A non-linear stochastic approach of ligaments and tendons fractional-order hereditariness

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Abstract

In this study the non-linear hereditariness of knee tendons and ligaments is framed in the context of stochastic mechanics.Without losing the possibility of generalization, this work was focused on knee Anterior Cruciate Ligament (ACL) and the tendons used in its surgical reconstruction. The proposed constitutive equations of fibrous tissues involves three material parameters for the creep tests and three material parameters for relaxation tests. One-to-one relations among material parameters estimated in creep and relaxations were established and reported in the paper. Data scattering, observed with a novel experimental protocol used to characterize the mechanics of the tissue, was modelled as the outcome of the random mechanical parameters. The numerical example proposed in the paper shows that for an assigned probability density function of the material random parameters, the parameters of the probability density function (pdf) may be obtained by a statistical analysis of the experimental data.

Keywords: Non-linear creep, Non-linear relaxation, Random hereditariness

1. Introduction

Surgical reconstruction of tendons and ligaments of the human knee yields successful outcomes as long-term optimal performances are achieved [1, 2]. Indeed, in surgical procedures used to treat Anterior Cruciate Ligament (ACL) injury, that autologous tissue replacements presents long-term drawback [3, 4, 5]. Such aspects may result in to over stiffness or excessive laxity of the joint, thus leading to sub-optimal outcomes of the surgery [6, 7] as well as to osteochondritis and early development of osteoarthritis [8, 9].

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The long-term outcome of tendons and ligaments reconstruction is ruled by the so called "Hereditariness" of fibrous biological tissues. This feature, beside the well known material elasticity, is a peculiar aspects of organic materials such as polymers, rubbers [10, 11, 12] among others and, more important, of biological tissues.

Several studies involving knee fibrous tissue hereditariness were reported [13, 14, 15, 16]. In such studies partial experimental campaign on small animal ligaments and tendons involved: i) long-standing displacement-control mechanical tests leading to the so-called relaxation function $G(t)$ or ii) long-standing force-control mechanical tests leading to well-known creep function $J(t)$. Additional experimental data on ligaments and tendons hereditariness showed that the relaxation function does depend on the reached strain during the experiment, so that $G(\varepsilon, t)$ can be defined as the strain-dependent relaxation function

[17, 18]. Similar outcomes were found also for the long-standing load-control tests, thus yielding to the stress-dependent creep function $J(t)$[17, 19].

The application of Boltzmann superposition integrals to strain/stress dependent relaxation $G(t)$/creep $J(t)$ function yields the so-called quasi-linear hereditariness [20]. Such a formulation is analogous to the well-known linear hereditariness expressed by the strain/stress independence of the relaxation $G(t)$ and creep $J(t)$ functions, respectively [21]. In uniaxial condition, several kind of expressions for relaxations and creep functions, namely $G(t)$ and $J(t)$ were proposed [22] but, in order to fit experimental data, no mathematical consistency was achieved. Recently, in the context of material hereditariness the so called fractional-order calculus has been introduced to describe linear hereditariness of different kind of materials [23, 24, 25].

Fractional calculus is considered as generalization of classical differential calculus involving real or complex differentiation orders. It is defined by means of convolution integrals with power-law kernels as $t^{\pm\beta}$ with $0 \leq \beta \leq 1$ representing the differentiation order [26]. Expression for relaxation as $G(t) \sim t^{-\beta}$ or creep $J(t) \sim t^{\beta}$ yields fractional-order hereditary materials (FHM) that were previously introduced [27, 28, 29, 30].

The use of fractional-order calculus for the analysis of hereditariness of ligaments and tendons of the knee was recently proposed showing that: i) a strong dependence of the relaxation/creep function on the level of reached strain/stress may be observed and ii) the order of relaxation decay, $t^{-\beta_r}$ is larger then creep increment, t^{β_c} so that $\beta_r > \beta_c$. Such inequality is not mathematical consistent by FHM [31], yielding that no prediction of relaxation order β_r by creep measures β_c and,conversely of β_c by measures of β_r can be provided.

In this study the authors aimed to model the presence of strain/stress dependence of relaxation $G(\varepsilon, t)$ and creep $J(\sigma, t)$ by means of quasi-linear fractional hereditary materials theory [32].This approach allows for a prediction of creep parameters from relaxation tests as well as for a prediction of the relaxation parameters from creep measures. The scattering of material parameters observed in experimental tests was modelled as outcomes of random variables and a phenomenological model of non-linear creep and relaxation functions with random parameters was introduced.

