IAC-22-C1.3.7

Multi-Revolution Low-Thrust Trajectory Optimisation using Differential Dynamic Programming in Orbital Element formulation

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Abstract

Space planetary missions' analysis with low-thrust propulsion includes orbit raising and de-orbiting manoeuvres which can involve multiple revolutions resulting in a spiralling motion of the satellite. The launch of large constellation satellites is increasing the number of satellites launched per month and the design of their trajectory to be positioned in their operational orbit. This problem is particularly relevant when low-thrust satellites are considered that are characterised by a continuous thrust and are getting more involved in the design of new missions since they grant a greater final operational mass thanks to their high specific impulse.

The optimisation of low-thrust trajectories involving a larger number of orbit revolutions is a challenging problem. Differential dynamic programming is one of the techniques that can be used to solve nonlinear optimal control problems. This method based on the application of Bellman principle of optimality defines a feedback control law solving necessary optimality conditions during the backward sweep discretising the overall problem in several decision steps and checks for the functional cost reduction during the forward integration to accept or reject the computed control law. In the last years, differential dynamic programming technique evolved thanks to the formulation of the hybrid differential dynamic programming proposed by Lantoine and Russell which maps the required derivatives recursively using state transition matrices and the stochastic differential dynamic programming which introduces random perturbations that can affect the dynamics. However, all past works deal with orbital dynamics expressed in terms of Cartesian coordinates and in only one paper orbital elements are used as state representation, but the rendezvous problem is not solved.

This paper presents a systematic procedure for the optimisation of multi-revolution low-thrust trajectories using the differential dynamic programming technique based on orbital elements as state rep-
representation of the dynamics. Lagrange and Gauss planetary equations are used to model the spacecraft dynamics to include both conservative and non-conservative accelerations.

Some planetary missions like orbit raising for large constellations considering the engine specifics of actual satellites are used to test the proposed approach including also J2 orbital perturbation.

Keywords: low-thrust, optimal control, orbital elements, differential dynamic programming.

Nomenclature

$\delta\boldsymbol{b}$	Lagrange multipliers variation
$\delta\boldsymbol{u}$	control variation
$\delta\boldsymbol{x}$	state variation
μ	gravitational parameter [$\mathrm{km}^3/\mathrm{s}^2$]
$\boldsymbol{\varphi}$	endpoint equality constraints
ω	pericentre anomaly [rad]
Ω	right ascension of the ascending node [rad]
a	semi-major axis [km]
$\tilde{a}$	adimensional semi-major axis
a_0	initial semi-major axis [km]
$\boldsymbol{b}$	Lagrange multipliers
$\bar{\boldsymbol{b}}$	nominal set of Lagrange multipliers
e	eccentricity
f	true anomaly
i	orbit inclination [rad]
Isp	specific impulse [s]
J	cost function
L_{ref}	reference length [km]
m	satellite mass [kg]
$\tilde{m}$	adimensional mass
m_0	initial satellite mass [kg]
m_{ref}	reference mass [kg]
r	satellite distance [km]
t	time variable [s]
$\tilde{t}$	adimensional time
t_{ref}	reference time [s]
$\boldsymbol{u}$	control thrust [N]
$\bar{\boldsymbol{u}}$	nominal control
$\tilde{\boldsymbol{u}}$	adimensional control thrust
$\boldsymbol{u}^*$	control minimising the Hamiltonian function

u_0	maximum control thrust magnitude [N]
u_h	out-of-plane control thrust [N]
$\tilde{u}_h$	adimensional out-of-plane control thrust
u_n	normal control thrust [N]
$\tilde{u}_n$	adimensional normal control thrust
u_t	tangential control thrust [N]
$\tilde{u}_t$	adimensional tangential control thrust
v	satellite velocity magnitude [km/s]
V	value function
$\boldsymbol{x}$	satellite state vector
$\bar{\boldsymbol{x}}$	nominal state vector
$\boldsymbol{x}_f$	final state vector

Acronyms/Abbreviations

ESA	European Space Agency
GNSS	Global Navigation Satellite Systems
HDPP	Hybrid Differential Dynamic Programming
SDDP	Stochastic Differential Dynamic Programming
DDP	Differential Dynamic Programming
PDE	Partial Differential Equation
HJB	Hamilton-Jacobi-Bellman
RAAN	Right Ascension of the Ascending Node
TOF	Time Of Flight

1. Introduction

Electric spacecraft are the new frontier of next space missions, not only for planetary missions, but also for interplanetary missions. This can be inferred by looking at the latest space missions like the ESA mission BepiColombo [1] towards Mercury, or the Galileo [2] satellites belonging to the GNSS services.

The low-thrust systems onboard of the electric spacecraft present the great advantage in maximising the final operational mass of the spacecraft but the design of the trajectories involving these systems are more involved because their dynamics cannot be represented by ballistic motion but it is a continuous dynamics where the thruster is always providing an acceleration.

