Robust non-convex optimization with structured constraints: complexity bounds and guaranteed reliability level of the scenario solution

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***Abstract*— In this paper, we show how a separable structure between decision and uncertain variables in the constraints of non-convex robust scenario optimization problems can be exploited to bound the complexity associated with the solution. The resulting bounds are easily computable, and can be solved prior to determining the solution to the non-convex scenario program. Leveraging the scenario approach theory, these bounds can be used to find suitable certifications of the risk (a posteriori, once the scenarios are collected). Furthermore, this result can be exploited to determine the size of the scenario sample necessary to provide a user-chosen reliability level of the solution, for which we discuss both a one-shot and an iterative resolution approach.**

I. INTRODUCTION

The scenario approach [1], [2], [3] is a framework for data-driven decision making in presence of uncertainty that has attracted significant attention in systems and control, with a large body of theoretical [4], [5], [6], [7], [8], [9], [10], [11] and application-driven [12], [13], [14], [15], [16] work. At its core, the scenario approach offers a tool at the decision maker's disposal to make a decision with certified robustness with respect to unseen realizations of the uncertainty. According to the scenario theory, the uncertainty is modeled as a random quantity δ taking value in a measurable space $(\Delta, \mathcal{D})$ with distribution $\mathbb{P}$. Rather than attempting to model the probability space $(\Delta, \mathcal{D}, \mathbb{P})$ itself, which may be intractable in complex problems, the scenario approach assumes that N independent identically distributed (i.i.d.) observations $\delta_1, \ldots, \delta_N$ of the uncertainty δ (the so-called *scenarios*) are available. Decisions are then based on $\delta_1, \ldots, \delta_N$ and the scenario approach provides results for evaluating their reliability, i.e., the probability that they remain *appropriate* when an unseen realization of the uncertainty occurs.

Initial results developed in the context of convex robust optimization problems showed that the number of observations could be appropriately chosen to ensure that the optimal solution has an *a priori* defined reliability, with a certain confidence, [4]. Recent results have improved such guarantees and extended the theory's applicability to non necessarily convex problems of various kind, by assessing *a posteriori* the reliability based on a suitably defined measure of complexity of the solution [8], [11], [17], [18]. The application of the scenario theory to non-convex problems presents two main practical issues: determining the complexity of the solution can be cumbersome, and the *a posteriori* guarantees can be turned into *a priori* ones only if the problem admits a computable upper-bound on its complexity.

In this paper, we consider a specific class of non-convex robust optimization problems, which is relevant to a number of design problems, and show that their structure eases the evaluation of the complexity as well as an upper bound on it, thus enhancing the applicability of the scenario approach results. Specifically, we are interested in robust non-convex optimization problems over the decision domain $\mathcal{X} \subseteq \mathbb{R}^{n_c} \times \{0,1\}^{n_l}$, minimizing a cost $f(x)$, $f : \mathcal{X} \to \mathbb{R}$, with constraints in the form $g(x) \leqslant b(\delta)$ where $g : \mathcal{X} \to \mathbb{R}^q$ and $b : \Delta \to \mathbb{R}^q$ are vector-valued functions. Here, the inequality is to be understood as taken elementwise, i.e., as q scalar constraints $g_\ell(x) \leqslant b_\ell(\delta), \ell = 1, \ldots, q$. Notice that δ appears only in the right-hand side, that is, the constraints have a separable structure where the decision x is decoupled from the uncertainty δ. Problems with constraints and $\mathcal{X}$ as above are of interest from a control perspective, as they include robust finite horizon optimal control problems for linear systems affected by an additive disturbance and with a control input with possibly some discrete components.

In robust optimization, the objective is to safeguard against the worst case. Given that the only available information about uncertainty comes from the scenarios $\delta_1, \ldots, \delta_N$, this leads to the following robust scenario optimization program:

$$\min_{x \in \mathcal{X}} \quad f(x) \tag{1a}$$

$$\text{s.t.} \quad g(x) \leqslant b(\delta_i), \quad i = 1, \ldots, N, \tag{1b}$$

whose solution is denoted by x_N^*. In this paper, we exploit the special structure of its constraints to show the following results for the class of problems described by (1):

i. there is an *a priori* known bound to the complexity associated with the problem;
ii. the complexity of the solution can be also upper bounded by a scenario dependent quantity, which is easily computed at low computational cost without the need of solving the scenario program itself;
iii. the aforementioned bounds can be used across various scenario approach results for the certification of the

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reliability of the optimal solution x_N^* with significant computational advantages;

iv. given the known bound on complexity, a minimum sample size needed to ensure the resulting risk does not exceed a user-chosen threshold can be determined.

