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# Quasi-invariant states

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In this paper, we develop the theory of quasi-invariant (respectively, strongly quasi-invariant) states under the action of a group $G$ of normal $*$-automorphisms of a $*$-algebra (or von Neumann algebra) $\mathcal{A}$. We prove that these states are naturally associated to left-$G$-1-cocycles. If $G$ is compact, the structure of strongly $G$-quasi-invariant states is determined. For any $G$-strongly quasi-invariant state $\varphi$, we construct a unitary representation associated to the triple $(\mathcal{A}, G, \varphi)$. We prove, under some conditions, that any quantum Markov chain with commuting, invertible and Hermitian conditional density amplitudes on a countable tensor product of type I factors is strongly quasi-invariant with respect to the natural action of the group $\mathcal{S}_\infty$ of local permutations and we give the explicit form of the associated cocycle. This provides a family of nontrivial examples of strongly quasi-invariant states for locally compact groups obtained as inductive limit of an increasing sequence of compact groups.

*Keywords*: Quasi-invariant states; group of $*$-automorphisms.

## 1. Introduction

Given a faithful state on $*$-algebra $\mathcal{A}$ and a group $G$ of $*$-automorphisms of $\mathcal{A}$, we say that $\varphi$ is $G$-invariant if

$$\varphi(g(a)) = \varphi(a), \quad \forall\, g \in G, \;\; a \in \mathcal{A}. \tag{1.1}$$

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This class of states is of paramount importance in many branches of mathematics and physics and it has been studied in different contexts (cf.[2–5, 7, 8] and references therein). However, invariant states are rare, both in classical and in quantum probability. Moreover, in classical probability, there is a much wider class of states, namely those associated to quasi-invariant measures. In particular, given a measurable space $(S, \mathcal{B})$ and a group $G$ acting on $S$ by measurable transformations

$$s \in S \mapsto gs \in S, \quad g \in G,$$

a measure $\mu$ on $(S, \mathcal{B})$ is said to be $G$-quasi-invariant if

$$\mu \circ g \sim \mu, \quad \forall\, g \in G,$$

where $\sim$ is meant in the sense of mutually absolute continuity of the two measures. It is worth noticing that quasi-invariant measures constitute the natural environment for the theory of dynamical systems and ergodic theory, but until recently, there was no quantum analogue of the quasi-invariant measures.

In this paper, we develop the theory of quasi-invariant states. In particular, we extend definition (1.1) as follows: we assume that for every $g \in G$ there exists $x_g \in \mathcal{A}$ such that

$$\varphi(g(a)) = \varphi(x_g a), \tag{1.2}$$

and we call $G$-quasi-invariant a state satisfying (1.2). Moreover, if in addition $x_g = x_g^*$ for every $g \in G$, the state $\varphi$ is said to be $G$-strongly quasi-invariant. In this case, we prove that the Radon–Nikodym derivatives $x_g$'s are invertible positive operators and satisfy the normalized left-$G$-1-cocycle property

$$x_{g_2 g_1} = x_{g_1}(g_1^{-1}(x_{g_2})), \quad x_e = 1.$$

Moreover, we show that the $*$-algebra generated by the Radon–Nikodym derivatives $x_g$'s is commutative. We further study the case of a compact group $G$ of normal $*$-automorphisms of a von Neumann algebra $\mathcal{A}$. In particular, we give the structure of a normal $G$-strongly quasi-invariant state $\varphi$, when $G$ is a compact group, and we prove that it has the following form:

$$\varphi(a) = \varphi_G(\kappa_G^{-1} a), \quad \forall\, a \in \mathcal{A}, \tag{1.3}$$

for some bounded invertible positive operator $\kappa_G$ in the centralizer of $\varphi$ whose structure is explicitly given (see Theorem 5). Conversely, if $\varphi_G$ is a $G$-invariant state and $\kappa_G$ is an invertible positive operator in the centralizer of $\varphi$, then the state $\varphi$, defined by the right-hand side of (1.3), is a $G$-quasi-invariant state with cocycle

$$x_g = \kappa_G g^{-1}(\kappa_G^{-1}), \quad \forall\, g \in G. \tag{1.4}$$

We study the inductive limit associated to a group $G$ which is the union of an increasing sequence of compact groups. For any $G$-strongly quasi-invariant state $\varphi$, the unitary representation of the group $G$ is also given. Moreover, we give some examples and we investigate the structure of a $\mathcal{S}_\infty$-quasi-invariant state $\varphi$, where

$\mathcal{S}_\infty = \bigcup_{N=1}^{\infty} \mathcal{S}_N$ is the group of local permutations on an infinite tensor product of a von Neumann algebra ($\mathcal{S}_N$ is the group of permutations on $\{1, 2, \ldots, N\}$).

## 2. Quasi-Invariant States Under a Group of ∗-Automorphisms

In the following, $\mathcal{A}$ will denote a $*$-algebra, $\varphi$ a *faithful* state on $\mathcal{A}$ and $G \subseteq \mathrm{Aut}(\mathcal{A})$ a group of $*$-automorphisms of $\mathcal{A}$.

**Proposition 1.** *Suppose that, for any $g \in G$, there exists a $x_g \in \mathcal{A}$ such that*

$$\varphi(g(a)) = \varphi(x_g a), \quad \forall\, a \in \mathcal{A}. \tag{2.1}$$

*Then $x_g$ is unique and the map $g \in G \mapsto x_g \in \mathcal{A}$ is a* normalized multiplicative left $G$-1-cocycle, *i.e. it satisfies the identities*

$$x_e = 1, \tag{2.2}$$

$$x_{g_2 g_1} = x_{g_1} g_1^{-1}(x_{g_2}), \quad \forall\, g_2, g_1 \in G. \tag{2.3}$$

*In particular, each $x_g$ is* invertible *and its inverse is*

$$x_g^{-1} = g^{-1}(x_{g^{-1}}) (\Leftrightarrow x_{g^{-1}} = g(x_g^{-1})). \tag{2.4}$$

**Proof.** Suppose that for any $g \in G$, there exist $x_g$, $y_g \in \mathcal{A}$ such that

$$\varphi(g(a)) = \varphi(x_g a) = \varphi(y_g a), \quad \forall\, a \in \mathcal{A}.$$

Therefore, for all $a \in \mathcal{A}$, $\varphi((x_g - y_g)a) = 0$. Since $\varphi$ is faithful, then by taking $a = (x_g - y_g)^*$ it follows that $x_g = y_g$.

Now note that

$$\varphi(a) = \varphi(e(a)) = \varphi(x_e a), \quad \forall\, a \in \mathcal{A}.$$

Since $\varphi$ is faithful, (2.2) follows. Now for any $g_2, g_1 \in G$, one has

$$\begin{aligned} \varphi(g_2 g_1(a)) &= \varphi(x_{g_2 g_1} a) = \varphi(x_{g_2}(g_1(a))) = \varphi(g_1(g_1^{-1}(x_{g_2})a)) \\ &= \varphi(x_{g_1}(g_1^{-1}(x_{g_2})a)) = \varphi((x_{g_1} g_1^{-1}(x_{g_2}))a). \end{aligned}$$

This is equivalent to

$$\begin{aligned} 0 &= \varphi(x_{g_2 g_1} a) - \varphi(x_{g_1}(g_1^{-1}(x_{g_2}))a) \\ &= \varphi((x_{g_2 g_1} - x_{g_1}(g_1^{-1}(x_{g_2})))a). \end{aligned}$$

Since $a$ is arbitrary, one can choose $a = (x_{g_2 g_1} - x_{g_1}(g_1^{-1}(x_{g_2})))^*$. This gives

$$0 = \varphi(|(x_{g_2 g_1} - x_{g_1}(g_1^{-1}(x_{g_2})))^*|^2).$$

Since $\varphi$ is faithful, (2.3) follows. Taking $g_2 := g_1^{-1}$ in (2.3), one finds, using $x_e = 1$

$$1 = x_e = x_{g_1}(g_1^{-1}(x_{g_1^{-1}})).$$

This implies that $x_{g_1}$ is invertible and its inverse is

$$x_{g_1}^{-1} = g_1^{-1}(x_{g_1^{-1}}),$$

since $g_1$ is arbitrary, the first identity in (2.4) follows. Multiplying both sides of it by $g_1$, one obtains

$$g_1(x_{g_1}^{-1}) = x_{g_1^{-1}},$$

which is equivalent to the second identity in (2.4). $\square$

The following result is known. We include a simple proof of it for completeness.

**Lemma 1.** *If $\mathcal{A}$ is a $C^*$-algebra, $g$ is a $*$-automorphism of $\mathcal{A}$ and if*

$$x = g(y),$$

*where $x$ and $y$ are positive self-adjoint, invertible in $\mathcal{A}$. Then*

$$x^s = g(y^s), \quad \forall\, s \in \mathbb{Q}. \tag{2.5}$$

**Proof.**

$$1 = g(yy^{-1}) = g(y)g(y^{-1}).$$

Therefore, $g(y)$ is invertible and

$$g(y)^{-1} = g(y^{-1}) = x^{-1}. \tag{2.6}$$

Thus

$$x^s = g(y^s), \quad \forall\, s \in \mathbb{Z}.$$

For $n \in \mathbb{N}$ one has

$$g(y^{\frac{1}{n}})^n = g((y^{\frac{1}{n}})^n) = g(y) = x.$$

Therefore

$$x^{\frac{1}{n}} = g(y^{\frac{1}{n}}), \quad \forall\, n \in \mathbb{N}.$$

Consequently

$$x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = g(y^{\frac{1}{n}})^m = g(y^{\frac{m}{n}}),$$

and, because of (2.6), $x^s = g(y^s)$ for all $s \in \mathbb{Q}$. $\square$

**Corollary 1.** *Under the assumptions of Lemma 1, if $x_g = x_g^*$, then for any $s \in \mathbb{Q}$*

$$x_g^{-s} = g^{-1}(x_{g^{-1}}^s). \tag{2.7}$$

**Proof.** From Proposition 1, $x_g$ is invertible. Moreover, by Lemma 4 which will be proved later, $x_g$ is positive. Now by using (2.4) and by replacing in (2.5) $g$ by $g^{-1}$, $x$ by $x_g^{-1}$ and $y$ by $x_{g^{-1}}$, one finds

$$x_g^{-s} = g^{-1}(x_{g^{-1}}^s),$$

which is (2.7). $\square$

**Definition 1.** If the pair $(G, x)$ satisfies condition (2.1) of Proposition 1 the state $\varphi$ is called $(G, x)$-*quasi-invariant*, or $G$-quasi-invariant with cocycle $x$.