There are a lot of existing techniques dealing with the problem of low-thrust trajectory optimisation in literature. One of the most interesting techniques for solving non-linear optimal control problems is DDP. This method is based on Bellman's principle of optimality [3] which states that an optimal policy has the property to be the same even if the optimal control is found starting from an intermediate state, and so it is independent on the initial guess used for the trajectory of the dynamics. This principle is mathematically expressed by a PDE which is the HJB equation. Unfortunately, this PDE has no analytical solution and the numerical solution cannot be provided since the dimension of the searching space is not finite. The DDP proposes to apply the dynamic programming in a neighbourhood of a nominal non-optimal trajectory. This method is more effective the closer the non-optimal trajectory is to the optimal solution. Colombo et al. [4] presented a modified DDP algorithm for the optimisation of low-thrust trajectories where the problem is discretised in several decision steps, so that the optimisation process requires the solution of a great number of small problems. Lantoine and Russell [5] developed a new second-order algorithm based on DDP, called HDDP, which maps the required derivatives recursively through first-order and second-order state transition matrices. Ozaki et al. [6] proposed a SDDP where random perturbations enter the dynamics of the problem and their expected values are computed by the unscented transform. However, this kind of technique has not been further explored and it has been used within the Cartesian framework.

This paper presents a low-thrust trajectory optimisation using a DDP algorithm which is based on Keplerian orbital elements as state representation to prove that the methodology can work also in a different framework like the one proposed by the orbital elements. The dynamics of the system will be expressed by Gauss' planetary equations in the $[\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{h}}]$ (tangential, normal, out-of-plane) reference frame because the low-thrust acceleration cannot be modelled as a conservative force.

The paper is structured as follows: Section 2 presents the general DDP, while in Section 3 the modification due to the new representation in terms of the orbital elements will be presented. The results of the optimisation will be shown in Section 4, whereas Section 5 is devoted to the discussion of the results and of the methodology. Finally, Section 6 concludes the paper.

2. Methodology and mathematical theory

In this section the main problem of finding an optimal control law for trajectory design will be presented together with the fundamental theory and methodology of the DDP algorithm.

2.1 Dynamics formulation

The optimisation problem consists in finding the a control law, $\boldsymbol{u}(t)$, that inserted in the system dynamics provides a trajectory resulting in the minimisation of a functional cost subject to some final equality constraints.

$$J = \int_{t_0}^{t_f} [u(t)]^2 dt \quad \text{subject to} \quad \boldsymbol{\varphi} = \boldsymbol{x}(t_f) - \boldsymbol{x}_f \tag{1}$$

In the Keplerian orbital elements framework, the equations of motions of an orbiting satellite are represented by Gauss' planetary equations which model both conservative and non-conservative accelerations. In this work Gauss' planetary equations are defined in $[\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{h}}]$ reference frame since. The set of Gauss' equations is reported from Battin [7] together with the mass variation equation that completes the dynamics of a low-thrust satellite:

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2a^2 v}{\mu}\frac{u_t}{m} \\
\frac{de}{dt} &= \frac{1}{v}\left[2\left(e+\cos f\right)\frac{u_t}{m} - \frac{r}{a}\sin f\,\frac{u_n}{m}\right] \\
\frac{di}{dt} &= \frac{r\cos\theta}{h}\frac{u_h}{m} \\
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i}\frac{u_h}{m} \\
\frac{d\omega}{dt} &= \frac{1}{ev}\left[2\sin f\,\frac{u_t}{m} + \left(2e+\frac{r}{a}\cos f\right)\frac{u_n}{m}\right] - \frac{r\sin\theta\cos i}{h\sin i}\frac{u_h}{m} \\
\frac{df}{dt} &= \frac{h}{r^2} - \frac{1}{ev}\left[2\sin f\,\frac{u_t}{m} + \left(2e+\frac{r}{a}\cos f\right)\frac{u_n}{m}\right] \\
\frac{dm}{dt} &= -\frac{1}{Isp\,g_0}\sqrt{u_t^2+u_n^2+u_h^2}
\end{aligned}
\tag{1}
$$

where a, e, i, Ω, ω, f are the osculating semi-major axis, eccentricity, inclination, Right Ascension of the Ascending Node (RAAN), pericentre anomaly, and true anomaly, respectively. The vector $[u_t, u_n, u_h]$ represent the components of the disturbing forces that are the sum of the control actions and the orbital perturbations while m id the mass of the satellite and I_{sp} the specific impulse.

The dynamics formulation can be rearranged to improve numerical integration making all the equations non dimensional so that the difference in terms of order of magnitudes between the orbital parameters is reduced. All the orbital elements and disturbing accelerations are allowed to range between [0,1]. The following set of reference quantities has been used for the adimensionalisation process:

$$
\begin{aligned}
L_{ref} &= \begin{cases} a_i & if \quad a_i > a_f \\ a_f & if \quad a_i \le a_f \end{cases} \\
t_{ref} &= \sqrt{\frac{L_{ref}^3}{\mu}} \qquad m_{ref} = \frac{m_0}{n} \\
v_{ref} &= \sqrt{\frac{\mu}{L_{ref}}} \qquad u_{ref} = m_{ref}\,\frac{\mu}{L_{ref}^2}
\end{aligned}
\tag{2}
$$

