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On inverses of Kreĭn's $\mathscr{Q}$-functions

Claudio Cacciapuoti, Davide Fermi and Andrea Posilicano

Dedicated to Gianfausto Dell'Antonio on the occasion of his 85th birthday

Abstract. *Let A_Q be the self-adjoint operator defined by the $\mathscr{Q}$-function $Q\colon z \mapsto Q_z$ through the Kreĭn-like resolvent formula*

$$(-A_Q + z)^{-1} = (-A_0 + z)^{-1} + G_z W Q_z^{-1} V G_{\bar z}^{*}\,, \quad z \in Z_Q\,,$$

where V and W are bounded operators and

$$Z_Q := \{z \in \rho(A_0) : Q_z \text{ and } Q_{\bar z} \text{ have a bounded inverse}\}\,.$$

We show that

$$Z_Q \neq \emptyset \quad \Longrightarrow \quad Z_Q = \rho(A_0) \cap \rho(A_Q)\,.$$

We do not suppose that Q is represented in terms of a uniformly strict, operator-valued Nevanlinna function (equivalently, we do not assume that Q is associated to an ordinary boundary triplet), thus our result extends previously known ones. The proof relies on simple algebraic computations stemming from the first resolvent identity.

1 Introduction

Let $A_0\colon \mathrm{dom}(A_0) \subseteq \mathsf{H} \to \mathsf{H}$ be a self-adjoint operator in the Hilbert space H and let $S\colon \mathrm{dom}(S) \subseteq \mathsf{H} \to \mathsf{H}$ be the symmetric operator given by the restriction of A_0 to the kernel (assumed to be dense) of the continuous (w.r.t. the graph norm) linear map $\tau\colon \mathrm{dom}(A_0) \to \mathsf{K}$, K being an auxiliary Hilbert space. By [32, Theorem 2.1] (see Theorem 2.4 in the next section), a family of self-adjoint extensions of S can be defined through the Kreĭn-like resolvent formula

$$(-A_Q + z)^{-1} = (-A_0 + z)^{-1} + G_z W Q_z^{-1} V G_{\bar z}^{*}\,, \quad z \in Z_Q\,, \tag{1.1}$$

where V and W are bounded operators,

$$Z_Q := \{z \in \rho(A_0) : Q_z \text{ and } Q_{\bar z} \text{ have a bounded inverse}\}$$

and Q_z is a family of (not necessarily bounded) densely defined, closed linear maps such that

$$Q_z - Q_w = (z - w) V \tau(-A_0 + w)^{-1} (\tau(-A_0 + \bar z)^{-1})^{*} W\,, \qquad w, z \in \rho(A_0)\,,$$

