal{G} : \ v^\Omega = 0\}, \quad \text{and} \quad \mathcal{V}_\Omega := \mathcal{G}_\Omega \cap \mathcal{V}, \tag{2.12}$$

and endow them with their natural topologies as subspaces of $\mathcal{G}$ and $\mathcal{V}$, respectively. Moreover, we define

$$\operatorname{dom}\mathcal{N} := \{v_* \in V^* : \ v_*^\Omega = 0\}, \quad \text{and} \quad \mathcal{N} : \operatorname{dom}\mathcal{N} \to \{v \in V : \ v^\Omega = 0\}, \tag{2.13}$$

as the map which assigns to every $v_* \in \operatorname{dom}\mathcal{N}$ the element $\mathcal{N}v_*$ which satisfies

$$\mathcal{N}v_* \in V, \quad (\mathcal{N}v_*)^\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla\mathcal{N}v_* \cdot \nabla z = \langle v_*, z\rangle \quad \text{for every } z \in V. \tag{2.14}$$

Hence, $\mathcal{N}v_*$ represents the solution v to the generalized Neumann problem for $-\Delta$ with datum v_* that in addition has to satisfy the zero mean value condition. In fact, if $v_* \in H$, the above conditions mean that $-\Delta\mathcal{N}v_* = v_*$ in Ω and $\partial_n(\mathcal{N}v_*) = 0$ on Γ. As far as Ω is bounded, smooth and connected, it follows that (2.14) yields a well-defined isomorphism which also satisfies

$$\mathcal{N}v_* \in H^{s+2}(\Omega), \quad \|\mathcal{N}v_*\|_{H^{s+2}(\Omega)} \le C_s\|v_*\|_{H^s(\Omega)},$$
$$\text{if } s \ge 0 \quad \text{and} \quad v_* \in H^s(\Omega) \cap \operatorname{dom}\mathcal{N}, \tag{2.15}$$

with a constant C_s that depends only on Ω and s. Moreover, we have the following properties

$$\langle u_*, \mathcal{N}v_*\rangle = \langle v_*, \mathcal{N}u_*\rangle = \int_\Omega (\nabla\mathcal{N}u_*) \cdot (\nabla\mathcal{N}v_*) \quad \text{for } u_*, v_* \in \operatorname{dom}\mathcal{N}, \tag{2.16}$$

whence also

$$2\langle \partial_t v_*(t), \mathcal{N} v_*(t)\rangle = \frac{d}{dt}\int_\Omega |\nabla \mathcal{N} v_*(t)|^2 = \frac{d}{dt}\,\|v_*(t)\|_*^2 \quad \text{for a.a. } t\in(0,T), \tag{2.17}$$

for every $v_* \in H^1(0,T;V^*)$ satisfying $(v_*)^\Omega = 0$ a.e. in $(0,T)$, where we have set $\|\cdot\|_* := \|\nabla\mathcal{N}(\cdot)\|_H$, which turns out to be a norm in V^* equivalent to the usual dual norm.

As the initial data are concerned, we require that

$$y_0 \in H^2(\Omega), \quad {y_0}_{|\Gamma} \in H^2(\Gamma), \quad \text{and} \quad \Delta y_0 \in V, \tag{2.18}$$

where the last condition has been already assumed in [19] to ensure good regularity results for the non-viscous system (see [19, Eq. (2.40), p. 978]). Even though we could write the equations and the boundary conditions in their strong forms, we however prefer to use the corresponding variational formulations. Hence, the problem we want to deal with consists of looking for a triplet (y, y_Γ, w) that satisfies the regularity

$$y \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.19}$$
$$y_\Gamma \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{2.20}$$
$$y_\Gamma(t) = y(t)_{|\Gamma} \quad \text{for a.a. } t\in(0,T), \tag{2.21}$$
$$w \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{2.22}$$

as well as, for almost every $t\in(0,T)$, the variational equalities

$$\langle \partial_t y(t), v\rangle + \int_\Omega \nabla w(t)\cdot\nabla v = 0 \quad \text{for every } v\in V, \tag{2.23}$$

$$\int_\Omega w(t)\, v = \int_\Gamma \partial_t y_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla y(t)\cdot \nabla v + \int_\Gamma \nabla_\Gamma y_\Gamma(t)\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega f'(y(t))\, v + \int_\Gamma \bigl(f_\Gamma'(y_\Gamma(t)) - u_\Gamma(t)\bigr)\, v_\Gamma \quad \text{for every } (v,v_\Gamma)\in\mathcal{V} \tag{2.24}$$

and the initial condition

$$y(0) = y_0. \tag{2.25}$$

Of course, (2.23)–(2.24) can be equivalently rewritten as follows

$$\int_0^T \langle \partial_t y, v\rangle + \int_Q \nabla w\cdot\nabla v = 0 \quad \text{for every } v\in L^2(0,T;V), \tag{2.26}$$

$$\int_Q wv = \int_\Sigma \partial_t y_\Gamma\, v_\Gamma + \int_Q \nabla y\cdot\nabla v + \int_\Sigma \nabla_\Gamma y_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_Q f'(y)\, v + \int_\Sigma \bigl(f_\Gamma'(y_\Gamma) - u_\Gamma\bigr)\, v_\Gamma \quad \text{for every } (v,v_\Gamma)\in L^2(0,T;\mathcal{V}). \tag{2.27}$$

We are now in a position to introduce our results. As far as the existence, the uniqueness, the regularity and the continuous dependence results are concerned, we can account for Theorems 2.2, 2.3, 2.4, and 2.6 of [19]. Hence, we have the following statement.

Theorem 2.2. *Assume that* (2.3)–(2.6), (2.18) *are fulfilled and let* $u_\Gamma \in H^1(0,T;H_\Gamma)$. *Then, system* (2.19)–(2.25) *admits a unique solution* (y, y_Γ, w) *which satisfies*

$$\|y\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))} + \|y_\Gamma\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))}$$
$$+ \|w\|_{L^\infty(0,T;V)\cap L^2(0,T;H^3(\Omega))} \le C_1, \tag{2.28}$$

from which, accounting for the Sobolev embedding, it also follows that

$$\|y\|_{L^\infty(Q)} + \|y_\Gamma\|_{L^\infty(\Sigma)} \le C_1, \tag{2.29}$$

for a positive constant C_1 that depends only on Ω, T, the shape of the nonlinearities f and f_Γ, the initial datum y_0, and on an upper bound for the norm of u_Γ in $H^1(0,T;H_\Gamma)$. Moreover, if $u_{\Gamma,i} \in H^1(0,T;H_\Gamma)$, $i = 1,2$, are two forcing terms and $(y_i, y_{\Gamma,i}, w_i)$ are the corresponding solutions, we have that

$$\begin{aligned} &\|y_1 - y_2\|^2_{L^\infty(0,T;V^*)} + \|y_{\Gamma,1} - y_{\Gamma,2}\|^2_{L^\infty(0,T;H_\Gamma)} + \|\nabla(y_1 - y_2)\|^2_{L^2(0,T;H)} \\ &\quad + \|\nabla_\Gamma(y_{\Gamma,1} - y_{\Gamma,2})\|^2_{L^2(0,T;H_\Gamma)} \le C_2 \, \|u_{\Gamma,1} - u_{\Gamma,2}\|^2_{L^2(0,T;H_\Gamma)}, \end{aligned} \tag{2.30}$$

where the constant C_2 depends only on Ω, T, and the shape of the nonlinearities f and f_Γ.

Once the well-posedness of the system (2.19)–(2.25) has been proved, we can address the corresponding control problem. As far as the assumptions on the cost functional are concerned, we postulate that

$$z_Q \in H^1(0,T;H), \quad z_\Sigma \in L^2(\Sigma), \quad z_\Omega \in H^1(\Omega), \quad z_\Gamma \in H^1(\Gamma). \tag{2.31}$$

$$b_Q,\ b_\Sigma,\ b_\Omega,\ b_\Gamma,\ b_0 \ \text{ are nonnegative constants, but not all zero.} \tag{2.32}$$

$$\begin{aligned} &M_0 > 0, \quad u_{\Gamma,\min},\, u_{\Gamma,\max} \in L^\infty(\Sigma), \quad \text{with} \quad u_{\Gamma,\min} \le u_{\Gamma,\max} \quad \text{a.e. on } \Sigma, \\ &\quad \text{in such a way that } \mathcal{U}_{\rm ad} \text{ turns out to be nonempty.} \end{aligned} \tag{2.33}$$

Below, the first fundamental result related to the existence of optimal controls can be found.

Theorem 2.3. *Assume that* (2.3)–(2.6), (2.18), *and* (2.31)–(2.33) *are in force. Then, there exists $\overline{u}_\Gamma \in \mathcal{U}_{\rm ad}$ such that*

$$\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) \le \mathcal{J}(y, y_\Gamma, u_\Gamma) \quad \textit{for every } u_\Gamma \in \mathcal{U}_{\rm ad}, \tag{2.34}$$

where $\overline{y}$, $\overline{y}_\Gamma$ and y, y_Γ are the components of the solutions $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ and (y, y_Γ, w) to the state system (2.19)–(2.25) *corresponding to the controls $\overline{u}_\Gamma$ and u_Γ, respectively. Such a control variable $\overline{u}_\Gamma$ is called optimal control.*

The well-posedness of the system (2.19)–(2.25), allows us to properly define the so-called control-to-state mapping. We set

$$\begin{aligned} &\mathcal{X} := H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \ \text{ and } \ \mathcal{Y} := H^1(0,T;\mathcal{G}) \cap L^\infty(0,T;\mathcal{V}), \\ &\mathcal{U} \text{ is an open bounded set in } \mathcal{X} \text{ that includes } \mathcal{U}_{\rm ad}, \\ &\mathcal{S} : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y} \ \text{ is defined by } \mathcal{S}(u_\Gamma) := (y, y_\Gamma), \\ &\quad \text{where } (y, y_\Gamma, w) \text{ is the solution to (2.19)–(2.25) corresponding to } u_\Gamma. \end{aligned}$$

Remark 2.4. Note that the existence of the superset $\mathcal{U}$ containing $\mathcal{U}_{\rm ad}$ is trivially satisfied. Indeed, for instance, we can take

$$\mathcal{U} := \bigl\{ u_\Gamma \in \mathcal{X} : \|u_\Gamma\|_{L^\infty(\Sigma)} < \|u_{\Gamma,\min}\|_{L^\infty(\Sigma)} + \|u_{\Gamma,\max}\|_{L^\infty(\Sigma)} + 1, \ \|\partial_t u_\Gamma\|_{L^2(\Sigma)} < M_0 + 1 \bigr\}.$$

Thus, we can express the cost functional $\mathcal{J}$ as a function of u_Γ by introducing the so-called reduced cost functional

$$\widetilde{\mathcal{J}} : \mathcal{U} \to \mathbb{R} \quad \text{which is defined by} \quad \widetilde{\mathcal{J}}(u_\Gamma) := \mathcal{J}(\mathcal{S}(u_\Gamma), u_\Gamma). \tag{2.35}$$

Formally, as $\mathcal{U}_{\rm ad}$ is convex, it is a standard matter to realize that the desired necessary condition for $\overline{u}_\Gamma$ is carried out by the following variational inequality

$$\langle D\widetilde{\mathcal{J}}(\overline{u}_\Gamma), v_\Gamma - \overline{u}_\Gamma \rangle \geq 0 \quad \text{for every } v_\Gamma \in \mathcal{U}_{\rm ad}, \tag{2.36}$$

where $D\widetilde{\mathcal{J}}(\overline{u}_\Gamma)$ denotes the derivative of $\widetilde{\mathcal{J}}$ at $\overline{u}_\Gamma$ in a suitable functional sense. The strategy we follow in order to obtain some optimality conditions consists in proving at first that $\mathcal{S}$ is Fréchet differentiable at $\overline{u}_\Gamma$, and then, accounting for the chain rule, developing the above inequality to get an explicit formulation which characterizes the optimality. As we shall see in Section 4, this procedure naturally leads to the linearized system, that we briefly introduce in the lines below. Let us fix $\overline{u}_\Gamma \in \mathcal{U}$, the corresponding state $(\overline{y}, \overline{y}_\Gamma) := \mathcal{S}(\overline{u}_\Gamma)$, and introduce the increment $h_\Gamma \in H^1(0,T;H_\Gamma)$. Moreover, we set for convenience

$$\lambda := f''(\overline{y}), \quad \text{and} \quad \lambda_\Gamma := f''_\Gamma(\overline{y}_\Gamma). \tag{2.37}$$

Then, the linearized system for (1.7)–(1.11) consists of finding a triplet (ξ, ξ_Γ, η) satisfying the analogue of (2.19)–(2.22), solving, for a.a. $t \in (0,T)$, the variational equations

$$\langle \partial_t \xi(t), v \rangle + \int_\Omega \nabla \eta(t) \cdot \nabla v = 0 \quad \text{for every } v \in V, \tag{2.38}$$

$$\int_\Omega \eta(t) v = \int_\Gamma \partial_t \xi_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla \xi(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega \lambda(t)\, \xi(t)\, v + \int_\Gamma \big(\lambda_\Gamma(t)\, \xi_\Gamma(t) - h_\Gamma(t) \big)\, v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V} \tag{2.39}$$

and satisfying the initial condition

$$\xi(0) = 0. \tag{2.40}$$

In order to obtain the well-posedness for the above system, we would be tempted to directly invoke [19, Thm. 2.4, p. 978]. However, from a careful investigation, we realized that the requirements on λ are not satisfied in our setting. Indeed, note that $\partial_t \lambda = f'''(\overline{y}) \partial_t \overline{y}$ and that $\partial_t \overline{y} \in L^\infty(0,T;V^*) \cap L^2(0,T;V)$ by virtue of Theorem 2.2. So, in our framework we cannot infer that $\lambda \in W^{1,\infty}(0,T;H)$. This lack of regularity, due to the absence of the viscous term, can be however overcome by applying a different estimate in the term involving λ. Therefore, modifying properly the proof of [19, Thm. 2.4, p. 978], the same result holds.