The results we present in this work are comparable in spirit to those presented in [19], [20], but differ in their theoretical foundations. Indeed, [19], [20] make reference to the convex case and exploit the structure of the constraints to find a bound on Helly's dimension lower than the number of decision variables, to then provide lower values of N for guaranteeing some pre-defined reliability of the scenario solution. These theoretical results rest on the *a priori* results in [4], which require convexity of the underlying scenario problem. Conversely, in this work we provide results for non-convex problems, resting on [11] as foundational work.

The remainder of this paper is structured as follows. Section II provides an overview of some fundamental concepts in the scenario approach, giving the theoretical foundations necessary for the main contributions of the work. In Section III, we present the key bounds to the complexity which follow from the separable structure of the non-convex constraints in (1). Subsequently, the implications of these bounds for the application of scenario approach results in the context of (1) are discussed in Sections IV and V. The theoretical results are illustrated through a simulation example in Section VI, where they are applied to a finite horizon optimal control problem for the regulation of a Heating, Ventilation, and Air Conditioning (HVAC) system. Some concluding remarks are given in Section VII.

II. The Scenario Approach: A Recap of Fundamental Concepts

We define $x_N^* \in \mathcal{X}$ as the optimal solution to (1), where the subscript is a shortcut for $x^*(\delta_1, \dots, \delta_N)$, to recall that the solution is obtained using N scenarios. In view of the i.i.d. assumption, the collection of scenarios $\delta_1, \dots, \delta_N$ can be thought of as a sample drawn from the product probability space $(\Delta^N, \mathcal{D}^{\otimes N}, \mathbb{P}^N)$.

Assumption 1. *For every N, with probability one, the scenario program* (1) *is feasible, and the optimizer x_N^* exists and is unique.*[1] ◇

In the context of the robust scenario optimization problem (1), a given decision x is inappropriate for an unseen outcome δ if the constraint (1b) is not satisfied, i.e., $g(x) \not\leq b(\delta)$ (meaning that $g_\ell(x) > b_\ell(\delta)$ for some ℓ). As such, we can define the *risk* associated with x as follows.

Definition 1 (Risk)**.** *The risk associated with any given $x \in \mathcal{X}$ is $V(x) = \mathbb{P}\{\delta \in \Delta : g(x) \not\leq b(\delta)\}$.* ◁

In the context of (1), we are interested in $V(x_N^*)$, which is $V(x)$ evaluated for the optimal solution of (1). This quantity cannot be directly computed because it requires the knowledge of $\mathbb{P}$, which we do not assume to possess. The idea of the scenario approach is thus to determine a bound on $V(x_N^*)$ which is guaranteed with a certain confidence, since x_N^* depends on the extracted scenarios.

In [11], it was shown that the risk $V(x_N^*)$ is highly correlated to an observable quantity linked to the scenarios called *complexity* s_N^*, from which certifiable bounds on $V(x_N^*)$ can be derived. The definition of s_N^* moves directly from that of the support list.

Definition 2 (Support list)**.** *Given a list of scenarios $\delta_1, \dots, \delta_N$, a support list is any sub-list $\mathcal{S}_N^k = \delta_{i_1}, \dots, \delta_{i_k}$, with $i_1 < i_2 < \dots < i_k$ such that:*

i. $x_k^* = x_N^*$, where x_k^* is the minimizer of (1) *with scenarios in $\mathcal{S}_N^k$ only.*

ii. $\mathcal{S}_N^k$ is irreducible, i.e., no element can be removed from $\mathcal{S}_N^k$ without changing the solution to (1). ◁

Definition 3 (Complexity)**.** *For any given $\delta_1, \dots, \delta_N$, its related complexity is defined as $s_N^* = |\mathcal{S}_N^*|$, where $\mathcal{S}_N^*$ is a support list of minimal size and $|\cdot|$ denotes its length.* ◁

The fundamental results in [11] relate to finding a $[0,1]$-valued function, $\epsilon_{N,\beta} : \{0, 1, \dots, N\} \to [0,1]$, such that the following is satisfied for any $\mathbb{P}$:

$$\mathbb{P}^N\{V(x_N^*) \leqslant \epsilon_{N,\beta}(s_N^*)\} \geqslant 1 - \beta, \tag{2}$$

for a given confidence level $\beta \in (0,1)$.[2] The interpretation of (2) is that $\epsilon_{N,\beta}(s_N^*)$ upper-bounds $V(x_N^*)$ with high confidence, justifying using $\epsilon_{N,\beta}(s_N^*)$ as a certificate for the risk, which is tight and informative. Further discussion can be found in [11].