The paper is organized as follows: In Section 2 the experimental campaign conducted at the Bio/NanoMechanics Laboratory of University of Palermo is shortly outlined; In Section 3 Power-law hereditariness is discussed for the linear hereditari-ness (Section 3.1), the non-linear hereditariness (Section 3.2) and the phenomenological random constitutive equation for the creep and relaxation is presented (Section 3.3). Conclusions have been withdrawn in Section 4.

2. Experimental campaign on ligaments and tendons hereditariness: Non-linear relaxation and creep

In this section the results of an experimental campaign conducted on ligaments and tendons of human knee is outlined. The experimental campaign involved thirty samples of human patellar tendons and hamstring ligament, subjected to simple uniaxial tensile. Several details about the used protocol as well as about the finding are provided in next subsections.

2.1. Materials and methods

The experimental campaign has involved two kind of human tissue, namely patellar (P) and hamstring (H) tendons. The human tissues were obtained by a tissue bank (Lifelegacy Foundation, Arizona, USA) with the requirements that each ensemble of P and H were obtained by the same human knee to avoid donor variability. Biological specimens were stored at 80 °C and thawed in a 37°C water bath for 15 min prior to testing [20], then prepared for the test and finally each specimen was there cut approximately at the same length before clamping for the uniaxial test. commercial electromechanic system (Electroforce, Bose 3330) was used to test both the tendons' groups. We have used a specific protocol for the repeatability of the experimental campaign. Initially, the samples were preconditioned by cycling between 20 and 100 N, for twenty cycles at 0.25 Hz to remove any crimping in the tendon fibrils [33]; after preconditioning, we performed relaxation test with prescribed values of the strain level in the range $1 - 5\%$

[34, 33]. We conducted the relaxation tests applying a linear ramp of displacement with speed 250 mm/s and after the hold fixed for 100s at the achievement of preselected value of strain, at the end of relaxation test resting the sample for 15 min in order to achieve the same length of the initial specimen measured at the end of the first phase. In last phase, the creep test obtained applying the same initial stress reached at the end of the relaxation test with a linear load ramp of 315 N/s and holding the load 100 s. During the test, the sample was continuously moistened with saline solution.

2.2. *Data analysis*

The experimental data in terms of the axial engineering strain $\varepsilon(\sigma, t)$ were averaged, for each level of applied stress. The averaged creep functions, namely $\mu_\varepsilon^{(P)}(\bar{\sigma}_i, t)$ and $\mu_\varepsilon^{(H)}(\bar{\sigma}_i, t)$ are reported in fig.(1), respectively. A more detailed representation of the averaged creep functions may be observed in a $log\mu_\varepsilon - logt$ plot reported in figs.(1,2) for the patellar and hamstring tendons, respectively.