Theorem 2.5. *Let $\overline{u}_\Gamma \in \mathcal{U}$, $(\overline{y}, \overline{y}_\Gamma) = \mathcal{S}(\overline{u}_\Gamma)$, and λ, λ_Γ be defined by* (2.37). *Then, for every $h_\Gamma \in H^1(0,T;H_\Gamma)$, there exists a unique triplet (ξ, ξ_Γ, η) satisfying the analogue of* (2.19)–(2.22) *and solving the linearized system* (2.38)–(2.40).

Next, we will show that $\mathcal{S}$ is Fréchet differentiable at $\overline{u}_\Gamma$, that $D\mathcal{S}(\overline{u}_\Gamma)$ is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$, and also that, for every h_Γ in $\mathcal{X}$, $[D\mathcal{S}(\overline{u}_\Gamma)](h_\Gamma) = (\xi, \xi_\Gamma)$, where the triplet (ξ, ξ_Γ, η) represents the unique solution to the linearized system associated to h_Γ. Here is the precise result.

Theorem 2.6. *Let $\overline{u}_\Gamma \in \mathcal{U}$, $(\overline{y}, \overline{y}_\Gamma) = \mathcal{S}(\overline{u}_\Gamma)$, and λ, λ_Γ be defined by* (2.37). *Then the control-to-state mapping $\mathcal{S} : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y}$ is Fréchet differentiable at $\overline{u}_\Gamma$. Moreover, its derivative $D\mathcal{S}(\overline{u}_\Gamma)$ is a linear operator from $\mathcal{U}$ to $\mathcal{Y}$ which is given as follows: whenever $h_\Gamma \in \mathcal{X}$ fulfills $\overline{u}_\Gamma + h_\Gamma \in \mathcal{U}$, the value of $D\mathcal{S}(\overline{u}_\Gamma)$ at h_Γ consists of the pair (ξ, ξ_Γ), where (ξ, ξ_Γ, η) is the unique solution to the linearized system* (2.38)–(2.40).

Then, by invoking the chain rule, we develop (2.36) in order to obtain the following explicit optimality condition

$$\begin{aligned} & b_Q \int_Q (\overline{y} - z_Q)\xi + b_\Sigma \int_\Sigma (\overline{y}_\Gamma - z_\Sigma)\xi_\Gamma + b_\Omega \int_\Omega (\overline{y}(T) - z_\Omega)\xi(T) \\ & \quad + b_\Gamma \int_\Gamma (\overline{y}_\Gamma(T) - z_\Gamma)\xi_\Gamma(T) + b_0 \int_\Sigma \overline{u}_\Gamma(v_\Gamma - \overline{u}_\Gamma) \geq 0 \quad \text{for every } v_\Gamma \in \mathcal{U}_{\mathrm{ad}}, \end{aligned} \tag{2.41}$$

where ξ and ξ_Γ are the first two components of the unique solution to the linearized system corresponding to $h_\Gamma = v_\Gamma - \overline{u}_\Gamma$.

Lastly, we try to eliminate the pair (ξ, ξ_Γ) from the above inequality. To overcome this issue, we introduce the so-called adjoint system. Namely, we are looking for a triplet (q, q_Γ, p) that fulfills the regularity requirements

$$(q, q_\Gamma) \in H^1(0,T; \mathcal{G}_\Omega) \cap L^\infty(0,T; \mathcal{V}_\Omega) \cap L^2(0,T; H^2(\Omega) \times H^2(\Gamma)), \tag{2.42}$$

$$p \in H^1(0,T; V) \cap L^2(0,T; H^4(\Omega)), \tag{2.43}$$

$$q_\Gamma(t) = q(t)_{|\Gamma} \text{ for a.a. } t \in (0,T), \tag{2.44}$$

and solves, for a.a. $t \in (0,T)$, the following backward-in-time problem

$$\int_\Omega q(t)\, v = \int_\Omega \nabla p(t) \cdot \nabla v \quad \text{for every } v \in V, \tag{2.45}$$

$$\begin{aligned} & - \int_\Omega \partial_t p(t)\, v + \int_\Omega \nabla q(t) \cdot \nabla v + \int_\Omega \lambda(t) q(t) v - \int_\Gamma \partial_t q_\Gamma(t)\, v_\Gamma + \int_\Gamma \nabla_\Gamma q_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma \\ & \quad + \int_\Gamma \lambda_\Gamma(t) q_\Gamma(t) v_\Gamma = b_Q \int_\Omega (\overline{y}(t) - z_Q(t)) v + b_\Sigma \int_\Gamma (\overline{y}_\Gamma(t) - z_\Sigma(t)) v_\Gamma \\ & \qquad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \end{aligned} \tag{2.46}$$

and the final condition

$$\begin{aligned} & \int_\Omega p(T) v + \int_\Gamma q_\Gamma(T) v_\Gamma = b_\Omega \int_\Omega (\overline{y}(T) - z_\Omega) v(T) + b_\Gamma \int_\Gamma (\overline{y}_\Gamma(T) - z_\Gamma) v_\Gamma(T) \\ & \qquad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \end{aligned} \tag{2.47}$$

In order to simplify the notation, let us convey to denote

$$\varphi_Q := b_Q(\overline{y} - z_Q), \quad \varphi_\Sigma := b_\Sigma(\overline{y}_\Gamma - z_\Sigma), \quad \varphi_\Omega := b_\Omega(\overline{y}(T) - z_\Omega), \quad \varphi_\Gamma := b_\Gamma(\overline{y}_\Gamma(T) - z_\Gamma).$$

Here, the well-posedness result follows.

Theorem 2.7. *Let $\overline{u}_\Gamma$ be an optimal control with the corresponding optimal state $(\overline{y}, \overline{y}_\Gamma)$. Moreover, let us postulate that φ_Ω and φ_Γ satisfy the following compatibility condition: there exists a couple $(\Phi, \varphi_\Gamma) \in \mathcal{V}_\Omega$ such that $\varphi_\Omega = \mathcal{N}(\Phi) + (\varphi_\Omega)^\Omega$. Then, the adjoint system* (2.45)–(2.47) *admits a unique solution (p, q, q_Γ) satisfying the regularity requirements* (2.42)–(2.43).

Let us underline that the above result is new with respect to [18], where just the existence, in a very weak setting, was proved. Here, the complete well-posedness of the adjoint system is now achievable under the enforced assumptions on the potential setting. Moreover, notice that the unique solution to (2.45)–(2.47) enjoys the strong regularity (2.42)–(2.43). Then, once that the adjoint variables are at our disposal, we are in a position to eliminate ξ and ξ_Γ from (2.41), thus leading to the following optimality condition.

Theorem 2.8. *Let $\overline{u}_\Gamma$ be an optimal control, $(\overline{y}, \overline{y}_\Gamma)$ be the corresponding optimal state, and (p, q, q_Γ) be the associated solution to the adjoint system* (2.43)*–*(2.46)*. Then, the first-order necessary condition for optimality is characterized by the following variational inequality*

$$\int_\Sigma (q_\Gamma + b_0 \overline{u}_\Gamma)(v_\Gamma - \overline{u}_\Gamma) \geq 0 \quad \textit{for every } v_\Gamma \in \mathcal{U}_{\rm ad}. \tag{2.48}$$

Moreover, whenever $b_0 > 0$, it turns out that

$$\overline{u}_\Gamma \ \textit{ is the orthogonal projection of } -q_\Gamma/b_0 \ \textit{ on } \ \mathcal{U}_{\rm ad} \tag{2.49}$$

with respect to the standard inner product of $L^2(\Sigma)$.

Remark 2.9. Of course, the condition (2.48) also entails that the element $-(q_\Gamma + b_0 \overline{u}_\Gamma)$ belongs to the *normal cone* of the closed and convex set $\mathcal{U}_{\rm ad}$ (defined in (1.13)) at $\overline{u}_\Gamma$ in the framework of the Hilbert space $L^2(\Sigma)$. Owing to the structure of the control-box $\mathcal{U}_{\rm ad}$, if $b_0 > 0$ then the projection $\overline{u}_\Gamma$ in (2.49) is the one among the elements $u_\Gamma \in H^1(0,T; L^2(\Gamma)) \cap L^\infty(\Sigma)$ satisfying the two constraints

$$u_{\Gamma,\min} \leq u_\Gamma \leq u_{\Gamma,\max} \quad \text{a.e. on } \Sigma, \qquad \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \leq M_0$$

that is closest to $-q_\Gamma/b_0$ in the sense of the norm in $L^2(\Sigma)$. In particular, if the function z_Γ defined by

$$z_\Gamma(x,t) = \max\bigl\{u_{\Gamma,\min}(x,t), \min\{u_{\Gamma,\max}(x,t), -q_\Gamma(x,t)/b_0\}\bigr\}, \quad (x,t) \in \Sigma,$$

belongs to $H^1(0,T; L^2(\Gamma))$ and its time derivative fulfills $\|\partial_t z_\Gamma\|_{L^2(\Sigma)} \leq M_0$, then we necessarily have that $\overline{u}_\Gamma = z_\Gamma$ a.e. on Σ.

In the remainder, we introduce further notation and recall some well-known inequalities and general facts which will be useful later on. First of all, we often owe to the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \tag{2.50}$$

Furthermore, we account for the Poincaré inequality

$$\|v\|_V^2 \leq C_\Omega \bigl(\|\nabla v\|_H^2 + |v^\Omega|^2\bigr) \quad \text{for every } v \in V, \tag{2.51}$$

where C_Ω depends only on Ω. Furthermore, we point out the following inequality, to which we will refer to as compactness inequality (see, e.g., [30, Lem. 5.1, p. 58]): for every $\delta > 0$ there exists $c_\delta > 0$ such that

$$\|v\|_H^2 \leq \delta \|\nabla v\|_H^2 + c_\delta \|v\|_{V^*}^2 \quad \text{for every } v \in V, \tag{2.52}$$

where the constant c_δ depends only on δ and Ω.

Lastly, let us point out a convention we use in the whole paper as far as the constants are concerned. We agree that the small-case symbol c stands for different constants depending only on the final time T, on Ω, the shape of the nonlinearities and the norms of functions involved in the assumptions of our statements. For this reason, its meaning might change from line to line and even in the same chain of calculations. Conversely, the capital letters are devoted to denote precise constants which we eventually will refer to.

3 Existence of an optimal control

From this section on, we will start with the proofs of the stated results. Here, we aim to prove the existence of optimal control. Before moving on, let us briefly remark that Theorem 2.2 is slightly stronger with respect to the result of [19], since by (2.22) we require additional space regularity for the variable w. As a matter of fact, it suffices to combine the original result with a comparison argument to realize that $w \in L^2(0,T;H^3(\Omega))$ as well.