A few remarks are in order to highlight some computational challenges in non-convex settings. Firstly, computing the minimal support list may be a very hard combinatorial problem in general. While useful guarantees can still be obtained by identifying any support list (not necessarily minimal), this can remain computationally challenging. Indeed, while for the convex case it is often the case that a support list is almost surely given by those constraints which are active at the solution, in non-convex programs non-active constraints may be part of support lists and finding a support list may require removing constraints one by one and repeatedly solve the non-convex program. Additionally, there is in general no *a priori* bound on s_N^*, which may be as large as N. This prevents one to determine a sufficiently large N guaranteeing that the resulting risk $V(x_N^*)$ does not exceed a user-chosen threshold with high confidence.

III. Bounding Complexity of Non-Convex Scenario Problems with Separable Constraints

The core of this paper relies on the two following propositions, where we exploit the separable structure of the constraints in (1) to provide two bounds on the complexity

1 Note that the uniqueness assumption could be released by introducing a suitable tie-break rule to univocally define the solution. However, to streamline the presentation we prefer to stick to Assumption 1.

2 We include both N and β in the notation of the function $\epsilon_{N,\beta}(\cdot)$ to highlight that its definition depends explicitly on the number of scenarios in (1), and the desired confidence. $\mathbb{P}^N$ is a product probability because it refers to the variability of $\delta_1, \dots, \delta_N$ on which x_N^* and s_N^* depend.

s_N^* of the solution x_N^*. Despite their simplicity, these results have practical significance and are instrumental in the developments proposed in the following sections.

Proposition III.1. *Given the scenario problem* (1)*, for any $\delta_1, \ldots, \delta_N \in \Delta^N$, let ς_N be the cardinality of the set of indices $\mathcal{I}_N = \{i_1, \ldots, i_q\}$, where for each $\ell = 1, \ldots, q$, i_ℓ is obtained by*

$$i_\ell = \arg \min_{i=1,\ldots,N} b_\ell(\delta_i).^3 \tag{3}$$

Then, the complexity satisfies $s_N^* \leqslant \varsigma_N$. □

Proof: We start by noting that, for each $\ell = 1, \ldots, q$, $g_\ell(x) \leqslant b_\ell(\delta_i), \quad i = 1, \ldots, N$ defines a set of nested constraints, with $b_\ell(\delta_{i_\ell})$ the one that dominates all others. It follows that $x_{\mathcal{I}_N}^* = x_N^*$, where $x_{\mathcal{I}_N}^*$ is the solution of (1) with only those scenarios indexed by $\mathcal{I}_N$ in place, as they are sufficient to define the feasibility region of the optimization problem with N scenarios. The list of scenarios indexed by $\mathcal{I}_N$ need not be irreducible, and therefore may not be a support list, but surely it contains one as a sub-list since $x_{\mathcal{I}_N}^* = x_N^*$. The statement follows from the definition of s_N^* as the cardinality of a *minimal* support list. ■

The result in Proposition III.1 holds *a posteriori*, once scenarios are collected. On the other hand, the following Proposition III.2 is a statement which can be made *a priori*, independently of the scenarios $\delta_i, i = 1, \ldots, N$.

Proposition III.2. *Given the robust scenario optimization problem* (1)*, for each sample of scenarios $\delta_1, \ldots, \delta_N \in \Delta^N$, the complexity satisfies $s_N^* \leqslant q$, where q is the number of scalar constraints in* (1b). □

The proof follows by noting that $\varsigma_N \leqslant q$, by definition. It may be that $\varsigma_N < q$ since, being a set, $\mathcal{I}_N$ does not include any repetitions, while there may be one same scenario δ_i dominating all others for two or more values of ℓ.

The importance of the bounds discussed in Propositions III.1 and III.2 is twofold. On the one hand, we have that both bounds can be found easily: ς_N can be easily counted after solving (3) by means of sorting procedures; q is known by construction of the constraints. On the other, the bound on the complexity ς_N can be obtained without finding the solution x_N^*, but right after the N scenarios are collected. Furthermore, as shown in the following sections, these propositions prove instrumental in evaluating *a posteriori* risk certificates, or in determining *a priori* the minimum sample size necessary to ensure the risk remain within a certain user-defined bound.