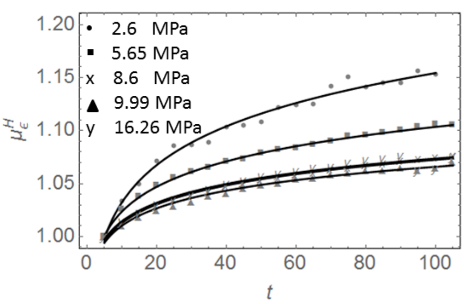

Figure 1: averaged creep functions hamstring ligaments

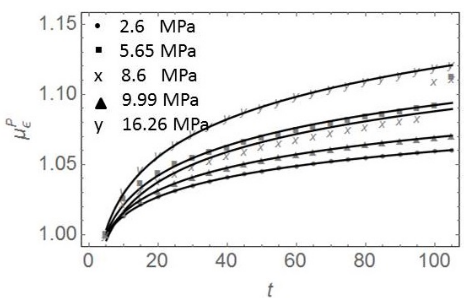

Figure 2: averaged creep functions Patellar tendons

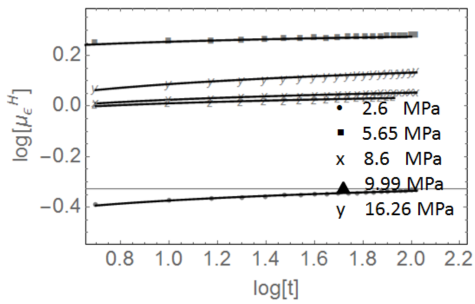

Figure 3: log-log plots averaged creep functions hamstring ligaments

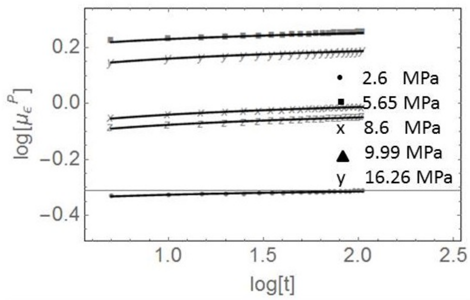

Figure 4: log-log plots averaged creep functions Patellar tendons

Data analysis reported in fig.(3,4) for the log-log plots reveals that good candidate to fit averaged values of creep functions $\mu_\varepsilon^{(P)}(\bar{\sigma}_i, t)$ and $\mu_\varepsilon^{(H)}(\bar{\sigma}_i, t)$ is the a linear model with equation:

$$\log \bar{\mu}_\varepsilon^{(j)}(\sigma, t) = \beta_j \log\left(\frac{t}{\tau_c^{(j)}}\right) + \alpha_j \log\left(\bar{\sigma}_j\right) \quad (1)$$

where $j = P, H$ denotes the specific tissue considered, $\tau_c^{(j)}$ and $\bar{\sigma}_j$ are respectively a characteristic time and the non-dimensional stress $\bar{\sigma}_j = \frac{\sigma_j}{E}$ where E is the tangent elastic modulus obtained at the origin of a monotone test. Straightforward manipulation of eq. (1) yields the relation for the average of the strain omitting j-dependence:

$$\mu_\varepsilon(\sigma, t) = \bar{\sigma}^\alpha \left(\frac{t}{\tau_c}\right)^\beta \quad (2)$$

with $0 \leq \beta \leq 1$ and $0 \leq \alpha \leq 1$ two material parameters and $[\tau_c] = [T]$ and additional material constant representing the characteristic time of the material observed in a creep test. It may be observed that values of α, β and τ_c are represented in figs.(3,4) for the considered tissues.

Solid lines in fig.(3) represents fits of the data with eq.(1) and excellent agreement among curves and data may be observed.

Previous considerations about the averaged values of the creep test results may be reported for the relaxation averaged data in figs.(5,6) and for the log-log plots reported in figs.(7,8) for the patellar and hamstring tendons, respectively. Solid lines in figs.(7,8) represents the linear fitting with equations (omitting j-dependence)

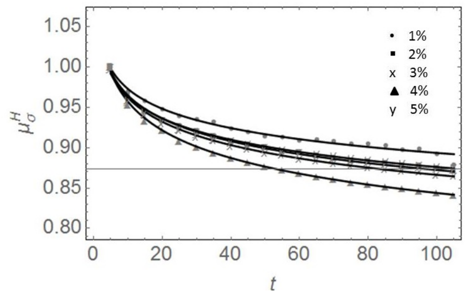

Figure 5: averaged relaxation functions hamstring ligaments

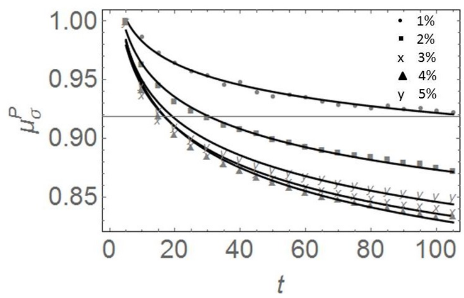

Figure 6: averaged relaxation functions Patellar tendons

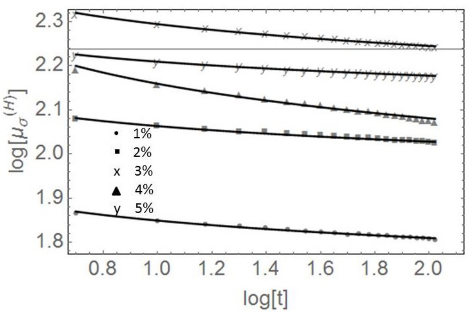

Figure 7: log-log plots averaged relaxation functions hamstring ligaments

$$\log\left[\mu_{\bar{\sigma}}(\varepsilon,t)\right] = -\delta \log\left(\frac{t}{\tau_r}\right) + \gamma \log(\varepsilon) \tag{3}$$

