Post-Design Verification in the Scenario Approach

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***Abstract*— The scenario approach is an established data-driven design framework equipped with a powerful theory that links design complexity to generalization properties. A key feature of this method is that training data are used both for design and for certifying the design's reliability, without resorting to additional data points. This paper takes a step further: the goal is not only to certify the properties considered at the design stage, but also to guarantee new properties of interest in post-design usage. To this end, we introduce a two-level framework of appropriateness: baseline appropriateness, which guides the design process, and post-design appropriateness, which serves as a criterion for *a posteriori* evaluation. The paper fully develops a post-design verification framework that works without resorting to additional test data and illustrates it on an H_2 problem.**

I. Introduction

The scenario approach is a well-established methodology in systems and control for making data-driven designs while also certifying the design's quality without resorting to test sets. Introduced around the mid-2000s (refer to the book [1] and the review article [2] for a general introduction), the scenario approach has thrived over the years, providing guaranteed methods for data-driven robust optimization, [3]–[5], optimization schemes incorporating relaxation, [6], [7], optimization based on risk measures like the Conditional Value at Risk (CVaR), [8], [9], and, beyond optimization, aiding data-driven designs based on general decision-making techniques, a task pursued in [6], [7], [10].

The theoretical advancement of scenario theory is due to the contributions of a vast community of researchers working in many groups worldwide, [11]–[33]. This theory addresses questions such as: if optimization is performed, what is the probabilistic guarantee that a certain performance level will be achieved, or that a given constraint will be satisfied, in new, as-yet unseen cases? Or, what is the probability that a given design, developed using a decision-making scenario technique, will fall short in a future operating condition?

In this paper, we specifically refer to two fundamental contributions in the context of the scenario approach, namely [6] and [7]. In [6], a broad framework called "*consistent decision-making*" is introduced and analyzed. Consistency relates the change of a decision to its *appropriateness* for new incoming observations, a notion that is recalled later in this paper as Assumption 1. In this context, upper and lower bounds on the *probability of inappropriateness* are established under an additional assumption of *nondegeneracy*. Depending on the problem at hand, this probability takes different interpretations, ranging from the failure to meet a certain control performance to the inability of a predictor to correctly forecast an event. For a comprehensive discussion of inappropriateness in various contexts, the reader is referred to the review paper [2]. These bounds are informative and tight, as demonstrated in [2], while an asymptotic analysis has appeared in [10]. The later paper [7] has freed the theory from the assumption of nondegeneracy, showing that an upper bound continues to hold in this extended setup.

This paper takes a step further: there are circumstances in which one is interested in a given notion of appropriateness, which is explicitly incorporated at the time a design is made, while also caring for an additional notion of appropriateness that plays an ancillary role. We call the first *baseline appropriateness* and the second *post-design appropriateness*. We show that the scope of the consistency framework can be broadened to provide tight and informative bounds for post-design inappropriateness as well. Typical situations in which these new findings can be used include:

(i) after designing under a baseline appropriateness condition, one is interested in securing additional properties related to potential, modified conditions. For example, while the presence of an adversary, or a hacker, is deemed unlikely, one may still want to assess how severely such an occurrence could impact a given design;

(ii) given a baseline performance, for example the return of an investment or the performance of a control loop, one may in post-design ask for the probability of meeting an even better performance, as defined by a threshold value;

(iii) yet another situation of use arises when the ultimate goal is to guarantee a certain property, but the corresponding appropriateness condition is difficult to enforce at the design stage due to computational issues. In such cases, one can envisage using a handier, heuristic approach for design based on a simplified notion of baseline appropriateness. The fulfillment of the original appropriateness condition can then be assessed in post-design, where it takes on the role of post-design appropriateness.

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In this paper, we introduce a theoretical framework for post-design verification. Specifically, upper bounds for the probability of not meeting post-design appropriateness are established in Section II. These theoretical achievements are demonstrated on an example in H_2 control in Section III: the example is related to the situation in point (ii) in the previous list. Conclusions are provided in Section IV.