It is possible to further exploit these results for computational advantages in the resolution of (1) itself. Indeed, as is clear from the proof of Proposition III.1, to obtain the optimal solution x_N^*, it is sufficient to solve the robust non-convex scenario problem with only the scenarios indexed by $\mathcal{I}_N$ in place. This provides a significant advantage in terms of computational effort as, in general, $\varsigma_N \leqslant q \ll N$.

3 In (3), if two (or more) scenarios δ_i, δ_j are such that $b_\ell(\delta_i) = b_\ell(\delta_j)$, we assume a unique i_ℓ is selected via a tie-break rule, e.g., $i_\ell = \min\{i, j\}$.

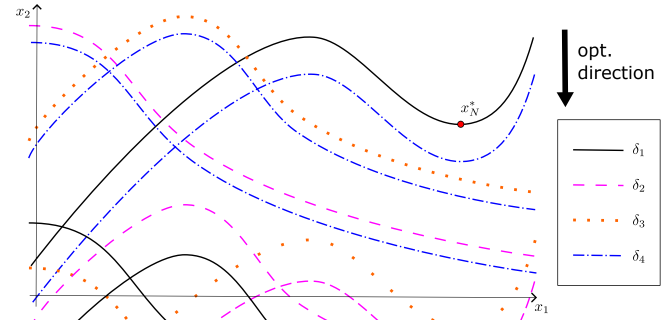

Fig. 1. Counterexample showing that the minimal support list needs not be contained in $\mathcal{I}_N$. The optimization direction is given by the cost $f(x) = x_2$, while scalar constraints are of the form $g_\ell(x_2) - x_1 \leqslant b_\ell(\delta)$, for $\ell = 1, \ldots, 3$. Different color and line types correspond to different scenarios $\delta_i, i = 1, \ldots, 4$. As it appears, while $\mathcal{I}_N = \{1, 2, 3\}$, $\mathcal{S}_N^* = (\delta_1, \delta_4)$.

Remark 1. *In defining a tie-break rule for* (3)*, it is possible to define one that minimizes ς_N, without violating Proposition III.1. Although this may reduce the gap between ς_N and s_N^*, its computation may become cumbersome.* ◁

Remark 2. *One might ask whether the minimal support list is to be found as a sublist of the scenarios indexed by $\mathcal{I}_N$, as this would provide computational advantages in finding $\mathcal{S}_N^*$, and would remove the need for a bound. This is not the case, as is illustrated in the counterexample presented in Fig. 1. For illustrative purposes, $q = 3$, and $N = 4$. From observation, $\mathcal{I}_N = \{1, 2, 3\}$, whilst the minimal support list is given by $\mathcal{S}_N^* = (\delta_1, \delta_4)$. Nonetheless, we expect that ς_N can be close to s_N^* in many problems of interest.* ◁

IV. Solution Certification

Let us now focus on the consequences of the bounds on the complexity presented in the previous section. The first, direct, consequence of the bound proposed in Proposition III.1 is the result in Theorem IV.1, where the bound ς_N to s_N^* together with [11, Thm. 6] is exploited to certify the risk $V(x_N^*)$.

Theorem IV.1. *Let x_N^* be the solution to* (1)*, and ς_N as in Proposition III.1. Then, for any $\mathbb{P}$, it holds that*

$$\mathbb{P}^N\{V(x_N^*) \leqslant \epsilon_{N,\beta}(\varsigma_N)\} \geqslant 1 - \beta$$

for any user defined $\beta \in (0, 1)$ and $(0, 1)$-valued function $\epsilon_{N,\beta}$ defined as $\epsilon_{N,\beta}(k) = 1 - t(k), k = 0, 1, \ldots, N-1$, with $t(k) \in (0, 1)$ the unique solution to

$$\frac{\beta}{N} \sum_{m=k}^{N-1} \binom{m}{k} t^{m-k} - \binom{N}{k} t^{N-k} = 0 \tag{4}$$

and $\epsilon_{N,\beta}(N) = 1$. □

Proof: The definition of $\epsilon_{N,\beta}$ is taken from [11, Thm. 6], in which it is shown that, for any non-convex robust scenario optimization problem $\mathbb{P}^N\{V(x_N^*) \leqslant \epsilon_{N,\beta}(s_N^*)\} \geqslant 1 - \beta$ is guaranteed for any $\beta \in (0, 1)$, and any $\mathbb{P}$.