**Lemma 2.** *If $\varphi$ is $(G, x)$-quasi-invariant, then for all $g \in G$*

$$a \in \mathcal{A}, \quad a \geq 0 \Rightarrow \varphi(x_g a) \geq 0, \quad 1 = \varphi(x_g), \tag{2.8}$$

$$\varphi(x_g a) = \varphi(a x_g^*), \quad \forall\, a \in \mathcal{A}. \tag{2.9}$$

**Proof.** If $a \in \mathcal{A}$, $a \geq 0$ and $g \in G$, one has $0 \leq \varphi(g(a)) = \varphi(x_g a)$ and this proves the inequality in (2.8). The equality follows from

$$1 = \varphi(1) = \varphi(g(1)) = \varphi(x_g), \quad \forall\, g \in G.$$

Moreover, one has

$$\varphi(g(a^*)) = \varphi(x_g a^*), \quad \forall\, a \in \mathcal{A}.$$

On the other hand

$$\varphi(g(a^*)) = \varphi((g(a))^*) = \overline{\varphi(g(a))} = \overline{\varphi(x_g a)} = \varphi((x_g a)^*) = \varphi(a^* x_g^*).$$

It follows that

$$\varphi(x_g a^*) = \varphi(a^* x_g^*), \quad \forall\, a \in \mathcal{A},$$

and, since $a \in \mathcal{A}$ is arbitrary, this is equivalent to (2.9). $\square$

**Remark 1.** Recall that the *centralizer* of $\varphi$, hereinafter denoted Centrz$(\varphi)$, is characterized by the property

$$c \in \mathrm{Centrz}(\varphi) \Leftrightarrow \varphi(ac) = \varphi(ca), \quad \forall\, a \in \mathcal{A}. \tag{2.10}$$

**Lemma 3.** *If $\varphi$ is $(G, x)$-quasi-invariant, for any $g \in G$, the following are equivalent*:

(i) *$x_g$ is a Hermitian element of $\mathcal{A}$.*
(ii) *$x_g$ is in the centralizer of $\varphi$.*

**Proof.** (i) $\Rightarrow$ (ii) If $x_g = x_g^*$, (ii) follows from (2.9).

(ii) $\Rightarrow$ (i) If $x_g$ is in the centralizer of $\varphi$, then the same is true for $x_g^*$. Therefore, (2.9) implies that for all $a \in \mathcal{A}$

$$\varphi(x_g a) = \varphi(a x_g^*) = \varphi(x_g^* a) \Leftrightarrow \varphi((x_g - x_g^*)a) = 0 \overset{a := (x_g - x_g^*)^*}{\Rightarrow} \varphi(|x_g - x_g^*|^2) = 0,$$

and (i) follows because $\varphi$ is faithful. $\square$

**Definition 2.** A $(G, x)$-quasi-invariant state $\varphi$ is called $(G, x)$-*strongly quasi-invariant* if, for any $g \in G$, $x_g$ is *Hermitian.*

**Remark 2.** From Lemma 3, it is worth noticing that in the classical setting any quasi-invariant state is strongly quasi-invariant.

**Lemma 4.** *If $\varphi$ is $(G,x)$-strongly quasi-invariant, $x$ satisfies*

$$x_{g_1}(g_1^{-1}(x_{g_2})) = (g_1^{-1}(x_{g_2}))x_{g_1}, \quad \forall\, g_2, g_1 \in G, \tag{2.11}$$

*which is equivalent to*

$$x_{g_2}x_{g_1} = x_{g_1}x_{g_2}, \quad \forall\, g_2, g_1 \in G. \tag{2.12}$$

*Moreover, for all $g \in G$, $x_g$ is an invertible positive operator.*

**Proof.** From the cocycle identity (2.3) and the Hermitianity of the $x_g$, one has, for all $g_2, g_1 \in G$

$$x_{g_1}(g_1^{-1}(x_{g_2})) = x_{g_2g_1} = x^*_{g_2g_1} = (g_1^{-1}(x_{g_2}))x_{g_1},$$

which is (2.11). (2.11) is equivalent to

$$\begin{aligned}(g_1^{-1}(x_{g_2}))x_{g_1}^{-1} = x_{g_1}^{-1}(g_1^{-1}(x_{g_2})) &\overset{(2.4)}{\Leftrightarrow} (g_1^{-1}(x_{g_2}))g_1^{-1}(x_{g_1}) = g_1^{-1}(x_{g_1})(g_1^{-1}(x_{g_2}))\\ &\Leftrightarrow g_1^{-1}(x_{g_2}x_{g_1}) = g_1^{-1}(x_{g_1}x_{g_2})\\ &\Leftrightarrow x_{g_2}x_{g_1} = x_{g_1}x_{g_2},\end{aligned}$$

which is (2.12).

Note that $x_g = x_g^* = x_{g,+} - x_{g,-}$, with $x_{g,\pm}$ are positive such $x_{g,+}x_{g,-} = 0$. Therefore, choosing in (2.9) $a = x_{g,-}$, one finds

$$\begin{aligned}0 &\le \varphi(x_g x_{g,-}) = \varphi((x_{g,+} - x_{g,-})x_{g,-})\\ &= -\varphi(x_{g,-}^2) \Leftrightarrow 0 = \varphi(x_{g,-}^2) \Leftrightarrow 0 = x_{g,-}.\end{aligned}$$

Thus $x_g = x_g^* = x_{g,+} \ge 0$ and by Proposition 1, $x_g$ is invertible. □

Let $\mathcal{C}$ be the $*$-algebra generated by the $x_g$ $(g \in G)$. It follows from Lemma 3 that $\mathcal{C} \subseteq \mathrm{Centrz}(\varphi)$ and $\mathcal{C}$ is abelian by Lemma 4.

**Lemma 5.** *If $\varphi$ is $(G,x)$-strongly quasi-invariant, then for all $g \in G$ and $a \in \mathcal{A}$*

$$\varphi(g(x)a) = \varphi(ag(x_g x x_g^{-1})), \quad \forall\, x \in \mathrm{Centrz}(\varphi). \tag{2.13}$$

*In particular, for all $g \in G$*

$$g(\mathcal{C}) = \mathcal{C} \subseteq \mathrm{Centrz}(\varphi), \quad \forall\, g \in G. \tag{2.14}$$

**Proof.** In the above notations, one has

$$\begin{aligned}\varphi(g(x)a) &= \varphi(g(xg^{-1}(a))) = \varphi(x_g x g^{-1}(a)) = \varphi(g^{-1}(a)x_g x)\\ &= \varphi(g^{-1}(ag(x_g x))) = \varphi(x_{g^{-1}}ag(x_g)g(x)) = \varphi(ag(x_g)g(x)x_{g^{-1}})\\ &\overset{(2.4)}{=} \varphi(ag(x_g)g(x)g(x_g^{-1})) = \varphi(ag(x_g x x_g^{-1})),\end{aligned}$$

which is (2.13). Now if $x \in \mathcal{C} \subset \mathrm{Centrz}(\varphi)$ and if $g \in G$, one has $x_g x x_g^{-1} = x$. This proves the inclusion in (2.14). The identity in (2.14) follows from

$$x_{g_2 g_1} \overset{(2.3)}{=} x_{g_1} g_1^{-1}(x_{g_2}) \Leftrightarrow x_{g_1}^{-1} x_{g_2 g_1} = g_1(x_{g_2}) \Leftrightarrow g_1(x_{g_2}) \in \mathcal{C},$$

because $g_1$ and $x_{g_2}$ are arbitrary and $g_1$ is invertible. □

### 2.1. *The case $\mathcal{A} = \mathcal{B}(\mathcal{H})$*

In the above notations, consider $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional Hilbert space.

**Proposition 2.** *If $\varphi(\cdot) = \mathrm{Tr}(W \cdot)$ is a faithful state on $\mathcal{A}$ and if $G$ is a group acting on $\mathcal{A}$, then $\varphi$ is $G$-quasi-invariant. Moreover, $\varphi$ is $G$-strongly quasi-invariant iff $g(W^{-1})W = W^{-1}g(W)$ for every $g \in G$.*

**Proof.** As any automorphism of $\mathcal{B}(\mathcal{H})$ is inner, for $g \in G$, there exists a unitary $U_g$ such that

$$g(a) = U_g a U_g^*, \quad \forall\, a \in \mathcal{A}.$$

Now we have

$$\varphi(g(a)) = \mathrm{Tr}(W U_g a U_g^*) = \mathrm{Tr}(U_g^* W U_g a) = \mathrm{Tr}(W x_g a),$$

where $x_g = W^{-1} U_g^* W U_g = W^{-1} g^{-1}(W)$. Moreover, $\varphi$ is $G$-strongly quasi-invariant iff $x_g = x_g^*$, that is, $W^{-1} U_g^* W U_g = U_g^* W^{-1} U_g W$, which is equivalent to $g(W^{-1})W = W^{-1} g(W)$. □

### 2.2. *Quasi-invariant states with trivial cocycles*

**Lemma 6.** *For any operator $\kappa \in \mathcal{A}$ invertible in $\mathcal{A}$, the map*

$$g \in G \mapsto x_g = \kappa g^{-1}(\kappa^{-1}), \tag{2.15}$$

*is a normalized left multiplicative $G$-1-cocycle.*

**Proof.** Condition (2.2) follows from

$$x_e = \kappa\kappa^{-1} = 1.$$

Condition (2.3) follows from

$$\begin{aligned} x_{g_2 g_1} &= \kappa (g_2 g_1)^{-1}(\kappa^{-1}) = \kappa g_1^{-1} g_2^{-1}(\kappa^{-1}) = \kappa g_1^{-1}(\kappa^{-1}(\kappa g_2^{-1}(\kappa^{-1}))) \\ &= \kappa g_1^{-1}(\kappa^{-1}) g_1^{-1}(\kappa g_2^{-1}(\kappa^{-1})) = x_{g_1} g_1^{-1}(x_{g_2}), \quad \forall\, g_2, g_1 \in G. \end{aligned}$$ □

**Definition 3.** Let $G$ be a group of $*$-automorphisms of $\mathcal{A}$. A multiplicative left $G$-1-cocycle of the form (2.15) is called *trivial*.