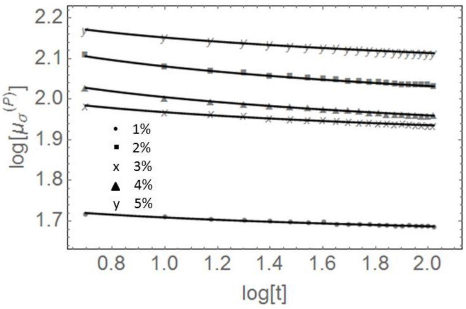

Figure 8: log-log plots averaged relaxation functions Patellar tendons

that corresponds, after straightforward manipulations to the stress average relaxation expressed as:

$$\mu_{\bar{\sigma}}(\varepsilon,t) = \varepsilon^{\gamma}\left(\frac{t}{\tau_r}\right)^{-\delta} \tag{4}$$

with γ, δ relaxation material parameters and $[\tau_r] = [T]$ the characteristic time of the fibrous tissue obtained in a relaxation test.

The observation of eqs.(2, 4) shows that both creep and relaxation functions of the fibrous tissue are non-linear functions of the stress and the strain respectively. Under the assumption that $\alpha = \gamma = 1$ a linear dependence is experienced so that the creep and relaxation may be expressed as:

$$\mu_{\varepsilon}(t) = \bar{\sigma}\left(\frac{t}{\tau_c}\right)^{\beta} = \bar{\sigma}J(t) \tag{5a}$$

$$\mu_{\bar{\sigma}}(t) = \varepsilon\left(\frac{t}{\tau_r}\right)^{-\delta} = \varepsilon G(t) \tag{5b}$$

with $J(t)$ and $G(t)$ the well-known creep and relaxation functions of linear hereditariness.

The non-linear dependence of the strain and the stress observed in the experimental campaign was extensively investigated in several papers on ligaments and tendons hereditariness [19, 20, 18]. Despite the large efforts in the description of material parameters observed in relaxation tests no relations among α, β, τ_c for the creep tests and γ, δ, τ_r for the relaxation could be observed as reported by several authors.

3. The stochastic non-linear hereditariness of fibrous tissues

Data analysis of the experimental campaign reported in previous section showed that constitutive equations for creep and relaxation involves three sets of parameters α, β, τ_c, and γ, δ, τ_r without any relation among them. In this section we aim to show that, under the formalism of fractional differential calculus, the explicit relations among the coefficients estimated in creep and relaxation tests may be established for the 1D case.

We introduce at first the linear 1D case in order to introduce the fundamental definitions of material hereditariness.

3.1. Fractional Hereditary Material (FHM)

The assumption of $\alpha = \gamma = 1$ reported in Sec.2.2 showed that creep and relaxations may be evaluated with the aid of the creep $J(t)$ and relaxation $G(t)$ functions multiplied by the non dimensional stress $\bar{\sigma}$ or strain ε. In such a case, it is well-known that the creep and relaxation functions, in Laplace domain must be compliant with the relation:

$$\widehat{G}(s)\,\widehat{J}(s) = \frac{1}{s^2} \tag{6}$$

where $[s] = T^{-1}$ is the Laplace parameter and $[\hat{\cdot}]$ denotes Laplace transforms.

Power-laws expressions, t^β or $t^{-\delta}$ for creep and relaxation, respectively, are assumed in this section as:

$$J(t) = \frac{1}{\Gamma(\beta+1)}\left(\frac{t}{\tau_c}\right)^{\beta} \tag{7a}$$

$$G(t) = \frac{1}{\Gamma(\delta)}\left(\frac{t}{\tau_r}\right)^{-\delta} \tag{7b}$$

where $\Gamma(\cdot)$ is the Euler-Gamma function.
Evaluation of Laplace transform of eqs.(7a, 7b) substitution into eq.(6) yields that the fundamental relation of linear hereditariness is fulfilled only if the time evolution orders $\beta = \delta$ and the characteristic times $\tau_c = \tau_r = \tau_l$ coalesce.
Under such circumstances, creep and relaxation function may be reported as:

$$J(t) = \frac{1}{\Gamma(\beta+1)}\left(\frac{t}{\tau_l}\right)^{\beta} \tag{8a}$$

$$G(t) = \frac{1}{\Gamma(\beta)}\left(\frac{t}{\tau_l}\right)^{-\beta} \tag{8b}$$