Note that $\epsilon_{N,\beta}(k)$ is strictly non-decreasing in $k = 0, 1, \ldots, N$, so that $\epsilon_{N,\beta}(s_N^*) \leqslant \epsilon_{N,\beta}(\varsigma_N)$ in view of

Proposition III.1. This yields $\mathbb{P}^N\{V(x_N^*) \leqslant \epsilon_{N,\beta}(\varsigma_N)\} \geqslant \mathbb{P}^N\{V(x_N^*) \leqslant \epsilon_{N,\beta}(s_N^*)\} \geqslant 1-\beta$. ■

The result of Theorem IV.1, evidently, provides bounds on risk which can be obtained from the problem data. This is advantageous, as it allows us to provide certificates efficiently computed from $b(\delta_i), i = 1, \ldots, N$, without needing to solve the optimization problem multiple times.

V. Experiment sizing for non-convex scenario problems with separable constraints

The result in Theorem IV.1 enables *a posteriori* risk certifications, where the risk certificate is computed only once scenarios are collected and depends on the proportion between the seen ς_N and N. In many design problems, it is often the case that the decision to be taken must guarantee a certain, user defined risk threshold $\bar{\epsilon} \in (0, 1)$. Throughout this section we provide an overview of how to size the scenario problem (i.e., we provide insight into how to pick N) such that the resulting solution x_N^* is guaranteed to satisfy, for some user-chosen $\beta \in (0, 1)$,

$$\mathbb{P}^N\{V(x_N^*) \leqslant \bar{\epsilon}\} \geqslant 1-\beta.$$

A. One-shot sample size selection

Theorem V.1. *For a given confidence parameter $\beta \in (0, 1)$ and a risk threshold $\bar{\epsilon} \in (0, 1)$, define*

$$\bar{N} = \min\big\{N \geqslant q : \quad \epsilon_{N,\beta}(q) \leqslant \bar{\epsilon}\big\}. \tag{5}$$

with $\epsilon_{N,\beta}(\cdot)$ defined as in Theorem IV.1. Then, it holds that

$$\mathbb{P}^{\bar{N}}\{V(x_{\bar{N}}^*) > \bar{\epsilon}\} \leqslant \beta, \tag{6}$$

for any $\mathbb{P}$. □

Proof: Theorem IV.1 and $\varsigma_{\bar{N}} \leqslant q$ yields $\mathbb{P}^{\bar{N}}\{V(x_{\bar{N}}^*) \leqslant \epsilon_{\bar{N},\beta}(q)\} \geqslant 1-\beta$. The statement in (6) follows, as $\epsilon_{\bar{N},\beta}(q) \leqslant \bar{\epsilon}$, from (5). ■

Although Theorem V.1 provides *a priori* guarantees that the risk associated with the solution $x_{\bar{N}}^*$ satisfies the bound on the risk (6) with some prescribed $\bar{\epsilon}, \beta$, it has been shown in the convex setup that in many cases the empirical risk is well below its predetermined bound $\bar{\epsilon}$, [8]. Therefore, it is not necessary to collect $\bar{N}$ observations to achieve a satisfactory result. Relying on a smaller number of scenarios may be advantageous for a number of reasons, as collecting data may be financially costly or time consuming. Indeed, the mantra "data is cheap" does not always hold. This motivates the development in the next section.

B. Iterative sample size selection

To overcome the possible conservativeness of the result in Theorem V.1, [21] proposes an iterative approach that merges an *a priori* computation of the scenarios necessary to guarantee a certain risk level, with an *a posteriori* evaluation of complexity. This provides a solution towards achieving the desired risk threshold $\bar{\epsilon}$ utilizing fewer scenarios than $\bar{N}$. In this section, we shall adapt [21, Alg. 1] to the peculiarity of the non-convex scenario program (1).