The n divisor used for definition of the reference mass is introduced to avoid that adimensional mass is close to zero leading to a divergence of the integration of the equations of motions. Using the reference quantities given by Eq. (2), Gauss' adimensional equations are formulated in the following manner:

$$
\begin{aligned}
\frac{d\tilde{a}}{d\tilde{t}} &= 2\sqrt{\frac{\tilde{a}^3}{1-e^2}\left(1+2e\cos f+e^2\right)}\frac{\tilde{u}_t}{\tilde{m}} \\
\frac{de}{d\tilde{t}} &= \sqrt{\frac{\tilde{a}\left(1-e^2\right)}{1+2e\cos f+e^2}}\left[2\left(e+\cos f\right)\frac{\tilde{u}_t}{\tilde{m}} - \frac{1-e^2}{1+e\cos f}\sin f\,\frac{\tilde{u}_n}{\tilde{m}}\right] \\
\frac{di}{d\tilde{t}} &= \sqrt{\tilde{a}\left(1-e^2\right)}\frac{\cos\left(\omega+f\right)}{1+e\cos f}\frac{\tilde{u}_h}{\tilde{m}} \\
\frac{d\Omega}{d\tilde{t}} &= \sqrt{\tilde{a}\left(1-e^2\right)}\frac{\sin\left(\omega+f\right)}{\sin i\left(1+e\cos f\right)}\frac{\tilde{u}_h}{\tilde{m}} \\
\frac{d\omega}{d\tilde{t}} &= \sqrt{\tilde{a}\left(1-e^2\right)}\left\{\frac{1}{e\sqrt{1+2e\cos f+e^2}}\left[2\sin f\,\frac{\tilde{u}_t}{\tilde{m}} + \frac{2e+\cos f+e^2\cos f}{1+e\cos f}\frac{\tilde{u}_n}{\tilde{m}}\right]+\right. \\
&\qquad \left. -\frac{\sin\left(\omega+f\right)}{1+e\cos f}\frac{\cos i}{\sin i}\frac{\tilde{u}_h}{\tilde{m}}\right\} \\
\frac{df}{d\tilde{t}} &= \frac{\left(1+e\cos f\right)^2}{\left[a\left(1-e^2\right)\right]^{\frac{3}{2}}} - \frac{1}{e}\sqrt{\frac{\tilde{a}\left(1-e^2\right)}{1+e\cos f+e^2}}\left[2\sin f\,\frac{\tilde{u}_t}{\tilde{m}} + \frac{2e+\cos f+e^2\cos f}{1+e\cos f}\frac{\tilde{u}_n}{\tilde{m}}\right] \\
\frac{d\tilde{m}}{d\tilde{t}} &= -\sqrt{\frac{\mu}{L_{ref}}}\frac{1}{Isp\,g_0}\sqrt{\tilde{u}_t^2+\tilde{u}_n^2+\tilde{u}_h^2}
\end{aligned}
\tag{3}
$$

The use of classic Keplerian elements as representation of the state dynamics introduces some limitations on the orbits that can be considered because of the singularities associated to the equations of motion. Gauss' variational equations in terms of classic Keplerian elements are singular for circular orbits ($e = 0$) and equatorial orbits ($i = 0$). These singularities are restricting the application of such orbital elements because both circular orbits and equatorial orbits are of particular interest for space missions. Modified equinoctial elements can be considered to eliminate the two singularities and include both circular and equatorial orbits in the analysis. The expression of non-dimensional Gauss' variational equations in terms of modified equinoctial elements considering the same set of reference variables in the radial-transversal-orthogonal $\left[\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\mathbf{h}}\right]$ reference frame is reported:

$$
\begin{aligned}
\frac{d\tilde{p}}{d\tilde{t}} &= 2\sqrt{\frac{\tilde{p}^3}{1+f\cos L+g\sin L}}\frac{\tilde{u}_\theta}{\tilde{m}} \\
\frac{df}{d\tilde{t}} &= \sqrt{\frac{\tilde{p}}{1+f\cos L+g\sin L}}\left\{\left(1+f\cos L+g\sin L\right)\sin L\frac{\tilde{u}_r}{\tilde{m}} + \left[f+\cos L\left(2+\right.\right.\right. \\
&\qquad \left.\left.\left. +f\cos L+g\sin L\right]\frac{\tilde{u}_\theta}{\tilde{m}} - g\left(h\sin L-k\cos L\right)\frac{\tilde{u}_h}{\tilde{m}}\right\} \\
\frac{dg}{d\tilde{t}} &= \sqrt{\frac{p}{1+f\cos L+g\sin L}}\left\{-\left(1+f\cos L+g\sin L\right)\cos L\frac{\tilde{u}_r}{\tilde{m}} + \left[g+\sin L\left(2+\right.\right.\right. \\
&\qquad \left.\left.\left. +f\cos L+g\sin L\right]\frac{\tilde{u}_\theta}{\tilde{m}} + f\left(h\sin L-k\cos L\right)\frac{\tilde{u}_h}{\tilde{m}}\right\} \\
\frac{dh}{d\tilde{t}} &= \frac{\sqrt{\tilde{p}}}{2}\frac{\left(1+h^2+k^2\right)}{1+f\cos L+g\sin L}\cos L\frac{\tilde{u}_h}{\tilde{m}} \\
\frac{dk}{d\tilde{t}} &= \frac{\sqrt{\tilde{p}}}{2}\frac{\left(1+h^2+k^2\right)}{1+f\cos L+g\sin L}\sin L\frac{\tilde{u}_h}{\tilde{m}} \\
\frac{dL}{d\tilde{t}} &= \frac{\left(1+f\cos L+g\sin L\right)^2}{\tilde{p}^{\frac{3}{2}}} + \frac{\sqrt{\tilde{p}}}{1+f\cos L+g\sin L}\left(h\sin L-k\cos L\right)\frac{\tilde{u}_h}{\tilde{m}} \\
\frac{d\tilde{m}}{d\tilde{t}} &= -\sqrt{\frac{\mu}{L_{ref}}}\frac{1}{Isp\,g_0}\sqrt{\tilde{u}_r^2+\tilde{u}_\theta^2+\tilde{u}_h^2}
\end{aligned}
\tag{4}
$$