**Theorem 1.** *Let $G$ be a group of $*$-automorphisms of $\mathcal{A}$ and let $\kappa \in \mathcal{A}$ be an invertible operator with inverse in $\mathcal{A}$. Then for any $G$-invariant state $\varphi_G$,* *the state*

$$\varphi(\,\cdot\,) := \varphi_G(\kappa\,\cdot\,), \tag{2.16}$$

*is* $(G,x)$*-quasi-invariant with cocycle*

$$x_g := \kappa^{-1}g^{-1}(\kappa), \quad \forall\, g \in G. \tag{2.17}$$

$\varphi$ *is strongly* $(G,x)$*-quasi-invariant with cocycle* (2.17) *iff*

$$\kappa = \kappa^*, \quad \kappa g(\kappa) = g(\kappa)\kappa, \quad \forall\, g \in G. \tag{2.18}$$

**Proof.** The first statement follows from the fact that, if $\varphi_G$, $\varphi$ and $\kappa^{-1}$ are as in the statement, then for any $a \in \mathcal{A}$ and $g \in G$, one has

$$\begin{aligned}\varphi(g(a)) &= \varphi_G(\kappa g(a)) = \varphi_G(g[g^{-1}(\kappa)a]) = \varphi_G([g^{-1}(\kappa)a]) \\ &= \varphi_G(\kappa\kappa^{-1}[g^{-1}(\kappa)a]) = \varphi((\kappa^{-1}g^{-1}(\kappa)a) = \varphi(x_g a).\end{aligned}$$

The first identity in (2.18) follows by taking adjoints of both sides of (2.16) as in the proof of Lemma 3. Given the validity of the first identity, the right-hand side of (2.17) is Hermitian iff, for all $g \in G$

$$\kappa^{-1}g^{-1}(\kappa) = (\kappa^{-1}g^{-1}(\kappa)^* = g^{-1}(\kappa)\kappa^{-1},$$

and, since $g^{-1}$ is arbitrary, this is equivalent to the second identity in (2.18). □

## 3. Examples of Quasi-Invariant States Under Permutations

In this section, we construct examples of quasi-invariant states on infinite tensor products of copies of a type I factor with the natural action given by the permutation group on $\mathbb{N}^* = \{1, 2, \ldots\}$. We show that, in this framework, examples of nontrivial cocycles naturally arise.

Let $\mathcal{B} := \mathcal{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$. Below the symbol "$\otimes$" will always refer to the minimal $C^*$-tensor product. Denote

$$\mathcal{A} = \bigotimes_{\mathbb{N}^*} \mathcal{B}, \tag{3.1}$$

the tensor product of countably many copies of $\mathcal{B}$ ($\bigotimes_{\mathbb{N}^*} \mathcal{B}$ is the inductive limit of $\bigotimes_{m=1}^n \mathcal{B}$) and let $j_n$ be the natural embedding of $\mathcal{B}$ onto the $n$th factor of $\mathcal{A}$ (we also use the notation $j_n(\mathcal{B}) =: \mathcal{A}_n \equiv \mathcal{B} \otimes \bigotimes_{\mathbb{N}^*\setminus\{n\}} 1_{\mathcal{B}}$). With these notations, introducing the *local algebras*

$$\mathcal{A}_{[M,N]} := \bigvee_{n\in[M,N]} \mathcal{A}_n := \text{algebra generated by the } j_n(\mathcal{B}) = \mathcal{A}_n : n \in [M,N], \tag{3.2}$$

$(M \le N \in \mathbb{N})$ whose elements are called *local elements*, one has

$$\mathcal{A} = \bigcup_{N\in\mathbb{N}^*} \mathcal{A}_{[1,N]}. \tag{3.3}$$

Denote by $\mathcal{S}_N$ the permutation group on $\{1,\dots,N\}$. Define $\mathcal{S}_\infty = \bigcup_{N\in\mathbb{N}^*}\mathcal{S}_N$. $\mathcal{S}_\infty$ has a natural action on $\mathcal{A}$ by $*$-automorphisms given by $g\circ j_n := j_{gn}$ where we use the same symbol for the actions of $\mathcal{S}_\infty$ on $\mathbb{N}^*$ and on $\mathcal{A}$. With this identification, $\mathcal{S}_\infty$ is called the group of local permutations on $\mathcal{A}$. Recall that, for each $g\in\mathcal{S}_\infty$, the support of $g$ is defined by

$$\Lambda_g := \mathrm{supp}(g) := \{n\in\mathbb{N}^* : gn\neq n\} = \mathrm{supp}(g^{-1}).$$

Denoting

$$\mathcal{S}_N := \{g\in\mathcal{S}_\infty : \mathrm{supp}(g)\subseteq[1,N]\},$$

one has

$$\mathcal{S}_\infty = \bigcup_{N\in\mathbb{N}^*}\mathcal{S}_N. \tag{3.4}$$

In the literature, $\mathcal{S}_\infty$-invariant states (or weights) are also called *exchangeable* and we will we use both terms in what follows.

### 3.1. *Product states*

In this section, we use the symbolic expressions $\mathrm{Tr}_{\mathbb{N}^*}$ and $\prod_{n\in\mathbb{N}^*} j_n(W_n)$ (see formula (3.6)). It is known that there is no general way to give a meaning to $\mathrm{Tr}_{\mathbb{N}^*}$ as a (non-identically infinite) weight on $\mathcal{A}$ or to $\prod_{n\in\mathbb{N}^*} j_n(W_n)$ as an element of a closure of $\mathcal{A}$ in some representation. However, the combined expression

$$\mathrm{Tr}_{\mathbb{N}^*}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\,\cdot\right), \tag{3.5}$$

is well defined even at algebraic level by the left-hand side of (3.6) and intuitively highlights the formal analogy with (2.16) in which $\mathrm{Tr}_{\mathbb{N}^*}$ plays the role of the state $\varphi_G$ and $\prod_{n\in\mathbb{N}^*} j_n(W_n)$ the role of $\kappa$.

In Sec. 3.2, we will see that this formal analogy can also be used for Markov states on $\mathcal{A}$. In both cases, the continuous classical analogues of (3.5) are widely used in quantum field theory (Feynman integral) and they also appear in the theory of quantum flows. Furthermore, once correctly defined by (3.6), the expression (3.5) can be used to obtain simplified proofs of several results. This statement is illustrated in the proofs of Proposition 3 and Theorem 3.

**Proposition 3.** *Any product state on* $\mathcal{A}$

$$\varphi = \bigotimes_{n\in\mathbb{N}^*}\varphi_n = \bigotimes_{n\in\mathbb{N}^*}\mathrm{Tr}(W_n\,\cdot) =: \mathrm{Tr}_{\mathbb{N}^*}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\,\cdot\right), \tag{3.6}$$

*with* $W_n$ *invertible for each* $n\in\mathbb{N}^*$ *and satisfying the condition*

$$\forall\, n\in\mathbb{N},\ \ \exists\, c_n>0 : W_n\geq c_n, \tag{3.7}$$

*is $\mathcal{S}_\infty$-quasi-invariant with cocycle*

$$x_g := \left(\prod_{n\in\Lambda_g} j_n(W_n^{-1})\right) g^{-1}\left(\prod_{n\in\Lambda_g} j_n(W_n)\right) = \prod_{n\in\Lambda_g} j_n(W_n^{-1}W_{g^{-1}n}), \quad \forall\, g\in\mathcal{S}_\infty, \tag{3.8}$$

*and it is $\mathcal{S}_\infty$-strongly quasi-invariant iff the $W_n$ mutually commute.*

**Proof.** Let $g \in \mathcal{S}_\infty$ with support $\Lambda_g$. Note that $g(\Lambda_g) = \Lambda_g$. Therefore, for any $n \in \Lambda_g$ and $a \in \mathcal{A}$ one has

$$\begin{aligned}
\varphi(g(a)) &= \mathrm{Tr}_{\mathbb{N}^*}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n) g(a)\right)\\
&= \mathrm{Tr}_{\mathbb{N}^*}\left(g\left(g^{-1}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\right)a\right)\right)\\
&= \mathrm{Tr}_{\mathbb{N}^*}\left(g^{-1}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\right)a\right)\\
&= \mathrm{Tr}_{\mathbb{N}^*}\left(\prod_{n\in\mathbb{N}^*} j_{g^{-1}n}(W_n)a\right)\\
&= \mathrm{Tr}_{\mathbb{N}^*}\left(\left(\prod_{n\in\Lambda_g^c} j_n(W_n)\right)\left(\prod_{n\in\Lambda_g} j_{g^{-1}n}(W_n)\right)a\right)\\
&= \mathrm{Tr}_{\mathbb{N}^*}\left(\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\right)\left(\prod_{n\in\Lambda_g} j_n(W_n^{-1})\right)\left(\prod_{n\in\Lambda_g} j_{g^{-1}n}(W_n)\right)a\right)\\
&= \varphi\left(\left(\prod_{n\in\Lambda_g} j_n(W_n^{-1})\right)\left(\prod_{n\in\Lambda_g} j_{g^{-1}n}(W_n)\right)a\right) = \varphi\left(x_g a\right).
\end{aligned}$$

Condition (3.7) guarantees that $x_g$ is bounded and the definition of $\Lambda_g$ that $x_g$ is Hermitian. Therefore, $\varphi$ is $\mathcal{S}_\infty$-quasi-invariant with cocycle given by (3.8). □

**Remark.** (1) Note that condition (3.7) is very strong. In fact, if $W_n = \sum_{n\in\mathbb{N}} w_n P_n$ is the spectral decomposition of $W_n$, due to the inequality

$$1 = \mathrm{Tr}(W_n) = \sum_{n\in\mathbb{N}} w_n \mathrm{Tr}(P_n) \geq \sum_{n\in\mathbb{N}} w_n,$$

condition (3.7) can be satisfied only if the spectrum of $W_n$ has finite cardinality.