The knowledge of creep and relaxation function in presence of linear dependence of strain $\varepsilon(t)$ on the stress $\bar{\sigma}(t)$ and of the stress $\bar{\sigma}(t)$ on the strain $\varepsilon(t)$ allows Boltzmann superposition as:

$$\begin{aligned}\varepsilon(t) &= \int_0^t J(t-\tau)\dot{\bar{\sigma}}(\tau)\,d\tau = \\ &= \frac{1}{\Gamma(\beta)\tau_l^{\beta}}\int_0^t (t-\tau)^{\beta-1}\bar{\sigma}(\tau)\,d\tau = \frac{1}{\tau_l{}^{\beta}}\left(I_{0^+}^{\beta}\bar{\sigma}\right)(t)\end{aligned} \tag{9}$$

$$\begin{aligned}\bar{\sigma}(t) &= \int_0^t G(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau = \\ &= \frac{\tau_l^{\beta}}{\Gamma(\beta)}\int_0^t (t-\tau)^{-\beta}\dot{\varepsilon}(\tau)\,d\tau = \tau_l^{\beta}\left(D_{0^+}^{\beta}\varepsilon\right)(t)\end{aligned} \tag{10}$$

where $D_{0^+}^{\beta}[\cdot](t)$ and $I_{0^+}^{\beta}[\cdot](t)$ are, respectively, the Caputo fractional derivative and Riemann-Liouville fractional integral of order β with $0 \le \beta \le 1$. Some details on fractional-order calculus.

3.2. The Non-linear Fractional Hereditary Materials (NFHM)

Different orders of the decaying functions measured by creep and relaxation tests in Sec.2 yield that fractional-order linear hereditariness is not appropriate to describe the long-term behaviour of biological fibrous tissue.

Such conclusion is further supported by direct inspection of eq.2, and eq.4 reporting the averaged values of the strain/stress functions, respectively.

In the following, NFHM will be defined with respect to strain/stress functions so that the evolution of the strain $\varepsilon(t)$ and of the stress $\sigma(t)$ is accounted by means of the phenomenological dependence observed in Sec.(2) as:

$$\varepsilon(t) = \frac{\bar{\sigma}^{\alpha}}{\Gamma(1+\beta)}\left(\frac{t}{\tau_c}\right)^{\beta} = \bar{\sigma}^{\alpha}J_c(t) \tag{11a}$$

$$\bar{\sigma}(t) = \frac{\varepsilon^{\gamma}}{\Gamma(\delta)}\left(\frac{t}{\tau_r}\right)^{-\delta} = \varepsilon^{\gamma}G_r(t) \tag{11b}$$

where the creep function $J_c(t)$ and $G_r(t)$ are, respectively, the non-linear creep and the non-linear relaxation functions that differs from their linear counterparts in eqs.(8a, 8b) for the different time-order (namely β and δ) and the different characteristic times, namely τ_c and τ_r, measured in creep and experimental tests.

Eq.(11a) eq.(11b) may be considered as generalization of the well-known Nutting relation [22] introduced at the beginning of the last century to describe the non-linear creep of polymers and rubbers. The non-linear relaxation model, instead, was used by several authors to represent the relaxation fibrous tissues [19, 35] but no relations among creep and relaxation parameters were established [18].

In the following the main relations among creep and relaxation parameters for the phenomenological model observed in (11a) eq.(11b) are established, for the first time, at the best of authors' knowledge. To this aim let us evaluate the strain $\varepsilon(t)$ at time instant $t = \tau_c$ yielding a one-to-one relation among the applied stress $\bar{\sigma}$ and the measured strain $\varepsilon(\tau_c)$, thus obtaining by, neglecting arguments:

$$\bar{\sigma} = (\varepsilon\Gamma(\beta+1))^{1/\alpha} \tag{12}$$

that, after substitution in eq.(11b) yields the equality:

$$(\varepsilon\Gamma(\beta+1))^{1/\alpha} = \frac{\varepsilon^{\delta}}{\Gamma(\delta)}\frac{\tau_c}{\tau_r} \tag{13}$$

that may be cast as:

$$\varepsilon^{\left(\frac{1}{\alpha}-\gamma\right)}\Gamma(\delta)\,\Gamma(1+\beta)^{1/\alpha} = \left(\frac{\tau_c}{\tau_r}\right)^{-\delta} \tag{14}$$

