Existence of weak solutions for models of general compressible viscous fluids with linear pressure

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Abstract

In this work we will focus on the existence of weak solutions for a system describing a general compressible viscous fluid in the case of the pressure being a linear function of the density and the viscous stress tensor being a non-linear function of the symmetric velocity gradient. More precisely, we will first prove the existence of dissipative solutions and study under which conditions it is possible to guarantee the existence of weak solutions.

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1 Introduction

The motion of fluids can be modelled through a system of partial differential equations

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \operatorname{div}_x \mathbb{S}, \end{aligned} \tag{1.1}$$

which can be seen as a mathematical transcription of mainly two physical conservation laws: *conservation of mass* and *conservation of momentum*. For a general non-Newtonian fluid, we can suppose the viscous stress tensor $\mathbb{S}$ to be related to the symmetric velocity gradient

$$\mathbb{D}_x \mathbf{u} = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u})$$

through an implicit rheological law of the type

$$\mathbb{S} : \mathbb{D}_x \mathbf{u} = F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}), \tag{1.2}$$

with F a proper lower semi-continuous function and F^* its conjugate. The physical background of writing the constitutive equation for $\mathbb{S}$ in this form is the fact that $\mathbb{S}$ is monotone in the velocity gradient and vice versa, as clearly explained in the recent survey on a new classification of incompressible fluids by Blechta, Málek and Rajagopal [3]. It is worth noticing that choosing

$$F(\mathbb{D}_x \mathbf{u}) = \frac{\mu}{2} |\mathbb{D}_x \mathbf{u}|^2 + \frac{\lambda}{2} |\operatorname{div}_x \mathbf{u}|^2, \quad \text{with } \mu > 0, \ \frac{2}{d} + \lambda \geq 0,$$

we obtain the compressible Navier-Stokes system. Even though in the latter case there are several results on the existence of global-in-time weak solutions (see e.g. [5], [7], [9], [10]), much less is known for the case where the viscous stress tensor is not a linear function of the velocity gradient: the existence of large-time weak solutions was proved by Feireisl, Liao and Málek [8] in the case where the bulk viscosity $\lambda = \lambda(|\operatorname{div}_x \mathbf{u}|)$ becomes singular for a finite value of $|\operatorname{div}_x \mathbf{u}|$; choosing a linear pressure

$$p(\varrho) = a\varrho, \tag{1.3}$$

the existence was proved by Mamontov [11], [12] in the context of exponentially growing viscosity coefficients, and by Matušů-Nečasová and Novotný [13], exploiting the concept of measure-valued solutions.

In this work, we are going to study under which hypothesis on the convex potential F appearing in (1.2) it is possible to guarantee the existence of global-in-time weak solutions for system (1.1),(1.2) and with a linear pressure of the type (1.3), cf. Theorem 5.2. The proof will done via the concept of *dissipative solutions*, satisfying system (1.1) in the distributional sense with an extra defect term in the second equation that we may call *Reynolds stress*. Recently, Abbatiello, Feireisl and Novotný [1] proved the existence of dissipative solutions for system (1.1) with $\mathbb{S}$ satisfying (1.2) and the isentropic pressure

$$p(\varrho) = a\varrho^\gamma \quad \text{with } \gamma > 1.$$

Our goal is to focus on the case $\gamma = 1$, for which we will prove the existence of dissipative solutions, cf. Theorem 4.9. The advantage of relaying on this very weak concept of solution is that they can be easily identified as limits of weakly convergent subsequences of approximate solutions, as we will see in Section 4.5. It is worth noticing that our approach represents an alternative and improvement to the "standard" measure–valued framework applied in this context by Matušů-Nečasová and Novotný [13].

The paper is organized as follows.

- In Section 2 we introduce the system we are going to study, fixing the necessary hypothesis on the pressure potential F appearing in (1.2).
- In Section 3 we provide the definition of dissipative solution for system (1.1)–(1.2) with the pressure being a linear function of the density, cf. Definition 3.1.
- Section 4 will be devoted to the proof of the existence of dissipative solutions, cf. Theorem 4.9. More precisely, we will perform a three-level approximation scheme: addition of artificial viscosity terms in the continuity equation and balance of momentum in order to convert the hyperbolic system into a parabolic one, regularization of the convex potential to make it continuously differentiable, approximation via the Faedo-Galerkin technique and a family of finite-dimensional spaces.
- In Section 5 we prove the existence of weak solutions for particular choices of the convex potential F, cf. Theorem 5.2.
- In the Appendix A we provide a slightly modified version of the De la Vallée–Poussin criterion as we require the stronger condition, with respect to the standard formulation, that the Young function satisfies the Δ_2-condition, cf. Theorem A.2, necessary to get the existence of weak solutions.