Proof of Theorem 2.2. Since the proof is the same as in [19], we can afford to be sketchy by just pointing out some highlights. From [19], it follows that there exists a positive constant c such that

$$\begin{aligned} &\|y\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))} + \|y_\Gamma\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))} \\ &\quad + \|w\|_{L^\infty(0,T;V)} \leq c. \end{aligned}$$

On the other hand, owing to the above estimate, by comparison in equation (1.7), we infer that $\Delta w \in L^2(0,T;V)$, and the classical elliptic regularity theory directly implies that $w \in L^2(0,T;H^3(\Omega))$. $\square$

Proof of Theorem 2.3. We proceed by employing the direct method. First, let us pick a minimizing sequence $\{u_{\Gamma,n}\}_n$ for the cost functional $\mathcal{J}$ and, for every n, let us denote by $(y_n, y_{\Gamma,n}, w_n)$ the corresponding solution to (2.19)–(2.25). Since, for every n, $u_{\Gamma,n}$ belongs to $\mathcal{U}_{\rm ad}$ and the triplet $(y_n, y_{\Gamma,n}, w_n)$ solves the state system, the bounds (2.28)–(2.29) are in force. Thus, for every n, we infer that

$$r_- \leq y_n \leq r_+ \quad \text{a.e. in } Q, \quad r_- \leq y_{\Gamma,n} \leq r_+ \quad \text{a.e. on } \Sigma, \tag{3.1}$$

for some r_-, r_+ satisfying $-\infty < r_- \leq r_+ < +\infty$. It is now a standard matter to show that, accounting for weak and weak-star compactness arguments (see, e.g., [40, Sect. 8, Cor. 4, p. 85]), up to a subsequence, the following convergences are verified

$$\begin{aligned} u_{\Gamma,n} \to \overline{u}_\Gamma \quad & \text{weakly star in } L^\infty(\Sigma) \cap H^1(0,T;H), \\ y_n \to \overline{y} \quad & \text{weakly star in } W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \\ & \quad \text{and strongly in } C^0([0,T];V), \\ y_{\Gamma,n} \to \overline{y}_\Gamma \quad & \text{weakly star in } W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)) \\ & \quad \text{and strongly in } C^0([0,T];V_\Gamma), \\ w_n \to \overline{w} \quad & \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)). \end{aligned}$$

In addition, since $\mathcal{U}_{\rm ad}$ is closed it follows that $\overline{u}_\Gamma \in \mathcal{U}_{\rm ad}$ and, from the strong convergences pointed out above that $\overline{y}(0) = y_0$. Moreover, the strong convergences of y_n and $y_{\Gamma,n}$, combined with the regularity of f and f_Γ, imply that

$$f'(y_n) \to f'(\overline{y}) \quad \text{strongly in } C^0([0,T];H),$$
$$f'_\Gamma(y_{\Gamma,n}) \to f'_\Gamma(\overline{y}_\Gamma) \quad \text{strongly in } C^0([0,T];H_\Gamma).$$

By virtue of all these convergences, we can easily pass to the limit in the integrated variational formulation (2.26)–(2.27) written for $(y_n, y_{\Gamma,n}, w_n)$ and $u_{\Gamma,n}$ to conclude that $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ solves (2.26)–(2.27) with $u_\Gamma := \overline{u}_\Gamma$. Lastly, accounting for the lower weak semicontinuity of $\mathcal{J}$, is straightforward to realize that $(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma)$ is indeed the minimizer we are looking for. □

4 The control-to-state mapping

In this section, we first prove Theorem 2.5, and then show the Fréchet differentiability of the control-to-state operator $\mathcal{S}$ between suitable Banach spaces.

Proof of Theorem 2.5. As sketched above, we would like to invoke [19, Thm. 2.4, p. 978]. Unfortunately, the assumption $\lambda \in W^{1,\infty}(0,T;H)$ fails to be satisfied. On the other hand, due to the regularity of the potential f, along with (2.29), we can easily check that $\partial_t \lambda = f'''(\overline{y})\partial_t \overline{y}$ belongs at least to $L^2(0,T;H)$. Let us claim that this regularity is actually sufficient in order to prove the same result as in [19, Thm. 2.4]. Since it consists of a minor change, let us proceed quite formally, leaving the details to the reader and avoiding to write the explicit dependence on the time variable for convenience. As a starting point let us assume that

$$\begin{aligned}(\xi, \xi_\Gamma) &\in H^1(0,T;\mathcal{G}) \cap L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;H^2(\Omega) \times H^2(\Gamma)),\\ \eta &\in L^2(0,T;V),\end{aligned}$$

which can be easily obtained applying [19, Thms. 2.2 and 2.3]. Moreover, note that the mean value of ξ is zero, thanks to (2.38) and (2.40). Then, let us formally differentiate (2.38)–(2.39) with respect to time and integrate over $(0,t)$ to get

$$\begin{aligned}&\int_0^t \langle \partial_{tt}\xi, v\rangle + \int_{Q_t} \nabla \partial_t \eta \cdot \nabla v = 0 \quad \text{for every } v \in V,\\ &\int_{Q_t} \partial_t \eta v = \int_{\Sigma_t} \partial_{tt}\xi_\Gamma \, v_\Gamma + \int_{Q_t} \nabla \partial_t \xi \cdot \nabla v + \int_{\Sigma_t} \nabla_\Gamma \partial_t \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_{Q_t} \lambda \, \partial_t \xi \, v\\ &\quad + \int_{Q_t} \partial_t \lambda \, \xi \, v + \int_{\Sigma_t} \lambda_\Gamma \partial_t \xi_\Gamma v_\Gamma + \int_{\Sigma_t} \partial_t \lambda_\Gamma \xi_\Gamma v_\Gamma - \int_{\Sigma_t} \partial_t h_\Gamma v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}.\end{aligned}$$

Next, we test the former by $\mathcal{N}(\partial_t \xi)$ and the latter by $-(\partial_t \xi, \partial_t \xi_\Gamma)$, add the resulting equalities and integrate by parts to obtain, after some simplifications, that

$$\begin{aligned}&\frac{1}{2}\|\partial_t \xi(t)\|_*^2 + \frac{1}{2}\int_\Gamma |\partial_t \xi(t)|^2 + \int_{Q_t} |\nabla \partial_t \xi|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t \xi_\Gamma|^2\\ &\quad = \frac{1}{2}\|\partial_t \xi(0)\|_*^2 + \frac{1}{2}\int_\Gamma |\partial_t \xi(0)|^2 - \int_{Q_t} \lambda |\partial_t \xi|^2 - \int_{\Sigma_t} \lambda_\Gamma |\partial_t \xi_\Gamma|^2\\ &\qquad - \int_{Q_t} \partial_t \lambda \, \xi \, \partial_t \xi - \int_{\Sigma_t} \partial_t \lambda_\Gamma \, \xi_\Gamma \, \partial_t \xi_\Gamma + \int_{\Sigma_t} \partial_t h_\Gamma \partial_t \xi_\Gamma. \end{aligned} \tag{4.1}$$

Let us denote the terms on the right-hand side by $I_1, ..., I_7$, in this order. Owing to the Young inequality and to the boundedness of λ_Γ, we easily handle the boundary terms as follows

$$|I_4| + |I_7| \le c \int_{\Sigma_t} |\xi_\Gamma|^2 + c \int_{\Sigma_t} |\partial_t \xi_\Gamma|^2 + c \int_{\Sigma_t} |\partial_t h_\Gamma|^2,$$

where let us remark that ξ_Γ has been already estimated in $H^1(0,T;H_\Gamma)$. Moreover, owing to the inequality (2.52) and to the Poincaré inequality (2.51), we have that

$$|I_3| \le c \int_{Q_t} |\partial_t \xi|^2 \le \frac{1}{4} \int_{Q_t} |\nabla \partial_t \xi|^2 + c \int_0^t \|\partial_t \xi\|_*^2.$$

Finally, using the Hölder inequality and (2.52), we obtain that

$$\begin{aligned}
|I_5| &\le c \int_0^t \|\partial_t \lambda\|_2 \|\xi\|_4 \|\partial_t \xi\|_4 \le c \int_0^t \|\partial_t \lambda\|_H \|\xi\|_V \|\partial_t \xi\|_V \\
&\le \frac{1}{8} \int_0^t \|\partial_t \xi\|_V^2 + c \int_0^t \|\partial_t \lambda\|_H^2 \|\xi\|_V^2 \le \frac{1}{8} \int_{Q_t} |\partial_t \xi|^2 + \frac{1}{8} \int_{Q_t} |\nabla \partial_t \xi|^2 + c \int_{Q_t} |\partial_t \lambda|^2 \\
&\le \frac{1}{4} \int_{Q_t} |\nabla \partial_t \xi|^2 + c \int_0^t \|\partial_t \xi\|_*^2 + c,
\end{aligned}$$

thanks to the Sobolev embedding $V \subset L^4(\Omega)$ and the fact that ξ has already been estimated in $L^\infty(0,T;V)$. In a similar way, we can deal with I_6 by using the Hölder and Young inequalities. In fact, we have that

$$\begin{aligned}
|I_6| &\le c \int_0^t \|\partial_t \lambda_\Gamma\|_{L^2(\Gamma)} \|\xi_\Gamma\|_{L^4(\Gamma)} \|\partial_t \xi_\Gamma\|_{L^4(\Gamma)} \le c \int_0^t \|\partial_t \lambda_\Gamma\|_{H_\Gamma} \|\xi_\Gamma\|_{V_\Gamma} \|\partial_t \xi_\Gamma\|_{V_\Gamma} \\
&\le \frac{1}{2} \int_0^t \|\partial_t \xi_\Gamma\|_{V_\Gamma}^2 + c \int_0^t \|\xi_\Gamma\|_{V_\Gamma}^2 \|\partial_t \lambda_\Gamma\|_{H_\Gamma}^2 \\
&\le \frac{1}{2} \int_{\Sigma_t} |\nabla \partial_t \xi_\Gamma|^2 + \frac{1}{2} \int_{\Sigma_t} |\partial_t \xi_\Gamma|^2 + c \|\xi_\Gamma\|_{L^\infty(0,T;V_\Gamma)}^2 \|\partial_t \lambda_\Gamma\|_{L^\infty(0,T;H_\Gamma)}^2 \\
&\le \frac{1}{2} \int_{\Sigma_t} |\nabla \partial_t \xi_\Gamma|^2 + c,
\end{aligned}$$

where we apply the Sobolev embedding $V_\Gamma \subset L^4(\Gamma)$, the fact that $\partial_t \lambda_\Gamma \in L^\infty(0,T;H_\Gamma)$, and that ξ_Γ has been already estimated in $H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma)$. Therefore, it suffices to show that I_1 and I_2 remained bounded. In this regards, we evaluate equations (2.38) and (2.39) at $t = 0$. Then, we test them by $\mathcal{N}(\partial_t \xi(0))$ and $-\partial_t \xi(0)$, add the resulting equalities and rearrange the terms to obtain that

$$\|\partial_t \xi(0)\|_*^2 + \|\partial_t \xi_\Gamma(0)\|_{L^2(\Gamma)}^2 = \int_\Gamma h_\Gamma(0) \partial_t \xi_\Gamma(0),$$

where the initial condition $\xi(0) = 0$ has been exploited. Hence, we use the Young inequality to handle the term on the right-hand side and infer that

$$\|\partial_t \xi(0)\|_*^2 + \frac{1}{2} \|\partial_t \xi_\Gamma(0)\|_{L^2(\Gamma)}^2 \le \frac{1}{2} \|h_\Gamma(0)\|_{H_\Gamma}^2 \le c \|h_\Gamma\|_{C^0([0,T];H_\Gamma)}^2 \le c \|h_\Gamma\|_{H^1(0,T;H_\Gamma)}^2 \le c.$$

Then, recalling (4.1) and collecting the previous estimates, we conclude the proof by applying the Gronwall lemma. ☐

We will see that, in order to directly check the definition of Fréchet differentiability for $\mathcal{S}$, some stronger continuous dependence results with respect to (2.30) need to be shown. Therefore, this is the task of the following lemmas. The first one is somehow the corresponding non-viscous version of [17, Lem. 4.1, p. 207].