that holds true, for any value of the strain ε as $\gamma = \dfrac{1}{\alpha}$ so that a relation among the material characteristic times observed in creep and relaxation may be established as:

$$\tau_r = \tau_c\Gamma(\beta+1)^{1/(\alpha\delta)}\Gamma(\delta)^{1/\delta} \tag{15}$$

that, in. conjunction with the relation $\gamma = \dfrac{1}{\alpha}$ allows to estimate the characteristic time of the relaxation upon measure of the characteristic time observed in creep once a relation among the decay δ and the order β has been established.

This latter condition may be obtained as we search the estimates of creep parameters with direct measures of the relaxation parameters, namely, γ, δ, τ_r. Under this condition the relation among the characteristic time in creep estimate τ_c and the characteristic time observed in relaxation reads:

$$\tau_c = \tau_r\left[\Gamma(\delta)^{\frac{1}{\gamma\beta}}\,\Gamma(\beta+1)^{\frac{1}{\beta}}\right]^{-1} \tag{16}$$

yielding:

$$\tau_r = \tau_c\left[\Gamma(\delta)^{\frac{1}{\gamma\beta}}\,\Gamma(\beta+1)^{\frac{1}{\beta}}\right] \tag{17}$$

Direct comparison of eq.(17) with eq.(15) yields the relation among the orders:

$$\beta = \alpha\delta \tag{18a}$$

$$\delta = \gamma\beta \tag{18b}$$

Eqs.(18) allows for a relation among the decaying order of the relaxation, given the creep parameters as:

$$\delta = \frac{\beta}{\alpha} \tag{19}$$

that corresponds, in conjunction with $\gamma = 1/\alpha$, to eq.(18b).

Time-order of relaxation δ of the stress $\bar{\sigma}(t)$ yields that under the condition $\alpha < 1$ the order of the relaxation $\delta \geq \beta$ according to the well-established paradigms that *relaxation run faster than creep* as reported by several authors [36, 37, 38].

Summing up, the results of the experimental campaign of creep and relaxation may be described as power-law relations containing each three material parameters: *i*) the order of the non-linearity of the materials; *ii*) the time-order and *iii*) the characteristic times. On one hand, if linear material behavior is considered, then the material parameters reduces to two and coalesces in creep and relaxation, that is linear-order hereditariness. On the other hand if material non-linearity is experienced then the three material parameters observed in creep and relaxations do not correspond each other and, in this study, the specific relations among them have been reported for the case of power-law non-linearity as in Nutting relations.

The results obtained in this section was listed in tables 1 and 2 reporting the averaged data of measured creep parameters for hamstring ligaments and patellar tendons in figs(3,4,7,8) at each level of the stress, namely α and β in column two and three. Measured values of the corresponding parameters for relaxations were reported in columns three and five whereas their analytic estimates have been reported in columns four and six.

$\sigma[MPa]$	α	β	δ	β/α
16.26	0.5945	0.02461	0.0414	0.0414
9.98	0.4249	0.0232	0.05473	0.0547
8.60	0.5112	0.0232	0.0455	0.0454
5.65	0.7349	0.0326	0.0437	0.0444
2.60	0.6459	0.0485	0.0369	0.0751

$\sigma[MPa]$	γ	$1/\alpha$
16.26	1.6819	1.6818
9.98	2.3511	2.2534
8.60	1.9571	1.9561
5.65	1.3698	1.3607
2.60	1.5481	1.5482

Table 1: parameters for Patellar tendons

$\sigma[MPa]$	α	β	δ	β/α
7.58	0.7112	0.0362	0.0500	0.0509
6.88	0.3841	0.0224	0.0566	0.0583
5.37	0.5448	0.0294	0.0528	0.0540
4.55	0.6228	0.0291	0.0426	0.0468
2.21	0.7705	0.0192	0.0277	0.0249

$\sigma[MPa]$	γ	$1/\alpha$
7.58	1.4197	1.4059
6.88	2.6108	2.6034
5.37	1.9029	1.8355
4.55	1.699	1.6056
2.21	1.2777	1.2978

Table 2: parameters for Patellar tendons

The observation of the data obtained with the proposed model of non-linear hereditariness showed that, beside a small difference unavoidable for data scattering, estimated and measurements presented fair agreement.