Algorithm 1: Incremental scenario resolution scheme

Input: $N_0, N_1, \ldots, N_q$ as in (7)
Output: x^*
1: $j \leftarrow 0$; $N_{-1} \leftarrow 0$
2: Collect i.i.d. samples $\delta_{N_{j-1}+1}, \delta_{N_{j-1}+2}, \ldots, \delta_{N_j}$, independent of previous scenarios
3: Find $\mathcal{I}_{N_j}$ as in Proposition III.1
4: $\varsigma_{N_j} \leftarrow |\mathcal{I}_{N_j}|$
5: **if** $\varsigma_{N_j} \leqslant j$ **then**
6: $\quad x^* \leftarrow \arg\min_{x\in\mathcal{X}} \; f(x)$ s.t. $g(x) \leqslant b(\delta_i), \quad i \in \mathcal{I}_{N_j}$
7: **else**
8: $\quad j \leftarrow j+1$; go to step 2
9: **end**

The algorithm, summarized in Algorithm 1, is as follows: for each iteration $j = 0, 1, \ldots, q$, using sufficiently many N_j scenarios (see Theorem V.2 below), find ς_{N_j} as in Proposition III.1, without solving (1); if $\varsigma_{N_j} \leqslant j$, the iterations are interrupted, (1) is solved with ς_{N_j} scenarios, and the minimizer $x^* = x_{N_j}^*$ is returned; if instead $\varsigma_{N_j} > j$, the procedure is repeated with $j = j+1$. The procedure terminates either when $\varsigma_{N_j} \leqslant j$, or $j = q$ (since $\varsigma_{N_q} \leqslant q$ is always guaranteed). The following theorem, which is a revisitation of [21, Thm. 1], specifies how to select N_j, $j = 0, 1, \ldots, q$, to ensure that the risk of x^* is below threshold $\bar{\epsilon}$ with confidence $1-\beta$.

Theorem V.2. *For a given confidence parameter $\beta \in (0, 1)$, and risk threshold $\bar{\epsilon} \in (0, 1)$, define N_j, $j = 0, 1, \ldots, q$ according to*

$$N_j = \min\left\{ N > \bar{M}_j : \tilde{\beta} \sum_{m=j}^{\bar{M}_j} \binom{m}{j} (1-\bar{\epsilon})^{m-j} \geqslant \binom{N}{j} (1-\bar{\epsilon})^{N-j}. \right\} \tag{7}$$

with $\tilde{\beta} = \beta/(q+1)(\bar{M}_j+1)$,

$$\bar{M}_j = \min\left\{ M \geqslant j : \sum_{i=0}^{j-1} \binom{M}{i} \bar{\epsilon}^i (1-\bar{\epsilon})^{M-i} \leqslant \beta \right\}, \quad j = 1, \ldots, q \tag{8}$$

and $\bar{M}_0 = \bar{M}_1$. Then, for the x^* returned by Algorithm 1, it holds that, for any $\mathbb{P}$,

$$\mathbb{P}^{N_q}\{V(x^*) \leqslant \bar{\epsilon}\} \geqslant 1-\beta. \tag{9}$$

□

Proof: Since x^* returned by Algorithm 1 is defined as $x_{N_j}^*$ for the lowest j such that $\varsigma_{N_j} \leqslant j$, the event $V(x^*) > \bar{\epsilon}$ occurs if and only if $\varsigma_{N_\ell} > \ell$ for $\ell < j$, $\varsigma_{N_j} \leqslant j$ and

$V(x^*_{N_j}) > \bar{\epsilon}$ for some j. Therefore:

$$
\begin{aligned}
&\mathbb{P}^{N_q}(V(x^*) > \bar{\epsilon}) \\
&= \mathbb{P}^{N_0}\{\varsigma_{N_0} \leqslant 0 \wedge V(x^*_{N_0}) > \bar{\epsilon}\} + \\
&\quad \mathbb{P}^{N_1}\{\varsigma_{N_0} > 0 \wedge \varsigma_{N_1} \leqslant 1 \wedge V(x^*_{N_1}) > \bar{\epsilon}\} + \cdots \\
&\quad \mathbb{P}^{N_q}\{\varsigma_{N_0} > 0 \wedge \varsigma_{N_1} > 1 \wedge \cdots \wedge \\
&\qquad \varsigma_{N_{q-1}} > q-1 \wedge \varsigma_{N_q} \leqslant q \wedge V(x^*_{N_d}) > \bar{\epsilon}\} \\
&\leqslant \sum_{j=0}^{q} \mathbb{P}^{N_j}\{\varsigma_{N_j} \leqslant j \wedge V(x^*_{N_j}) > \bar{\epsilon}\} \\
&\leqslant \sum_{j=0}^{q} \mathbb{P}^{N_j}\{s^*_{N_j} \leqslant j \wedge V(x^*_{N_j}) > \bar{\epsilon}\}
\end{aligned}
$$

where the first inequality follows from dropping the conditions $\varsigma_{N_0} > 0 \wedge \cdots \wedge \varsigma_{N_{j-1}} > j-1$, for each $j = 0, \ldots, q$, while the second because $s^*_{N_j} \leqslant \varsigma_{N_j}$ (see Proposition III.1).