(2) Note that the cocycle $x_g$ is nontrivial because it has the following form:

$$x_g = \prod_{n\in\Lambda_g} j_n(W_n^{-1})g^{-1}\left(\prod_{n\in\Lambda_g} j_n(W_n)\right) =: \kappa_g g^{-1}\left(\kappa_g^{-1}\right), \quad \forall\, g\in\mathcal{S}_\infty, \qquad (3.9)$$

so (2.15) is not satisfied because $\kappa_g$ depends on $g$. The only way to make it formally independent on $g$ is to use the identity

$$x_g = \prod_{n\in\mathbb{N}^*} j_n(W_n^{-1})g^{-1}\left(\prod_{n\in\mathbb{N}^*} j_n(W_n)\right) = \prod_{n\in\mathbb{N}^*} j_n(W_n^{-1}) \prod_{n\in\mathbb{N}^*} j_{g^{-1}(n)}(W_n)$$
$$= \prod_{n\in\mathbb{N}^*} j_n(W_n^{-1}) j_{g^{-1}(n)}(W_n) = \prod_{n\in\Lambda_g} j_n(W_n^{-1}) j_{g^{-1}(n)}(W_n),$$

which is formally of the form (2.15) with $\kappa := \prod_{n\in\mathbb{N}^*} j_n(W_n^{-1})$. But this is only a formal expression because, in general, there is no reasonable representation $\pi$ of $\mathcal{A}$ in the bounded operators on some Hilbert space $\mathcal{H}$ such that expressions like $\prod_{n\in\mathbb{N}^*} j_n(W_n^{-1})$ or $\prod_{n\in\mathbb{N}^*} j_n(W_n)$ can be interpreted as operators in the closure of $\pi(\mathcal{A})$ for some topology on $\mathcal{B}(\mathcal{H})$.

The above discussion suggests the following definition.

**Definition 4.** Let $\mathcal{A}$, $(\mathcal{A}_{[1,N]})_{N\in\mathbb{N}^*}$ and $\mathcal{S}_\infty$ be as above. An $\mathcal{S}_\infty$-cocycle $x : \mathcal{S}_\infty \to \mathcal{A}$ is called *locally trivial* if, denoting for $N\in\mathbb{N}^*$

$$\mathcal{S}_{[1,N]} := \{g\in\mathcal{S}_\infty : g[1,N] = [1,N]\},$$

there exists $\kappa_{[1,N]}\in\mathcal{A}_{[1,N]}$, with inverse in $\mathcal{A}_{[1,N]}$ such that, for each $g\in\mathcal{S}_{[1,N]}$

$$x_g = \kappa_{[1,N]} g^{-1}\big(\kappa_{[1,N]}^{-1}\big). \qquad (3.10)$$

**Theorem 2.** *Let $\varphi := \bigotimes_{\mathbb{N}^*} \mathrm{Tr}(W_k\,\cdot)$ be an arbitrary product state on $\mathcal{A}$ and let $W_\infty\in\mathcal{B}$ be a density operator bounded away from zero. Denote*

$$\psi := \bigotimes_{\mathbb{N}^*} \psi_0 = \bigotimes_{\mathbb{N}^*} \mathrm{Tr}_{\mathbb{N}^*}(W_\infty\,\cdot), \qquad (3.11)$$

*the exchangeable state on $\mathcal{A}$ defined by $W_\infty$ and $(\mathcal{H}_\psi, \pi_\psi, \Psi)$ the cyclic representation of $\psi$. Then*

(i) *The state $\hat{\psi}$ on $\pi_\psi(\mathcal{A})$ obtained by restriction on $\pi_\psi(\mathcal{A})$ of the state*

$$\bar{a}\in\pi_\psi(\mathcal{A})'' \mapsto \langle\Psi, \bar{a}\Psi\rangle, \quad \bar{a}\in\pi_\psi(\mathcal{A})'', \qquad (3.12)$$

*is exchangeable, for the action of $\mathcal{S}_\infty$ on $\mathcal{A}$ defined by $g\pi_\psi(\,\cdot\,) := \pi_\psi(g(\,\cdot\,))$ $(g\in\mathcal{S}_\infty)$.*

(ii) *$\varphi$ is $\mathcal{S}_\infty$-quasi-invariant with cocycle*

$$x_g := \left(\prod_{n\in\Lambda_g} j_n(W_\infty^{-1}W_n^{-1})\right) g^{-1} \left(\prod_{n\in\Lambda_g} j_n(W_\infty^{-1}W_n)\right) \in \mathcal{A}_{\Lambda_g}, \quad (3.13)$$

*and it is $\mathcal{S}_\infty$-strongly quasi-invariant iff all $W_\infty$ and $W_n$'s commute.*

(iii) *The weak limit*

$$\lim_{n\to+\infty} \pi_\psi\left(\prod_{k\in[1,n]} j_k(W_\infty^{-1}W_k)\right) =: X_{\psi,\infty}, \quad (3.14)$$

*exists in the strongly finite sense on the norm-dense subspace $\bigcup_{n\in\mathbb{N}^*} \pi_\psi \times (\mathcal{A}_{[1,M]})'' \cdot \Psi \subset \mathcal{H}_\psi$ and satisfies*

$$\varphi(a) = \hat{\psi}(X_{\psi,\infty}\pi_\psi(a)), \quad \forall\, a \in \mathcal{A}. \quad (3.15)$$

*Moreover, $X_{\psi,\infty} \geq 0$ if all $W_\infty$ and the $W_k$ commute.*

(iv) *If the sequence $(\prod_{k=1}^N j_k(W_\infty^{-1}W_k))_{N\in\mathbb{N}^*}$ is norm bounded, the weak limit* (3.14) *exists on $\mathcal{H}_\psi$.*

(v) *Define, for $I \subset_{\text{fin}} \mathbb{N}^*$*

$$x_I := \prod_{k\in I} j_k(W_\infty^{-1}W_k). \quad (3.16)$$

*If the series*

$$\sum_{k=1}^\infty \|W_\infty^{-1}W_k - 1\|, \quad (3.17)$$

*converges, then the sequence $(x_{[1,N]})$ is Cauchy in norm.*

**Proof.** (i) follows from the fact that $\psi$ is exchangeable by construction and, for any $g \in \mathcal{S}_\infty$ and $a \in \mathcal{A}$, one has

$$\hat{\psi}(g\pi_\psi(a)) = \hat{\psi}(\pi_\psi(ga)) = \langle\Psi, \pi_\psi(ga)\Psi\rangle = \psi(ga) = \psi(a) = \hat{\psi}(\pi_\psi(a)).$$

(ii) Note that the map $I \subset_{\text{fin}} \mathbb{N}^* \mapsto x_I$, defined by (3.16), is a multiplicative functional, i.e. $I \cap J = \emptyset \Rightarrow x_{I\cup J} = x_I x_J = x_J x_I$.

For any $N \in \mathbb{N}^*$, $g \in \mathcal{S}_N$ and $a_{[1,N]} \in \mathcal{A}_{[1,N]}$, one has

$$\varphi(g(a))\mathrm{Tr}_{[1,N]}\left(\prod_{k=1}^N j_k(W_k)g(a)\right)\mathrm{Tr}_{[1,N]}\left(\prod_{k=1}^N j_k(W_\infty)\prod_{k=1}^N j_k(W_\infty^{-1}W_k g(a))\right)$$

$$= \psi(x_{[1,N]}g(a)) \overset{\psi\circ g=\psi}{=} \psi(g^{-1}((x_{[1,N]})g(a))) = \psi(g^{-1}(x_{[1,N]})a)$$

$$= \psi(x_{[1,N]}x_{[1,N]}^{-1}g^{-1}(x_{[1,N]})a) = \varphi(x_{[1,N]}^{-1}g^{-1}(x_{[1,N]})a).$$

Since $a$ and $N$ are arbitrary, this proves that $\varphi$ is $\mathcal{S}_\infty$-quasi-invariant with cocycle given by (3.13). If $W_\infty$ commutes with all the $W_n$'s, (3.13) implies that $x_g$ is Hermitian, i.e. $\varphi$ is $\mathcal{S}_\infty$-strongly quasi-invariant.

To prove (iii) note that, if $a_{[1,M]}, b_{[1,M]} \in \pi_\psi(\mathcal{A})''$ (one can always suppose that $M$ is the same for both), then for any $M < N \in \mathbb{N}^*$, one has

$$\begin{aligned}
&\left\langle a_{[1,M]} \cdot \Psi, \pi_\psi\left(\prod_{k=1}^{N} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \left\langle a_{[1,M]} \cdot \Psi, \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) \pi_\psi\left(\prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \left\langle \Psi, \pi_\psi\left(\prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \left\langle \Psi, \pi_\psi\left(\prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) \cdot \Psi\right\rangle \\
&\qquad \times \left\langle \Psi, a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle, \qquad (3.18)
\end{aligned}$$

where in the last two equalities, we have used the fact that, for local algebras localized on disjoint sets, both their commutativity and the factorizability of the state $\langle \Psi, \cdot\, \Psi\rangle|_{\pi_\psi(\mathcal{A})}$ are preserved under weak closures. So one has

$$\begin{aligned}
&\left\langle \Psi, \pi_\psi\left(\prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) \cdot \Psi\right\rangle \left\langle \Psi, a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \psi\left(\prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) \left\langle \Psi, a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \mathrm{Tr}_{\mathbb{N}^*}\left(\prod_{k=M+1}^{N} j_k(W_\infty) \prod_{k=M+1}^{N} j_k(W_\infty^{-1}W_k)\right) \\
&\qquad \times \left\langle \Psi, a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle \\
&\quad = \left\langle \Psi, a_{[1,M]}^* \pi_\psi\left(\prod_{k=1}^{M} j_k(W_\infty^{-1}W_k)\right) b_{[1,M]} \cdot \Psi\right\rangle,
\end{aligned}$$

because the $W_k$'s are density operators and $\mathrm{Tr}_{\mathbb{N}^*}$ is factorizable in the given localization. This proves that the limit (3.14) exists in the strongly finite sense on $\pi_\psi(\mathcal{A}_{[1,M]})'' \cdot \Psi$ and, since $M \in \mathbb{N}^*$ is arbitrary, (i) follows.