3.3. *The stochastic model of non-linear hereditariness*

Results collected in experimental campaign reported in Sec.2 showed that data scattering observed in experimental tests result in averaged expressions of the strain evolution $\mu_\varepsilon(t)$ and $\mu_{\bar{\sigma}}(t)$ as well as in standard deviation, namely $S_\varepsilon(t)$ and $S_{\bar{\sigma}}(t)$.

In this section we assume that the source of data scattering is due to the outcomes of the characteristic times of the material, considered a random variables namely, $\tau_c \to T_c$ and $\tau_r \to T_r$ with prescribed probability density functions $p_{\tau_c}(\tau_c)$ and $p_{\tau_r}(\tau_r)$, respectively for the characteristic times in creep and relaxations.

Under these circumstances the random description of the stress/strain evolution equations read, in creep and relaxation, respectively:

$$\mathrm{E}(t) = \left(\frac{1}{T_c}\right)^{\beta} \frac{\bar{\sigma}^{\alpha} t^{\beta}}{\Gamma(\beta+1)} \tag{20a}$$

$$\Sigma(t) = \left(\frac{1}{T_r}\right)^{-\delta} \frac{\varepsilon^{\gamma} t^{-\delta}}{\Gamma(\delta)} \tag{20b}$$

Eqs.(20a,20b)) allow for the evaluation of the averages of the stress and strain functions as:

$$\mu(t) =< \left(\frac{1}{T_c}\right)^{\beta} > \frac{\bar{\sigma}^{\alpha} t^{\beta}}{\Gamma(\beta+1)} \tag{21a}$$

$$\mu_{\bar{\sigma}}(t) =< \left(\frac{1}{T_r}\right)^{-\delta} > \frac{\varepsilon^{\gamma} t^{-\delta}}{\Gamma(\delta)} \tag{21b}$$

where $< \bullet >$ denotes the mathematical expectation operator that reads:

$$< \left(\frac{1}{T_c}\right)^{\beta} >= \int_{-\infty}^{\infty} \left(\frac{1}{\tau_c}\right)^{\beta} p_c(\tau_c)\, d\tau_c \tag{22a}$$

$$< \left(\frac{1}{T_r}\right)^{-} \delta >= \int_{-\infty}^{\infty} \left(\frac{1}{\tau_r}\right)^{\delta} p_r(\tau_r)\, d\tau_r \tag{22b}$$

Similar comments hold true also for the mean square error of the random functions E (t) and $\Sigma(t)$ resulting into:

$$S_{\varepsilon}(t) = <(\mathrm{E}(t) - \mu_{\varepsilon}(t))^2> = <\mathrm{E}(t)^2> - \mu_{\varepsilon}(t)^2 \tag{23a}$$

$$S_{\bar{\sigma}}(t) = <(\Sigma(t) - \mu_{\bar{\sigma}}(t))^2> = <\Sigma(t)^2> - \mu_{\bar{\sigma}}(t)^2 \tag{23b}$$

with second-order moments:

$$<\mathrm{E}(t)^2> = <\left(\frac{1}{T_c}\right)^{2\beta}> \frac{\bar{\sigma}^{2\alpha} t^{2\beta}}{\Gamma(1+\beta)^2} \tag{24a}$$

$$<\Sigma(t)^2> = <\left(\frac{1}{T_r}\right)^{-2\delta}> \frac{\varepsilon^{2\gamma} t^{-2\delta}}{\Gamma(\delta)^2} \tag{24b}$$

and the mathematical expectation reads:

$$<\left(\frac{1}{T_c}\right)^{2\beta}> = \int_{-\infty}^{\infty} \left(\frac{1}{\tau_c}\right)^{2\beta} p_c(\tau_c)\, d\tau_c \tag{25a}$$

$$<\left(\frac{1}{T_r}\right)^{2\delta}> = \int_{-\infty}^{\infty} \left(\frac{1}{\tau_r}\right)^{2\delta} p_r(\tau_r)\, d\tau_r \tag{25b}$$

In the following we assume that the probability density functions $p_{\tau_c}(\tau_c)$ and $p_{\tau_r}(\tau_c)$ are described by uniform density in the interval $[\bar{\tau}_c - a_c; \bar{\tau}_c + a_C]$ and $[\bar{\tau}_r - a_r; \bar{\tau}_r + a_r]$ with $2a_r$ and $2a_c$ the amplitude of the interval representing the boundary of the characteristic times.