2.2 DDP theory and fundamental algorithm

Differential dynamic programming is a numerical technique for the resolution of non-linear optimal control problems, and it is a simplification of the most general concept of dynamic programming. It is based on Bellman's principle of optimality [3] that can be mathematically represented using Hamilton-Jacobi-Bellman (HJB) equations in its continuous version:

$$\frac{\partial V(\mathbf{x},t)}{\partial t} = -\min_{\mathbf{u}(t)}\left[J(\mathbf{x},\mathbf{u},t) + \mathbf{f}(\mathbf{x},\mathbf{u},t)\cdot\nabla V(\mathbf{x},t)\right] \tag{5}$$

where V is the value function, J is the functional cost and $\mathbf{f}$ represents the equations of motion describing the dynamics of the system. The HJB equation is a partial differential equation with no analytical solution. However, no numerical solution can be as well obtained because the dimension of the search space is not finite. This is defined in literature as the "curse of dimensionality". The DDP offers a solution to the HJB equation representing a linear-quadratic expansion starting from a nonoptimal solution used as first guess. This way it is possible to apply Bellman's principle of optimality to obtain at least a local optimal solution, because the global optimality is lost due to the application of the linear-quadratic expansion. The HJB equation also admits a discrete version that can be coupled with the numerical integration schemes:

$$V_k^*\left(\mathbf{x}_k,\mathbf{b},t_k\right) = \min_{\mathbf{u}_k}\left[J_k\left(\mathbf{x}_k,\mathbf{u}_k,t_k\right) + V_{k+1}^*\left(\mathbf{x}_{k+1},\mathbf{b},t_{k+1}\right)\right] \tag{6}$$

where V^*_{k+1} represents the optimal value function obtained at the successive step t_{k+1}, and $\mathbf{b}$ is the vector of Lagrange multipliers used to adjoint the endpoint constraints to the cost function, J, for the definition of the value function, V. In this work the discrete version of the DDP algorithm is used. Before Taylor expansion is applied, each variable in Eq. is reformulated as the sum of a nominal initial guess and a small variation in the following way:

$$\begin{aligned} &\mathbf{x}_k = \bar{\mathbf{x}}_k + \delta\mathbf{x}_k, \quad \mathbf{u}_k = \bar{\mathbf{u}}_k + \delta\mathbf{u}_k, \\ &\mathbf{x}_{k+1} = \bar{\mathbf{x}}_{k+1} + \delta\mathbf{x}_{k+1}, \quad \mathbf{b} = \bar{\mathbf{b}} + \delta\mathbf{b} \end{aligned} \tag{7}$$

Each term is expanded in Taylor series starting from the initial nominal guess stopping at the second-order term to be consistent with the linear-quadratic expansion assumption. At this point the algorithm can be formulated in its "local" or "global" version:

- the local version keeps the nominal control $\mathbf{u}_k$ as starting point of the Taylor expansions.
- the global version uses as initial guess for the expansions the optimal control $\mathbf{u}^*$ obtained from the minimization of the expanded HJB equation with all the variations equal to zero.

The attribute "local" or "global" refers to the magnitude of the control variation that can be applied and not to the final minimum solution which will be always a local optimal solution. In the "global" version of the algorithm the initial guess is the result of an optimisation problem. The consequence is that the search space for the overall optimal control problem is increased.

Once each term in the HJB equation is expanded in Taylor series the optimal control variation that minimizes the right-hand side of HJB equation must be computed. The evaluation of the optimal solution is carried out differentiating with respect to the control variation leading to a linear feedback control law.

$$\delta\mathbf{u}_k = \beta\delta\mathbf{x}_k + \gamma\delta\mathbf{b} \tag{8}$$

The feedback control law is replaced in the expanded HJB equation to obtain the following set of backward difference equations:

$$\begin{aligned} P_k &= A_k - \frac{1}{4}B_k C_k^{-1} B_k \\ Q_k &= -\frac{1}{2}D_k C_k^{-1} B_k + E_k^T \\ S_k &= H_k - \frac{1}{2}B_k^T C_k^{-1} K_k \\ R_k &= R_{k+1} - \frac{1}{4}K_k^T C_k^{-1} K_k \\ Z_k &= Z_{k+1} - \frac{1}{2}D_k^T C_k^{-1} K_k \end{aligned} \tag{9}$$

The initial condition for the backward difference equations is given by the partials of the cost function evaluated at the final state.