II. Problem statement and technical result

A. A scenario decision framework with two notions of appropriateness

We first recall how the process that goes from data to a decision is modelled in the framework of the scenario approach, [6], [7]. A list $\delta_1, \ldots, \delta_N$ of observations of an uncertain variable δ is modelled as an independent and identically distributed (i.i.d.) sample from a probability space $(\Delta, \mathcal{D}, \mathbb{P})$. Each observed δ_i is called a "scenario", and, borrowing a machine learning terminology, the list of scenarios $\delta_1, \ldots, \delta_N$ can be also called a training set. The probability $\mathbb{P}$ is typically unknown to the user, so the framework is often called *agnostic*, or *distribution-free*.

Let $\mathcal{Z}$ be a generic set, which is interpreted as the domain from which a decision z has to be chosen. From an abstract perspective, a data-driven decision scheme corresponds to a family of maps from lists of scenarios to decisions, namely

$$M_m : \Delta^m \to \mathcal{Z}, \quad m = 0, 1, 2, \ldots.$$

When $m = 0$, $\delta_1, \ldots, \delta_m$ is meant to be the empty list and M_0 returns the decision that is made when no scenarios are available. We will often denote the decision returned by $M_m(\delta_1, \ldots, \delta_m)$ as z_m^*.

A further ingredient of the standard scenario approach is a criterion of appropriateness, based on which a given decision $z \in \mathcal{Z}$ is deemed appropriate or inappropriate for a given $\delta \in \Delta$. The main objective of the theory of the scenario approach is to certify the level of appropriateness of the decision delivered by $M_N(\delta_1, \ldots, \delta_N)$ (N is the actual number of scenarios used to design). This goal is pursued in [6], [7] under the requirement that the maps M_m satisfy certain consistency properties, that we describe below in Assumption 1.

In this paper, we consider two different kinds of appropriateness:

- *baseline appropriateness*: this is the "standard" kind of appropriateness, with respect to which the decision map is consistent in the sense of Assumption 1;
- *post-design appropriateness*: this is a further, desirable notion of appropriateness for which M_m need not be consistent.

The two notions of appropriateness are formally introduced in the following definitions, which are abstracted from the concrete appropriateness criteria found in applications.

Definition 1 (baseline appropriateness): For every δ, there corresponds a set $\mathcal{Z}'_\delta \subseteq \mathcal{Z}$, which represents the decisions that are **baseline appropriate** for δ.[1] $\star$

[1] Measurability of sets $\{\delta \in \Delta : z \in \mathcal{Z}'_\delta\}$, and of analogue sets defined throughout the paper, is tacitly assumed.

Definition 2 (post-design appropriateness): For every δ, there corresponds a set $\mathcal{Z}''_\delta \subseteq \mathcal{Z}$, which represents the decisions that are **post-design appropriate** for δ. $\star$

The requirement of baseline consistency is specified in the following assumption.

Assumption 1: The maps M_m satisfy the consistency requirements as per Property 1 in [7] with respect to the baseline appropriateness. That is, for every integers $m \geq 0$ and $n > 0$, and every $\delta_1, \ldots, \delta_m, \delta_{m+1}, \ldots, \delta_{m+n}$, the following three conditions, called *permutation invariance*, *confirmation under appropriateness* and *responsiveness to inappropriateness*, are satisfied:

- *permutation invariance*:
  given any permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$, it holds that $M_m(\delta_1, \ldots, \delta_m) = M_m(\delta_{i_1}, \ldots, \delta_{i_m})$;
- *confirmation under appropriateness*:
  if $M_m(\delta_1, \ldots, \delta_m) \in \mathcal{Z}'_{\delta_{m+i}}$, $\forall i \in \{1, \ldots, n\}$, then $M_{m+n}(\delta_1, \ldots, \delta_{m+n}) = M_m(\delta_1, \ldots, \delta_m)$;
- *responsiveness to inappropriateness*:
  if $\exists\, i \in \{1, \ldots, n\} : \quad M_m(\delta_1, \ldots, \delta_m) \notin \mathcal{Z}'_{\delta_{m+i}}$, then $M_{m+n}(\delta_1, \ldots, \delta_{m+n}) \neq M_m(\delta_1, \ldots, \delta_m)$.