(iv) follows from the fact that weak convergence of a norm bounded sequence of operators on a dense subspace of a Hilbert space implies weak convergence on the whole space.

To prove (v), we consider

$$\|x_{[1,N]} - x_{[1,M]}\| \overset{(3.16)}{=} \left\| \prod_{k=1}^{N} W_\infty^{-1} W_k - \prod_{k=1}^{M} W_\infty^{-1} W_k \right\|$$
$$\leq \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \left\| \prod_{k=M+1}^{N} W_\infty^{-1} W_k - 1 \right\|. \tag{3.19}$$

Now we use the following identity, valid on any multiplicative semi-group $S$ with distinguished element 1: for any $M \leq N \in \mathbb{N}^*$ and any sequence $(a_h)_{h=M}^{N}$, in $S$, one has

$$\prod_{h=M}^{N} a_h - 1 = \sum_{h=M}^{N} \left( \prod_{j=M}^{h-1} a_j \right) (a_h - 1), \tag{3.20}$$

with the convention that $\prod_{j=M}^{M-1} a_j := 1$. The identity (3.20) is proved by induction. It clearly holds for $M = N$. Supposing that it holds for $N - 1 > M$, one has

$$\prod_{h=M}^{N} a_h - 1 = \prod_{j=M}^{N-1} a_j (a_N - 1) + \prod_{j=M}^{N-1} a_j - 1$$
$$\overset{\text{induction}}{=} \prod_{j=M}^{N-1} a_j (a_N - 1) + \sum_{h=M}^{N-1} \left( \prod_{j=M}^{h-1} a_j \right) (a_h - 1)$$
$$= \sum_{h=M}^{N} \left( \prod_{j=M}^{h-1} a_j \right) (a_h - 1).$$

Therefore, by induction, (3.20) holds for any $N \in \mathbb{N}$. Using (3.20), one obtains

$$\|x_{[1,N]} - x_{[1,M]}\|$$
$$\leq \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \left\| \prod_{k=M+1}^{N} W_\infty^{-1} W_k - 1 \right\|$$
$$= \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \left\| \sum_{k=M+1}^{N} \left( \prod_{j=M+1}^{k-1} W_\infty^{-1} W_j \right) (W_\infty^{-1} W_k - 1) \right\|$$
$$\leq \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \sum_{k=M+1}^{N} \left\| \left( \prod_{j=M+1}^{k-1} W_\infty^{-1} W_j \right) (W_\infty^{-1} W_k - 1) \right\|$$
$$\leq \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \sum_{k=M+1}^{N} \left\| \prod_{j=M+1}^{k-1} W_\infty^{-1} W_j \right\| \|W_\infty^{-1} W_k - 1\|$$

$$\le \left\| \prod_{k=1}^{M} W_\infty^{-1} W_k \right\| \left( \sup_{k\ge M+2} \left\| \prod_{j=M+1}^{k-1} W_\infty^{-1} W_j \right\| \right) \sum_{k=M+1}^{N} \|W_\infty^{-1} W_k - 1\|$$

$$\le \prod_{k=1}^{M} \|W_\infty^{-1} W_k\| \left( \sup_{k\ge M+2} \prod_{j=M+1}^{k-1} \|W_\infty^{-1} W_j\| \right) \sum_{k=M+1}^{N} \|W_\infty^{-1} W_k - 1\| . \tag{3.21}$$

By assumption, the series (3.17) converges. Hence, due to the inequality $|\|W_\infty^{-1}W_k\| - 1| \le \|W_\infty^{-1}W_k - 1\|$, condition (3.17) implies that

$$\sum_{k=1}^{\infty} |\|W_\infty^{-1}W_k\| - 1| < +\infty, \tag{3.22}$$

and, since by assumption $\|W_\infty^{-1}W_k\| \ne 0$ for each $k \in \mathbb{N}^*$, it is known that condition (3.22) is necessary and sufficient for the existence of a number $P > 0$ such that

$$\lim_{M\to+\infty} \prod_{k=1}^{M} \|W_\infty^{-1}W_k\| = P. \tag{3.23}$$

Let us prove that, if condition (3.17) is satisfied, for $M$ large enough, the sup in the right-hand side of (3.21) can be bounded by a number independent of $N$. (3.23) implies that, for any fixed $M$ and $N \ge M+1$

$$\prod_{k=M+1}^{N} \|W_\infty^{-1}W_k\| = \frac{\prod_{k=1}^{N} \|W_\infty^{-1}W_k\|}{\prod_{k=1}^{M} \|W_\infty^{-1}W_k\|} \overset{\text{as } N\to+\infty}{\longrightarrow} \frac{P}{\prod_{k=1}^{M} \|W_\infty^{-1}W_k\|} > 0. \tag{3.24}$$

It is convenient to introduce the notations

$$P_{M]} := \prod_{k=1}^{M} \|W_\infty^{-1}W_k\|, \quad P_{(M,N]} := \prod_{k=M+1}^{N} \|W_\infty^{-1}W_k\|.$$

From (3.23), we know that, for any $\varepsilon \in (0,1)$ there exists $M_\varepsilon$ such that, for $M \ge M_\varepsilon$, and for any $N \ge M+1$

$$|P - P_{M]}| < \varepsilon, \qquad \sum_{k=M+1}^{N} \left\|W_\infty^{-1}W_k - 1\right\| < \varepsilon \left( 2\|P\| \left( \varepsilon + \frac{P}{P-\varepsilon} \right) \right)^{-1} . \tag{3.25}$$

Similarly, (3.24) implies that there exists $N_{M,\varepsilon}$ such that, for $N \ge N_{M,\varepsilon}$

$$\left| P_{(M,N]} - \frac{P}{P_{M]}} \right| < \varepsilon.$$

Therefore

$$|P_{(M,N]}| \le \left| P_{(M,N]} - \frac{P}{P_{M]}} \right| + \frac{P}{P_{M]}} < \varepsilon + \frac{P}{P - (P - P_{M]})}.$$

On the other hand, since

$$P - (P - P_{M]}) \ge P - |(P - P_{M]})| > P - \varepsilon,$$

one has

$$\sup_{M\geq M_\varepsilon, N\geq N_\varepsilon} |P_{(M,N]}| < \varepsilon + \frac{P}{P-\varepsilon}. \tag{3.26}$$

So that $M \geq M_\varepsilon$ and $N \geq M+2$

$$\begin{aligned}\|x_{[1,N]} - x_{[1,M]}\| &\overset{(3.21)}{\leq} \prod_{k=1}^{M} \|W_\infty^{-1} W_k\| \left( \sup_{k\geq M+2} \prod_{j=M+1}^{k-1} \|W_\infty^{-1} W_j\| \right) \\ &\quad \times \sum_{k=M+1}^{N} \|W_\infty^{-1} W_k - 1\| \\ &\overset{(3.23),(3.26)}{\leq} 2\|P\| \left(\varepsilon + \frac{P}{P-\varepsilon}\right) \sum_{k=M+1}^{N} \|W_\infty^{-1} W_k - 1\| \overset{(3.25)}{\leq} \varepsilon,\end{aligned}$$

i.e. the sequence $(x_{[1,N]})$ is Cauchy in norm. □

**Remark.** The identity (3.15) is the first example of a decomposition of the form (5.8) for a locally compact but non-compact group.

### 3.2. *Markov Chains with commuting conditional density amplitudes*

Recall that, for any exchangeable state $\psi$ on $\mathcal{A}$, all local sub-algebras of $\mathcal{A} = \bigotimes_{n\in\mathbb{N}^*} \mathcal{B}$ are $\psi$-expected. For $I \subseteq \mathbb{N}^*$, the sub-algebra of $\mathcal{A}$ localized in $I$ is denoted $\mathcal{A}_I$. In this section we fix an exchangeable state $\psi$ on $\mathcal{A}$, we denote $E_I^\psi$ the (unique) Umegaki conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_I$ satisfying

$$\psi \circ E_I^\psi = \psi \quad (\mathcal{A}_I\ \psi\text{-expected}).$$

Recall that, in the notation (3.2), for $n \in \mathbb{N}^*$, $j_{[n,n+1]} := j_n \otimes j_{n+1}$, one has $j_{[n,n+1]}(\mathcal{B} \otimes \mathcal{B}) = \mathcal{A}_{[n,n+1]}$ and that an $E_I^\psi$-*conditionally density amplitude (CDA)* localized in $\mathcal{A}_{[n,n+1]}$ is an operator $K_{n,n+1} \in \mathcal{B} \otimes \mathcal{B}$ satisfying

$$E_{\{n\}}^\psi(j_{[n,n+1]}(K_{n,n+1}^* K_{n,n+1})) = E_{n]}^\psi(j_{[n,n+1]}(K_{n,n+1}^* K_{n,n+1})) = 1_{\mathcal{A}_n}, \tag{3.27}$$

and that the family $(j_{[n,n+1]}(K_{n,n+1}))_{n\in\mathbb{N}^*}$ of CDA is called *commutative* if the $C^*$-algebra generated by it is commutative. With this notation, for any $g \in \mathcal{S}_\infty$ and any $a, b \in \mathcal{B}$, one has

$$gj_{[n,n+1]}(a \otimes b) := \begin{cases} j_{[gn,g(n+1)]}(a \otimes b), & \text{if } g(n+1) > gn, \\ j_{g(n+1)}(a) \otimes j_{gn}(b), & \text{if } gn > g(n+1). \end{cases}$$

**Remark 3.** In the following proposition we use the formal language introduced in the remark after Proposition 3. More explicitly, extending to infinite sets the notion of *right-product*

$$K_{[1,n]} := \overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})$$
$$:= j_{[1,2]}(K_{1,2}) j_{[2,3]}(K_{2,3}) \cdots j_{[N,N+1]}(K_{N,N+1}), \tag{3.28}$$

(and similarly for *left-product* $\overleftarrow{\prod}_{n\in[1,N]}$) one obtains products like

$$K_{\mathbb{N}^*} := \overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K_{n,n+1})$$
$$:= j_{[1,2]}(K_{1,2}) j_{[2,3]}(K_{2,3}) \cdots j_{[n,n+1]}(K_{n,n+1}) \cdots, \tag{3.29}$$

which are formal expressions whose precise meaning is given by the general theory of quantum Markov chains (see for example[1] for more details). On the other hand, the use of these formal expressions clearly indicates that, dealing with quantum Markov chains, and without the assumption that the CDAs are *commutative*, the natural cocycles have the form $\varphi(g(a)) = \varphi(y_g^* a y_g)$ and not $\varphi(g(a)) = \varphi(x_g a)$, so that the natural notion of quasi-invariant state is more general than the one discussed in the present paper. This more general notion will be discussed in a paper in preparation.