$$\begin{aligned} P_{N+1} &= \frac{1}{2}V_{xx}\left(\mathbf{x}_{N+1},\bar{\mathbf{b}}\right) \\ Q_{N+1} &= V_x\left(\mathbf{x}_{N+1},\bar{\mathbf{b}}\right) \\ S_{N+1} &= V_{xb}\left(\mathbf{x}_{N+1},\bar{\mathbf{b}}\right) \\ R_{N+1} &= \frac{1}{2}V_{bb}\left(\mathbf{x}_{N+1},\bar{\mathbf{b}}\right) \\ Z_{N+1} &= V_b\left(\mathbf{x}_{N+1},\bar{\mathbf{b}}\right) \end{aligned} \tag{10}$$

The matrices appearing in the backward equations are obtained plugging together the coefficients multiplying the same differentials in the expanded HJB equation.

$$
\begin{aligned}
A_k &= \frac{1}{2}\left(J_{xx}^k + 2 f_x^{k^T} P_{k+1} f_x^k + Q_{k+1} f_{xx}^k\right) \\
B_k &= \left(J_{xu}^k + 2 f_x^{k^T} P_{k+1} f_u^k + Q_{k+1} f_{xu}^k\right)^T \\
C_k &= \frac{1}{2}\left(J_{uu}^k + 2 f_u^{k^T} P_{k+1} f_u^k + Q_{k+1} f_{uu}^k\right) \\
D_k &= \left(J_u^k + Q_{k+1} f_u^k\right)^T \\
E_k &= \left(J_x^k + Q_{k+1} f_x^k\right)^T \\
H_k &= f_x^{k^T} S_{k+1} \\
K_k &= f_u^{k^T} S_{k+1}
\end{aligned}
\tag{11}
$$

Therefore, the DDP can be summarised as a technique that is divided in a first backward sweep where the optimal control feedback law is computed and a second forward integration where the control law is applied to check if a minimisation of the functional cost is achieved. The process iterates until no further reduction is obtained leading to the minimum optimal solution. A schematic procedure of DDP algorithm is reported in Fig. 1.

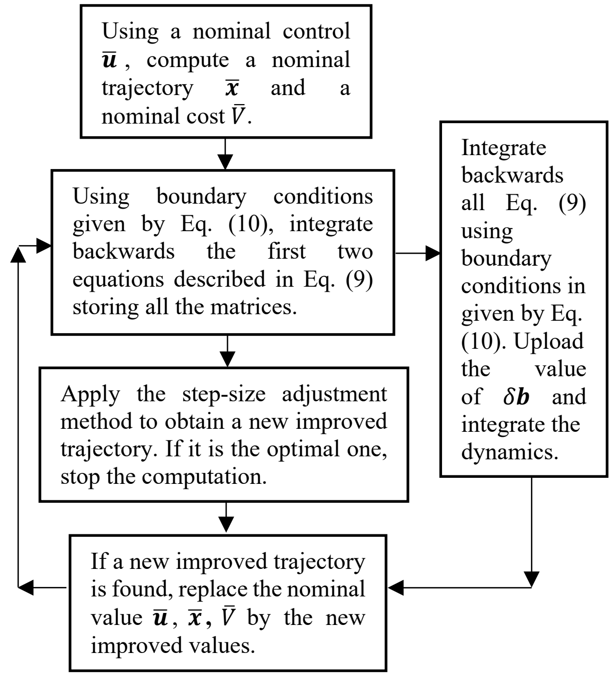

Fig. 1. Schematic procedure of DDP algorithm

3. DDP based on Keplerian orbital elements

In the previous section the standard DDP algorithm has been described. Some adjustments must be carried out when orbital elements are used as state representation of the system dynamic. Indeed, the use of orbital elements is introducing two main features which are not present when Cartesian coordinates are considered:

- The largest part of orbital elements is made of angles which are limited in the interval [0, 2π].
- The orbital elements are divided into "slow" variables and "fast" variables".

The first feature impose a strict control on the variation of the orbital elements associated with angles. A miscalculation of the angular state variables can lead to a wrong optimisation of the dynamics causing the DDP algorithm to diverge. Such control can be performed introducing two indices which are storing for each numerical integration of the trajectory if the angle is inside or outside the interval [0, 2π]. If the angle is inside the interval the index assumes value equal to 0. If the angle is larger than 2π the index assumes value equal to +1, while if the angle is negative the index is equal to -1. This way the variation of the angular state variables can be generalised considering the following formulation:

$$
\delta \mathbf{x}_k = \left(\mathbf{x}_k + 2\pi i_{opt}\right) - \left(\bar{\mathbf{x}}_k + 2\pi i_{nom}\right) \tag{12}
$$

A schematic procedure of the definition of the two indices for the evaluation of the angular state variables variations is reported in Fig 2.