$\star$

It is important to remark that no assumption linking M_m and post-design appropriateness is instead introduced.

B. The risk of inappropriateness - review and new goal

The notion of level of appropriateness for both baseline and post-design criteria is formalized as follows.

Definition 3 (baseline and post-design risk): For a given decision $z \in \mathcal{Z}$, we define the risk of baseline inappropriatedness, or *baseline risk*, $R'(z)$ as the probability that a newly sampled δ is baseline inappropriate for the decision, i.e.,

$$R'(z) := \mathbb{P}\{\delta \in \Delta \,:\, z \notin \mathcal{Z}'_\delta\},$$

while the risk of post-design inappropriateness, or *post-design risk*, $R''(z)$ is the probability that a new δ is post-design inappropriate for the decision, i.e.,

$$R''(z) := \mathbb{P}\{\delta \in \Delta \,:\, z \notin \mathcal{Z}''_\delta\}.$$

$\star$

The main contribution of the present paper is the certification of the baseline and post-design risks of $z_N^* = M_N(\delta_1, \ldots, \delta_N)$, the decision constructed from the training set. Although $R'(z_N^*)$ and $R''(z_N^*)$ cannot be directly computed because $\mathbb{P}$ is unknown, the goal is to demonstrate that certain statistics of the data can be used to bound $R'(z_N^*)$ and $R''(z_N^*)$ with high confidence $1-\beta$ with respect to the variability of $\delta_1, \ldots, \delta_N$. The statistics that serve to certify $R'(z_N^*)$ and $R''(z_N^*)$ are constructed from the same dataset used for design, avoiding any loss of data.

Thanks to Assumption 1, all the results of [6] and [7] can be directly used "off-the-shelf" to scrutinize $R'(z_N^*)$. To this purpose we have first to borrow the following notion of complexity from [7].

Definition 4 (baseline support list, baseline complexity): For the decision map M_m, given the list of scenarios

$\delta_1, \dots, \delta_m$, a baseline support list is a sub-list, say $\delta_{i_1}, \dots, \delta_{i_k}$ with $i_1 < i_2 < \dots < i_k$ such that:
(a) $M_m(\delta_1, \dots, \delta_m) = M_k(\delta_{i_1}, \dots, \delta_{i_k})$;
(b) $\delta_{i_1}, \dots, \delta_{i_k}$ is irreducible, that is, no element can be further removed from $\delta_{i_1}, \dots, \delta_{i_k}$ while leaving the decision unchanged.
For a given $\delta_1, \dots, \delta_m$ there can be more than one selection of the indexes $i_1, i_2, \dots, i_k$, possibly with different cardinality k, that give a baseline support list. The minimal cardinality among all baseline support lists is called the *baseline complexity* and is denoted by $s_m^{b,*}$. ⋆

For any given choice of the confidence parameter $\beta \in (0, 1)$, a direct application of Theorem 4 in [7] ensures that

$$\mathbb{P}^N\{R'(z_N^*) \leq \epsilon(s_N^{b,*})\} \geq 1 - \beta, \tag{1}$$

with $\epsilon(N) = 1$ and, for $k = 0, 1, \dots, N-1$, $\epsilon(k) := 1 - t(k)$ where $t(k)$ is the only solution in the interval $(0, 1)$ of the equation in the t variable

$$\frac{\beta}{N} \sum_{m=k}^{N-1} \binom{m}{k} t^{m-k} - \binom{N}{k} t^{N-k} = 0. \tag{2}$$

Likewise, Theorem 2 in [6], which under additional conditions provides not only an upper bound, but also a lower bound, as well as other results in [6], [7] can be directly applied to certify $R'(z_N^*)$.