2 The system

We are going to study the system described by the following couple of equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{2.1}$$
$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}. \tag{2.2}$$

The unknown variables are the density $\varrho = \varrho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of the fluid, while the viscous stress tensor $\mathbb{S}$ is assumed to be connected to the symmetric velocity gradient $\mathbb{D}_x\mathbf{u}$ through an implicit rheological law of the type

$$\mathbb{S} : \mathbb{D}_x \mathbf{u} = F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}). \tag{2.3}$$

where, denoting with $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ the space of d-dimensional real symmetric tensors,

$$F : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty) \text{ is convex and lower semi-continuous with } F(0) = 0, \tag{2.4}$$

and F^* is its conjugate. As clearly motivated in [1], Section 2.1.2, we will suppose F to satisfy relation

$$F(\mathbb{D}) \geq \mu \left| \mathbb{D} - \frac{1}{d}\operatorname{Tr}[\mathbb{D}]\mathbb{I} \right|^q - c \quad \text{for all } \mathbb{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{2.5}$$

for some $\mu > 0$, $c > 0$ and $q > 1$. Notice that condition (2.3) is equivalent in requiring

$$\mathbb{S} \in \partial F(\mathbb{D}\mathbf{u}),$$

where ∂ denotes the subdifferential of a convex function. Furthermore, we will consider a linear barotropic pressure

$$p(\varrho) = a\varrho, \quad a > 0; \tag{2.6}$$

the pressure potential P, satisfying the ODE

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho),$$

will be then of the form

$$P(\varrho) = a\ \varrho \log \varrho, \tag{2.7}$$

which implies that P is a strictly convex superlinear continuous function on $[0,\infty)$. We will study the system on the set

$$(t,x) \in (0,T) \times \Omega,$$

where the time $T > 0$ can be chosen arbitrarily large and the physical domain $\Omega \subset \mathbb{R}^d$ is assumed to be bounded and Lipschitz, on the boundary of which we impose the no–slip condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{2.8}$$

Finally, we fix the initial conditions

$$\varrho(0,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0. \tag{2.9}$$

We conclude this section with the following result, collecting the significant properties of the conjugate function F^*.

Proposition 2.1. *Let the function F satisfy conditions* (2.4)*. Then, its conjugate*

$$F^* : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,\infty] \textit{ is convex, lower semi-continuous and superlinear.} \tag{2.10}$$

Proof. First of all, we recall that F^* is defined for every $\mathbb{A} \in \mathbb{R}^{d\times d}_{\rm sym}$ as

$$F^*(\mathbb{A}) := \sup_{\mathbb{B}\in\mathbb{R}^{d\times d}_{\rm sym}} \{\mathbb{A} : \mathbb{B} - F(\mathbb{B})\}.$$

The non-negativity of F^* is trivial if $F(0) = 0$ since

$$F^*(\mathbb{A}) \geq \mathbb{A} : 0 - F(0) = 0 \quad \text{for every } \mathbb{A} \in \mathbb{R}^{d\times d}_{\rm sym}.$$

It is also well-know that the conjugate is convex and lower semi-continuous as it is the supremum of a family of affine functions. It remains to prove the superlinearity:

$$\lim_{|\mathbb{A}|\to\infty} \frac{F^*(\mathbb{A})}{|\mathbb{A}|} = +\infty. \tag{2.11}$$

Let $B_R(0)$ be the ball centred at origin and radius $R > 0$; using the fact that for any $\mathbb{A} \in \mathbb{R}^{d\times d}_{\rm sym}$

$$\sup_{\mathbb{B}\in B_R(0)} \mathbb{A} : \mathbb{B} = \sup_{\mathbb{B}\in B_R(0)} \{\mathbb{A} : \mathbb{B} - F(\mathbb{B}) + F(\mathbb{B})\} \leq F^*(\mathbb{A}) + \sup_{\mathbb{B}\in B_R(0)} F(\mathbb{B})$$

we have

$$\frac{F^*(\mathbb{A})}{|\mathbb{A}|} \geq \sup_{\substack{0<r\leq R\\ |\mathbb{V}|\leq 1}} \left\{ r\frac{\mathbb{A}}{|\mathbb{A}|} : \mathbb{V} \right\} - \frac{1}{|\mathbb{A}|} \sup_{\mathbb{B}\in B_R(0)} F(\mathbb{B}) \geq R - \frac{c}{|\mathbb{A}|},$$

where we used the fact that $F(\mathbb{B})$ is finite for any $\mathbb{B} \in \mathbb{R}^{d\times d}_{\rm sym}$. We conclude that

$$\liminf_{|\mathbb{A}|\to\infty} \frac{F^*(\mathbb{A})}{|\mathbb{A}|} \geq R,$$

and, since R can be chosen arbitrarily large, we obtain (2.11). □

3 Dissipative solution

Following [1], we introduce to the concept of *dissipative solutions*, which satisfy the system in the distributional sense but with an extra "turbulent" term $\mathfrak{R}$ in the balance of momentum (2.2) that we may call *Reynolds stress*. As pointed out in [2], Section 4.1.1, in this context, i.e. when the pressure is a linear function of the density, it is only the possible concentrations and/or oscillations in the convective term that contributes to $\mathfrak{R}$. It is worth noticing that when $\mathfrak{R} \equiv 0$, we get the standard notion of weak solution. From now on, it is better to consider the density ϱ and the momentum $\mathbf{m} = \varrho\mathbf{u}$ as state variables, since they are at least weakly continuous in time.