Lemma 4.1. *Let $u_{\Gamma,i} \in \mathcal{U}$ for $i = 1, 2$ and let $(y_i, y_{\Gamma,i}, w_i)$ be the corresponding solutions to* (2.19)*–*(2.25)*. Then, it follows that*

$$\|(y_1, y_{\Gamma,1}) - (y_2, y_{\Gamma,2})\|_{\mathcal{Y}} \le C_3 \|u_{\Gamma,1} - u_{\Gamma,2}\|_{L^2(0,T;H_\Gamma)}, \tag{4.2}$$

for a positive constant C_3 that may depend on Ω, T, the shape of the nonlinearities f and f_Γ, and on the initial datum y_0.

Proof. To begin with, let us fix for convenience the notation

$$u_\Gamma := u_{\Gamma,1} - u_{\Gamma,2}, \quad y := y_1 - y_2, \quad y_\Gamma := y_{\Gamma,1} - y_{\Gamma,2}, \quad \text{and} \quad w := w_1 - w_2. \tag{4.3}$$

Then, we write the system (2.23)–(2.25) for both the solutions $(y_i, y_{\Gamma,i}, w_i)$ for $i = 1, 2$, and take the difference to obtain, for a.a. $t \in (0, T)$, that

$$\langle \partial_t y(t), v \rangle + \int_\Omega \nabla w(t) \cdot \nabla v = 0 \quad \text{for every } v \in V, \tag{4.4}$$

$$\begin{aligned}
&\int_\Omega w(t)\, v = \int_\Gamma \partial_t y_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla y(t) \cdot \nabla v \\
&\quad + \int_\Gamma \nabla_\Gamma y_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma + \int_\Omega \big(f'(y_1(t)) - f'(y_2(t))\big)\, v \\
&\quad + \int_\Gamma \big(f_\Gamma'(y_{\Gamma,1}(t)) - f_\Gamma'(y_{\Gamma,2}(t)) - u_\Gamma(t)\big)\, v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}
\end{aligned} \tag{4.5}$$

and $y(0) = 0$. Moreover, we point out that $\partial_t y$ has zero mean value since (1.4) holds for both $\partial_t y_1$ and $\partial_t y_2$ so that $\mathcal{N}(\partial_t y)$ can be considered as a test function. So, we subtract to both sides of (4.5) the terms $\int_\Omega y(t)\, v$ and $\int_\Gamma y_\Gamma(t)\, v_\Gamma$, write the above equations at the time s, test (4.4) by $\mathcal{N}(\partial_t y(s))$, the new (4.5) by $-\partial_t(y, y_\Gamma)(s)$, add the resulting equalities and integrate over $(0, t)$ for an arbitrary $t \in (0, T)$. We obtain that

$$\begin{aligned}
&\int_0^t \langle \partial_t y, \mathcal{N}(\partial_t y) \rangle + \int_{Q_t} \nabla w \cdot \nabla \mathcal{N}(\partial_t y) - \int_0^t \langle \partial_t y, w \rangle + \int_{\Sigma_t} |\partial_t y_\Gamma|^2 \\
&\quad + \frac{1}{2}\|y(t)\|_V^2 + \frac{1}{2}\|y_\Gamma(t)\|_{V_\Gamma}^2 = -\int_0^t \langle \partial_t y, f'(y_1) - f'(y_2) - y \rangle \\
&\quad - \int_{\Sigma_t} \big(f_\Gamma'(y_{\Gamma,1}) - f_\Gamma'(y_{\Gamma,2}) - y_\Gamma\big) \partial_t y_\Gamma + \int_{\Sigma_t} u_\Gamma\, \partial_t y_\Gamma,
\end{aligned} \tag{4.6}$$

where we also invoke the fact that $y(0) = 0$, $y_\Gamma(0) = 0$ since y_1 and y_2 have the same initial value y_0. The first three integrals of the above equality can be treated with the help of (2.14) and (2.16) as follows

$$\int_0^t \langle \partial_t y, \mathcal{N}(\partial_t y) \rangle + \int_{Q_t} \nabla w \cdot \nabla \mathcal{N}(\partial_t y) - \int_0^t \langle \partial_t y, w \rangle = \int_0^t \|\partial_t y\|_*^2 \ge 0\,.$$

Furthermore, all the other contributions on the left-hand side are nonnegative, so we are reduced to control the integrals on the right-hand side. On the other hand, both y_1 and

y_2, as solutions to (1.7)–(1.11), satisfy (2.28) and (2.29). Using the Young inequality, we estimate the first term of the right-hand side by

$$
\begin{aligned}
-\int_0^t \langle \partial_t y, f'(y_1) - f'(y_2) - y \rangle &\leq \int_0^t \|\partial_t y\|_* \|f'(y_1) - f'(y_2) - y\|_V \\
&\leq \frac{1}{2}\int_0^t \|\partial_t y\|_*^2 + \frac{1}{2}\int_0^t \|f'(y_1) - f'(y_2) - y\|_V^2. \qquad (4.7)
\end{aligned}
$$

By invoking the Lipschitz continuity of f' and f'', and the Sobolev embedding $V \subset L^4(\Omega)$, we are able to bound the last term of the previous estimate as follows

$$
\begin{aligned}
\|f'(y_1) - f'(y_2) - y\|_V^2 &\leq c\|y\|_V^2 + c\|f'(y_1) - f'(y_2)\|_V^2 \\
&\leq c\|y\|_V^2 + c\|f'(y_1) - f'(y_2)\|_H^2 + c\|\nabla\big(f'(y_1) - f'(y_2)\big)\|_H^2 \\
&\leq c\|y\|_V^2 + c\|y\|_H^2 + \big(\|f''(y_1)\nabla y\|_H^2 + \|\big(f''(y_1) - f''(y_2)\big)\nabla y_2\|_H^2\big) \\
&\leq c\|y\|_V^2 + c\|\nabla y\|_H^2 + c\int_\Omega |y|^2 |\nabla y_2|^2 \\
&\leq c\|y\|_V^2 + c\|y\|_4^2 \|\nabla y_2\|_4^2 \leq c\|y\|_V^2 + c\|y\|_V^2 \|y_2\|_{H^2(\Omega)}^2,
\end{aligned}
$$

where in the last inequality we invoke the fact that, as a solution, y_2 satisfies (2.28) so that y_2 is bounded in $L^\infty(0,T;H^2(\Omega))$. Summing up, the estimate

$$
-\int_0^t \langle \partial_t y, f'(y_1) - f'(y_2) - y \rangle \leq \frac{1}{2}\int_0^t \|\partial_t y\|_*^2 + c\int_0^t \|y\|_V^2
$$

has been shown. The boundary integrals can be easily handled owing to (2.50) and the Lipschitz continuity of f_Γ. Indeed, we have that

$$
-\int_{\Sigma_t} \big(f'_\Gamma(y_{\Gamma,1}) - f'_\Gamma(y_{\Gamma,2}) - y_\Gamma\big)\partial_t y_\Gamma \leq \frac{1}{4}\int_{\Sigma_t} |\partial_t y_\Gamma|^2 + c\int_{\Sigma_t} |y_\Gamma|^2,
$$

and

$$
\int_{\Sigma_t} u_\Gamma\, \partial_t y_\Gamma \leq \frac{1}{4}\int_{\Sigma_t} |\partial_t y_\Gamma|^2 + c\int_{\Sigma_t} |u_\Gamma|^2,
$$

respectively. Lastly, upon collecting all the previous estimates, we realize that

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \|\partial_t y\|_*^2 + \frac{1}{2}\int_{\Sigma_t} |\partial_t y_\Gamma|^2 + \frac{1}{2}\|y(t)\|_V^2 + \frac{1}{2}\|y_\Gamma(t)\|_{V_\Gamma}^2 \\
&\quad \leq c\int_0^t \|y\|_V^2 + c\int_{\Sigma_t} |y_\Gamma|^2 + c\int_{\Sigma_t} |u_\Gamma|^2,
\end{aligned}
$$

whence the standard Gronwall lemma yields the stability inequality we are looking for. □

Unfortunately, we will see that in order to prove the Fréchet differentiability of $\mathcal{S}$, the above lemma turns out to be insufficient. Then, in the lines below we present an improvement. Notice that the following result is a novelty in comparison to [17]. On the other hand, it turns out to be necessary in order to handle the control problem we are dealing with.

Lemma 4.2. *Let $u_{\Gamma,i} \in \mathcal{U}$ for $i = 1, 2$ and let $(y_i, y_{\Gamma,i}, w_i)$ be the corresponding solutions to* (1.7)–(1.11)*. Then, there exists a positive constant C_4 such that*

$$\begin{aligned} &\|(y_1, y_{\Gamma,1}) - (y_2, y_{\Gamma,2})\|_{W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V}) \cap L^\infty(0,T;H^2(\Omega)\times H^2(\Gamma))} \\ &\quad \leq C_4 \|u_{\Gamma,1} - u_{\Gamma,2}\|_{H^1(0,T;H_\Gamma)}, \end{aligned} \tag{4.8}$$

where C_4 is a positive constant which depends only on Ω, T, the shape of the nonlinearities f and f_Γ, the initial datum y_0.

Proof. In what follows, to keep the proof as easy as possible, we proceed formally. The justification can be carried out rigorously, e.g., within a time-discretization scheme. Then, providing to show some estimates for the differences, one has to pass to the limit in suitable topologies.

To begin with, we write the problem (2.23)–(2.25) for both the solutions $(y_i, y_{\Gamma,i}, w_i)$, $i = 1, 2$, take the difference and use the notation set by (4.3). Then, we differentiate the equations with respect to the time variable to obtain that, for a.a. $t \in (0, T)$, the following are satisfied

$$\langle \partial_{tt} y(t), v\rangle + \int_\Omega \nabla \partial_t w(t) \cdot \nabla v = 0 \quad \text{for every } v \in V, \tag{4.9}$$

$$\begin{aligned} &\int_\Omega \partial_t w(t)\, v = \int_\Gamma \partial_{tt} y_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla \partial_t y(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma \partial_t y_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \int_\Omega f''(y_1(t)) \partial_t y(t)\, v + \int_\Omega \big(f''(y_1(t)) - f''(y_2(t))\big) \partial_t y_2(t)\, v \\ &\quad + \int_\Gamma f''_\Gamma(y_{\Gamma,1}(t)) \partial_t y_\Gamma(t)\, v_\Gamma + \int_\Gamma \big(f''_\Gamma(y_{\Gamma,1}(t)) - f''_\Gamma(y_{\Gamma,2}(t))\big) \partial_t y_{\Gamma,2}(t)\, v_\Gamma \\ &\quad - \int_\Gamma \partial_t u_\Gamma(t)\, v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \end{aligned} \tag{4.10}$$

Again, $\partial_t y$ possesses zero mean value. Taking into account the previous equations at the time s, testing (4.9) by $\mathcal{N}(\partial_t y(s))$, (4.10) by $-\partial_t (y, y_\Gamma)(s)$, integrating over $(0, t)$ with respect to s, and adding the resulting equations leads to

$$\begin{aligned} &\int_0^t \langle \partial_t(\partial_t y), \mathcal{N}(\partial_t y)\rangle + \int_{Q_t} \nabla(\partial_t w) \cdot \nabla \mathcal{N}(\partial_t y) - \int_{Q_t} \partial_t w\, \partial_t y + \int_{\Sigma_t} \partial_t(\partial_t y_\Gamma) \partial_t y_\Gamma \\ &\quad + \int_{Q_t} |\nabla \partial_t y|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t y_\Gamma|^2 = -\int_{Q_t} \big(f''(y_1) - f''(y_2)\big) \partial_t y_2\, \partial_t y - \int_{Q_t} f''(y_1) |\partial_t y|^2 \\ &\quad - \int_{\Sigma_t} \big(f''_\Gamma(y_{\Gamma,1}) - f''_\Gamma(y_{\Gamma,2})\big) \partial_t y_{\Gamma,2}\, \partial_t y_\Gamma - \int_{\Sigma_t} f''_\Gamma(y_{\Gamma,1}) |\partial_t y_\Gamma|^2 + \int_{\Sigma_t} \partial_t u_\Gamma\, \partial_t y_\Gamma, \end{aligned} \tag{4.11}$$

where the first three terms can be treated, using (2.14) and (2.16), as follows

$$\int_0^t \langle \partial_t(\partial_t y), \mathcal{N}(\partial_t y)\rangle + \int_{Q_t} \nabla(\partial_t w) \nabla \mathcal{N}(\partial_t y) - \int_{Q_t} \partial_t w \partial_t y = \frac{1}{2} \int_0^t \left(\frac{d}{dt} \|\partial_t y\|_*^2 \right).$$