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and

$$V^*(Q_z^*)^{-1}W^* = WQ_{\bar z}^{-1}V\,, \qquad z \in Z_Q\,. \tag{1.2}$$

By a slight abuse of terminology, we call such a map $Q\colon z \mapsto Q_z$ a $\mathscr{Q}$-function; for the definition (in the case $V = \mathbb{1}$ and $W = \mathbb{1}$, where (1.2) reduces to $Q_z^* = Q_{\bar z}$) of a "$\mathscr{Q}$-function of S belonging to A_0" (with values in the space of bounded operators) we refer to [14, Definition 3] and to the original papers [24] (defect indices $n_\pm(S) = 1$), [25] (finite defect indices), [37] (infinite defect indices). Evidently the above definition of A_Q by (1.1) only requires $Z_Q \neq \emptyset$. However, taking into account formula (1.1), one would expect $\rho(A_0) \cap \rho(A_Q) \subseteq Z_Q$ (hence $Z_Q = \rho(A_0) \cap \rho(A_Q)$ since $Z_Q \subseteq \rho(A_0) \cap \rho(A_Q)$ by $Z_Q \subseteq \rho(A_0)$ and (1.1)); moreover, in order to treat scattering theory for the couple (A_Q, A_0) through a limiting absorption principle (see [30], [31], [28], [10]), one at least would need $\mathbb{C}\backslash\mathbb{R} \subseteq Z_Q$. The aim of this work is to show that if Z_Q is not empty then it necessarily coincides with $\rho(A_0) \cap \rho(A_Q)$ (and so it always contains the whole $\mathbb{C}\backslash\mathbb{R}$). In the case the map τ is surjective, i.e., $\operatorname{ran}(\tau) = \mathsf{K}$, and $V = \pi$, $W = \pi^*$, π an orthogonal projector onto a closed subspace of K (coinciding with K itself in the case $\pi = \mathbb{1}$), then (see [34], [35, Section 4]) the construction provided in [32] is equivalent to the one given by boundary triplet theory (we refer to [14], [8, Section 1], [15, Section 7.3], [38, Section 14] and references therein for an introduction to such a theory). Thus, in this case, Q can be expressed in terms of a self-adjoint operator and an holomorphic function $M : z \mapsto M_z$ with values in the space of bounded operators such that $M_z = M_{\bar z}^*$ and $0 \in \rho(M_z - M_z^*)$ (see [14, Theorem 1], [15, Theorem 7.15]), i.e., M is a uniformly strict Nevanlinna operator function. Hence, whenever $\operatorname{ran}(\tau) = \mathsf{K}$, $V = \pi$, $W = \pi^*$, one gets $Z_Q = \rho(A_0) \cap \rho(A_Q)$ by standard arguments (see [14, Theorem 2], [15, Theorem 7.16]; see also [32, Proposition 2.1], [35, Theorem 2.1]). Since, by the correspondence with von Neumann's theory (see [33], [35]), any self-adjoint extension of S can be defined through (1.1) assuming the hypothesis $\operatorname{ran}(\tau) = \mathsf{K}$ (equivalently, using the corresponding ordinary boundary triplet, see [14], [38, Theorem 14.7]), these results seem to settle down our questions about Z_Q (at least in the case $V = \pi$, $W = \pi^*$). However, in cases where the defect indices of S are not finite, in particular in applications to partial differential operators, it can be much more convenient to do not require $\operatorname{ran}(\tau) = \mathsf{K}$ (and sometimes $V \neq \mathbb{1}$, $W \neq \mathbb{1}$) and so to do not use ordinary boundary triplets (see, e.g., [12], [7], [13], [32], [23], [16], [17], [18], [2], [3], [4], [6], [28], [9], [10]). While some results regarding the validity of (1.1) for any $z \in \rho(A_0) \cap \rho(A_Q)$ are known even for not ordinary boundary triplets (as generalized boundary triplets and quasi-boundary triplets, see e.g., [3], [15], [5]), some additional hypotheses are required in these cases (which moreover do not necessarily conform to our framework). Here, see Theorem 2.19 in the next section, we provide a simple proof of

$$Z_Q \neq \emptyset \quad \Rightarrow \quad Z_Q = \rho(A_0) \cap \rho(A_Q)$$

in the case $\operatorname{ran}(\tau) \neq \mathsf{K}$, $V \neq \pi$, $W \neq \pi^*$, without further hypotheses.

2 Inverses of Kreĭn's $\mathscr{Q}$-functions

Let H and K be Hilbert spaces with scalar products (which we assume to be conjugate-linear w.r.t. the first variable) $\langle\cdot,\cdot\rangle_{\mathsf{H}}$ and $\langle\cdot,\cdot\rangle_{\mathsf{K}}$. In the following, for notational convenience, we do not identify K with its dual K^*; however we use $\mathsf{K}^{**}\equiv\mathsf{K}$. We denote by $\langle\cdot,\cdot\rangle_{\mathsf{K}^*,\mathsf{K}}$ the K^*-K duality (conjugate-linear w.r.t. the first variable) defined by $\langle\psi,\phi\rangle_{\mathsf{K}^*,\mathsf{K}}:=\langle J^{-1}\psi,\phi\rangle_{\mathsf{K}}$, where $J:\mathsf{K}\to\mathsf{K}^*$ is the duality mapping given by the differential of $\phi\mapsto\frac{1}{2}\langle\phi,\phi\rangle_{\mathsf{K}}$.

Given the self-adjoint operator

$$A_0\colon \mathrm{dom}(A_0)\subseteq\mathsf{H}\to\mathsf{H}\,,$$

we consider a continuous (w.r.t. the graph norm in $\mathrm{dom}(A_0)$) linear map

$$\tau\colon \mathrm{dom}(A_0)\to\mathsf{K}$$

such that

$$\ker(\tau)\text{ is dense in }\mathsf{H}. \tag{2.1}$$

Remark 2.1. Notice that we do not suppose that $\mathrm{ran}(\tau)=\mathsf{K}$. This means that the corresponding (accordingly to [34]) boundary triplet is not an ordinary boundary triplet. See the successive Remark 2.20 for the case in which $\ker(\tau)$ is not dense.