Definition 3.1. The pair of functions $[\varrho, \mathbf{m}]$ constitutes a *dissipative solution* to the problem (2.1)–(2.9) with initial data

$$[\varrho_0, \mathbf{m}_0] \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^d)$$

if the following holds:

(i) $\varrho \geq 0$ in $(0, T) \times \Omega$ and

$$[\varrho, \mathbf{m}] \in C_{\rm weak}([0, T]; L^1(\Omega)) \times C_{\rm weak}([0, T]; L^1(\Omega; \mathbb{R}^d));$$

(ii) the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\ \mathrm{d}x\mathrm{d}t \tag{3.1}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C_c^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_0$;

(iii) there exist

$$\mathbb{S} \in L^1(0,T;L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \quad \text{and} \quad \mathfrak{R} \in L^\infty_{\mathrm{weak}}(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

such that the integral identity

$$\begin{aligned}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t\end{aligned} \tag{3.2}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\overline{\Omega};\mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, with $\mathbf{m}(0,\cdot) = \mathbf{m}_0$;

(iv) there exists

$$\mathbf{u} \in L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^d)) \text{ such that } \mathbf{m} = \varrho\mathbf{u} \text{ a.e. in } (0,T)\times\Omega;$$

(v) there exists a constant $\lambda > 0$ such that the energy inequality

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + a\varrho\log\varrho\right](\tau,\cdot)\ \mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}(\tau)] + \int_0^\tau \int_\Omega [F(\mathbb{D}\mathbf{u}) + F^*(\mathbb{S})]\ \mathrm{d}x\mathrm{d}t \\ \leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + a\varrho_0\log\varrho_0\right]\ \mathrm{d}x\end{aligned} \tag{3.3}$$

holds for a.e. $\tau \in (0,T)$.

Remark 3.2. Here and in the sequel, $\mathcal{M}^+(\overline{\Omega})$ represents the space of all the positive Borel measures on $\overline{\Omega}$, while $\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ denotes the space of tensor–valued (signed) Borel measures $\mathfrak{R}$ such that

$$\mathfrak{R} : (\xi\otimes\xi) \in \mathcal{M}^+(\overline{\Omega}),$$

for all $\xi \in \mathbb{R}^d$, and with components $\mathfrak{R}_{i,j} = \mathfrak{R}_{j,i}$. $L^\infty_{\mathrm{weak}}(0,T;\mathcal{M}(\overline{\Omega}))$ denotes the space of all the weak–$*$ measurable mapping $\nu : [0,T] \to \mathcal{M}(\overline{\Omega})$ such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\nu(t,\cdot)\|_{\mathcal{M}(\overline{\Omega})} < \infty,$$

which can also be identified as the dual space of $L^1(0,T;C(\overline{\Omega}))$.

4 Existence of dissipative solutions

As in [1] Abbatiello, Feireisl and Novotný proved the existence of dissipative solutions of system (2.1)–(2.9) with $p(\varrho) = a\varrho^\gamma$ and $\gamma > 1$, in this section we aim to show existence for $\gamma = 1$. We employ an approximation scheme based on

(i) addition of an artificial viscosity term of the type $\varepsilon\Delta_x\varrho$ in the continuity equation (2.1) in order to convert the hyperbolic equation into a parabolic one and thus recover better regularity properties of ϱ;

(ii) addition of an extra term of the type $\varepsilon\nabla_x\mathbf{u}\cdot\nabla_x\varrho$ in the balance of momentum (2.2) in order to eliminate the extra terms arising in the energy inequality to save the a priori estimates;

(iii) regularization of the convex potential F through convolution with a family of regularizing kernels to make it continuously differentiable.

More precisely, we will study the following system:

- **continuity equation**

$$\partial_t\varrho + \operatorname{div}_x(\varrho\mathbf{u}) = \varepsilon\Delta_x\varrho, \tag{4.1}$$

on $(0,T)\times\Omega$, with $\varepsilon>0$, the homogeneous Neumann boundary condition

$$\nabla_x\varrho\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{4.2}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_{0,n} \quad \text{on } \Omega, \quad \varrho_{0,n}\to\varrho_0 \text{ in } L^1(\Omega) \text{ as } n\to\infty, \tag{4.3}$$

with $\varrho_{0,n}\in C(\overline{\Omega})$, $\varrho_{0,n}>0$ for all $n\in\mathbb{N}$.

- **momentum equation**

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + a\nabla_x\varrho + \varepsilon\nabla_x\mathbf{u}\cdot\nabla_x\varrho = \operatorname{div}_x\mathbb{S} \tag{4.4}$$

on $(0,T)\times\Omega$, with $\varepsilon>0$, the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \tag{4.5}$$

and the initial condition

$$(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0 \quad \text{on } \Omega. \tag{4.6}$$

- **convex potential**

$$F_\delta(\mathbb{D}) = (\xi_\delta * F)(\mathbb{D}) - \inf_{\mathbb{D}\in\mathbb{R}^{d\times d}_{\text{sym}}}(\xi_\delta * F) \tag{4.7}$$

for any $\mathbb{D}\in\mathbb{R}^{d\times d}_{\text{sym}}$, with $\{\xi_\delta\}_{\delta>0}$ a family of regularizing kernels in $\mathbb{R}^{d\times d}_{\text{sym}}$, the function F satisfying (2.4)–(2.5), and such that

$$\mathbb{S} : F_\delta(\mathbb{D}_x\mathbf{u}) = F_\delta(\mathbb{D}_x\mathbf{u}) + F^*_\delta(\mathbb{S}). \tag{4.8}$$