**Theorem 3.** *In the above notations, given a sequence $K_{n,n+1} \in \mathcal{B} \otimes \mathcal{B}$ of invertible CDAs and an exchangeable state $\psi$ on $\mathcal{A}$, the state $\varphi$ on $\mathcal{A}$ defined, in the notation* (3.29), *by*

$$\varphi(a) := \psi\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K^*_{n,n+1})\right) a \left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K_{n,n+1})\right)\right), \tag{3.30}$$

*is well defined and satisfies, $\forall\, N \in \mathbb{N}^*$, $\forall\, g \in \mathcal{S}_\infty$, such that $\Lambda_g \subseteq [1,N]$ and $\forall\, a \in \mathcal{A}_{[1,N]}$*

$$\varphi(g(a)) = \varphi(y^*_{[1,N];g}\, a\, y_{[1,N];g}), \tag{3.31}$$

*where*

$$y_{[1,N];g} := \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K_{n,n+1})\right)$$
$$\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)^{-1} \in \mathcal{A}_{[1,N]}. \tag{3.32}$$

*In particular, if all the $K_{n,n+1}$ are in the centralizer of $\psi$, then $\varphi$ is a $\mathcal{S}_\infty$-strongly quasi-invariant state with cocycle*

$$x_g = |(y^*_{[1,N];g}|^2$$
$$= \left| \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right)^{-1} \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K^*_{n,n+1})\right)\right|^2. \tag{3.33}$$

*If the $K_{n,n+1}$ are in the centralizer of $\psi$ and mutually commute, then $\forall\, N \in \mathbb{N}^*$, $\forall\, g \in \mathcal{S}_\infty$, such that $\Lambda_g \subseteq [1,N]$, the cocycle* (3.33) *takes the following form*:

$$x_g = \prod_{n\in[1,N]} j_{[n,n+1]}(|K_{n,n+1}|^2)^{-1}) j_{[g^{-1}n,g^{-1}(n+1)]}(|K_{n,n+1}|^2), \quad \forall\, g \in \mathcal{S}_\infty. \tag{3.34}$$

**Proof.** The first statement follows from the commutativity of local algebras localized on disjoint sets of $\mathbb{N}^*$ and from (3.27). In fact, if $g \in \mathcal{S}_\infty$, $N \in \mathbb{N}^*$ and $a \in \mathcal{A}_{[1,N]}$, one can always suppose that $\Lambda_g \subseteq [1,N]$. Under this condition, one has

$$\begin{aligned}
\varphi(a) &\overset{(3.30)}{=} \psi\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K^*_{n,n+1})\right) a \left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K_{n,n+1})\right)\right)\\
&= \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]^c} j_{[n,n+1]}(K^*_{n,n+1})\right)\left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right) a\right.\\
&\quad \times \left.\left(\overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)\left(\overrightarrow{\prod}_{n\in[1,N]^c} j_{[n,n+1]}(K_{n,n+1})\right)\right)\\
&\overset{(3.27)}{=} \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right) a \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)\right).
\end{aligned} \tag{3.35}$$

So the right-hand side of (3.30) is well defined on all local elements of $\mathcal{A}$. With the same notations, and using the fact that $g\Lambda_g = \Lambda_g \subseteq [1,N]$, (3.31) follows from

$$\begin{aligned}
\varphi(g(a)) &\overset{(3.30)}{=} \psi\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K^*_{n,n+1})\right) g(a) \left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K_{n,n+1})\right)\right)\\
&= \psi\left(g\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} g^{-1}(j_{[n,n+1]}(K^*_{n,n+1}))\right)\right.\right.\\
&\quad \times a \left.\left.\left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} g^{-1}(j_{[n,n+1]}(K_{n,n+1}))\right)\right)\right)\\
&= \psi\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} j_{[g^{-1}n,g^{-1}(n+1)]}(K^*_{n,n+1})\right)\right.\\
&\quad \times a \left.\left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[g^{-1}n,g^{-1}(n+1)]}(K_{n,n+1})\right)\right)\\
&= \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]^c} j_{[n,n+1]}(K^*_{n,n+1})\right)\right.
\end{aligned}$$

$$\times \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K^*_{n,n+1})\right) a$$
$$\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K_{n,n+1})\right)$$
$$\times \left(\overrightarrow{\prod}_{n\in[1,N]^c} j_{[n,n+1]}(K_{n,n+1})\right)\Bigg)$$
$$\overset{(3.27)}{=} \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K^*_{n,n+1})\right)\right.$$
$$\left.\times\, a\left(\overrightarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K_{n,n+1})\right)\right) \tag{3.36}$$

$$= \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right)\right.$$
$$\times \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right)^{-1}$$
$$\times \left(\overleftarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K^*_{n,n+1})\right) a$$
$$\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[g^{-1}n,g^{-1}(n+1)]}(K_{n,n+1})\right)$$
$$\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)^{-1}$$
$$\left.\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)\right), \tag{3.37}$$

using (3.32), (3.37) becomes

$$\varphi(g(a)) = \psi\left(\left(\overleftarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K^*_{n,n+1})\right) y^*_{[1,N];g}\, a\, y_{[1,N];g}\right.$$
$$\left.\times \left(\overrightarrow{\prod}_{n\in[1,N]} j_{[n,n+1]}(K_{n,n+1})\right)\right)$$

$$= \psi\left(\left(\overleftarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K^*_{n,n+1})\right) y^*_{[1,N];g} a y_{[1,N];g}\right.$$
$$\left.\times \left(\overrightarrow{\prod}_{n\in\mathbb{N}^*} j_{[n,n+1]}(K_{n,n+1})\right)\right)$$
$$= \varphi(y^*_{[1,N];g} a y_{[1,N];g}),$$

which, given (3.32), is (3.31). (3.33) and (3.34) are simple consequences of (3.31). □

**Remark.** Theorem 3, in particular (3.33) and (3.34), gives another example of locally trivial, in the sense of Definition 4, but in general nontrivial cocycle for the action of $\mathcal{S}_\infty$ on $\mathcal{A}$.

## 4. Unitary Representations Associated to Strongly Quasi-Invariant States

Let $G \subseteq \mathrm{Aut}(\mathcal{A})$ be a group of $*$-automorphisms of $\mathcal{A}$. In the following the cyclic representation of $\{\mathcal{A}, \varphi\}$ shall be denoted $\{\mathcal{H}_\varphi, \pi, \Phi\}$.

**Theorem 4.** *If $\varphi$ is $(G, x)$-strongly quasi-invariant, there exists a unique unitary representation $U$ of $G$ on $\mathcal{H}_\varphi$ characterized by the property*

$$U_g\pi(a)\Phi = \pi(g(a)x^{1/2}_{g^{-1}}))\Phi, \quad \forall\, a \in \mathcal{A}, \tag{4.1}$$

*where $x^{1/2}_{g^{-1}}$ is the square root of $x_{g^{-1}}$. Moreover*

$$u_g(\pi(a)) := U^*_g\pi(a)U_g = \pi(g^{-1}(a)), \quad \forall\, g \in G, \ \forall\, a \in \mathcal{A}. \tag{4.2}$$

**Proof.** In the above notations, for any $a, b \in \mathcal{A}$ and $g \in G$, one has

$$\begin{aligned}
\langle U_g\pi(a)\Phi, U_g\pi(b)\Phi\rangle &\overset{(4.1)}{=} \langle \pi(g(a)x^{1/2}_{g^{-1}})\Phi, \pi(g(b)x^{1/2}_{g^{-1}})\Phi\rangle \\
&= \langle \Phi, \pi(x^{1/2}_{g^{-1}}g(a^*b)x^{1/2}_{g^{-1}})\Phi\rangle \\
&= \varphi(x^{1/2}_{g^{-1}}g(a^*b)x^{1/2}_{g^{-1}}) \\
&= \varphi(g(a^*b)x_{g^{-1}}) \\
&= \varphi(g(a^*b\cdot g^{-1}(x_{g^{-1}}))) \\
&\overset{(2.1)}{=} \varphi(x_g a^*bg^{-1}(x_{g^{-1}})) \\
&= \varphi(a^*bg^{-1}(x_{g^{-1}})x_g) \\
&\overset{(2.4)}{=} \varphi(a^*b) = \langle \pi(a)\Phi, \pi(b)\Phi\rangle.
\end{aligned}$$

Thus $U_g$ is isometric. Now if $g, g' \in G$, then

$$\begin{aligned} U_g U_{g'} \pi(a)\Phi &= U_g \pi(g'(a) x_{g'^{-1}}^{1/2})\Phi \\ &= \pi(g[g'(a) x_{g'^{-1}}^{1/2}] x_{g^{-1}}^{1/2})\Phi \\ &= \pi(gg'(a) g(x_{g'^{-1}}^{1/2}) x_{g^{-1}}^{1/2})\Phi \\ &= \pi(gg'(a) g(x_{g'^{-1}}^{1/2}) x_{g^{-1}}^{1/2})\Phi \\ &\overset{(2.5)}{=} \pi(gg'(a) g(x_{g'^{-1}})^{1/2} x_{g^{-1}}^{1/2})\Phi \\ &\overset{(2.12),(2.11)}{=} \pi(gg'(a)(g(x_{g'^{-1}}) x_{g^{-1}})^{1/2})\Phi \\ &\overset{(2.11)}{=} \pi(gg'(a)(x_{g^{-1}} g(x_{g'^{-1}}))^{1/2})\Phi \\ &\overset{(2.3)}{=} \pi(gg'(a) x_{g'^{-1}g^{-1}}^{1/2})\Phi \\ &= \pi(gg'(a) x_{(gg')^{-1}}^{1/2})\Phi \\ &= U_{gg'}\pi(a)\Phi. \end{aligned}$$

In particular, putting first $g' = g^{-1}$ and then $g = g'^{-1}$, one finds $U_g U_{g^{-1}} = U_{g^{-1}} U_g = U_e = 1$, so that

$$U_g^* = U_{g^{-1}}. \tag{4.3}$$

Thus $U$ is a unitary representation.