For any $z\in\rho(A_0)$ we define $R^0_z\in\mathscr{B}(\mathsf{H},\mathrm{dom}(A_0))$ by $R^0_z:=(-A_0+z)^{-1}$ and $G_z\in\mathscr{B}(\mathsf{K}^*,\mathsf{H})$ by

$$G_z\colon\mathsf{K}^*\to\mathsf{H}\,,\quad G_z:=(\tau R^0_{\bar z})^*\,,$$

i.e.,

$$\langle G_z\phi,u\rangle_{\mathsf{H}}=\langle\phi,\tau(-A_0+\bar z)^{-1}u\rangle_{\mathsf{K}^*,\mathsf{K}}\quad \phi\in\mathsf{K}^*\,,\ u\in\mathsf{H}\,.$$

By (2.1), one has (see [32, Remark 2.9]),

$$\mathrm{ran}(G_z)\cap\mathrm{dom}(A_0)=\{0\}$$

and, by the resolvent identity,

$$G_z-G_w=(w-z)R^0_wG_z\,, \tag{2.2}$$

so that

$$\mathrm{ran}(G_z-G_w)\subset\mathrm{dom}(A_0)\,. \tag{2.3}$$

Remark 2.2. Notice that (2.2) is equivalent to

$$G_z=\big(1+(w-z)R^0_z\big)\,G_w\,. \tag{2.4}$$

Let X and Y be two Hilbert spaces and let V and W be two bounded operators, $V \in \mathscr{B}(\mathsf{K},\mathsf{X})$ and $W \in \mathscr{B}(\mathsf{Y},\mathsf{K}^*)$. Given a not empty set $Z_\Lambda \subseteq \rho(A_0)$, symmetric with respect to the real axis (i.e., $z \in Z_\Lambda \Rightarrow \bar{z} \in Z_\Lambda$), we consider a map

$$\Lambda\colon Z_\Lambda \to \mathscr{B}(\mathsf{X},\mathsf{Y})\,, \qquad z \mapsto \Lambda_z\,,$$

such that

$$V^*\Lambda_z^* W^* = W\Lambda_{\bar z}V\,, \tag{2.5}$$

$$\Lambda_z - \Lambda_w = (w-z)\Lambda_w V G_{\bar w}^* G_z W \Lambda_z\,. \tag{2.6}$$

Remark 2.3. Notice that (2.6) is equivalent to

$$\Lambda_z = (1 + (w-z)\Lambda_z V G_{\bar z}^* G_w W)\,\Lambda_w\,. \tag{2.7}$$

Notice that, by (2.5) and (2.6), the map $\widetilde{\Lambda}_z := W\Lambda_z V : \mathsf{K} \to \mathsf{K}^*$ satisfies the relations

$$\widetilde{\Lambda}_z^* = \widetilde{\Lambda}_{\bar z}$$

and

$$\widetilde{\Lambda}_z - \widetilde{\Lambda}_w = (w-z)\widetilde{\Lambda}_w G_{\bar w}^* G_z \widetilde{\Lambda}_z\,,$$

see [32, equations (2) and (4)]. Hence, building on [32, Theorem 2.1], one has (see [28, Theorem 2.4 and Remark 2.5]; our $\widetilde{\Lambda}_z = W\Lambda_z V$ corresponds to the operator there denoted by Λ_z)

Theorem 2.4. *Let $\Lambda : Z_\Lambda \to \mathscr{B}(\mathsf{X},\mathsf{Y})$ satisfy (2.5) and (2.6). Then there exists a unique self-adjoint extension A_Λ of the closed symmetric operator $S := A_0|\ker(\tau)$ such that $Z_\Lambda \subseteq \rho(A_0) \cap \rho(A_\Lambda)$ and*

$$(-A_\Lambda + z)^{-1} = R_z^0 + G_z W \Lambda_z V G_{\bar z}^*\,, \qquad z \in Z_\Lambda\,. \tag{2.8}$$

Remark 2.5. Any self-adjoint extension of S is of the kind provided by the previous theorem (see [33, 35]).

From now on we use the shorthand notation

$$R_z^\Lambda := (-A_\Lambda + z)^{-1}\,, \qquad z \in \rho(A_\Lambda)\,.$$

Lemma 2.6. *For any w and z in Z_Λ one has*

$$\Lambda_z - \Lambda_w = (w-z)\Lambda_w V G_{\bar w}^* \left(1 + (w-z)R_z^\Lambda\right) G_w W \Lambda_w\,. \tag{2.9}$$

Proof. Taking into account relations (2.6), (2.7), (2.4) and (2.8), one gets

$$\begin{aligned}
&\Lambda_z - \Lambda_w \\
&\quad = (w-z)\Lambda_w V G_{\bar w}^* \left(G_z + (w-z) G_z W \Lambda_z V G_{\bar z}^* G_w\right) W \Lambda_w \\
&\quad = (w-z)\Lambda_w V G_{\bar w}^* \left(1 + (w-z)R_z^0 + (w-z) G_z W \Lambda_z V G_{\bar z}^*\right) G_w W \Lambda_w \\
&\quad = (w-z)\Lambda_w V G_{\bar w}^* \left(1 + (w-z)R_z^\Lambda\right) G_w W \Lambda_w\,.
\end{aligned}$$