Even if system (4.1)–(4.8) is of parabolic type, we are forced to perform a further approximation known as *Faedo-Galerkin technique*. The reason is that the unknown state variable $\mathbf{u}$ appears multiplied by ϱ in (4.4), which prevents us from applying the already existing results for parabolic systems that can be found in literature. The idea is to consider a family $\{X_n\}_{n\in\mathbb{N}}$ of finite-dimensional spaces $X_n\subset L^2(\Omega;\mathbb{R}^d)$, such that

$$X_n := \operatorname{span}\{\mathbf{w}_i|\ \mathbf{w}_i\in C^\infty_c(\Omega;\mathbb{R}^d),\ i=1,\dots,n\},$$

where $\mathbf{w}_i$ are orthonormal with respect to the standard scalar product in $L^2(\Omega;\mathbb{R}^d)$, and to look for approximated velocities

$$\mathbf{u}_n\in C([0,T];X_n).$$

Solvability of the approximated problem will be discussed in the following sections.

4.1 On the approximated continuity equation

Given $\mathbf{u} \in C([0,T]; X_n)$, let us focus on identifying that unique solution

$$\varrho = \varrho[\mathbf{u}]$$

of system (4.1)–(4.3). As our domain Ω is merely Lipschitz, we cannot simply repeat the same passages performed for instance by Feireisl [7] in the context of the compressible Navier-Stokes system since better regularity for the domain would be required. However, since X_n is finite-dimensional, all the norms on X_n induced by $W^{k,p}$-norms, with $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, are equivalent and thus, we deduce that

$$\mathbf{u} \in L^\infty(0,T; W^{1,\infty}(\Omega; \mathbb{R}^d)),$$

and there exist two constants $0 < \underline{n} < \overline{n} < \infty$, depending solely on the dimension n of X_n, such that for any $t \in [0,T]$

$$\underline{n} \|\mathbf{u}(t,\cdot)\|_{W^{1,\infty}(\Omega)} \leq \|\mathbf{u}(t,\cdot)\|_{X_n} \leq \overline{n} \|\mathbf{u}(t,\cdot)\|_{W^{1,\infty}(\Omega)}. \tag{4.9}$$

It is now enough to apply the following result to get the existence of weak solutions and the necessary bounds to recover the existence of the corresponded velocity $\mathbf{u}$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For any given $\mathbf{u} \in C([0,T]; X_n)$ and $\varepsilon > 0$, there exists a unique weak solution*

$$\varrho = \varrho_{\varepsilon,n} \in L^2((0,T); W^{1,2}(\Omega)) \cap C([0,T]; L^2(\Omega))$$

of system (4.1)–(4.3) in the sense that the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi - \varepsilon \nabla_x \varrho \cdot \nabla_x \varphi) \, \mathrm{d}x,$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T] \times \overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$. Moreover,

(i) (bound from above - maximum principle) the weak solution ϱ satisfies

$$\|\varrho\|_{L^\infty((0,\tau)\times\Omega)} \leq \overline{\varrho} \exp\left(\tau \|\operatorname{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{4.10}$$

for any $\tau \in [0,T]$, with

$$\overline{\varrho} := \max_\Omega \varrho_{0,n}; \tag{4.11}$$

(ii) (bound from below) the weak solution ϱ satisfies

$$\operatorname*{ess\,inf}_{(0,\tau)\times\Omega} \varrho(t,x) \geq \underline{\varrho} \exp\left(-\tau \|\operatorname{div}_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{4.12}$$

for any $\tau \in [0,T]$, with

$$\underline{\varrho} := \min_\Omega \varrho_{0,n}; \tag{4.13}$$

(iii) let $\mathbf{u}_1, \mathbf{u}_2 \in C([0,T]; X_n)$ be such that

$$\max_{i=1,2} \|\mathbf{u}_i\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))} \leq K,$$

and let $\varrho_i = \varrho[\mathbf{u}_i]$, $i = 1,2$ be the weak solutions of the approximated problem (4.1)–(4.3) sharing the same initial data $\varrho_{0,n}$ in (4.3). Then, for any $\tau \in [0,T]$ we have

$$\|(\varrho_1 - \varrho_2)(\tau,\cdot)\|_{L^2(\Omega)} \leq c_1 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,\tau;W^{1,\infty}(\Omega;\mathbb{R}^d))} \tag{4.14}$$

with $c_1 = c_1(\varepsilon, \varrho_0, T, K)$.

Proof. For the existence of weak solutions and for (i) see Crippa, Donadello and Spinolo [6], Lemmas 3.2 and 3.4, for (ii) see Abbatiello, Feireisl and Novotný [1], Corollary 3.4, and for (iii) see Chang, Jin and Novotný [5], Lemma 4.3 point 3. □

4.2 On the approximated balance of momentum

Let us now turn our attention to the approximated problem (4.4)–(4.8). Following the same approach performed by Feireisl [7], we will first solve the problem on a time interval $[0, T(n)]$ via a fixed point argument, where $T(n)$ depends on the dimension n of the finite-dimensional space X_n. Subsequently we will establish estimates independent of time and iterate the same procedure to finally obtain, after a finite number of steps, our solution $\mathbf{u}$ on the whole time interval $[0, T]$.