These were known results. The goal of this paper is to upper bound the risk of z_N^* also with respect to a *post-design appropriateness*. Bounding $R''(z_N^*)$ is nontrivial because, in general, the maps M_m *do not* satisfy the consistency requirements as per Property 1 in [7] in relation to post-design appropriateness: the results in [7] and [6] cannot be applied and a new theoretical development is required to pursue this goal.

C. Novel results: post-design risk certification

In order to certify the post-design risk $R''(z_N^*)$, we define new, instrumental, decision maps from scenarios to an augmented decision space $\mathcal{Z}^+ := \mathcal{Z} \times \mathbb{N}$, and an instrumental notion of appropriateness for these maps.

Definition 5 (instrumental decision map): For any $m = 0, 1, \dots$ and any list $\delta_1, \dots, \delta_m$ of scenarios, the *instrumental decision map* is defined as $M_m^+(\delta_1, \dots, \delta_m) := (z_m^*, c_m^*)$, where $z_m^* = M_m(\delta_1, \dots, \delta_m)$ and $c_m^* = \#\{i : i \in \{1, \dots, m\}$ and $z_m^* \notin \mathcal{Z}''_{\delta_i}\}$ (the symbol "#" denotes cardinality). ⋆

Definition 6 (instrumentally appropriate decisions): For every δ, we define the set of the *instrumentally appropriate decisions* for δ as $\mathcal{Z}_\delta^+ := (\mathcal{Z}'_\delta \times \mathbb{N}) \cap (\mathcal{Z}''_\delta \times \mathbb{N})$. ⋆

The following lemma is key in the development of the theory.

Lemma 1: The maps M_m^+, with respect to *instrumental appropriateness* $\mathcal{Z}_\delta^+$, satisfy the consistency requirements as per Property 1 in [7] (or, which is the same, Assumption 1 stated above with obvious notational shifts). ⋆

Proof: We recall that $M_m^+(\delta_1, \dots, \delta_m) = (z_m^*, c_m^*)$, where $z_m^* = M_m(\delta_1, \dots, \delta_m)$ and $c_m^* = \#\{i : i \in \{1, \dots, m\}$ and $z_m^* \notin \mathcal{Z}''_{\delta_i}\}$. Clearly, M_m^+ is *permutation invariant* because z_m^* is permutation invariant by Assumption 1, and c_m^* does not depend on the order in which data points appear in the sample. To check *confirmation under appropriateness*, observe that $M_m^+(\delta_1, \dots, \delta_m) \in \mathcal{Z}^+_{\delta_{m+i}}$ implies that (a) $z_m^* \in \mathcal{Z}'_{\delta_{m+i}}$ and (b) $z_m^* \in \mathcal{Z}''_{\delta_{m+i}}$. Since M_m satisfies *confirmation under appropriateness* w.r.t. $\mathcal{Z}'_\delta$, condition (a) entails that $z_{m+n}^* = z_m^*$; moreover, this fact along with condition (b) entails that $c_{m+n}^* = c_m^*$. Regarding *responsiveness to inappropriateness*, we observe that if $M_m^+(\delta_1, \dots, \delta_m) \notin \mathcal{Z}^+_{\delta_{m+i}}$ for some i, then either (a) $z_m^* \notin \mathcal{Z}'_{\delta_{m+i}}$ or (b) $z_m^* \notin \mathcal{Z}''_{\delta_{m+i}}$. If (a), then $z_{m+n}^* \neq z_m^*$ because M_m is responsive w.r.t. $\mathcal{Z}'_{\delta_{m+i}}$, and therefore (z_{m+n}^*, c_{m+n}^*) differs from (z_m^*, c_m^*) in its first component; on the other hand, if $z_{m+n}^* = z_m^*$ because only (b) occurs, then (z_{m+n}^*, c_{m+n}^*) and (z_m^*, c_m^*) must differ in their second component, thus concluding the proof. ■