The proof of [21, Thm. 1] rests on showing that the final term in the above chain of inequalities is no bigger than β.[4] This directly implies that $\mathbb{P}^{N_q}(V(x^*) > \bar{\epsilon}) \leqslant \beta$, which is equivalent to the theorem statement. ∎

As $N_j \ll \bar{N}$ for small values of j, whenever the algorithm halts at the first iterations a great saving of scenarios is achieved. A fundamental difference between the algorithm proposed here and [21, Alg. 1] is that in our context it is not necessary to solve the scenario optimization problem at each iteration, but rather (1) must be solved *only once*, once the suitable N_j is identified. Additionally, it is not necessary to solve the scenario program with N_j scenarios, but only those in $\mathcal{I}_{N_j}$, as this suffices to reconstruct $x^*_{N_j} = x^*$ (see Section III). These offer significant computational advantages, given the non-convex nature of (1).

VI. Numerical Example

We consider the setting in [22], where a heating, ventilation, and air conditioning (HVAC) system is described, with a single rooftop unit (RTU). The RTU is fitted with a two-stage compressor, a multi-speed supply fan, and a modulating economizer. A controller regulates the building's temperature and humidity, with measurements of these variables available via dedicated sensors. The system, under some nominal conditions, is described via the following discrete system dynamics

$$\xi_{\tau+1} = A\xi_\tau + B_u u_\tau + B_d d_\tau$$

where $\xi \in \mathbb{R}^4$ is the system state, with ξ_1 the temperature difference away from the operating point of the zone to be regulated, in [°C], and ξ_3 its difference with respect to the nominal humidity, in [%]; $u \in \mathbb{R}^2$ is the input, and $d \in \mathbb{R}^2$ some unknown disturbance. The control input, including the supply fan's speed and the compressor, is quantized to three possible levels: {OFF, LOW, HIGH}, with these values coded as $\{-50, 0, 50\}$, respectively. The system and problem parameters are taken from [22] where a sampling period $T_s = 300\,\text{s}$ is used.

4 Although [21, Thm. 1] is proven in a convex setting, the argument relies solely on the property that $s^*_N \leqslant q, \forall N$. Thus, the same argument remains valid in the present context in view of Proposition III.2.

The control objective is to stabilize ξ_1 and ξ_3 around 0, which corresponds to the given operating point. We propose a solution based on a finite horizon optimal control problem (FHOCP) using N scenarios. Specifically, we formulate the following robust scenario optimization problem over the time horizon $[0, T]$:

$$
\begin{aligned}
\min_{u \in \{0,50,-50\}^{2\times T}} \quad & \frac{1}{T}\sum_{k=0}^{T} w_T(h_{T,k})^2 + w_H(h_{H,k})^2 \qquad (10)\\
\text{s.t.} \quad & \bar{\xi}^{(i)}_k = A\bar{\xi}^{(i)}_{k-1} + B_u u_{k-1} + B_d d^{(i)}_{k-1} \\
& |\bar{\xi}^{(i)}_{1,k}| \leqslant h_{T,k} \\
& |\bar{\xi}^{(i)}_{3,k}| \leqslant h_{H,k} \\
& h_{T,k} \geqslant \bar{T} \\
& h_{H,k} \geqslant \bar{H} \\
& k = 1, \ldots, T, \ i = 1, \ldots, N
\end{aligned}
$$

and each scenario defined as $\delta_i = [\bar{\xi}^{(i)\top}_0, d^{(i)\top}_0, \cdots, d^{(i)\top}_{T-1}]^\top$. The bounds $h_{T,k}$ and $h_{H,k}$ are absolute bounds on the temperature and humidity variation with respect to their operational set point. They are introduced to act as soft constraints compared to using bounds $\bar{T} = 1^\circ$C and $\bar{H} = 5\%$. The realizations of the disturbance in $\delta_i, i = 1, \ldots, N$, are obtained according to the autoregressive process model

$$d_{\tau+1} = A_d d_\tau + e_\tau \qquad (11)$$

where $A_d = 0.9835 I_2$, and $e_\tau \sim \mathcal{N}(0, R_e)$ is a white Gaussian noise given by a sequence of i.i.d. variables with zero mean and covariance matrix $R_e = 0.6147 I_2$. The initial condition $[\bar{\xi}^{(i)\top}_0 d^{(i)\top}_0]^\top$ is taken as a realization of a Gaussian random variable with distribution $\mathcal{N}(0, P)$, where $P = \text{diag}(0.17, 1.7, 1.7, 1.7, 0.6147, 0.6147)$.