□

Obviously, by (2.8), $\rho(A_\Lambda) \ni z \mapsto R_z^\Lambda$ is a $\mathscr{B}(\mathsf{H})$-valued analytic extension of $Z_\Lambda \ni z \mapsto R_z^0 + G_z W \Lambda_z V G_{\bar z}^*$. Thus, given $w \in Z_\Lambda$, relation (2.9) suggests to define an analytic extension of Λ by

$$\widehat{\Lambda}^{(w)} \colon \rho(A_\Lambda) \to \mathscr{B}(\mathsf{X},\mathsf{Y})\,,$$

$$\widehat{\Lambda}_z^{(w)} := \Lambda_w + (w-z)\Lambda_w V G_{\bar w}^* \left(1 + (w-z)R_z^\Lambda\right) G_w W \Lambda_w\,. \tag{2.10}$$

Lemma 2.7. *Suppose that Z_Λ contains at least an accumulation point. Then $\widehat{\Lambda}^{(w)}$ is w-independent.*

Proof. Let $w_1 \neq w_2$. At first suppose that A_Λ has a spectral gap, equivalently $\rho(A_\Lambda)$ is a connected subset of $\mathbb{C}$. Since $\widehat{\Lambda}^{(w_1)} = \widehat{\Lambda}^{(w_2)}$ on Z_Λ by (2.9), then $\widehat{\Lambda}^{(w_1)} = \widehat{\Lambda}^{(w_2)}$ on the whole $\rho(A_\Lambda)$ by the Identity Theorem for analytic functions. Conversely suppose that $\rho(A_\Lambda) = \mathbb{C}_- \cup \mathbb{C}_+$, where $\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm\mathrm{Im}(z) > 0\}$. Then the thesis is consequence of the same argument separately applied to the connected sets $\mathbb{C}_-$ and $\mathbb{C}_+$. □

Remark 2.8. Suppose that $\widehat{\Lambda}^{(w)}$ in (2.10) does not depend on the choice of $w \in Z_\Lambda$, $\widehat{\Lambda}_z \equiv \widehat{\Lambda}_z^{(w)}$; then $V^*\widehat{\Lambda}_z^* W^* = W\widehat{\Lambda}_{\bar z} V$: by (2.5) and $(R_z^\Lambda)^* = R_{\bar z}^\Lambda$, one has

$$V^*\widehat{\Lambda}_z^* W^* = W\Lambda_{\bar w} V + (\bar w - \bar z) W \Lambda_{\bar w} V G_w^* \left(1 + (\bar w - \bar z) R_{\bar z}^\Lambda\right) G_{\bar w} W \Lambda_{\bar w} V = W\widehat{\Lambda}_{\bar z} V\,.$$

The previous lemma suggests that the Kreĭn-like resolvent formula (2.8) could hold on a larger set, i.e.,

$$(-A_\Lambda + z)^{-1} = R_z^0 + G_z W \widehat{\Lambda}_z V G_{\bar z}^*\,, \qquad z \in \rho(A_0) \cap \rho(A_\Lambda)\,.$$

Let us consider a map

$$Q \colon \rho(A_0) \to \mathscr{C}(\mathsf{Y},\mathsf{X})\,, \qquad z \mapsto Q_z\,,$$

(here $\mathscr{C}(\mathsf{Y},\mathsf{X})$ denotes the set of closed linear operators) such that

$$\mathrm{dom}(Q_z) \text{ is } z\text{-independent, } \mathrm{dom}(Q_z) \equiv \mathsf{D}, \text{ and dense, } \overline{\mathsf{D}} = \mathsf{Y}, \tag{2.11}$$

$$Q_z = Q_w + (z-w) V G_{\bar w}^* G_z W \qquad z, w \in \rho(A_0)\,. \tag{2.12}$$

Defining

$$\begin{aligned} Z_Q := \{z \in \rho(A_0) : \; & Q_z \text{ and } Q_{\bar z} \text{ are bijections from } \mathsf{D} \\ & \text{onto } \mathsf{X} \text{ with inverses in } \mathscr{B}(\mathsf{X},\mathsf{Y})\}\,, \end{aligned}$$

we suppose that

$$Z_Q \neq \emptyset \tag{2.13}$$

and

$$V^*(Q_z^*)^{-1} W^* = W Q_{\bar z}^{-1} V\,, \qquad z \in Z_Q\,. \tag{2.14}$$

Remark 2.9. Notice that the left hand side of (2.14) is well defined: since $z \in Z_Q$, Q_z^{-1} is bounded and so its adjoint exists (and is bounded); moreover $\ker(Q_z^*) = \operatorname{ran}(Q_z)^\perp = \mathsf{X}^\perp = \{0\}$ and so Q_z^* is invertible and $(Q_z^*)^{-1} = (Q_z^{-1})^*$.