Integrating by parts and invoking the boundedness and the Lipschitz continuity of f'' and

f''_Γ, we infer that

$$\frac{1}{2}\|\partial_t y(t)\|_*^2 + \frac{1}{2}\int_\Gamma |\partial_t y_\Gamma(t)|^2 + \int_{Q_t} |\nabla \partial_t y|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t y_\Gamma|^2$$
$$\leq \frac{1}{2}\|\partial_t y(0)\|_*^2 + \frac{1}{2}\int_\Gamma |\partial_t y_\Gamma(0)|^2 + c\int_{Q_t} |y||\partial_t y_2||\partial_t y| + c\int_{Q_t} |\partial_t y|^2$$
$$+ c\int_{\Sigma_t} |y_\Gamma||\partial_t y_{\Gamma,2}||\partial_t y_\Gamma| + c\int_{\Sigma_t} |\partial_t y_\Gamma|^2 + c\int_{\Sigma_t} |\partial_t u_\Gamma||\partial_t y_\Gamma|, \tag{4.12}$$

where the terms on the right-hand side are denoted by $I_1, ..., I_7$, in this order. By considering the variational formulation (4.4)–(4.5) and putting $t = 0$, we deduce that

$$\langle \partial_t y(0), v\rangle + \int_\Omega \nabla w(0)\cdot\nabla v = 0 \quad \text{for every } v \in V,$$
$$\int_\Omega w(0)\, v = \int_\Gamma \partial_t y_\Gamma(0)\, v_\Gamma - \int_\Gamma u_\Gamma(0) v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}.$$

Then, we test the former by $\mathcal{N}(\partial_t y(0))$, the latter by $-\partial_t(y, y_\Gamma)(0)$, and add the resulting equalities to obtain that

$$\int_\Omega \partial_t y(0)\mathcal{N}\partial_t y(0) + \int_\Omega \nabla w(0)\cdot\nabla\mathcal{N}(\partial_t y(0)) - \int_\Omega w(0)\partial_t y(0)$$
$$+ \int_\Gamma \partial_t y_\Gamma(0)\partial_t y_\Gamma(0) = \int_\Gamma u_\Gamma(0)\partial_t y_\Gamma(0). \tag{4.13}$$

Note that the second and third terms cancel out. Moreover, owing to the Young inequality we can estimate the integral on the right-hand side realizing that

$$\int_\Omega \partial_t y(0)\mathcal{N}\partial_t y(0) + \int_\Gamma |\partial_t y_\Gamma(0)|^2 = \int_\Gamma u_\Gamma(0)\partial_t y_\Gamma(0) \leq \frac{1}{2}\int_\Gamma |\partial_t y_\Gamma(0)|^2 + \frac{1}{2}\int_\Gamma |u_\Gamma(0)|^2.$$

Rearranging the terms, we deduce that

$$|I_1| + |I_2| \leq \frac{1}{2}\int_\Gamma |u_\Gamma(0)|^2 \leq \frac{1}{2}\|u_\Gamma\|^2_{C^0([0,T];H_\Gamma)} \leq c\|u_\Gamma\|^2_{H^1(0,T;H_\Gamma)},$$

where the standard embedding $H^1(0,T;H_\Gamma) \subset C^0([0,T];H_\Gamma)$ is also taken into account. Coming back to inequality (4.12), we continue the analysis focusing on the third integral, which can be managed as follows

$$|I_3| \leq c\int_{Q_t} |y||\partial_t y_2||\partial_t y| \leq c\int_0^t \|y\|_4\|\partial_t y_2\|_4\|\partial_t y\|_2 \leq c\int_0^t \|y\|_V\|\partial_t y_2\|_V\|\nabla\partial_t y\|_H$$
$$\leq \frac{1}{4}\int_0^t \|\nabla\partial_t y\|_H^2 + c\|y\|^2_{L^\infty(0,T;V)}\|\partial_t y_2\|^2_{L^2(0,T;V)} \leq \frac{1}{4}\int_0^t \|\nabla\partial_t y\|_H^2 + c\|u_\Gamma\|^2_{L^2(\Sigma)},$$

where we applied the Hölder, Poincaré and Young inequalities, the Sobolev embedding of $V \subset L^4(\Omega)$, and at the end also the stability estimate (4.2) along with (2.28) for y_2. Moreover, combining the compactness inequality (2.52) with the Poincaré inequality (2.51) and (4.2), we get that

$$|I_4| \leq \frac{1}{4}\int_0^t \|\nabla\partial_t y\|_H^2 + c\int_0^t \|\partial_t y\|_*^2 \leq \frac{1}{4}\int_0^t \|\nabla\partial_t y\|_H^2 + c\|u_\Gamma\|^2_{L^2(\Sigma)}. \tag{4.14}$$

The boundary terms can be dealt in a similar way as follows

$$
\begin{aligned}
|I_5| &\leq c \int_0^t \|y_\Gamma\|_4 \|\partial_t y_{\Gamma,2}\|_4 \|\partial_t y_\Gamma\|_2 \leq c \int_0^t \|y_\Gamma\|_{V_\Gamma} \|\partial_t y_{\Gamma,2}\|_{V_\Gamma} \|\partial_t y_\Gamma\|_{H_\Gamma} \\
&\leq c \int_0^t \|\partial_t y_\Gamma\|_{H_\Gamma}^2 + c\|\partial_t y_{\Gamma,2}\|_{L^2(0,T;V_\Gamma)}^2 \|y_\Gamma\|_{L^\infty(0,T;V_\Gamma)}^2 \\
&\leq c \int_0^t \|\partial_t y_\Gamma\|_{H_\Gamma}^2 + c\|u_\Gamma\|_{L^2(\Sigma)}^2, \qquad (4.15)
\end{aligned}
$$

where the fact that y_2 is a solution to system (1.7)–(1.11) and the inequality (4.2) turn out to be fundamental. Finally, using (4.2) once more, we infer that

$$
|I_7| \leq \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \|\partial_t y_\Gamma\|_{L^2(\Sigma)} \leq c\|\partial_t u_\Gamma\|_{L^2(\Sigma)} \|u_\Gamma\|_{L^2(\Sigma)} \leq c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}^2. \qquad (4.16)
$$

Then, upon collecting the above estimates, we rearrange (4.12) to realize that

$$
\frac{1}{2}\|\partial_t y(t)\|_*^2 + \frac{1}{2}\int_\Gamma |\partial_t y_\Gamma(t)|^2 + \frac{1}{2}\int_{Q_t} |\nabla \partial_t y|^2 + \frac{1}{2}\int_{\Sigma_t} |\nabla_\Gamma \partial_t y_\Gamma|^2 \leq c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}^2,
$$

which allows us to conclude that

$$
\|(y_1, y_{\Gamma,1}) - (y_2, y_{\Gamma,2})\|_{W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;\mathcal{V})} \leq c\|u_{\Gamma,1} - u_{\Gamma,2}\|_{H^1(0,T;H_\Gamma)}. \qquad (4.17)
$$

Now, it remains to show that

$$
\|y\|_{L^\infty(0,T;H^2(\Omega))} + \|y_\Gamma\|_{L^\infty(0,T;H^2(\Gamma))} \leq c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}
$$

is satisfied for some positive constant c. To this aim, we test (4.4) by $w(t) - (w(t))^\Omega$ and integrate over Ω to get

$$
\begin{aligned}
\int_\Omega \left|\nabla w(t)\right|^2 &= -\langle\, \partial_t y(t), w(t) - (w(t))^\Omega \rangle \leq c\|\partial_t y(t)\|_* \|w(t) - (w(t))^\Omega\|_V \\
&\leq \frac{1}{2}\int_\Omega \left|\nabla w(t)\right|^2 + c\|\partial_t y(t)\|_*^2,
\end{aligned}
$$

thanks to the Hölder, Young and Poincaré inequalities. Hence, applying (4.17) we find out that

$$
\|\nabla w\|_{L^\infty(0,T;H)} \leq c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}.
$$

Next, we would like to recover the full norm of w in $L^\infty(0,T;V)$. In this direction, we will show a bound for its mean value, and then apply the Poincaré inequality (2.51) to conclude. Thus, we test equation (4.5) by 1 and integrate over Ω to obtain that

$$
\int_\Omega w(t) = \int_\Gamma \partial_t y_\Gamma(t) + \int_\Omega \big(f'(y_1(t)) - f'(y_2(t))\big) + \int_\Gamma \big(f_\Gamma'(y_{\Gamma,1}(t)) - f_\Gamma'(y_{\Gamma,2}(t)) - u_\Gamma(t)\big),
$$

from which, owing to (2.51), we deduce that

$$
\|w\|_{L^\infty(0,T;V)} \leq c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}. \qquad (4.18)
$$

In order to apply a comparison principle, we consider the variational formulation (4.4)–(4.5), and integrate by parts so to derive the corresponding strong formulation, that holds at least in a distributional sense. It reads as follows

$$\partial_t y - \Delta w = 0 \quad \text{in } Q, \tag{4.19}$$
$$w = -\Delta y + f'(y_1) - f'(y_2) \quad \text{in } Q. \tag{4.20}$$

Then, comparison in (4.20) yields that

$$\|\Delta y\|_{L^\infty(0,T;H)} \le \|w\|_{L^\infty(0,T;H)} + c\|y\|_{L^\infty(0,T;H)} \le c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)},$$

accounting for the previous estimates, along with the regularity of f. Next, (4.17) and the regularity theory for elliptic equation (see, e.g., [32, Thms. 7.3 and 7.4, pp. 187-188] or [1, Thm. 3.2, p. 1.79, and Thm. 2.27, p. 1.64]) give us

$$\|y\|_{L^\infty(0,T;H^{3/2}(\Omega))} + \|\partial_n y\|_{L^\infty(0,T;H_\Gamma)} \le c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)},$$

which allows us to write the boundary conditions in the following form

$$\partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + f'_\Gamma(y_{\Gamma,1}) - f'_\Gamma(y_{\Gamma,2}) = u_\Gamma \quad \text{on } \Sigma, \tag{4.21}$$
$$\partial_n w = 0 \quad \text{on } \Sigma. \tag{4.22}$$

Arguing in a similar manner, we can use a comparison principle in the boundary equation (4.21) to infer that

$$\|\Delta_\Gamma y_\Gamma\|_{L^\infty(0,T;H_\Gamma)} \le c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}.$$

The boundary version of the regularity results for elliptic equations entails that

$$\|y_\Gamma\|_{L^\infty(0,T;H^2(\Gamma))} \le c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)},$$

which in turn, together with (4.17), implies that

$$\|y\|_{L^\infty(0,T;H^2(\Omega))} \le c\|u_\Gamma\|_{H^1(0,T;H_\Gamma)}. \tag{4.23}$$

Then, (4.8) is completely proved. □

With these stability results at disposal, we are now in a position to show the Fréchet differentiability of $\mathcal{S}$, that is, to check Theorem 2.6. Let us also point out that, owing to the different approaches employed in [18], the Fréchet differentiability of the control-to-state operator was not analyzed there.