The state inequality constraints in (10) can be readily rewritten in the form in (1), $g(x) \leqslant b(\delta_i)$, by algebraic manipulation. Setting $T = 12$, then, the constraint dimension is $q = 48$. This results in a Mixed Integer Quadratic Problem, that is solved in Matlab using Yalmip and Mosek.

By (5) we obtain that the number of scenarios necessary to ensure that the resulting risk is bounded by $\bar{\epsilon} = 0.1$ with confidence larger or equal to $1 - \beta = 1 - 10^{-6}$ is $\bar{N} = 951$.

In Figure 2 we plot the results of 50 runs where the scenario problem (10) is solved using $\bar{N}$ scenarios. Each run is associated with a segment with two extrema: above, represented as a black cross, the value of the *a posteriori* bound $\epsilon_{\bar{N},\beta}(\varsigma_{\bar{N}})$ obtained by first computing $\varsigma_{\bar{N}}$ and then solving (4) with $\beta = 10^{-6}$ and $k = \varsigma_{\bar{N}}$; below, represented with a blue circle, the value of the empirical risk, obtained by drawing 10^4 additional independent realizations of δ, and evaluating the fraction of instances when the constraint is violated.

Similarly, in Figure 3, we plot the results obtained by using Algorithm 1 to determine the number of scenarios. From the comparison of Figure 2 and Figure 3, we see that the empirical risk is closer to the bound, while not exceeding $\bar{\epsilon}$,

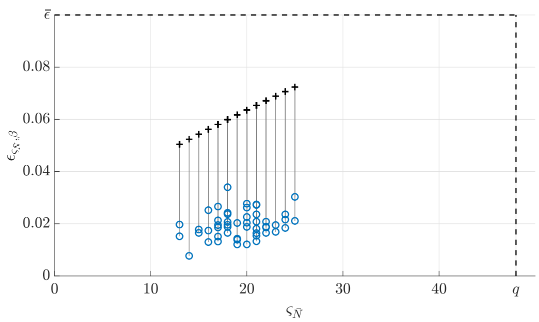

Fig. 2. Comparison of empirical risk vs. complexity using $\bar{N}$ scenarios.

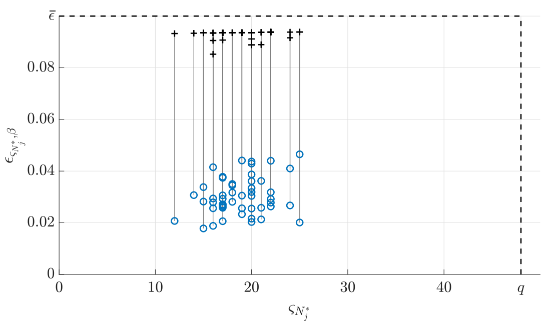

Fig. 3. Comparison of empirical risk vs. complexity using Algorithm 1 to determine the number of scenarios.

with a lower number of scenarios, as seen in the histogram in Figure 4.

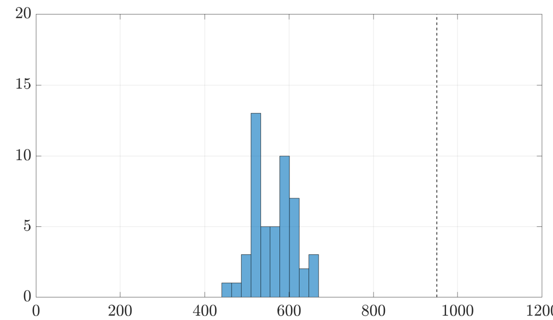

Fig. 4. Histogram of the number of scenarios used in 50 runs of Algorithm 1. The dotted line corresponds to the *a priori* bound $\bar{N} = 951$.

VII. Conclusion

In this paper, we consider a class of non-convex scenario optimization problems and show that the structure of the constraints can be exploited to find *a priori* and *a posteriori* bounds on the complexity of the solution. This enables a computationally efficient use of the scenario approach to provide guaranteed bounds on the risk associated with the solution, as well as to design a mechanism for the sizing of the scenario sample necessary to ensure that the risk remain below a user defined value.

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