Remark 2.10. Notice that Q_w, $w \in Z_Q$, is closed since it is the inverse of a bounded (hence closed) operator. Then Q_z, $z \in \rho(A_0)$, is closed since, by (2.12), it differs from Q_w by a bounded operator.

Remark 2.11. Notice that if $V = \mathbb{1}$ (or $W = \mathbb{1}$) then (2.14) follows from $Q_z^* W = W^* Q_{\bar z}$ (or $V Q_z^* = Q_{\bar z} V^*$).

The set of maps satisfying (2.11)-(2.14) is not void, we give some examples. Below we consider a Weyl function $M\colon \rho(A_0) \to \mathscr{B}(\mathsf{K}^*, \mathsf{K})$, $z \mapsto M_z$, i.e., a $\mathscr{B}(\mathsf{K}^*, \mathsf{K})$-valued map such that

$$M_z^* = M_{\bar z}\,, \qquad M_z - M_w = (z-w) G_{\bar w}^* G_z\,. \tag{2.15}$$

The canonical representation is $M_z := \tau((G_{z_0} + G_{\bar z_0})/2 - G_z)$, $z_0 \in \rho(A_0)$, (see [32, Lemma 2.2]; it is well defined thanks to (2.3)). In the case τ has a bounded extension to $\operatorname{ran}(G_z)$ (eventually considering a range space for τ larger than the original K), one can take $M_z := -\tau G_z$.

Example 2.12. Let X be a closed subspace of K and let $\pi\colon \mathsf{K} \to \mathsf{K}$, $\operatorname{ran}(\pi) = \mathsf{X}$, be the corresponding orthogonal projector. Then $\pi^*\colon \mathsf{K}^* \to \mathsf{K}^*$ is an orthogonal projector as well. Let us set $\mathsf{Y} := \mathsf{X}^* = \operatorname{ran}(\pi^*)$, $V := \pi\colon \mathsf{K} \to \mathsf{X}$, $W := \pi^*\colon \mathsf{Y} \to \mathsf{K}^*$. Given $\Theta\colon \operatorname{dom}(\Theta) \subseteq \mathsf{X}^* \to \mathsf{X}$ self-adjoint and a Weyl function $M\colon \rho(A_0) \to \mathscr{B}(\mathsf{K}^*, \mathsf{K})$, $z \mapsto M_z$, we define $Q_z\colon \operatorname{dom}(\Theta) \subseteq \mathsf{Y} \to \mathsf{X}$ by $Q_z := \Theta + V M_z W$. If one further supposes that τ is surjective, i.e., $\operatorname{ran}(\tau) = \mathsf{K}$, then $\mathbb{C}\backslash\mathbb{R} \subseteq Z_Q$ (see [32, Proposition 2.1], [35, Theorem 2.1]). $Q : z \mapsto Q_z$ satisfies (2.11), (2.12) and $Q_z^* = Q_{\bar z}$ by (2.15). So $(Q_z^{-1})^* = (Q_z^*)^{-1} = Q_{\bar z}^{-1}$, $z \in Z_Q$. Since V and W are orthogonal projectors, this gives (2.14). For explicit examples where such kind of maps appear in applications to partial differential operators, see [20], [35], [36], [27], [19], [21], [29], [11], [30] and references therein. As Theorem 2.19 below shows, it is not necessary to suppose $\operatorname{ran}(\tau) = \mathsf{K}$ whenever one knows that $Z_Q \neq \emptyset$.

Example 2.13. Let $\alpha \in \mathscr{B}(\mathsf{K}, \mathsf{K}^*)$, $\alpha^* = \alpha$, and let $M\colon \rho(A_0) \to \mathscr{B}(\mathsf{K}^*, \mathsf{K})$, be a Weyl function. Suppose that there exists $c > 0$ such that $\|M_z\|_{\mathscr{B}(\mathsf{K}^*,\mathsf{K})} < \|\alpha\|^{-1}_{\mathscr{B}(\mathsf{K},\mathsf{K}^*)}$ whenever $|\operatorname{Im}(z)| > c$. Then define $Q_z \in \mathscr{B}(\mathsf{K}^*)$ by $Q_z := -(\mathbb{1} - \alpha M_z)$. It is immediate to check (also use Remark 2.11) that $Q\colon z \mapsto Q_z$ satisfies (2.11)-(2.14) with $\mathsf{X} = \mathsf{Y} = \mathsf{K}^*$, $V = \alpha$, $W = \mathbb{1}$ and $Z_Q = \{z \in \rho(A_0) : |\operatorname{Im}(z)| > c\}$. Such kind of maps appears in the definition of Laplacians with δ-type potentials supported on a compact hypersurface (see [4], [28, Section 5.4], [31] and references therein); in such references it is proven that $\mathbb{C}\backslash\mathbb{R} \subseteq Z_Q$ by analytic Fredholm theory (M_z is a compact operators in these examples). As Theorem 2.19 below shows, this is not necessary, $Z_Q \neq \emptyset$ suffices. In the not compact case, for Laplacians with δ-type potentials supported on a deformed plane, in [10, Lemma 3.6] it is proven