Finally, for $a, b \in \mathcal{A}$ and $g \in G$, one has

$$\begin{aligned} U_g^* \pi(a) U_g \pi(b)\Phi &= U_{g^{-1}} \pi(a \cdot g(b) \cdot x_{g^{-1}}^{1/2})\Phi \\ &= \pi(g^{-1}[ag(b) x_{g^{-1}}^{1/2}] \cdot x_g^{1/2})\Phi \\ &= \pi(g^{-1}(a) b \cdot [g^{-1}(x_{g^{-1}}^{1/2}) x_g^{1/2}])\Phi, \end{aligned}$$

and, by (2.7) with $s = 1/2$, the right-hand side of (4.4) is equal to

$$\pi(g^{-1}(a)b)\Phi = \pi(g^{-1}(a))\pi(b)\Phi,$$

which implies (4.2) by the cyclicity of $\Phi$. □

## 5. Structure of Strongly Quasi-Invariant States

In this section, we assume that $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $G$ is a group of normal $*$-automorphisms of $\mathcal{A}$ such that the action $g \mapsto \tau_g$ on $\mathcal{A}$ is strongly continuous, $\varphi$ is a normal faithful $G$-strongly quasi-invariant on $\mathcal{A}$ and the map $g \in G \mapsto x_g$ is strongly continuous. Moreover, the commutative $C^*$-algebra generated by the $x_g$ will be denoted by $\mathcal{C}$. In the following if no confusion can arise $\tau_g$ will be simply denoted by $g$.

### 5.1. *The case of compact G*

In this section, we assume that $G$ is a *compact* group of normal $*$-automorphisms of $\mathrm{Aut}(\mathcal{A})$ with *normalized* Haar measure $dg$.

**Lemma 7.** *If $G$ is a compact group and the map $g \in G \mapsto x_g$ is strongly continuous, then $\{x_g\}_{g\in G}$ is uniformly bounded. In particular, there exists $0 < S_1 < S_2$ such that*

$$\mathrm{Spec}(x_g) \subset [S_1, S_2]. \tag{5.1}$$

**Proof.** Since $G$ is compact and for all $\xi \in \mathcal{H}$, $g \mapsto x_g\xi$ is continuous, then $\mathrm{Sup}_{g\in G}\|x_g\xi\| < \infty$. Therefore, by Banach–Steinhaus theorem, $\{x_g\}_{g\in G}$ is uniformly bounded. Finally, (5.1) follows from the fact that for all $g \in G$, $x_g$ is an invertible positive operator. □

**Theorem 5.** *In the above assumptions, the element*

$$\kappa := \int_G x_g dg \in \mathcal{C} \subseteq \mathrm{Centrz}(\varphi) \subseteq \mathcal{A}. \tag{5.2}$$

*$\kappa$ is an invertible operator with bounded inverse (hence $\kappa^{-1}$ belongs to $\mathcal{C}$).*

**Proof.** Note that the $C^*$-algebra $\mathcal{C}$ is abelian and contains the identity of $\mathcal{A}$ ($1_{\mathcal{A}} = x_e$). Therefore, by Gelfand theorem (see [6, Sec. 1.1.9]) $\mathcal{C}$ can be identified to the algebra

$$\mathcal{C}_{\mathbb{C}}(\mathcal{S}) := \{\text{continuous complex-valued functions on a compact space } \mathcal{S}\}. \tag{5.3}$$

Since $\mathcal{S}$ is compact, in this identification, the $x_g$, being positive and invertible, become strictly positive functions on a compact set. Using the functional realization of the algebra $\mathcal{C}$, we realize the $x_g's$ as continuous strictly positive functions on the compact Hausdorff space $\mathcal{S}$. Then each function $s \in \mathcal{S} \mapsto x_g(s)$ is bounded away from zero and from Lemma 7

$$\kappa(s) = \int_G x_g(s)dg > 0, \quad \forall\, s \in \mathcal{S},$$

is continuous and bounded away from zero. Hence $\kappa^{-1}$ is bounded, strictly positive and $\kappa^{-1}$ belongs to $\mathcal{C}$. □

**Theorem 6.** *Define*

$$E_G = \int_G g dg. \tag{5.4}$$

*Then*

(i) *For any $g \in G$, $E_G$ satisfies the following identity:*

$$gE_G = E_G = E_G g. \tag{5.5}$$

*In particular*

$$\mathrm{Range}(E_G) = \mathrm{Fix}(G) := \{a \in \mathcal{A} : g(a) = a, \forall\, g \in G\}, \tag{5.6}$$

*and $E_G(\mathcal{A})$ is a $W^*$-sub-algebra of $\mathcal{A}$.*

(ii) *$E_G$ is a faithful Umegaki conditional expectation from $\mathcal{A}$ onto $E_G(\mathcal{A}) = \mathrm{Fix}(G)$.*



**Proof.** For each $a \in \mathcal{A}$

$$\mathcal{A} \ni (E_G(a))^* = \left(\int_G g(a)dg\right)^* = \int_G g(a)^* dg = \int_G g(a^*)dg = E_G(a^*).$$

Therefore, $E_G(\mathcal{A})$ is a $*$-sub-space of $\mathcal{A}$. This proves (i).

Left-invariance of the Haar measure implies that, for each $a \in \mathcal{A}$ and $g \in G$

$$gE_G(a) = \int_G gh(a)dh = \int_G gh(a)d(gh) = \int_G g'(a)dg' = E_G(a).$$

This proves the first identity in (5.5). Similarly, right-invariance of the Haar measure of $G$ implies that, for each $a \in \mathcal{A}$ and $g \in G$

$$E_G(g(a)) = \int_G hg(a)dh = \int_G hg(a)dhg = \int_G g'(a)dg' = E_G(a),$$

which is the second identity in (5.5). The first identity in (5.5) implies that $\mathrm{Range}(E_G) \subseteq \mathrm{Fix}(G)$. The converse inclusion is clear because, if $a \in \mathrm{Fix}(G)$, then

$$a = \int_G g(a)dg = E_G(a),$$

$E_G(\mathcal{A}) = \mathrm{Fix}(G)$ which is a $W^*$-sub-algebra of $\mathcal{A}$. This proves (i).

To prove (ii) note that $E_G$ is a completely positive, identity preserving, $*$-map from $\mathcal{A}$ to $\mathcal{A}$ being a convex combination of automorphisms.

Moreover, for any $a \in \mathcal{A}$, one has

$$E_G^2(a) = \int_G dg g(E_G(a)) \overset{(5.5)}{=} \int_G dg E_G(a) = E_G(a),$$

due to the normalization of the Haar measure of $G$. So $E_G$ is a completely positive norm-1 projector on $\mathcal{A}$ and, by Tomijama's theorem,[9] $E_G(\mathcal{A})$ is a Umegaki conditional expectation from $\mathcal{A}$ to $E_G(\mathcal{A})$. Finally, if $a$ is an element of $\mathcal{A}$ such that $E_G(a^*a) = 0$, then

$$0 = \varphi(E_G(a * a)) = \varphi(\kappa a^* a) = \varphi(\kappa^{1/2} a^* a \kappa^{1/2}).$$

Since $\kappa$ is invertible and $\varphi$ is faithful, it follows that $a = 0$. □

The structure of strongly quasi-invariant faithful normal states with respect to compact groups is described by the following theorem.