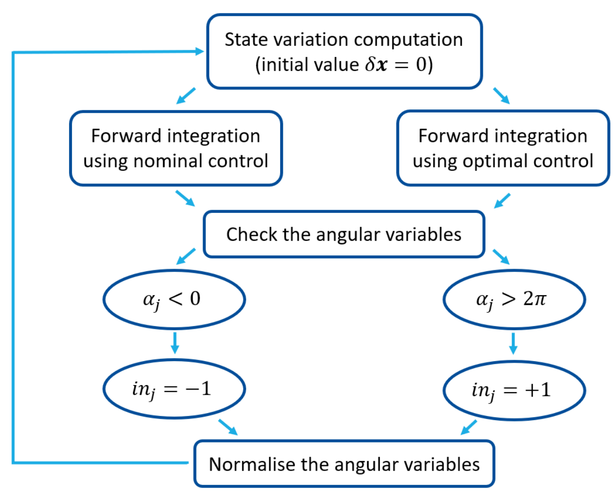

Fig. 2. Index definition for the angular state variations

The second feature is peculiar of orbit dynamics expressed in terms of orbital elements. Indeed, while Cartesian coordinates associated to a revolution are just random variables which are oscillating, the orbital elements are attached to the geometry of the satellite orbit. The consequence is that 5 main orbital elements which describe the size, shape, orbit inclination and orientation (a, e, i, Ω, ω) are constant in case any perturbations is affecting the satellite, and the only

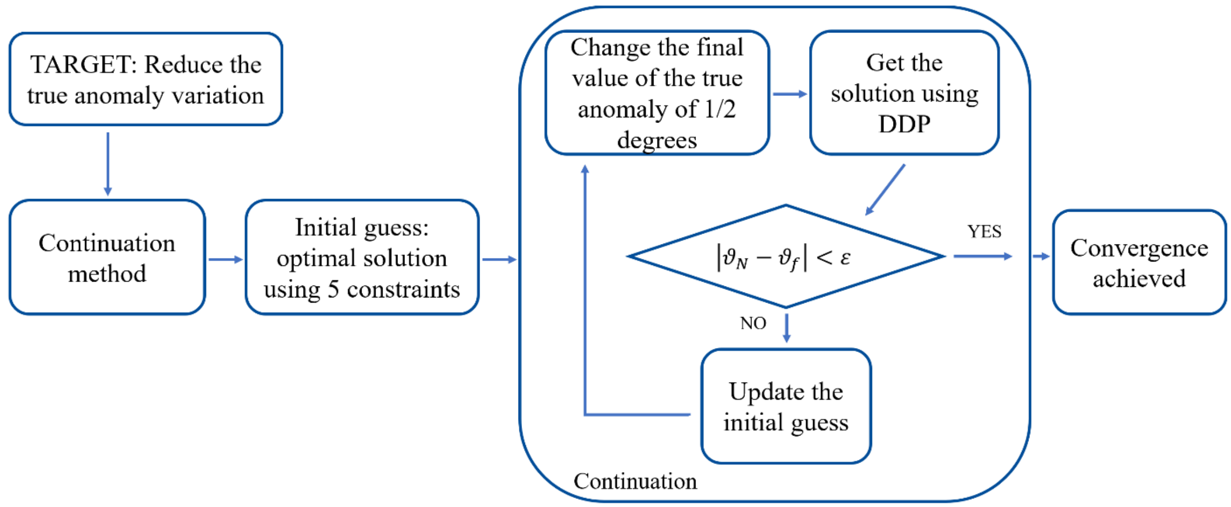

Fig. 3 Continuation scheme procedure

orbital parameter changing with time is the true anomaly *f*. When orbital perturbations or disturbing accelerations are introduced like the one provided by a low-thrust engine, the variation of orbital elements will be "slow" with respect to the variation of the true anomaly which will be "fast".

The DDP algorithm is structure so that the largest is the variation of a state variable the stronger will be the action provided by the control law to reduce that constraint violation. However, this way of optimizing the problem can lead to a wrong solution when a rendezvous is considered. If the largest constraint violation is represented by the true anomaly, the DDP algorithm will update the control law so that the difference between the actual true anomaly and the final prescribed one is reduced without changing the geometry of the orbit due to the large difference in variation in the constraint violation. This problem cannot be solved introducing weights that are tuning the magnitude of the constraint violations because it is a feature attached to the dynamics represented by orbital parameters.

Another result that can be obtained when working with orbital elements is that the control cannot optimise all the variables at the same time. Looking at Eq. (1) the pericenter anomaly rate and true anomaly rate equations present the coefficients multiplying the control actions which are opposite. If the gradient of the two equations with respect to the control action is considered the result will be equal and opposite because the first term in the true anomaly rate is independent on the control action. Therefore, if a given control action is used to maximise the pericenter anomaly, the true anomaly will be minimised and the other way around. The two variables cannot be maximised or minimised at the same time considering the same optimal control thrust. This phenomenon is again explained remembering that a variation of the true anomaly can be obtained not only thanks to the satellite motion but also changing the direction of the eccentric vector which is the reference line where the true anomaly is measured. Therefore, while for all the orbital parameters there is a direct effect when a control action is applied, for the true anomaly there is not this cause-effect relationship leading to the impossibility to optimise all the 6 orbital elements together.

The following strategy is proposed for the application of DDP algorithm when orbital elements are used as state representation of the dynamics. The overall problems is divided in two main parts:

- A first optimisation of the overall problem considering only the 5 slow orbital elements is considered.
- The optimal solution of the previous sub-problem is used as initial guess for the overall problem.

The decomposition of the overall problem in a first sub-problem where only the constraints related to the slow orbital elements are considering is ensuring that the final target orbit will be exactly the prescribed one. In the second part, the optimal thrust is adjusted so that the final position of the target orbit is corrected to match the prescribed true anomaly. However, the difference between the true anomaly variation and the other orbital parameters variation can lead the DDP algorithm to change the control thrust to depart from the target orbit. This means that a strategy for the reduction of the true anomaly variation must be pursued for the correct implementation of the algorithm.