An *instrumental risk* can be attributed to an instrumental decision $(z, \ell) \in \mathcal{Z} \times \mathbb{N}$, as follows:

Definition 7 (instrumental risk): For an instrumental decision $(z, \ell) \in \mathcal{Z} \times \mathbb{N}$, we define the *instrumental risk*, $R^+(z, \ell)$, as the probability that a newly sampled δ is instrumentally inappropriate for the decision, i.e.,

$$R^+(z, \ell) := \mathbb{P}\{\delta \in \Delta : (z, \ell) \notin \mathcal{Z}_\delta^+\}.$$

Crucially, $\ell \in \mathbb{N}$ does not play any role in the instrumental risk; therefore, instrumental risk can be attributed to a decision $z \in \mathcal{Z}$, i.e.,

$$R^+(z) := \mathbb{P}\{\delta \in \Delta : z \notin \mathcal{Z}'_\delta \cap \mathcal{Z}''_\delta\}. \tag{3}$$

⋆

The key observation to proceed is that the instrumental risk can be kept under control by considering the complexity of the training set with respect to the instrumental map. The formal definitions of support lists and of complexity for the instrumental map are the same as in Definition 4, where: the map M_m is replaced by instrumental map M_m^+; the word "baseline" is replaced by "instrumental"; the symbol "$s_m^{+,*}$" is used in place of "$s_m^{b,*}$" to denote the *instrumental complexity*.

Remark 1 (on computing $s_N^{+,*}$): All the possible instrumental support lists must contain a baseline support list and all the δ_i's in the training set for which z_N^* is post-design inappropriate.[2] However, while appending to a given baseline support list all the δ_i's that are not already in the baseline support list and for which z_N^* is post-design inappropriate surely suffices to preserve (z_N^*, c_N^*), this constructed list may be reducible if the baseline support list is not carefully selected at the beginning. This suggests that the cardinality of any such constructed list provides in general an upper bound on the instrumental complexity $s_N^{+,*}$. Plainly, the constructed

[2]Indeed, without a baseline support list z_N^* could not be reconstructed in view of the irreducibility of baseline support lists (see (b) in Definition 4), while c_N^* would be underestimated without all the δ_i's for which z_N^* is inappropriate.

list is instead irreducible and the computation of $s_N^{+,*}$ is exact if there is a unique baseline support list. ⋆

Thanks to Lemma 1, all the results of the scenario approach can be applied to certify the instrumental risk of $M_N^+(\delta_1,\dots,\delta_N)$ by means of the observable $s_N^{+,*}$. The significance of these results for the instrumental decision lies in the fact that they can be in turn used to certify the post-design risk of $M_N(\delta_1,\dots,\delta_N)$ as shown by the following theorem.

Theorem 1 (upper bound for post-design risk): For any given confidence parameter $\beta \in (0,1)$, let $\epsilon(k) := 1 - t(k)$ for $k = 0,\dots,N-1$, and $\epsilon(N) = 1$, where $t(k)$ is the only solution of (2) in the interval $(0,1)$. Then, under Assumption 1, it holds for every $\mathbb{P}$ that

$$\mathbb{P}^N\{R''(z_N^*) \le \epsilon(s_N^{+,*})\} \ge 1-\beta. \tag{4}$$

Proof: Thanks to Lemma 1, we can apply Theorem 4 in [7] to the instrumental maps M_m^+ with instrumental appropriateness $\mathcal{Z}_\delta^+$, to conclude that

$$\mathbb{P}^N\{R^+(z_N^*) \le \epsilon(s_N^{+,*})\} \ge 1-\beta.$$

Equation (4) simply follows by observing that $R''(z_N^*) \le R^+(z_N^*)$ in view of (3). ■

Remark 2 (on computing $s_N^{+,*}$ – follow-up): The choice of $\epsilon(k)$ based on (2) is such that $\epsilon(0) \le \epsilon(1) \le \dots \le \epsilon(N)$. Thus, the complexity $s_N^{+,*}$ in the statement of Theorem 1 can be replaced by any valid upper bound on $s_N^{+,*}$. This possibility comes handy in the case where computing the minimum cardinality of the instrumental support list is computationally demanding. ⋆

III. Example

We illustrate the two-level framework developed in this paper, as well as an application of Theorem 1, using a textbook example: *probabilistically robust LQ control*.