**Theorem 7.** *Let $G$ be a compact group of normal $*$-automorphisms of $\mathcal{A}$ and let $\varphi$ be a $G$-strongly quasi-invariant faithful normal state on $\mathcal{A}$. Denote $x : g \in G \to x_g \in \mathcal{A}$ the (strongly continuous, Hermitian) left-$G$-1-cocycle associated to $\varphi$ and let $\kappa$ be as in Theorem 5, namely*

$$\kappa = \int_G x_g dg. \tag{5.7}$$

*Then $\varphi$ can be written in the following form:*

$$\varphi(a) = (\varphi \circ E_G)(\kappa^{-1}a) =: \varphi_G(\kappa^{-1}a), \quad \forall\, a \in \mathcal{A}. \tag{5.8}$$

*In particular, $\varphi_G$ is a faithful $G$-invariant state and*

$$x_g = \kappa g^{-1}(\kappa^{-1}). \tag{5.9}$$

*Moreover*

$$E_G(\kappa^{-1}) = 1, \tag{5.10}$$

$$\kappa^* = \kappa, \quad \kappa g^{-1}(\kappa^{-1}) = g^{-1}(\kappa^{-1})\kappa, \quad \forall\, g \in G. \tag{5.11}$$

*Conversely, let $G$ and $\mathcal{A}$ be as in the statement of the theorem. Then for any $\kappa \in \mathcal{A}$, invertible with inverse in $\mathcal{A}$ and for any $G$-invariant faithful normal state $\varphi_G$ on $\mathcal{A}$, the state*

$$\varphi(\,\cdot\,) := \varphi_G(\kappa^{-1}\,\cdot),$$

*is a $G$-quasi-invariant faithful normal state on $\mathcal{A}$ with cocycle given by* (5.9). *If $\kappa$ also satisfies* (5.11), *then $\varphi(\,\cdot\,)$ is also $G$-strongly quasi-invariant.*

**Proof.** If $\varphi$ is a $(G, x_\cdot)$-strongly quasi-invariant state, define $\varphi_G := \varphi \circ E_G$. $\varphi_G$ is clearly a state and is faithful because $\varphi$ and $E_G$ are faithful. It is $G$-invariant because of (5.5). From Theorem 5, one knows that $\kappa$ is an invertible element of $\mathcal{A}$ with inverse in $\mathcal{A}$. Moreover, for any $a \in \mathcal{A}$, one has

$$\begin{aligned}\varphi_G(\kappa^{-1}a) &= \varphi(E_G(\kappa^{-1}a)) = \int_G dg\varphi(g(\kappa^{-1}a)) = \int_G dg\varphi(x_g\kappa^{-1}a)\\ &= \varphi(\kappa\kappa^{-1}a) = \varphi(a),\end{aligned}$$

which is (5.8). In particular

$$\begin{aligned}\varphi(x_g a) = \varphi(g(a)) &\overset{(5.8)}{=} \varphi_G(\kappa^{-1}g(a)) = \varphi_G(g(\kappa^{-1})a)\\ &= \varphi_G(\kappa^{-1}\kappa g(\kappa^{-1})a) = \varphi(\kappa g(\kappa^{-1})a),\end{aligned}$$

and (5.9) follows because $\varphi$ is faithful. $\varphi_G$ is also faithful because, if $0 \neq a$ is a positive element in $\mathcal{A}$ such that $\varphi_G(a) = 0$, then

$$0 \le \varphi(a^{1/2}) \overset{(5.8)}{=} \varphi_G(\kappa^{-1}a^{1/2}) \le \varphi_G(\kappa^{-2})\varphi_G(a) = 0,$$

against the assumption that $\varphi$ is faithful. Since the Haar measure on a unimodular group (in particular on a compact group) is invariant by inversion, one gets

$$E_G(\kappa^{-1}) = \int_G g(\kappa^{-1})dg \overset{(5.9)}{=} \int_G \kappa^{-1}x_{g^{-1}}dg = \kappa^{-1}\int_G x_{g^{-1}}dg = \kappa^{-1}\kappa = 1,$$

which is (5.10).

Conversely, that is $(G,x)$-quasi-invariant (strongly $(G,x)$-quasi-invariant) with cocycle (5.9) is known from Theorem 1. Faithfulness and normality follow from arguments similar to those used in the first part of the proof. □

**Remark.** If the algebra of fixed points of the action of $G$ on $\mathcal{A}$ is nontrivial, the decomposition (5.8), i.e. $\varphi = \varphi_G(\kappa^{-1}\,\cdot\,)$, is not unique. In fact, in this case, there exists a positive invertible fixed point $k$ which is not a multiple of the identity and such that $\varphi_G(k\,\cdot\,)$ is an invariant state on $\mathcal{A}$. In this case

$$\varphi = \varphi_G(\kappa^{-1}\,\cdot\,) = \varphi_G(k(\kappa k)^{-1}\,\cdot\,).$$

We have seen, in Sec. 3, that non-uniqueness of the decomposition (5.8) can still take place when the fixed point algebra of the action of $G$ is trivial, but its dual action has a convex set of invariant states of cardinality $> 1$.

### 5.2. *Inductive limits of compact groups*

In this section, $(G_N)_{N\in\mathbb{N}}$ will be an increasing sequence of compact sub-groups of a group $G$ of normal $*$-automorphisms of a von Neumann algebra $\mathcal{A}$ such that

$$G := \bigcup_{N\in\mathbb{N}} G_N, \tag{5.12}$$

and, for $N\in\mathbb{N}$, we denote $d_N g$ is the Haar measure on $G_N$ and

$$E_{G_N} = \int_{G_N} g d_N g. \tag{5.13}$$

**Proposition 4.** *The family* $(E_{G_N})_{N\in\mathbb{N}}$ *is a projective family of conditional expectations, i.e.*

$$E_{G_{N+1}}E_{G_N} = E_{G_{N+1}}, \quad \forall\, N\in\mathbb{N}. \tag{5.14}$$

**Proof.** Since $G_N \subseteq G_{N+1}$, one has

$$\mathrm{Range}(E_{G_{N+1}}) = \mathrm{Fix}(G_{N+1}) \subseteq \mathrm{Fix}(G_N) = \mathrm{Range}(E_{G_N}). \tag{5.15}$$

Therefore, projectivity is equivalent to (5.14). Moreover

$$E_{G_{N+1}}E_{G_N} = E_{G_{N+1}}\int_{G_N} g d_N g = \int_{G_N} d_N g E_{G_{N+1}} g$$
$$\overset{(5.5)}{=} \int_{G_N} d_N g E_{G_{N+1}} = E_{G_{N+1}}.$$
□

**Lemma 8.** *Suppose that a state $\varphi$ on $\mathcal{A}$ is $G$-quasi-invariant and denote, for each $g \in G$, $g \in G \mapsto x_g$ the $G$-cocycle and*

$$N_g := \min\{N \in \mathbb{N} : g \in G_N\}. \tag{5.16}$$

*Then for each $N \in \mathbb{N}$, $\varphi$ is $G_N$-quasi-invariant with cocycle*

$$g \in G_N \mapsto x_{G_N;g}, \tag{5.17}$$

*satisfying*

$$x_g = x_{G_{N_g};g}, \quad \forall\, g \in G_N. \tag{5.18}$$

*In particular, $\varphi$ is $G$-strongly quasi-invariant if and only if it is $G_N$-strongly quasi-invariant for each $N \in \mathbb{N}$.*

**Proof.** Since any $g \in G$ belongs to some $G_N$, $N_g$ is well defined by (5.16). Since, for each $N \in \mathbb{N}$, $G_N \subseteq G$, it is clear that, if $\varphi$ is $G$-quasi-invariant, it is also $G_N$-quasi-invariant. To prove that (5.18) holds note that, for any $N \in \mathbb{N}$ such that $g \in G_N$

$$\varphi(x_g a) = \varphi(g(a)) = \varphi(x_{G_N;g} a).$$

The faithfulness of $\varphi$ then implies that $x_g = x_{G_N;g}$ for any $N \in \mathbb{N}$ such that $g \in G_N$. Thus in particular (5.18) holds.

Conversely, suppose that $\varphi$ is $G_N$-quasi-invariant for each $N \in \mathbb{N}$ and note that, since any $g \in G$ belongs to some $G_{N_0}$. Then one has, for any $N \geq N_g$

$$\varphi(x_{G_{N_g};g} a) = \varphi(g(a)) = \varphi(x_g a).$$

The faithfulness of $\varphi$ then implies that, defining $x_g$ by the right-hand side of (5.18), one has

$$\varphi(g(a)) = \varphi(x_g a), \quad \forall\, g \in G,$$

i.e. that $\varphi$ is $G$-quasi-invariant with cocycle $g \in G \mapsto x_g$. This proves the first statement of the lemma. Given this and (5.18), the second statement is clear. □

**Proposition 5.** *In the notations of Sec.* 4, *the family $(u_g)_{g\in G}$ of $*$-automorphisms of $\pi(\mathcal{A})$ defined by* (4.2) *extends uniquely to a representation of $G$ into the normal (or equivalently strongly continuous) $*$-automorphisms of $\pi(\mathcal{A})''$, still denoted with the same symbol. Denoting for each $N \in \mathbb{N}$*

$$\overline{E}_{G_N} := \int_{G_N} u_g d_N g, \tag{5.19}$$

*the family $(\overline{E}_{G_N})$ is a projective family of normal Umegaki conditional expectations. Moreover, for each $N \in \mathbb{N}$ and with $E_{G_N}$ defined by* (5.13), *one has*

$$\overline{E}_{G_N}(\pi(a)) = \pi(E_{G_N}(a)), \quad \forall\, a \in \mathcal{A}. \tag{5.20}$$

*Each $\overline{E}_{G_N}$ can be extended by continuity to a Umegaki conditional expectation, denoted with the same symbol $\overline{E}_{G_N}$, onto the weak closure of*

$$\mathrm{Fix}(u(G_N)) := \text{weak closure of } \mathrm{Fix}(u(G_N)) \subset \pi(\mathcal{A})''. \tag{5.21}$$

*The family $(\overline{E}_{G_N})_{N\in\mathbb{N}}$ is a projective decreasing family of normal Umegaki conditional expectations.*

**Proof.** Each $\overline{E}_{G_N}$ is a normal map because each $u_g$ $(g \in G_N)$ is normal and this property is preserved under integration over a compact set. By normality, it can be extended by continuity to $\pi(\mathcal{A})''$. (5.20) holds because

$$\overline{E}_{G_N}(\pi(a)) = \int_{G_N} d_{G_N} g u_g(\pi(a)) = \int_{G_N} d_{G_N} g U_g^*(\pi(a))U_g \overset{(4.2)}{=} \int_{G_N} d_{G_N}(\pi(g^{-1}a))$$
$$= \int_{G_N} d_{G_N}(\pi(ga)) = \pi\left(\int_{G_N} d_{G_N} ga\right) = \pi(E_{G_N}(a)).$$

Therefore, $\overline{E}_{G_N}$ is a Umegaki conditional expectation from $\pi(\mathcal{A})$ onto $\mathrm{Fix}(G_N)$. This implies that its extension is a Umegaki conditional expectation from $\pi(\mathcal{A})''$ onto $\mathrm{Fix}(u(G_N))$. Since $(E_{G_N})_{N\in\mathbb{N}}$ is a projective decreasing family, the same is true for $(\overline{E}_{G_N})_{N\in\mathbb{N}}$. □

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