4.2.1 Technical preliminaries

For any $\varrho \in L^1(\Omega)$, consider the operator $\mathcal{M}[\varrho] : X_n \to X_n^*$ such that

$$\langle \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w}\rangle \equiv \int_\Omega \varrho \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x, \tag{4.15}$$

with $\langle \cdot, \cdot \rangle$ the L^2-standard scalar product. In particular, we have

$$\|\mathcal{M}[\varrho]\|_{\mathcal{L}(X_n, X_n^*)} = \sup_{\|\mathbf{v}\|_{X_n}, \|\mathbf{w}\|_{X_n} \leq 1} |\langle \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w}\rangle| \leq c(n) \|\varrho\|_{L^1(\Omega)}, \tag{4.16}$$

It is easy to see that the operator $\mathcal{M}$ is invertible provided ϱ is strictly positive on Ω, and in particular we have

$$\|\mathcal{M}^{-1}[\varrho]\|_{\mathcal{L}(X_n^*; X_n)} = \frac{1}{\inf\{\|\mathcal{M}[\varrho]\mathbf{v}\|_{X_n^*} : \mathbf{v} \in X_n, \ \|\mathbf{v}\|_{X_n} = 1\}} \leq \frac{c(n)}{\inf_\Omega \varrho}.$$

Moreover, the identity

$$\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2] = \mathcal{M}^{-1}[\varrho_2] \left(\mathcal{M}[\varrho_2] - \mathcal{M}[\varrho_1]\right) \mathcal{M}^{-1}[\varrho_1]$$

can be used to obtain

$$\left\|\mathcal{M}^{-1}[\varrho_1] - \mathcal{M}^{-1}[\varrho_2]\right\|_{\mathcal{L}(X_n^*; X_n)} \leq c\left(n, \inf_\Omega \varrho_1, \inf_\Omega \varrho_2\right) \|\varrho_1 - \varrho_2\|_{L^1(\Omega)} \tag{4.17}$$

for any $\varrho_1, \varrho_2 > 0$.

4.2.2 Fixed point argument

The approximate velocities $\mathbf{u} \in C([0, T]; X_n)$ are looked for to satisfy the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho \mathbf{u}(t, \cdot) \cdot \boldsymbol{\psi} \, \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [(\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\psi} + a\varrho \, \mathrm{div}_x \, \boldsymbol{\psi}] \, \mathrm{d}x \mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega [\partial F_\delta(\mathbb{D}_x \mathbf{u}) : \nabla_x \boldsymbol{\psi} + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\psi}] \, \mathrm{d}x \mathrm{d}t\end{aligned} \tag{4.18}$$

for any test function $\boldsymbol{\psi} \in X_n$ and all $\tau \in [0, T]$. Now, the integral identity (4.18) can be rephrased for any $\tau \in [0, T]$ as

$$\langle \mathcal{M}[\varrho(\tau, \cdot)](\mathbf{u}(\tau, \cdot)), \boldsymbol{\psi}\rangle = \langle \mathbf{m}_0^*, \boldsymbol{\psi}\rangle + \langle \int_0^\tau \mathcal{N}[\varrho(s, \cdot), \mathbf{u}(s, \cdot)] \, \mathrm{d}s, \boldsymbol{\psi}\rangle$$

with $\mathcal{M}[\varrho] : X_n \to X_n^*$ defined as in (4.15), $\mathbf{m}_0^* \in X_n^*$ such that

$$\langle \mathbf{m}_0^*, \boldsymbol{\psi}\rangle := \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\psi} \, \mathrm{d}x$$

and $\mathcal{N}[\varrho(s,\cdot),\mathbf{u}(s,\cdot)] \in X_n^*$ such that

$$\langle \mathcal{N}[\varrho(s,\cdot),\mathbf{u}(s,\cdot)],\boldsymbol{\psi}\rangle := \int_\Omega \left[(\varrho\mathbf{u}\otimes\mathbf{u} - \partial F_\delta(\mathbb{D}_x\mathbf{u})) : \nabla_x\boldsymbol{\psi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\psi}\right](s,\cdot)\ \mathrm{d}x$$
$$- \varepsilon\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\psi}(s,\cdot)\ \mathrm{d}x.$$

Here, $\varrho = \varrho[\mathbf{u}]$ is the weak solution uniquely determined by $\mathbf{u}$ and thus by Lemma 4.1, conditions (i) and (ii), for any $t \in [0,T]$ we have

$$0 < \underline{\varrho}\exp\left(-t\|\,\mathrm{div}_x\,\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right) \le \varrho(t,x) \le \overline{\varrho}\exp\left(t\|\,\mathrm{div}_x\,\mathbf{u}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{4.19}$$

where $\overline{\varrho}$, $\underline{\varrho}$ are defined as in (4.11), (4.13) respectively. In particular, the operator $\mathcal{M}$ is invertible and hence, for any $\tau \in [0,T]$, we can write

$$\mathbf{u}(\tau,\cdot) = \mathcal{M}^{-1}[\varrho(\tau,\cdot)]\left(\mathbf{m}_0^* + \int_0^\tau \mathcal{N}[\varrho(s,\cdot),\mathbf{u}(s,\cdot)]\ \mathrm{d}s\right).$$