Proof of Theorem 2.6. For the sake of simplicity, we fix $u_\Gamma \in \mathcal{U}$ (instead of $\overline{u}_\Gamma$ used in the statement). Then, since $\mathcal{U}$ is open, provided we take $h_\Gamma \in \mathcal{X}$ sufficiently small, we also have that $u_\Gamma + h_\Gamma \in \mathcal{U}$. From now on, we tacitly assume that this is the case. Moreover, for every given $h_\Gamma \in \mathcal{X}$, let us set

$$\begin{gathered}(y, y_\Gamma, w) := \text{solution to system (1.7)–(1.11) corresponding to } u_\Gamma, \\ (y^h, y^h_\Gamma, w^h) := \text{solution to system (1.7)–(1.11) corresponding to } u_\Gamma + h_\Gamma, \\ \text{where } (y, y_\Gamma) = \mathcal{S}(u_\Gamma), \quad \text{and} \quad (y^h, y^h_\Gamma) = \mathcal{S}(u_\Gamma + h_\Gamma).\end{gathered} \tag{4.24}$$

For convenience, we use the following notation

$$\vartheta^h := y^h - y - \xi, \quad \vartheta^h_\Gamma := y^h_\Gamma - y_\Gamma - \xi_\Gamma, \quad \text{and} \quad z^h := w^h - w - \eta\,,$$

where (ξ, ξ_Γ, η) is the solution to the linearized system (2.38)–(2.40) corresponding to h_Γ. We aim to verify the Fréchet differentiability of $\mathcal{S}$ by checking the definition. Namely, we should find a linear operator $[D\mathcal{S}(u_\Gamma)](h_\Gamma)$ such that

$$\mathcal{S}(u_\Gamma + h_\Gamma) = \mathcal{S}(u_\Gamma) + [D\mathcal{S}(u_\Gamma)](h_\Gamma) + o(\|h_\Gamma\|_{\mathcal{X}}) \quad \text{in } \mathcal{Y} \ \text{ as } \ \|h_\Gamma\|_{\mathcal{X}} \to 0.$$

We claim that $[D\mathcal{S}(\overline{u}_\Gamma)](h_\Gamma) = (\xi, \xi_\Gamma)$. Accounting for the above notation, we realize that the above condition is equivalent to show that

$$\frac{\|(\vartheta^h, \vartheta^h_\Gamma)\|_{\mathcal{Y}}}{\|h_\Gamma\|_{\mathcal{X}}} \to 0 \quad \text{as} \quad \|h_\Gamma\|_{\mathcal{X}} \to 0.$$

Furthermore, a sufficient condition consists in proving that

$$\|(\vartheta^h, \vartheta^h_\Gamma)\|_{\mathcal{Y}} \le c\|h_\Gamma\|^2_{L^2(\Sigma)}, \tag{4.25}$$

which is the estimate we are going to check. To this aim, let us consider the variational formulations for the triplets (y^h, y^h_Γ, w^h) and (y, y_Γ, w) satisfying problem (2.23)–(2.25) with data $u_\Gamma + h_\Gamma$ and u_Γ, and the one for (ξ, ξ_Γ, η) that solves the linearized system (2.38)–(2.40). Then, we take the difference to obtain, for a.a. $t \in (0, T)$, that

$$\langle \partial_t \vartheta^h(t), v\rangle + \int_\Omega \nabla z^h(t) \cdot \nabla v = 0 \quad \text{for every } v \in V, \tag{4.26}$$

$$\begin{aligned} \int_\Omega z^h(t)\, v &= \int_\Gamma \partial_t \vartheta^h_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla \vartheta^h(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma \vartheta^h_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma \\ &\quad + \int_\Omega \bigl(f'(y^h(t)) - f'(y(t)) - f''(y(t))\xi(t)\bigr)\, v \\ &\quad + \int_\Gamma \bigl(f'_\Gamma(y^h_\Gamma(t)) - f'_\Gamma(y_\Gamma(t)) - f''_\Gamma(y_\Gamma(t))\xi_\Gamma(t)\bigr)\, v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V} \end{aligned} \tag{4.27}$$

and that $\vartheta^h(0) = 0$. To perform our estimate, we first add to both sides of (4.27) the term $\int_\Omega \vartheta^h(t)\, v$ and the corresponding boundary contribution $\int_\Gamma \vartheta^h_\Gamma(t)\, v_\Gamma$. Then, we test (4.26) and this new (4.27), written at the time s, by $\mathcal{N}(\partial_t \vartheta^h(s))$ and $-\partial_t(\vartheta^h, \vartheta^h_\Gamma)(s)$, respectively. Adding the resulting equalities and integrating over $(0, t)$ for an arbitrary $t \in (0, T)$, leads to

$$\begin{aligned} &\int_0^t \langle \partial_t \vartheta^h, \mathcal{N}(\partial_t \vartheta^h)\rangle + \int_{Q_t} \nabla z^h \cdot \nabla \mathcal{N}(\partial_t \vartheta^h) - \int_0^t \langle \partial_t \vartheta^h, z^h\rangle + \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 \\ &\quad + \frac{1}{2}\|\vartheta^h(t)\|^2_V + \frac{1}{2}\|\vartheta^h_\Gamma(t)\|^2_{V_\Gamma} = -\int_0^t \langle \partial_t \vartheta^h, f'(y^h) - f'(y) - f''(y)\xi - \vartheta^h\rangle \\ &\quad - \int_{\Sigma_t} \bigl(f'_\Gamma(y^h_\Gamma) - f'_\Gamma(y_\Gamma) - f''_\Gamma(y_\Gamma)\xi_\Gamma - \vartheta^h_\Gamma\bigr)\, \partial_t \vartheta^h_\Gamma. \end{aligned} \tag{4.28}$$

As before, the first three integrals on the left-hand side can be easily handled with a cancellation, so that

$$\int_0^t \langle \partial_t \vartheta^h, \mathcal{N}(\partial_t \vartheta^h)\rangle + \int_{Q_t} \nabla z^h \cdot \nabla \mathcal{N}(\partial_t \vartheta^h) - \int_0^t \langle \partial_t \vartheta^h, z^h\rangle = \int_0^t \|\partial_t \vartheta^h\|^2_* \ge 0.$$

Note that the other terms of the left-hand side are nonnegative. Owing to the regularity of the potentials, we can invoke the Taylor formula with integral remainder for the function f' at y. Recalling that $y^h - y = \xi + \vartheta^h$, we have

$$f'(y^h) - f'(y) - f''(y)\xi = f''(y)\,\vartheta^h + \int_0^1 f'''(y + \zeta(y^h - y))(1-\zeta)(y^h - y)^2 d\zeta. \quad (4.29)$$

As the right-hand side of (4.28) is concerned, let us estimate the first integral as follows

$$\begin{aligned} &- \int_0^t \langle \partial_t \vartheta^h, f'(y^h) - f'(y) - f''(y)\xi - \vartheta^h \rangle \\ &\quad \leq \frac{1}{2}\int_0^t \|\partial_t \vartheta^h\|_*^2 + c\int_0^t \Big\| f'(y^h) - f'(y) - f''(y)\xi - \vartheta^h \Big\|_V^2 \\ &\quad \leq \frac{1}{2}\int_0^t \|\partial_t \vartheta^h\|_*^2 + c\int_0^t \|\vartheta^h\|_V^2 \\ &\qquad + c\int_0^t \Big\| f''(y)\,\vartheta^h + \int_0^1 f'''(y + \zeta(y^h - y))(1-\zeta)(y^h - y)^2 d\zeta \Big\|_V^2, \end{aligned} \quad (4.30)$$

thanks to (2.50) and (4.29). Moreover, the last term can be dealt as follows

$$\begin{aligned} &c\int_0^t \Big\| f''(y)\,\vartheta^h + \int_0^1 f'''(y + \zeta(y^h - y))(1-\zeta)(y^h - y)^2 d\zeta \Big\|_V^2 \\ &\quad \leq c\int_0^t \|f''(y)\,\vartheta^h\|_V^2 + c\int_0^t \Big\| \int_0^1 f'''(y + \zeta(y^h - y))(1-\zeta)(y^h - y)^2 d\zeta \Big\|_V^2. \end{aligned}$$

Proceeding with a separate analysis, we obtain that

$$\begin{aligned} \int_0^t \Big\| f''(y)\,\vartheta^h \Big\|_V^2 &= \int_0^t \Big(\|f''(y)\,\vartheta^h\|_H^2 + \|\nabla(f''(y)\,\vartheta^h)\|_H^2 \Big) \\ &\leq \int_0^t \Big(\|f''(y)\|_\infty^2 \|\vartheta^h\|_H^2 + \|f''(y)\|_\infty^2 \|\nabla \vartheta^h\|_H^2 + \|f'''(y)\|_\infty^2 \|\nabla y \cdot \vartheta^h\|_H^2 \Big) \\ &\leq c\int_0^t \|\vartheta^h\|_V^2 + c\int_0^t \|\nabla y\|_4^2 \|\vartheta^h\|_4^2 \leq c\int_0^t \|\vartheta^h\|_V^2 + c\int_0^t \|\nabla y\|_V^2 \|\vartheta^h\|_V^2 \\ &\leq c\int_0^t \|\vartheta^h\|_V^2 + c\|y\|_{L^\infty(0,T;H^2(\Omega))}^2 \int_0^t \|\vartheta^h\|_V^2, \end{aligned}$$

owing to the fact that y, as a solution to (2.19)–(2.25), satisfies (2.28) and thanks to the

Sobolev embedding $V \subset L^4(\Omega)$. Furthermore, the second part can be handled as follows

$$
\begin{aligned}
&\int_0^t \Big\| \int_0^1 f'''(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 d\zeta \Big\|_V^2 \\
&\quad \le \int_0^t \int_0^1 \Big\| f'''(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 \Big\|_V^2 d\zeta \\
&\quad \le \int_0^t \int_0^1 \Big\| f'''(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 \Big\|_H^2 d\zeta \\
&\qquad\quad + \int_0^t \int_0^1 \Big\| \nabla\Big(f'''(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 \Big) \Big\|_H^2 d\zeta \\
&\quad \le \sup_{0\le\zeta\le1} \| f'''(y+\zeta(y^h-y))\|_\infty \int_0^t \Big\| (y^h-y)^2 \Big\|_H^2 \\
&\qquad\quad + \int_0^t \int_0^1 \Big\| f^{(iv)}(y+\zeta(y^h-y)) \nabla(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 \Big\|_H^2 d\zeta \\
&\qquad\quad + \int_0^t \int_0^1 \Big\| f'''(y+\zeta(y^h-y))(1-\zeta) 2(y^h-y)\nabla(y^h-y) \Big\|_H^2 d\zeta.
\end{aligned}
$$

Consequently, we have that

$$
\begin{aligned}
&\int_0^t \Big\| \int_0^1 f'''(y+\zeta(y^h-y))(1-\zeta)(y^h-y)^2 d\zeta \Big\|_V^2 \\
&\le c \int_0^t \Big\| (y^h-y)^2 \Big\|_H^2 + \sup_{0\le\zeta\le1} \| f^{(iv)}(y+\zeta(y^h-y))\|_\infty \int_0^t \Big\| (|\nabla y| + |\nabla y^h|)(y^h-y)^2 \Big\|_H^2 \\
&\qquad + 2 \sup_{0\le\zeta\le1} \| f'''(y+\zeta(y^h-y))\|_\infty \int_0^t \Big\| (y^h-y)\nabla(y^h-y) \Big\|_H^2 \\
&\le c \int_0^t \Big\| y^h - y \Big\|_4^4 + c \int_0^t \Big(\|\nabla y\|_6^2 + \|\nabla y^h\|_6^2 \Big) \Big\| y^h - y \Big\|_6^4 + c \int_0^t \Big\| y^h - y \Big\|_4^2 \Big\| \nabla(y^h-y) \Big\|_4^2 \\
&\le c \int_0^t \Big\| y^h - y \Big\|_V^4 + c \int_0^t \Big(\|\nabla y\|_V^2 + \|\nabla y^h\|_V^2 \Big) \Big\| y^h - y \Big\|_V^4 \\
&\qquad + c \int_0^t \Big\| y^h - y \Big\|_V^2 \Big\| \nabla(y^h-y) \Big\|_V^2 \\
&\le c \|h_\Gamma\|_{H^1(0,T;H_\Gamma)}^4,
\end{aligned}
$$

where the Sobolev embeddings $V \subset L^4(\Omega)$ and $V \subset L^6(\Omega)$, and the stability estimate (4.8) have been used along with the fact that y and y^h, as solutions to (1.7)–(1.11), satisfy (2.28). Summarizing, we have just shown that

$$
c \int_0^t \Big\| f''(y)\,\vartheta^h + \int_y^{y^h} f'''(\gamma)(y^h-\gamma)^2 d\gamma \Big\|_V^2 \le c \int_0^t \|\vartheta^h\|_V^2 + c\|h_\Gamma\|_{H^1(0,T;H_\Gamma)}^4.
$$

Using the Taylor formula corresponding to (4.29) for the the nonlinearity f_Γ', combined with the Young inequality and the stability estimate (4.2), we control the last term of