$\mathbb{C}\backslash\mathbb{R} \subseteq Z_Q$ whenever the deformation is in $C_0^{1,1}(\mathbb{R}^2)$, while $Z_Q \neq \emptyset$ whenever the deformation is in $C_0^{0,1}(\mathbb{R}^2)$, i.e., is Lipschitz continuous (see [10, Lemma 3.5]). By Theorem 2.19, the latter hypothesis suffices to prove that $Z_Q = \rho(A_0) \cap \rho(A_\Lambda)$.

Example 2.14. Let $V \in \mathscr{B}(\mathsf{K}, \mathsf{K}^*)$, $W \in \mathscr{B}(\mathsf{K}^*, \mathsf{K})$ such that $V^*W^* = WV$ and let $M\colon \rho(A_0) \to \mathscr{B}(\mathsf{K}^*, \mathsf{K})$, $z \mapsto M_z$, be a Weyl function. Suppose that there exists $c > 0$ such that $\|M_z\|_{\mathscr{B}(\mathsf{K}^*,\mathsf{K})} < \|V\|^{-1}_{\mathscr{B}(\mathsf{K},\mathsf{K}^*)}\|W\|^{-1}_{\mathscr{B}(\mathsf{K}^*,\mathsf{K})}$ whenever $|\mathrm{Im}(z)| > c$. Then define $Q_z \in \mathscr{B}(\mathsf{K}^*)$ by $Q_z := -(\mathbb{1} - VM_zW)$. It is immediate to check that $Q : z \mapsto Q_z$ satisfies (2.11), (2.12) and $Z_Q = \{z \in \rho(A_0) : |\mathrm{Im}(z)| > c\}$ with $\mathsf{X} = \mathsf{Y} = \mathsf{K}^*$. As regards (2.14), it holds by

$$\begin{aligned}V^*(Q_z^*)^{-1}W^* &= -V^*(\mathbb{1} - W^*M_{\bar z}V^*)^{-1}W^* = -V^*\left(\sum_{n=0}^{\infty}(W^*M_{\bar z}V^*)^n\right)W^*\\ &= -\sum_{n=0}^{\infty} V^*\,\underbrace{W^*M_{\bar z}V^*\dots W^*M_{\bar z}V^*}_{n\text{-times}}\,W^* = -\sum_{n=0}^{\infty} W\,\underbrace{VM_{\bar z}W\dots VM_{\bar z}W}_{n\text{-times}}\,V\\ &= -W(\mathbb{1} - VM_{\bar z}W)^{-1}V = WQ_{\bar z}^{-1}V\,.\end{aligned}$$

Alike maps appear in [1, Appendix B] and produce resolvent formulae similar to the (Kato-)Konno-Kuroda one (see [22, 26]). However in [1, Appendix B] it is assumed that the map E^*F, where $F := V\tau$, $E := W^*\tau$, is infinitesimally bounded with respect to $|A_0|^{1/2}$ and that M_z is compact. As Theorem 2.19 below shows, these hypotheses are not necessary, $Z_Q \neq \emptyset$ suffices.

Example 2.15. Let $Q\colon \rho(A_0) \to \mathscr{C}(\mathsf{Y}, \mathsf{X})$ be any map satisfying (2.11)-(2.14) with $V = \mathbb{1}$ (or $W = \mathbb{1}$) and let $B \in \mathscr{B}(\mathsf{Y}, \mathsf{X})$ such that $B^*W = W^*B$ (or $VB^* = BV^*$). Define $\widetilde{Q}_z := B + Q_z$. For any $z \in Z_Q$ one has $\widetilde{Q}_z = (1 + BQ_z^{-1})Q_z$. Suppose that

$$\widetilde{Z}_Q := \{z \in Z_Q : 1 + BQ_z^{-1} \text{ and } 1 + BQ_{\bar z}^{-1} \text{ are continuous bijections from } \mathsf{X} \text{ onto } \mathsf{X}\}$$

is not void. Then $\widetilde{Q} : z \mapsto \widetilde{Q}_z$ satisfies (2.11)-(2.14). A map of such kind is used in [28, section 5.5] to describe Laplacians with δ'-type potentials supported on compact Lipschitz hypersurfaces. There Q_z^{-1} is compact and it is proven that $\mathbb{C}\backslash\mathbb{R} \subseteq \widetilde{Z}_Q$ by analytic Fredholm theory. As Theorem 2.19 below shows, $\widetilde{Z}_Q \neq \emptyset$ suffices to prove that $Z_{\widetilde{Q}} = \rho(A_0) \cap \rho(A_\Lambda)$.