For K and $T(n)$ to be fixed, consider a bounded ball $\mathcal{B}(0,\underline{n}K)$ in the space $C([0,T(n)];X_n)$, with $\underline{n}$ defined as in (4.9),

$$\mathcal{B}(0,\underline{n}K) := \left\{\mathbf{v}\in C([0,T(n)];X_n)\big|\ \sup_{t\in[0,T(n)]}\|\mathbf{v}(t,\cdot)\|_{X_n} \le \underline{n}K\right\},$$

and define a mapping

$$\mathcal{F} : \mathcal{B}(0,\underline{n}K) \to C([0,T(n)];X_n)$$

such that for all $\tau \in [0,T(n)]$

$$\mathcal{F}[\mathbf{u}](\tau,\cdot) := \mathcal{M}^{-1}[\varrho(\tau,\cdot)]\left(\mathbf{m}_0^* + \int_0^\tau \mathcal{N}[\varrho(s,\cdot),\mathbf{u}(s,\cdot)]\ \mathrm{d}s\right).$$

Notice that for every $\mathbf{u} \in \mathcal{B}(0,\underline{n}K)$, from (4.9) we obtain in particular that for all $t \in [0,T(n)]$

$$\|\mathbf{u}(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le K$$

and thus, from (4.19) we obtain that for all $t \in [0,T(n)]$

$$\underline{\varrho}e^{-Kt} \le \varrho(t,x) \le \overline{\varrho}e^{Kt}.$$

Moreover, it is easy to deduce that for every $\mathbf{u} \in \mathcal{B}(0,\underline{n}K)$, $\varrho = \varrho[\mathbf{u}]$ and every $t \in [0,T(n)]$

$$\|\mathcal{N}(\varrho(t,\cdot),\mathbf{u}(t,\cdot))\|_{X_n^*} \le c_2(\overline{\varrho},K,T),$$

and for every $\mathbf{u}_1,\mathbf{u}_2 \in \mathcal{B}(0,\underline{n}K)$, $\varrho_i = \varrho[\mathbf{u}_i]$, $i=1,2$ and $t \in [0,T(n)]$, making use of (4.14),

$$\|\mathcal{N}(\varrho_1(t,\cdot),\mathbf{u}_1(t,\cdot)) - \mathcal{N}(\varrho_2(t,\cdot),\mathbf{u}_2(t,\cdot))\|_{X_n^*} \le c_3(\overline{\varrho},K,T)\|\mathbf{u}_1(t,\cdot)-\mathbf{u}_2(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)}.$$

Then, for every $\mathbf{u} \in \mathcal{B}(0,\underline{n}K)$, $\varrho = \varrho[\mathbf{u}]$ and every $t \in [0,T(n)]$

$$\|\mathcal{F}(\mathbf{u})(t,\cdot)\|_{X_n} \le \|\mathcal{M}^{-1}[\varrho(t,\cdot)]\|_{\mathcal{L}(X_n^*;X_n)}(\|\mathbf{m}_0^*\|_{X_n^*} + \|\mathcal{N}(\varrho(t,\cdot),\mathbf{u}(t,\cdot))\|_{X_n^*}\ t)$$
$$\le \frac{c(n)}{\underline{\varrho}}\ e^{KT(n)}\ \left(\|\mathbf{m}_0^*\|_{X_n^*} + c_2\ T(n)\right),$$

and for every $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{B}(0, \underline{n}K)$, $\varrho_i = \varrho[\mathbf{u}_i]$, $i = 1, 2$ and $t \in [0, T(n)]$,

$$
\begin{aligned}
&\|\mathcal{F}(\mathbf{u}_1)(t,\cdot) - \mathcal{F}(\mathbf{u}_2)(t,\cdot)\|_{X_n} \\
&\quad \leq \left\| \left(\mathcal{M}^{-1}[\varrho_1(t,\cdot)] - \mathcal{M}^{-1}[\varrho_2(t,\cdot)]\right) \left[\int_0^t \mathcal{N}(\varrho_1(s,\cdot), \mathbf{u}_1(s,\cdot))\, \mathrm{d}s\right] \right\|_{X_n} \\
&\qquad + \left\| \mathcal{M}^{-1}[\varrho_2(t,\cdot)] \left[\int_0^t [\mathcal{N}(\varrho_1(s,\cdot), \mathbf{u}_1(s,\cdot)) - \mathcal{N}(\varrho_2(s,\cdot), \mathbf{u}_2(s,\cdot))]\, \mathrm{d}s\right] \right\|_{X_n} \\
&\quad \leq t \left\| \mathcal{M}^{-1}[\varrho_1(t,\cdot)] - \mathcal{M}^{-1}[\varrho_2(t,\cdot)] \right\|_{\mathcal{L}(X_n^*; X_n)} \|\mathcal{N}(\varrho_1(t,\cdot), \mathbf{u}_1(t,\cdot))\|_{X_n^*} \\
&\qquad + t \left\| \mathcal{M}^{-1}[\varrho_2(t,\cdot)] \right\|_{\mathcal{L}(X_n^*, X_n)} \|\mathcal{N}(\varrho_1(t,\cdot), \mathbf{u}_1(t,\cdot)) - \mathcal{N}(\varrho_2(t,\cdot), \mathbf{u}_2(t,\cdot))\|_{X_n^*} \\
&\quad \leq c(n)\, \frac{c_2}{(\underline{\varrho})^2}\, e^{2Kt} t\, \|\varrho_1(t,\cdot) - \varrho_2(t,\cdot)\|_{L^1(\Omega)} + c(n)\, \frac{c_3}{\underline{\varrho}}\, e^{Kt} t\, \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \\
&\quad \leq c(n) \left(\frac{c_1 c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}} \right) e^{2Kt} t\, \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \\
&\quad \leq T(n)\, \frac{c(n)}{\underline{n}} \left(\frac{c_1 c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}} \right) e^{2KT(n)}\, \|\mathbf{u}_1(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{X_n}.
\end{aligned}
$$