(4.28) by

$$
\begin{aligned}
&- \int_{\Sigma_t} \big(f'_\Gamma(y^h_\Gamma) - f'_\Gamma(y_\Gamma) - f''_\Gamma(y_\Gamma)\xi_\Gamma - \vartheta^h_\Gamma \big)\, \partial_t \vartheta^h_\Gamma \\
&\quad \le \frac12 \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 + \frac12 \int_{\Sigma_t} |f'_\Gamma(y^h_\Gamma) - f'_\Gamma(y_\Gamma) - f''_\Gamma(y_\Gamma)\xi_\Gamma - \vartheta^h_\Gamma|^2 \\
&\quad \le \frac12 \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 \\
&\qquad\quad + \frac12 \int_{\Sigma_t} \Big| f''_\Gamma(y^h_\Gamma)\vartheta^h_\Gamma + \int_0^1 f'''_\Gamma\big(y_\Gamma + \zeta(y^h_\Gamma - y_\Gamma)\big)(1-\zeta)(y^h_\Gamma - y_\Gamma)^2 d\zeta - \vartheta^h_\Gamma \Big|^2 \\
&\quad \le \frac12 \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 + c \int_{\Sigma_t} |\vartheta^h_\Gamma|^2 + c \| f''_\Gamma(y^h_\Gamma)\|^2_\infty \int_{\Sigma_t} |\vartheta^h_\Gamma|^2 \\
&\qquad\quad + c \sup_{0 \le \zeta \le 1} \Big\| f'''_\Gamma\big(y_\Gamma + \zeta(y^h_\Gamma - y_\Gamma)\big)\Big\|_\infty \int_{\Sigma_t} |y^h_\Gamma - y_\Gamma|^4 \\
&\quad \le \frac12 \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 + c \int_{\Sigma_t} |\vartheta^h_\Gamma|^2 + c\|h_\Gamma\|^4_{L^2(\Sigma)}.
\end{aligned}
$$

Summing up, upon collecting all the above estimates, we realize that the inequality

$$
\begin{aligned}
&\frac12 \int_0^t \|\partial_t \vartheta^h\|^2_* + \frac12 \int_{\Sigma_t} |\partial_t \vartheta^h_\Gamma|^2 + \frac12 \|\vartheta^h(t)\|^2_V + \frac12 \|\vartheta^h_\Gamma(t)\|^2_{V_\Gamma} \\
&\quad \le c \int_0^t \|\vartheta^h\|^2_V + c \int_{\Sigma_t} |\vartheta^h_\Gamma|^2 + c\|h_\Gamma\|^4_{H^1(0,T;H_\Gamma)},
\end{aligned}
$$

has been proved whence a Gronwall argument directly yields (4.25). □

5 Optimality conditions

5.1 The adjoint system

This section is completely devoted to the investigation of the adjoint system and to the necessary conditions for optimality. Let us begin with the task of ensuring the well-posedness of system (2.45)–(2.47), that is checking Theorem 2.7. Before going on, it is worth mentioning here that the technique below differs from the one employed in [17]. Indeed, the key argument to prove the well-posedness of the adjoint problem in [17] relies on the interpretion of the adjoint system as a suitable abstract Cauchy problem in a general mathematical framework: then, after proving that the involved operators verify some properties like coercivity and continuity, the existence and uniqueness of the solution are deduced from some classical results. In this direction, let us emphasize that the outlined analysis suggested the authors to confine themselves to the investigation to the case $b_\Omega = b_\Gamma = 0$ (see also [17, Rem. 5.6], where a possible way to overcome this restriction is explained by involving weighted Lebesgue spaces).

Proof of Theorem 2.7. We will tackle the proof in two steps. In the first one, we will check the existence of a solution with the required regularity, whereas in the second step, we will point out that such a solution is indeed unique. From now on, let us convey that $\overline{u}_\Gamma$ and $(\overline{y}, \overline{y}_\Gamma)$ stand for an optimal control with the corresponding optimal state, respectively.

Existence The key idea for the existing part is showing that system (2.45)–(2.47) can be rewritten as an initial boundary value problem which complies with the framework of [19, Thm. 2.3, p. 977].

Moreover, since we are going to reverse the time with the following change of variable $t \mapsto T - t$, it turns out to be useful to set

$$\widetilde{\varphi}_Q(t) := \varphi_Q(T-t), \quad \widetilde{\varphi}_\Sigma(t) := \varphi_\Sigma(T-t), \widetilde{q}(t) := q(T-t), \quad \widetilde{p}(t) := p(T-t),$$
$$\widetilde{\lambda}(t) := \lambda(T-t), \quad \widetilde{\lambda}_\Gamma(t) := \lambda_\Gamma(t), \quad \widetilde{q}_\Gamma(t) := \widetilde{q}_{|\Gamma}(t) \quad \text{for a.a. } t \in (0,T).$$

Therefore, after substituting t with $T - t$, we realize that system (2.45)–(2.47) can be reformulated as the initial boundary value problem

$$\int_\Omega \widetilde{q}(t) v = \int_\Omega \nabla \widetilde{p}(t) \cdot \nabla v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0,T), \tag{5.1}$$

$$\int_\Omega \partial_t \widetilde{p}(t) v + \int_\Omega \nabla \widetilde{q}(t) \cdot \nabla v + \int_\Omega \widetilde{\lambda}(t) \widetilde{q}(t) v + \int_\Gamma \partial_t \widetilde{q}_\Gamma(t) v_\Gamma + \int_\Gamma \nabla_\Gamma \widetilde{q}_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Gamma \widetilde{\lambda}_\Gamma(t) \widetilde{q}_\Gamma(t) v_\Gamma = \int_\Omega \widetilde{\varphi}_Q(t) v + \int_\Gamma \widetilde{\varphi}_\Sigma(t) v_\Gamma$$
$$\text{for every } (v, v_\Gamma) \in \mathcal{V}, \text{ for a.a. } t \in (0,T), \tag{5.2}$$

$$\int_\Omega \widetilde{p}(0) v + \int_\Gamma \widetilde{q}_\Gamma(0) v_\Gamma = \int_\Omega \varphi_\Omega v + \int_\Gamma \varphi_\Gamma v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{5.3}$$

We claim that (5.1)–(5.3) can be studied with the help of [19, Thm. 2.3]. In this direction, let us proceed indirectly. Hence, we pick a function $\Phi \in V$ such that $(\Phi, \varphi_\Gamma) \in \mathcal{V}$, i.e. $\Phi_{|\Gamma} = \varphi_\Gamma$. Then, we take into account the problem of looking for a triplet (r, r_Γ, μ) which satisfies the following problem:

$$r_\Gamma(t) = r(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T), \tag{5.4}$$

$$\langle \partial_t r(t), v \rangle = \int_\Omega \nabla \mu(t) \cdot \nabla v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0,T), \tag{5.5}$$

$$\int_\Omega \mu(t) v = \int_\Omega \nabla r(t) \cdot \nabla v + \int_\Omega \widetilde{\lambda}(t) r(t) v + \int_\Gamma \partial_t r_\Gamma(t) v_\Gamma + \int_\Gamma \nabla_\Gamma r_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Gamma \widetilde{\lambda}_\Gamma(t) r_\Gamma(t) v_\Gamma = \int_\Omega \widetilde{\varphi}_Q(t) v + \int_\Gamma \widetilde{\varphi}_\Sigma(t) v_\Gamma$$
$$\text{for every } (v, v_\Gamma) \in \mathcal{V}, \text{ for a.a. } t \in (0,T), \tag{5.6}$$

$$r(0) = \Phi \quad \text{in } \Omega, \tag{5.7}$$

where the functions $\widetilde{\lambda}, \widetilde{\lambda}_\Gamma, \widetilde{\varphi}_Q, \widetilde{\varphi}_\Sigma$ are the same as above. Furthermore, the previous investigation, along with (2.31), leads us to realize that

$$\widetilde{\lambda} \in L^\infty(Q) \cap L^\infty(0,T; W^{1,3}(\Omega)), \quad \widetilde{\lambda}_\Gamma \in L^\infty(\Sigma),$$
$$\widetilde{\varphi}_Q \in H^1(0,T;H), \quad \widetilde{\varphi}_\Sigma \in L^2(0,T;H_\Gamma), \quad (\Phi, \varphi_\Gamma) \in \mathcal{V}.$$

Therefore, the assumptions of [19, Thm. 2.3, p. 977] are satisfied so that the existence of a triplet (r, r_Γ, μ), which solves (5.4)–(5.7) and enjoys the following regularity

$$(r, r_\Gamma) \in H^1(0,T;\mathcal{G}) \cap L^\infty(0,T;\mathcal{V}) \cap L^2(0,T; H^2(\Omega) \times H^2(\Gamma)),$$
$$\mu \in L^2(0,T;V),$$

directly follows. We are then reduced to show that system (5.1)–(5.3) can be written in the form of (5.4)–(5.7). We claim that the following choice realizes this goal:

$$\widetilde{q} := r, \quad \widetilde{q}_\Gamma := r_\Gamma, \quad \widetilde{p}(t) := \varphi_\Omega - \int_0^t \mu(s)ds \quad \text{for a.a. } t \in (0,T). \tag{5.8}$$

In fact, by differentiating the last term, we deduce that $\mu = -\partial_t \widetilde{p}$ a.e. in Q, so that (5.6) implies (5.2). Moreover, integrating (5.5) with respect to t and using (5.7) yield

$$\int_\Omega r(t)v + \int_\Omega \nabla \int_0^t \mu(s)ds \cdot \nabla v = \int_\Omega \Phi\, v \quad \text{for every } v \in V,$$

which, owing to (5.8), entails that

$$\int_\Omega \widetilde{q}(t)v + \int_\Omega \nabla(-\widetilde{p}(t) + \varphi_\Omega) \cdot \nabla v = \int_\Omega \Phi\, v \quad \text{for every } v \in V.$$

Hence, provided we require that

$$\int_\Omega \nabla \varphi_\Omega \cdot \nabla v = \int_\Omega \Phi\, v \quad \text{for every } v \in V, \tag{5.9}$$

(5.1) follows from (5.5). Besides, (5.4), (5.7) and (5.8) imply that

$$\widetilde{p}(0) = \varphi_\Omega, \quad \widetilde{q}_\Gamma(0) = \varphi_\Gamma \quad \text{in } \Omega,$$

whence (5.3) immediately follows by testing by $(v, v_\Gamma) \in \mathcal{V}$ and integrating over Ω. Summing up, equation (5.9) gives, in turn, that $(\Phi)^\Omega = 0$ and also that φ_Ω solves the following elliptic problem

$$\begin{cases} -\Delta \varphi_\Omega &= \Phi \quad \text{in } \Omega, \\ \partial_n \varphi_\Omega &= 0 \quad \text{on } \Gamma, \end{cases}$$

which entails that $\varphi_\Omega = \mathcal{N}(\Phi) + (\varphi_\Omega)^\Omega$. If all these compatibility conditions on Φ and φ_Ω are in force, we have just checked that system (5.1)–(5.3) can be rewritten in the form of (5.4)–(5.7). Thus, owing to [19, Thm. 2.3, p. 977], there exists a triplet $(\widetilde{q}, \widetilde{q}_\Gamma, \widetilde{p})$, which solves (5.1)–(5.3) and possesses the following regularity

$$\begin{aligned} &(\widetilde{q}, \widetilde{q}_\Gamma) \in H^1(0,T; \mathcal{G}_\Omega) \cap L^\infty(0,T; \mathcal{V}_\Omega) \cap L^2(0,T; H^2(\Omega) \times H^2(\Gamma)), \\ &\quad \widetilde{p} \in H^1(0,T; V). \end{aligned}$$

Lastly, owing to the above regularity, along with comparison in the strong formulation of (2.45), we easily realize that $\Delta \widetilde{p} \in L^2(0,T; H^2(\Omega))$, so that the elliptic regularity theory ensures that $\widetilde{p} \in L^2(0,T; H^4(\Omega))$.

Remark 5.1. Let us point out that in [17], where the analogous control problem for the viscous case was treated, the conditions $b_\Omega = b_\Gamma = 0$ have been required in order to handle the adjoint system. Note that this restriction leads to consider $\varphi_\Omega = 0$ in Ω, $\varphi_\Gamma = 0$ in Γ, $\Phi = 0$ in Ω, which surely fulfill our requirements.