We consider the H_2 control of the lateral motion of an aircraft, a classic example introduced in [34] and revisited in several papers and textbooks, [1], [35]–[37]. The numerical values that we consider are taken from [34] and the uncertainty intervals are from [1].

The lateral motion of the aircraft can be described by the equation:

$$\begin{aligned}\dot{x}(t) \;&=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ 0.086 & 0 & -0.11 & -1 \\ 0.086 N_{\dot\beta} & N_p & N_\beta + 0.11 N_{\dot\beta} & N_r \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)\end{aligned} \tag{5}$$

where the four state variables are the bank angle, the derivative of the bank angle, the sideslip angle, and the yaw rate. The two inputs are the rudder deflection and the aileron deflection. The various parameters that appear in the state matrix are uncertain, i.e. the uncertain variable is $\delta = [L_p, L_\beta, L_r, N_p, N_\beta, N_{\dot\beta}, N_r]$, so Δ is a subset of $\mathbb{R}^7$. For short, the state and input matrices will be written as A_δ and B.

Given two symmetric positive definite matrices S and R and an initial condition $x_0 \in \mathbb{R}^4$, we aim to design the state feedback law $u(t) = Kx(t)$ that minimizes the cost function

$$J(K,\delta) = \int_0^\infty \left[x(t)^T S x(t) + u(t)^T R u(t)\right] dt,$$

while ensuring that the closed-loop system is quadratically stable within a high-probability subset of Δ. To proceed, we need to recall some well-known facts.

A. Review of linear-quadratic control and problem formulation

For a given δ, given the system $\dot{x}(t) = A_\delta x(t) + Bu(t)$ with $x(0) = x_0$, and a stabilizing control law $u(t) = Kx(t)$, the system state evolution is given by $x(t) = e^{(A_\delta + BK)t} x_0$, $t \ge 0$. Substituting the expression for $x(t)$ into the cost function yields $J(K,\delta) = x_0^T P x_0$, with $P = \int_0^\infty \left[e^{(A_\delta^T + K^T B^T)t}(S + K^T R K) e^{(A_\delta + BK)t}\right] dt$. By an application of the fundamental theorem of calculus, and recalling that the state feedback law is stabilizing, one gets that P satisfies the continuous-time Lyapunov equation $(A_\delta + BK)^T P + P(A_\delta + BK) = -(S + K^T R K)$. So, a stabilizing K yields $J(K,\delta) = x_0^T P x_0$, with $P \succ 0$ that satisfies the continuous-time Lyapunov equation. Viceversa, for a pair of matrices (P, K), with $P \succ 0$ satisfying the continuous-time Lyapunov equation, it is straightforward to check that $V(x) = x^T P x$ is a Lyapunov function, and therefore K is stabilizing, and that $J(K,\delta) = x_0^T P x_0$. Thus, the problem

$$\begin{aligned} &\min_{K, P \succ 0} \quad x_0^T P x_0 \\ &\text{subject to} \\ &\quad (A_\delta + BK)^T P + P(A_\delta + BK) = -(S + K^T R K) \end{aligned}$$

is equivalent to minimizing $J(K,\delta)$ w.r.t. to any stabilizing K.