Now, taking $K > 0$ sufficiently large and $T(n)$ sufficiently small, so that

$$
\frac{c(n)}{\underline{\varrho}}\, e^{KT(n)} \left(\|\mathbf{m}_0^*\|_{X_n^*} + c_2\, T(n) \right) \leq \underline{n}K,
$$

and

$$
T(n)\, \frac{c(n)}{\underline{n}} \left(\frac{c_1 c_2}{(\underline{\varrho})^2} + \frac{c_3}{\underline{\varrho}} \right) e^{2KT(n)} < 1,
$$

we obtain that $\mathcal{F}$ is a contraction mapping from the closed ball $\mathcal{B}(0, \underline{n}K)$ into itself. From the Banach-Cacciopoli fixed point theorem, we recover that $\mathcal{F}$ admits a unique fixed point $\mathbf{u} \in C([0, T(n)]; X_n)$, which in particular solves the integral identity (4.18).

This procedure can be repeated a finite number of times until we reach $T = T(n)$, as long as we have a bound on $\mathbf{u}$ independent of $T(n)$. the next section will be dedicated to establish all the necessary estimates.

4.2.3 Estimates independent of time

We start with the *energy estimates*. It follows from (4.18) that $\mathbf{u}$ is continuously differentiable and, consequently, the integral identity

$$
\begin{aligned}
\int_\Omega \partial_t(\varrho\mathbf{u}) \cdot \boldsymbol{\psi}\, \mathrm{d}x = &\int_\Omega [\varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\psi} + a\varrho \operatorname{div}_x \boldsymbol{\psi} - \partial F_\delta(\mathbb{D}_x \mathbf{u}) : \nabla_x \boldsymbol{\psi}]\, \mathrm{d}x \\
&- \varepsilon \int_\Omega [\nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\psi}]\, \mathrm{d}x
\end{aligned}
\tag{4.20}
$$

holds on $(0, T(n))$ for any $\boldsymbol{\psi} \in X_n$, with $\varrho = \varrho[\mathbf{u}]$. We recall that in this context the pressure potential $P = P(\varrho)$ satisfies the following identity

$$
a\varrho \operatorname{div}_x \mathbf{u} = -\partial_t P(\varrho) - \operatorname{div}_x(P(\varrho)\mathbf{u}) + \varepsilon\, a(\log\varrho + 1)\Delta_x \varrho.
$$

Now, taking $\boldsymbol{\psi} = \mathbf{u}$ in (4.20) and noticing that

$$
\begin{aligned}
\int_\Omega [\partial_t(\varrho\mathbf{u}) \cdot \mathbf{u} - \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{u}]\, \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2\, \mathrm{d}x + \frac{1}{2} \int_\Omega (\partial_t \varrho + \operatorname{div}_x(\varrho\mathbf{u}))|\mathbf{u}|^2\, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 \mathrm{d}x + \frac{\varepsilon}{2} \int_\Omega \Delta_x \varrho |\mathbf{u}|^2\, \mathrm{d}x,
\end{aligned}
$$

where, using the boundary condition (4.2),

$$
\begin{aligned}
\frac{\varepsilon}{2}\int_\Omega |\mathbf{u}|^2\Delta_x\varrho \,\mathrm{d}x &= \frac{\varepsilon}{2}\int_\Omega |\mathbf{u}|^2 \operatorname{div}_x \nabla_x\varrho \,\mathrm{d}x \\
&= \frac{\varepsilon}{2}\int_\Omega \operatorname{div}_x(|\mathbf{u}|^2\nabla_x\varrho)\,\mathrm{d}x - \frac{\varepsilon}{2}\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot 2\mathbf{u}\,\mathrm{d}x \\
&= \frac{\varepsilon}{2}\int_{\partial\Omega} |\mathbf{u}|^2\nabla_x\varrho\cdot\mathbf{n}\,\mathrm{d}S_x - \varepsilon\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x \\
&= -\varepsilon\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega (\log\varrho+1)\Delta_x\varrho\,\mathrm{d}x &= \int_\Omega (\log\varrho+1)\operatorname{div}_x\nabla_x\varrho\,\mathrm{d}x \\
&= \int_\Omega \operatorname{div}_x\left[(\log\varrho+1)\nabla_x\varrho\right]\mathrm{d}x - \int_\Omega \nabla_x(\log\varrho+1)\cdot\nabla_x\varrho\,\mathrm{d}x \\
&= \int_{\partial\Omega} (\log\varrho+1)\nabla_x\varrho\cdot\mathbf{n}\,\mathrm{d}S_x - \int_\Omega \frac{d}{d\varrho}(\log\varrho+1)|\nabla_x\varrho|^2\,\mathrm{d}x \\
&= -\int_\Omega \frac{1}{\varrho}|\nabla_x\varrho|^2\,\mathrm{d}x = -\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\,\mathrm{d}x,
\end{aligned}
$$

we finally obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)\right]\mathrm{d}x = -\varepsilon\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\,\mathrm{d}x - \int_\Omega \partial F_\delta(\mathbb{D}_x\mathbf{u}):\nabla_x\mathbf{u}\,\mathrm{d}x. \tag{4.21}
$$