Given Q which satisfies (2.11)-(2.14), it is immediate to check (also use Remark 2.9) that

$$\Lambda^Q\colon Z_Q \to \mathscr{B}(\mathsf{X}, \mathsf{Y})\,, \qquad \Lambda^Q_z := Q_z^{-1}\,,$$

satisfies (2.5) and (2.6) and thus we can apply Theorem 2.4. From now on we use the notations

$$A_Q := A_{\Lambda^Q}\,, \qquad R^Q_z := (-A_Q + z)^{-1}\,, \quad z \in \rho(A_Q)\,.$$

According to (2.10), we can introduce the analytic extension of Λ^Q given by

$$\widehat{\Lambda}^Q\colon \rho(A_Q) \to \mathscr{B}(\mathsf{X},\mathsf{Y})\,,$$

$$\widehat{\Lambda}^Q_z := Q_w^{-1} + (w-z)Q_w^{-1}VG^*_{\bar{w}}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}\,, \quad w\in Z_Q\,. \tag{2.16}$$

Remark 2.16. Notice that, since we are not supposing that Z_Q contains an accumulation point, the extension $\widehat{\Lambda}^Q$ could depend on the choice of the point $w\in Z_\Lambda$. This is not the case, as Theorem 2.19 shows.

At first we provide the following

Lemma 2.17. *Let* $\Lambda\colon Z_\Lambda \to \mathscr{B}(\mathsf{X},\mathsf{Y})$ *be as in Theorem 2.4. Then, for any* $w\in Z_\Lambda$ *and for any* $z\in\rho(A_0)\cap\rho(A_\Lambda)$, *one has*

$$R^\Lambda_z - R^0_z = \left(1+(w-z)R^\Lambda_z\right)G_wW\Lambda_wVG^*_{\bar{w}}\left(1+(w-z)R^0_z\right). \tag{2.17}$$

Proof. In the case $z=w$, (2.17) reduces to (2.8). Hence it suffices to prove the thesis in the case $z\neq w$. By functional calculus, it is immediate to check that

$$(w-z)\left(1+(w-z)R_z\right) = \left(-R_w+\frac{1}{w-z}\right)^{-1} \tag{2.18}$$

for any $w,z\in\rho(A)$, $w\neq z$, where $R_z := (-A+z)^{-1}$ is the resolvent of a self-adjoint operator A. Thus, by (2.18) and (2.8),

$$\begin{aligned}
&(w-z)^2(R^\Lambda_z - R^0_z)\\
&\quad= (w-z)\left(1+(w-z)R^\Lambda_z\right) - (w-z)\left(1+(w-z)R^0_z\right)\\
&\quad= \left(-R^\Lambda_w+\frac{1}{w-z}\right)^{-1} - \left(-R^0_w+\frac{1}{w-z}\right)^{-1}\\
&\quad= \left(-R^\Lambda_w+\frac{1}{w-z}\right)^{-1}(R^\Lambda_w - R^0_w)\left(-R^0_w+\frac{1}{w-z}\right)^{-1}\\
&\quad= \left(-R^\Lambda_w+\frac{1}{w-z}\right)^{-1}G_wW\Lambda_wVG^*_{\bar{w}}\left(-R^0_w+\frac{1}{w-z}\right)^{-1}\\
&\quad= (w-z)^2\left(1+(w-z)R^\Lambda_z\right)G_wW\Lambda_wVG^*_{\bar{w}}\left(1+(w-z)R^0_z\right).
\end{aligned}$$

□

Remark 2.18. Notice that by the exchange $R^\Lambda_z \leftrightarrow R^0_z$ in the above proof one gets the alternative identity

$$R^\Lambda_z - R^0_z = \left(1+(w-z)R^0_z\right)G_wW\Lambda_wVG^*_{\bar{w}}\left(1+(w-z)R^\Lambda_z\right). \tag{2.19}$$

The previous lemma provides an essential ingredient in the proof of our main result:

Theorem 2.19. *Let* $Z_Q\neq\emptyset$, $Q\colon\rho(A_0)\to\mathscr{C}(\mathsf{Y},\mathsf{X})$ *a map satisfying* (2.11), (2.12), *and* (2.14). *Then* $Z_Q=\rho(A_0)\cap\rho(A_Q)$ *and for any* $z\in\rho(A_0)\cap\rho(A_Q)$ *one has* $Q_z^{-1}=\widehat{\Lambda}^Q_z$. *Moreover the resolvent formula*

$$(-A_Q+z)^{-1}=R^0_z+G_zWQ_z^{-1}VG^*_{\bar z}\,,\qquad z\in\rho(A_0)\cap\rho(A_Q)\,,$$

holds true.