Problem (III-A) applies to any system with matrices A_δ and B, and solving it returns a controller matrix K and a P that depend on δ. We next consider a scenario-robust version of (III-A), namely

$$\begin{aligned} &\min_{K, P \succ 0} \quad x_0^T P x_0 \\ &\text{subject to} \\ &\quad (A_{\delta_i} + BK)^T P + P(A_{\delta_i} + BK) \preceq -(S + K^T R K), \\ &\qquad\quad i = 1,\dots,N. \end{aligned}$$

The K^* that solves (III-A) is stabilizing for all the scenarios $\delta_1,\dots,\delta_N$, and the resulting P^* defines a common Lyapunov function for all of the A_{δ_i}'s, while $x_0^T P^* x_0$ serves as a common upper bound on the cost function: this can be seen by rewriting any satisfied constraint as an equality $(A_{\delta_i} + BK^*)^T P^* + P^*(A_{\delta_i} + BK^*) = -(S+K^{*T}RK^*)-\Xi_i$, where $\Xi_i \succ 0$ is a slack matrix; then, we have $J(K^*, \delta_i) = \int_0^\infty x(t)^T(S + K^{*T}RK^*)x(t)dt \leq \int_0^\infty x(t)^T(S + K^{*T}RK^* + \Xi_i)x(t)dt = x_0^T P^* x_0$ for every δ_i.

B. Baseline and post-design appropriateness

Problem (III-A) defines a decision map from $\delta_1, \dots, \delta_N$ to (P^*, K^*). We say that the solution (P^*, K^*) is baseline appropriate for a $\delta \in \Delta$ if

$$(A_\delta + BK)^T P^* + P^*(A_\delta + BK^*) \preceq -(S + K^{*T}RK^*).$$

Since $-(S + K^{*T}RK^*) \prec 0$, the closed-loop system with the designed K^* is quadratically stable for all values δ for which (P^*, K^*) is baseline appropriate. Moreover, the cost of the designed controller is at most $x_0^T P^* x_0$ for all these values of δ.

It is well-known that the decision map defined by a robust optimization problem like (III-A), where constraint satisfaction plays the role of baseline appropriateness, is consistent in the sense of Assumption 1 (see [7][Section 3] for a proof). In post-design, we consider the obtained solution (P^*, K^*) to be *post-design* appropriate if the *tighter* constraint

$$(A_\delta + BK^*)^T P^* + P^*(A_\delta + BK^*) \preceq -\frac{1}{\gamma}(S + K^{*T}RK^*)$$

is satisfied for a predefined parameter $\gamma \in (0, 1)$. By introducing a slack matrix as above, it is immediate to see that if (P^*, K^*) is post-design appropriate for δ, then $J(K^*, \delta) \leq \gamma x_0^T P^* x_0$. Thanks to the theory of this paper, the risk of post-design inappropriateness can be tightly bounded without any extra data.

C. Equivalent formulation as an LMI

Following [38][Chapter 7.2.1] and [1], the scenario problem (III-A) can be reformulated as a convex program with LMIs (linear matrix inequalities) as follows. Since P is invertible, we rewrite the constraint in (III-A) as

$$P^{-1}(A_{\delta_i}+BK)^T + (A_{\delta_i}+BK)P^{-1} \preceq -P^{-1}(S+K^TRK)P^{-1},$$

define $Q := P^{-1}$ and $X := KP^{-1}$, and solve the scenario problem (III-A) in the decision variables Q and X (clearly, K^* can be recovered by the formula $K^* = X^*(Q^*)^{-1}$). Now, by an application of the Schur complement, the mathematical program becomes

$$\min_{X, Q \succ 0} \quad x_0^T Q^{-1} x_0$$

subject to

$$\begin{bmatrix} -(QA_{\delta_i}^T + A_{\delta_i}Q + X^TB^T + BX) & \begin{bmatrix} X^T & Q \end{bmatrix} \\ \begin{bmatrix} X \\ Q \end{bmatrix} & \begin{bmatrix} R^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \end{bmatrix} \preceq 0,$$
$$i = 1, \dots, N,$$

where the constraint is a convex LMI, [38], and $x_0^T Q^{-1} x_0$ is a convex cost function[3].