Uniqueness We proceed by contradiction assuming the existence of, at least, two solutions $(\widetilde{q}_i, \widetilde{q}_{\Gamma,i}, \widetilde{p}_i)$, $i = 1, 2$, to system (2.45)–(2.47). Then, we set

$$\widetilde{q} := \widetilde{q}_1 - \widetilde{q}_2, \quad \widetilde{q}_\Gamma := \widetilde{q}_{\Gamma,1} - \widetilde{q}_{\Gamma,2}, \quad \widetilde{p} := \widetilde{p}_1 - \widetilde{p}_2,$$

and we are going to show that the only possibility is $\widetilde{p} = \widetilde{q} = \widetilde{q}_\Gamma = 0$. In this direction, we write system (5.1)–(5.3) for both the solutions $(\widetilde{q}_i, \widetilde{q}_{\Gamma,i}, \widetilde{p}_i)$, $i = 1, 2$, and take the difference. Note that, taking $(v, 0) \in \mathcal{V}$ in (5.3), we get $\widetilde{p}_1(0) = \widetilde{p}_2(0) = \varphi_\Omega$ in Ω and by comparison also that $\widetilde{q}_{\Gamma,1}(0) = \widetilde{q}_{\Gamma,2}(0) = \varphi_\Gamma$. Thus, we have that

$$\int_\Omega \widetilde{q}(t)v = \int_\Omega \nabla\widetilde{p}(t)\cdot\nabla v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0,T), \tag{5.10}$$

$$\int_\Omega \partial_t\widetilde{p}(t)v + \int_\Omega \nabla\widetilde{q}(t)\cdot\nabla v + \int_\Omega \widetilde{\lambda}(t)\widetilde{q}(t)v + \int_\Gamma \partial_t\widetilde{q}_\Gamma(t)v_\Gamma + \int_\Gamma \nabla_\Gamma\widetilde{q}_\Gamma(t)\cdot\nabla_\Gamma v_\Gamma$$
$$+ \int_\Gamma \widetilde{\lambda}_\Gamma(t)\widetilde{q}_\Gamma(t)v_\Gamma = 0 \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \text{ for a.a. } t \in (0,T), \tag{5.11}$$

$$\widetilde{p}(0) = 0, \quad \widetilde{q}_\Gamma(0) = 0 \quad \text{in } \Omega. \tag{5.12}$$

Next, we test equation (5.10) by $-\partial_t\widetilde{p}$, (5.11) by $(\widetilde{q}, \widetilde{q}_\Gamma)$, and (5.10) once more by $K\widetilde{q}$, for a constant K, yet to be determined. Summing the obtained equalities and rearranging the terms lead to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\widetilde{p}|^2 + K\int_\Omega |\widetilde{q}|^2 + \int_\Omega |\nabla\widetilde{q}|^2 + \frac{1}{2}\frac{d}{dt}\int_\Gamma |\widetilde{q}_\Gamma|^2 + \int_\Gamma |\nabla_\Gamma\widetilde{q}_\Gamma|^2$$
$$= K\int_\Omega \nabla\widetilde{p}\cdot\nabla\widetilde{q} - \int_\Omega \widetilde{\lambda}|\widetilde{q}|^2 - \int_\Gamma \widetilde{\lambda}_\Gamma|\widetilde{q}_\Gamma|^2 \quad \textit{a.e. } \text{in } (0,T),$$

where the integrals on the right-hand side are denoted by I_1, I_2 and I_3, respectively. Using the Young inequality and the boundedness of $\widetilde{\lambda}_\Gamma$, we deduce that

$$|I_1| + |I_3| \le \frac{1}{2}\int_\Omega |\nabla\widetilde{q}|^2 + \frac{K^2}{2}\int_\Omega |\nabla\widetilde{p}|^2 + c\int_\Gamma |\widetilde{q}_\Gamma|^2 \quad \textit{a.e. } \text{in } (0,T).$$

Moreover, the boundedness of $\widetilde{\lambda}$ allows us to infer that

$$|I_2| \le \|\widetilde{\lambda}\|_\infty \int_\Omega |\widetilde{q}|^2,$$

and we move it to the left-hand side. Finally, we rearrange the terms and integrate over $(0, t)$ to obtain that

$$\frac{1}{2}\int_\Omega |\nabla\widetilde{p}(t)|^2 + (K - \|\widetilde{\lambda}\|_\infty)\int_{Q_t} |\widetilde{q}|^2 + \frac{1}{2}\int_{Q_t} |\nabla\widetilde{q}|^2 + \frac{1}{2}\int_\Gamma |\widetilde{q}_\Gamma(t)|^2 + \int_{\Sigma_t} |\nabla_\Gamma\widetilde{q}_\Gamma|^2$$
$$\le \frac{K^2}{2}\int_{Q_t} |\nabla\widetilde{p}|^2 + c\int_{\Sigma_t} |\widetilde{q}_\Gamma|^2 \quad \text{for all } t \in (0,T).$$

Hence, taking the constant K large enough, we apply the Gronwall lemma to conclude that

$$\|\nabla\widetilde{p}\|_{L^\infty(0,T;H)} + \|\widetilde{q}\|_{L^2(0,T;V)} + \|\widetilde{q}_\Gamma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le 0,$$

which yields

$$\widetilde{q} = 0, \quad \widetilde{q}_\Gamma = 0, \quad \nabla\widetilde{p} = 0.$$

Hence, we realize that $\widetilde{p}$ has to be constant with respect to the space variable. On the other hand, comparison in (5.11) produces $\partial_t\widetilde{p} = 0$, so that $\widetilde{p}$ has also to be constant in time and such that (5.12) is verified. Therefore, we infer that $\widetilde{p} = 0$ and Theorem 2.7 is completely proved. □

5.2 Necessary optimality conditions

The final step of the work consists in proving Theorem 2.8 by deriving the first-order optimality conditions. This will also point out that (2.45)–(2.47) yields the adjoint system for (2.19)–(2.25).

Proposition 5.2. *Let $\overline{u}_\Gamma$ and $(\overline{y}, \overline{y}_\Gamma)$ be an optimal control with the corresponding state. Then, inequality* (2.41) *holds true.*

Proof. In order to prove (2.41), we essentially make use of (2.36). In fact, we make explicit (2.36) exploiting the Fréchet differentiability of $\mathcal{S}$ and the chain rule. As a matter of fact, denoting by $\widetilde{\mathcal{S}} : \mathcal{U} \to \mathcal{Y} \times \mathcal{X}$ the function defined by $\widetilde{\mathcal{S}}(u_\Gamma) := (\mathcal{S}(u_\Gamma), u_\Gamma)$, we realize that Theorem 2.6 yields

$$D\widetilde{\mathcal{S}}(u_\Gamma) : h_\Gamma \mapsto \big([D\mathcal{S}(u_\Gamma)](h_\Gamma), h_\Gamma\big) = (\xi, \xi_\Gamma, h_\Gamma) \quad \text{for the admissible } h_\Gamma \in \mathcal{X},$$

where (ξ, ξ_Γ, η) is the solution to the linearized system (2.38)–(2.40) corresponding to h_Γ. On the other hand, if we consider the cost functional $\mathcal{J}$ as a mapping from $\mathcal{Y} \times \mathcal{X}$ to $\mathbb{R}$, its Fréchet derivative at $(y, y_\Gamma, u_\Gamma) \in \mathcal{Y} \times \mathcal{X}$ is straightforwardly given by

$$\begin{aligned}
[D\mathcal{J}(y, y_\Gamma, u_\Gamma)](k, k_\Gamma, h_\Gamma) &= b_Q \int_Q (y - z_Q)k + b_\Sigma \int_\Sigma (y_\Gamma - z_\Sigma)k_\Gamma + b_\Omega \int_\Omega (y(T) - z_\Omega)k(T) \\
&+ b_\Gamma \int_\Gamma (y(T) - z_\Gamma)k_\Gamma(T) + b_0 \int_\Sigma u_\Gamma h_\Gamma \quad \text{for } (k, k_\Gamma) \in \mathcal{Y} \text{ and } h_\Gamma \in \mathcal{X}.
\end{aligned}$$

Hence, since $\widetilde{\mathcal{J}} = \mathcal{J} \circ \widetilde{\mathcal{S}}$, the chain rule implies that

$$\begin{aligned}
[D\widetilde{\mathcal{J}}(u_\Gamma)](h_\Gamma) &= [D\mathcal{J}(\widetilde{\mathcal{S}}(u_\Gamma)]\big([D\widetilde{\mathcal{S}}(u_\Gamma)](h_\Gamma)\big) = [D\mathcal{J}(y, y_\Gamma, u_\Gamma)](\xi, \xi_\Gamma, h_\Gamma) \\
&= b_Q \int_Q (y - z_Q)\xi + b_\Sigma \int_\Sigma (y_\Gamma - z_\Sigma)\xi_\Gamma + b_\Omega \int_\Omega (y(T) - z_\Omega)\xi(T) \\
&\quad + b_\Gamma \int_\Gamma (y(T) - z_\Gamma)\xi_\Gamma(T) + b_0 \int_\Sigma u_\Gamma h_\Gamma.
\end{aligned}$$

Therefore, (2.41) immediately follows from (2.36) by choosing in the above calculations $(y, y_\Gamma, u_\Gamma) = (\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma)$. □

Proof of Theorem 2.8. For the sake of simplicity, we will avoid writing explicitly the time variable in the calculations below. Moreover, for the reader's convenience, we rewrite the variational formulation of the linearized and the adjoint system, respectively. They read as follows

$$-\int_\Omega \partial_t \xi v - \int_\Omega \nabla\eta\cdot\nabla v = 0 \quad \text{for every } v\in V,$$

$$\int_\Omega \eta v = \int_\Gamma \partial_t\xi_\Gamma\, v_\Gamma + \int_\Omega \nabla\xi\cdot\nabla v + \int_\Gamma \nabla_\Gamma\xi_\Gamma\cdot\nabla_\Gamma v_\Gamma + \int_\Omega \lambda\,\xi\, v + \int_\Gamma \big(\lambda_\Gamma\,\xi_\Gamma - h_\Gamma\big)\, v_\Gamma \quad \text{for every } (v,v_\Gamma)\in\mathcal{V},$$

$$-\int_\Omega q\, v + \int_\Omega \nabla p\cdot\nabla v = 0 \quad \text{for every } v\in V,$$

$$-\int_\Omega \partial_t p\, v + \int_\Omega \nabla q\cdot\nabla v + \int_\Omega \lambda q v - \int_\Gamma \partial_t q_\Gamma\, v_\Gamma + \int_\Gamma \nabla_\Gamma q_\Gamma\cdot\nabla_\Gamma v_\Gamma + \int_\Gamma \lambda_\Gamma q_\Gamma v_\Gamma$$
$$= b_Q \int_\Omega (\overline{y}-z_Q)v + b_\Sigma \int_\Gamma (\overline{y}_\Gamma - z_\Sigma)v_\Gamma \quad \text{for every } (v,v_\Gamma)\in\mathcal{V},$$

with the corresponding initial conditions

$$(\xi,\xi_\Gamma)(0) = (0,0) \quad \text{in } \Omega,$$

and final conditions

$$\int_\Omega p(T)v + \int_\Gamma q_\Gamma(T)v_\Gamma = b_\Omega\int_\Omega (\overline{y}(T)-z_\Omega)v(T) + b_\Gamma\int_\Gamma (\overline{y}_\Gamma(T)-z_\Gamma)v_\Gamma(T) \quad \text{for every } (v,v_\Gamma)\in\mathcal{V},$$

respectively. Then, we test these formulations by $p, (q,q_\Gamma), \eta$ and (ξ,ξ_Γ), in this order. Adding the resulting equalities, integrating over $(0,t)$ and by parts, and using the initial condition for ξ and the final ones for p and q_Γ, lead us to infer that the most of the terms cancel out and it remains

$$\int_\Sigma q_\Gamma h_\Gamma = \int_Q b_Q(\overline{y}-z_Q)\xi + \int_\Sigma b_\Sigma(\overline{y}_\Gamma - z_\Sigma)\xi_\Gamma$$
$$+ \int_\Omega b_\Omega(\overline{y}(T)-z_\Omega)\xi(T) + \int_\Gamma b_\Gamma(\overline{y}_\Gamma(T)-z_\Gamma)\xi_\Gamma(T),$$

which is the desired conclusion since it allows us to obtain (2.48), where $h_\Gamma = v_\Gamma - \overline{u}_\Gamma$, from (2.41). □

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