The results of the proposed model of random hereditariness was reported in figs.(9,10,11,12) with solid lines for the averaged and the second-order statistics of the strain evolution and stress decay in conjunction with the amplitude of the interval of the pdf obtained by best fitting of the data to characterize the density function.

Observation of figs.(9,10,11,12) shows that the second-order moments of data scattering is well described by the proposed random model of the characteristic times reported in this section.

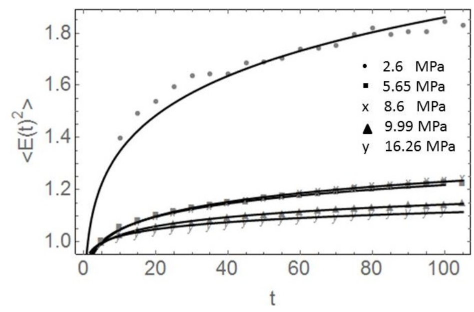

Figure 9: second-order moment creep hamstring ligaments

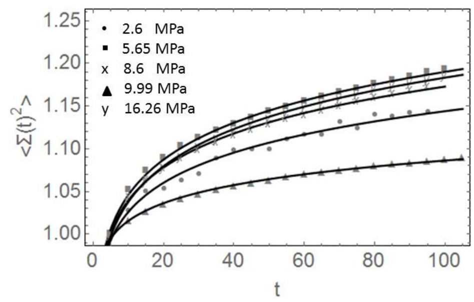

Figure 10: second-order moment creep Patellar tendons

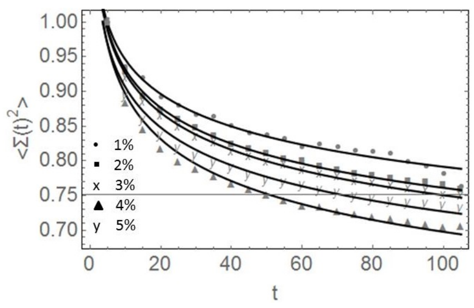

Figure 11: second-order moment relaxation hamstring ligaments

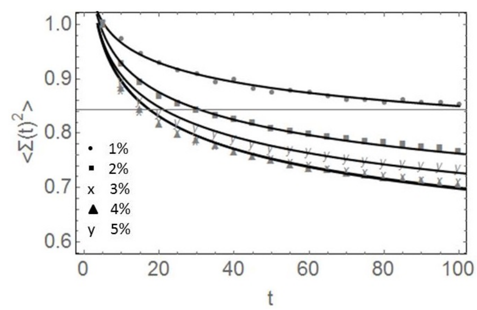

Figure 12: second-order moment relaxation Patellar tendons

In passing we observe also that the relation among the characteristic times τ_r and τ_c provided in sec.(3.2) holds true also with the random description of the characteristic times as:

$$< T_c >=< T_r > \Gamma(\beta + 1)^{1/\alpha\delta}\Gamma(\delta)^{1/\delta} \tag{26a}$$

$$< T_c{}^2 >=< T_r{}^2 > \Gamma(\beta + 1)^{2/\alpha\delta}\Gamma(\delta)^{2/\delta} \tag{26b}$$

allowing to define the statistics of the characteristic times with only the relaxation or the creep tests.

4. Conclusions

In this study a random model for the 1D non-linear hereditariness of human tendons and ligaments was proposed.

Non-linear hereditariness was observed by data obtained during the experimental campaign on ligaments and tendons of the human knee showing a marked stress-dependence and the strain influence on creep and relaxations of fibrous tissues respectively. This study specifically showed that averaged values and standard deviations of the experimental data yielded to averaged material parameters that were related in order to achieve creep parameters from relaxation measures as well as relaxation parameters on creep measures.

Data scattering involved in the experimental measures have been represented with a random model assuming that the characteristic times in creep and relaxation are modelled as random variables with prescribed probability density. The parameters of the density may be obtained by the measured first and second-order statistics of the creep and relaxation obtained from the experimental campaign.

A monte-carlo simulation conducted with the proposed random model shows that first-order statistics obtained with the proposed approach coalesces with the measured data allowing to use the random approach introduced in sec.3.3 for the prediction of the mechanical outcomes in terms of increments of the strain and the decaying of the stress in tendons and ligaments.

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