D. Numerical results

We solved the problem with $x_0 = [1, 1, 1, 1]^T$ and $N = 2000$ scenarios[4], and we set $\beta = 10^{-7}$. We solved the convex program in MATLAB© with the CVX package, [39], [40], and we obtained

$$P^* = \begin{bmatrix} 0.2678 & 0.1462 & -1.1967 & 0.3250 \\ 0.1462 & 0.1323 & -1.1120 & 0.3413 \\ -1.1967 & -1.1120 & 22.2723 & -4.7196 \\ 0.3250 & 0.3413 & -4.7196 & 1.7959 \end{bmatrix}$$

$$K^* = \begin{bmatrix} 0.8641 & 0.9024 & -12.7202 & 4.7089 \\ 0.4709 & 0.4114 & -2.8849 & 0.7778 \end{bmatrix}$$

yielding $x_0^T P^* x_0 = 12.0367$. The cardinality of the baseline support set was $s_N^{b,*} = 4$, so, according to (1), the *baseline risk* of our design is at most 1.53% ($\epsilon(s_N^{b,*})$) with practical certainty (confidence $1 - 10^{-7}$). On the other hand, there were 71 scenarios for which the tighter constraint

$$(A_{\delta_i} + BK)^T P^* + P^*(A_{\delta_i} + BK^*) \preceq -\frac{1}{0.8}(S + K^{*T}RK^*)$$

is violated, yielding $s_N^{+,*} = 75$ (the baseline support list in our simulation was unique). Applying Theorem 1, the risk that this constraint is violated by the infinitely many δ's that were not included in the training set (*post-design risk*) can be bounded by 6.94% ($\bar{\epsilon}(s_N^{+,*})$). This implies that no more than 6.94% of the uncertain plants will result in a cost larger than $0.8 \cdot 12.0367 = 9.6294$.

IV. Conclusions

In this paper, we introduced a novel post-design verification framework for data-driven design, distinguishing two levels of appropriateness: *baseline appropriateness*, which guides the design process, and *post-design appropriateness*, which serves as a criterion for *a posteriori* evaluation. These notions correspond to two different kinds of risk. The first, termed *baseline risk*, is the traditional subject of the scenario approach, while the second, termed *post-design risk*, was not previously covered by the theory of the scenario approach. In alternative approaches, the post-design risk is typically evaluated using test sets. We showed that, building on the theory of the scenario approach,

[3]Convexity of $x_0^T Q^{-1} x_0$ can be seen as follows: for $Q \succ 0$, $x_0^T Q^{-1} x_0$ is convex if and only if $\{Q, h : x_0^T Q^{-1} x_0 \leq h\}$ is a convex set. The latter set, by an application of the Schur complement, can be written as $\{Q, h : h - x_0^T Q^{-1} x_0 \geq 0\} = \{Q, h : \begin{bmatrix} h & x_0^T \\ x_0 & Q \end{bmatrix} \succeq 0\}$, which is the convex set defined by an LMI.

[4]For reproducibility, we describe how the scenarios were generated in our simulation. Following [1], the uncertain variables were independently and uniformly generated from the following intervals: $L_p \in [-2.93, -1]$, $L_\beta \in [-73.14, -4.75]$, $L_r \in [0.78, 3.18]$, $N_p \in [-0.042, 0.086]$, $N_\beta \in [2.59, 8.94]$, $N_{\dot\beta} \in [0, 0.1]$, $N_r \in [-0.39, -0.29]$. We remark that our approach does not make use of this knowledge, which is here provided only for reproducibility.

effective bounds on the post-design risk can be derived via an extended notion of complexity, computable directly from the training set, without the need for any additional test set.

Remarkably, the upper bounds derived in this paper can be shown to be tight in many relevant circumstances: this claim will be substantiated in future work by deriving complexity-based valid lower bounds on the post-design risk.

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