A continuation scheme is proposed as technique for the control of the true anomaly variation magnitude.

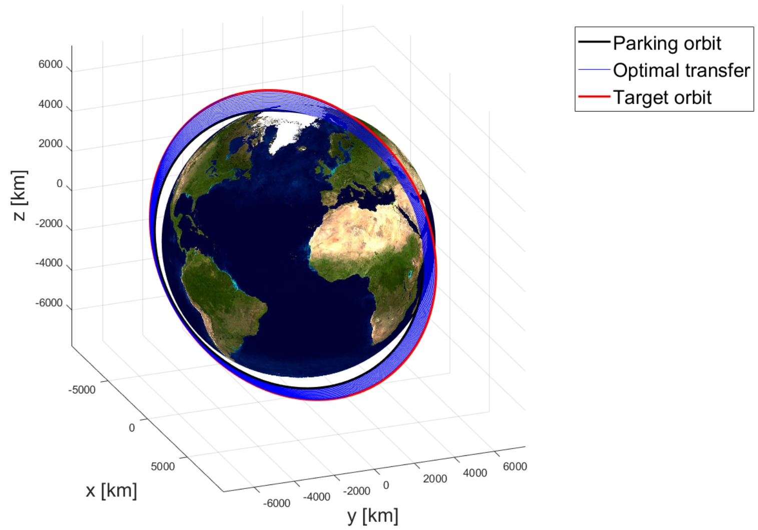

Fig. 4. Optimal vs nominal trajectory

A vector of true anomalies [$f_0, f_1, \ldots, f_{req}$] is generated starting from the true anomaly of the optimal solution of the first sub-problem and the prescribed final true anomaly. A series of optimal control problems is solved considering as target true anomaly each element of the vector. The optimal solution of each problem is used as initial guess for the next one. This way, the variation of the true anomaly we are asking to the DDP algorithm is reduced and the other orbital elements will not be changed. The continuation scheme is shown in Fig. *4*.

4. Results

The new DDP algorithm has been applied to a planetary orbit raising. The parameters of the problem are summarised in Table 1 and Table 2.

The problem has been solved including in the analysis also the effect of J_2 orbital perturbation which represents the strongest action when low Earth orbit satellite are considered.

Table 1. Initial data for the orbit raising

Initial Data	Value
m_0	150 kg
Altitude	450 km
Time of flight	24 h
Specific impulse	1500
Inclination	87.4 deg
Nominal control	[0.6,0,0] N

Table 2. Final data for the orbit raising

Final orbital elements	Value
Altitude	1200 km
Eccentricity	0.05
Inclination	87.9 deg
RAAN	40 deg
Pericenter Anomaly	40 deg
True Anomaly	120 deg

In Fig. 5 the magnitude of the nominal and optimal control law are shown while in Fig. 6 the time history of the 3 components is presented. The initial guess is a tangential control thrust.

The multiple revolutions associated to the final orbit are represented in Fig. 4 where the typical spiralling behaviour of low-thrust trajectories is shown.

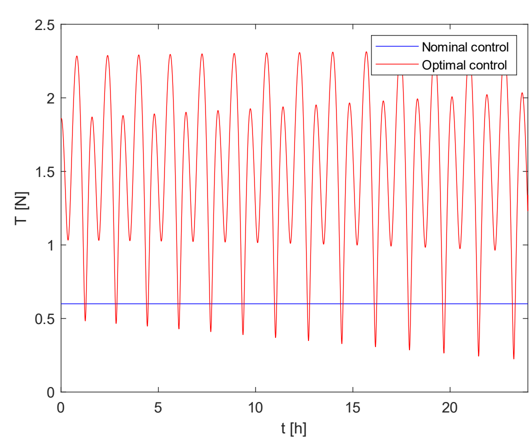

Fig. 5. Initial nominal control guess for the DDP

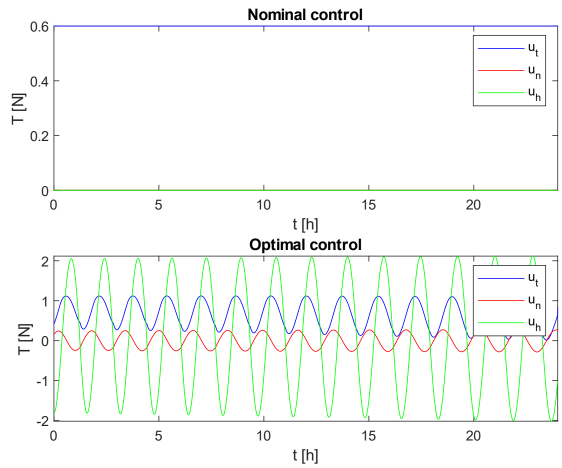

Fig. 6. Optimal control law after the DDP optimisation

It is possible to also plot the variation of the orbital elements associate to the optimal control thrust in Fig. *7*.

5. Conclusions

In this paper a DDP algorithm based on Keplerian elements as state representation of the problem has been investigated. The structure of the algorithm has been kept unchanged from the traditional one, but a new formulation for the dynamics has been provided through Gauss' planetary equations and a new algorithm strategy has been proposed. The method has been assessed testing a planetary orbit raising. A future work will include the effects of the perturbations expressed analytically thanks to the semi-analytical formulations inside the dynamics of the system to enhance the effects of the orbital perturbations.