Proof. The first statement of the theorem is equivalent to show that the two identities $\widehat{\Lambda}^Q_zQ_z=1_{\mathsf{Y}}$ and $Q_z\widehat{\Lambda}^Q_z=1_{\mathsf{X}}$ hold true for any $z\in\rho(A_0)\cap\rho(A_Q)$, $z\neq w\in Z_Q$.

By (2.16) and (2.12), one gets

$$\begin{aligned}
&\widehat{\Lambda}^Q_zQ_z\\
&\quad=\big(Q_w^{-1}+(w-z)Q_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}\big)\left(Q_w+(Q_z-Q_w)\right)\\
&\quad=1+(w-z)Q_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wW-(w-z)Q_w^{-1}VG^*_{\bar w}G_zW\\
&\qquad-(w-z)^2Q_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}VG^*_{\bar w}G_zW\,.
\end{aligned}$$

Hence, by (2.4) and (2.17),

$$\begin{aligned}
&(w-z)^{-2}\big(\widehat{\Lambda}^Q_zQ_z-1\big)\\
&\quad=(w-z)^{-1}Q_w^{-1}VG^*_{\bar w}\left(\left(1+(w-z)R^Q_z\right)-\left(1+(w-z)R^0_z\right)\right)G_wW\\
&\qquad-Q_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^0_z\right)G_wW\\
&\quad=Q_w^{-1}VG^*_{\bar w}\big((R^Q_z-R^0_z)\\
&\qquad\quad-\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^0_z\right)\big)G_wW\\
&\quad=0\,.
\end{aligned}$$

The proof of the other identity is almost the same. At first let us notice that $Q_z\widehat{\Lambda}^Q_z$ is well defined since, by definition (2.16) and (2.12),

$$\operatorname{ran}(\widehat{\Lambda}^Q_z)\subseteq\operatorname{ran}(Q_w^{-1})=\operatorname{dom}(Q_w)=\mathsf{D}=\operatorname{dom}(Q_z)\,.$$

By (2.16) and (2.12), one gets

$$\begin{aligned}
&Q_z\widehat{\Lambda}^Q_z\\
&\quad=\left(Q_w+(Q_z-Q_w)\right)\big(Q_w^{-1}+(w-z)Q_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}\big)\\
&\quad=1+(w-z)VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}-(w-z)VG^*_{\bar w}G_zWQ_w^{-1}\\
&\qquad-(w-z)^2VG^*_{\bar w}G_zWQ_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}\,.
\end{aligned}$$

Hence, by (2.4) and (2.19),

$$\begin{aligned}
&(w-z)^{-2}\big(Q_z\widehat{\Lambda}^Q_z-1\big)\\
&=(w-z)^{-1}VG^*_{\bar w}\left(\left(1+(w-z)R^Q_z\right)-\left(1+(w-z)R^0_z\right)\right)G_wWQ_w^{-1}\\
&\qquad-VG^*_{\bar w}\left(1+(w-z)R^0_z\right)G_wWQ_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)G_wWQ_w^{-1}\\
&=VG^*_{\bar w}\left((R^Q_z-R^0_z)-\left(1+(w-z)R^0_z\right)G_wWQ_w^{-1}VG^*_{\bar w}\left(1+(w-z)R^Q_z\right)\right)G_wWQ_w^{-1}\\
&=0\,.
\end{aligned}$$

To conclude the proof of the theorem we must show that $\widehat{\Lambda}^Q_z$ satisfies the identities (2.5) and (2.6) for all $z, w \in \rho(A_0) \cap \rho(A_Q)$. These are immediate consequences of Remark 2.8 ($\widehat{\Lambda}^Q_z = Q_z^{-1}$ does not depend on w) and (2.12). □

Remark 2.20. Notice that in the proof of the previous theorem we did not use (2.1). This hypothesis is only needed in the proof of Theorem 2.4. In case (1.1) still holds, then the statements in Theorem 2.19 retain their validity without requiring $\overline{\ker(\tau)} = \mathsf{H}$.

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Claudio Cacciapuoti
DiSAT, Sezione di Matematica, Università dell'Insubria, via Valleggio 11, I-22100 Como, Italy.
claudio.cacciapuoti@uninsubria.it

Davide Fermi
Dipartimento di Matematica, Università di Milano, Via Cesare Saldini 50, I-20133 Milano, Italy.
davide.fermi@unimi.it

Andrea Posilicano
DiSAT, Sezione di Matematica, Università dell'Insubria, via Valleggio 11, I-22100 Como, Italy.
andrea.posilicano@uninsubria.it