Acknowledgements

The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program as part of project COMPASS (Grant agreement No. 679086).

I would like to take this opportunity to express my gratitude to the Agenzia Spaziale Italiana (ASI) and Space Generation Advisory Council (SGAC) for their scholarship to attend the International Astronautical Congress.

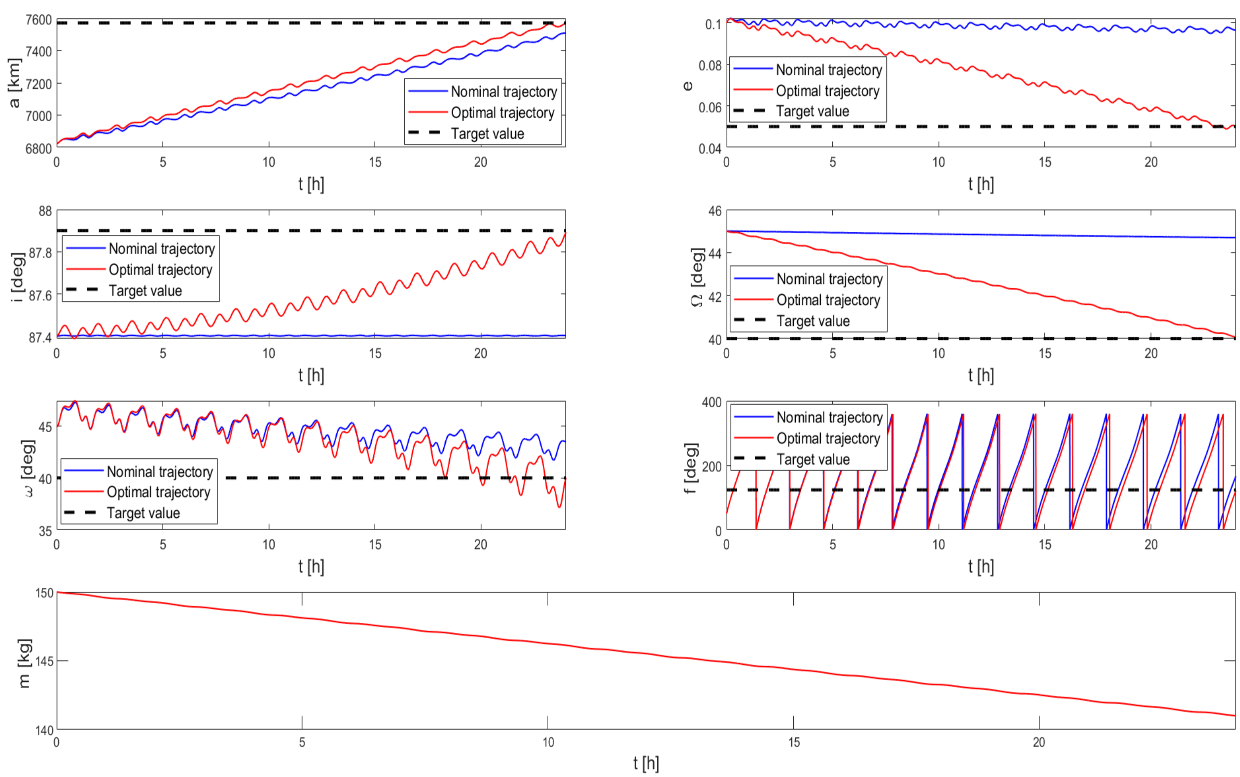

Fig. 7. Orbital elements time variation

References

[1] European Space Agency, BepiColombo, 1 September 2019, https://sci.esa.int/s/89z3z2A, (accessed 25.09.19).
[2] ESA Navipedia, Galileo General Introduction, 4 Marc, https://gssc.esa.int/navipedia/index.php/Galileo_General_Introduction, (accessed 25.09.19).
[3] R. Bellman, Dynamic Programming, Princeton University Press, 1957.
[4] C. Colombo, M. Vasile, G. Radice, Optimal low-thrust trajectories to asteroids through an algorithm based on differential dynamic programming, Celestial Mechanics and Dynamical Astronomy, 2009, pp. 75-112.
[5] G. Lantoine, R. P. Russell, A Hybrid Differential Dynamic Programming Algorithm for Constrained Optimal Control Problems, Part 1: Theory, Journal of Optimization Theory and Applications, Vol. 154, No. 2, 2012, pp. 382-417.
[6] N. Ozaki, S. Campagnola, R. Funase, Yam C. H., Stochastic Differential Dynamic Programming with Unscented Transform for Low-Thrust Trajectory Design, Journal of Guidance, Control, and Dynamics, Vol. 41, No. 2, 2018, pp. 377-387.
[7] R. Battin, An Introduction to the Mathematics and Methods of Astrodynamics, American Institute of Aeronautics and Astronautics, 1999.
[8] D. H. Jacobson, D. Q. Mayne, Differential Dynamic Programming, Elsevier, 1970.
[9] S. Gershwin, D. H. Jacobson, A Discrete-Time Differential Dynamic Programming Algorithm with Application to Optimal Orbit Transfer, AIAA Journal, Vol. 8, 1970, pp. 1616-1626.