Note that we got rid of the integral $\frac{\varepsilon}{2}\int_\Omega |\mathbf{u}_n|^2\Delta_x\varrho_n\mathrm{d}x$ thanks to the extra term $\varepsilon\nabla_x\varrho_n\cdot\nabla_x\mathbf{u}_n$ in (4.4). Since all the quantities involved are at least continuous in time, we may integrate (4.21) over $(0,\tau)$ in order to get the following energy equality

$$
\begin{aligned}
\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)\right](\tau,\cdot)\,\mathrm{d}x + \int_0^\tau\int_\Omega\left[\partial F_\delta(\mathbb{D}_x\mathbf{u}):\nabla_x\mathbf{u}+\varepsilon P''(\varrho)|\nabla_x\varrho|^2\right]\mathrm{d}x\mathrm{d}t \\
= \int_\Omega\left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_{0,n}}+P(\varrho_{0,n})\right]\mathrm{d}x,
\end{aligned} \tag{4.22}
$$

for any time $\tau\in[0,T(n)]$. In particular, if we suppose the initial value of the (modified) total energy

$$
\int_\Omega\left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_{0,n}}+P(\varrho_{0,n})\right]\mathrm{d}x \le \overline{E} \tag{4.23}
$$

where the constant $\overline{E}$ is independent of $n>0$, the term on the right-hand side of (4.22) is bounded.

Now, the following result, collecting all the significant properties of the regularized potential F_δ, is needed.

Proposition 4.2. *For every fixed $\delta>0$ and F satisfying hypothesis (2.4)–(2.5), the function F_δ defined in (4.7) is convex, non-negative, infinitely differentiable and such that*

$$
F_\delta(\mathbb{D}) \ge \nu\left|\mathbb{D}-\frac{1}{d}\operatorname{Tr}[\mathbb{D}]\mathbb{I}\right|^q - c \quad \textit{for all } \mathbb{D}\in\mathbb{R}^{d\times d}_{\mathrm{sym}} \tag{4.24}
$$

with $\nu>0$, $c>0$, $q>1$ independent of δ.

Proof. For every fixed $\delta > 0$, the non-negativity of F_δ is trivial while smoothness follows from the fact that each derivative can be transferred to the mollifiers ξ_δ. Moreover, for every $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}_{\rm sym}$ and every $t \in [0,1]$, denoting

$$C_1 := \inf_{\mathbb{D}\in\mathbb{R}^{d\times d}_{\rm sym}} \int_{\mathbb{R}^{d\times d}_{\rm sym}} \xi_\delta(|\mathbb{D}-\mathbb{Z}|)F(\mathbb{Z})\ \mathrm{d}\mathbb{Z}$$

we have

$$\begin{aligned}
&F_\delta(t\mathbb{A} + (1-t)\mathbb{B}) \\
&\quad = \int_{\mathbb{R}^{d\times d}_{\rm sym}} F\big(t(\mathbb{A}+\mathbb{Z}) + (1-t)(\mathbb{B}+\mathbb{Z})\big)\ \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} + tC_1 - (1-t)C_1 \\
&\quad \le t\left(\int_{\mathbb{R}^{d\times d}_{\rm sym}} F(\mathbb{A}+\mathbb{Z})\ \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} + C_1\right) + (1-t)\left(\int_{\mathbb{R}^{d\times d}_{\rm sym}} F(\mathbb{B}+\mathbb{Z})\ \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} + C_1\right) \\
&\quad = tF_\delta(\mathbb{A}) + (1-t)F_\delta(\mathbb{B}),
\end{aligned}$$

where we have simply summed and subtracted terms $t\mathbb{Z}$, tC_1 in the second line and used the convexity of F in the third line. In particular, we get that for every fixed $\delta > 0$, $F_\delta : \mathbb{R}^{d\times d}_{\rm sym} \to [0,\infty)$ is convex.

Let now $\mathbb{D} \in \mathbb{R}^{d\times d}_{\rm sym}$ be fixed. From (2.5), we have

$$\begin{aligned}
F_\delta(\mathbb{D}) &= \int_{\mathbb{R}^{d\times d}_{\rm sym}} F(\mathbb{D}-\mathbb{Z})\xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} - C_1 \\
&\ge \mu \int_{\mathbb{R}^{d\times d}_{\rm sym}} \left|\left(\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right) - \left(\mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right)\right|^q \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} - C_1.
\end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{d\times d}_{\rm sym}} \left|\left(\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right) - \left(\mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right)\right|^q \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} \\
&\quad \ge \left[\left(\int_{\mathbb{R}^{d\times d}_{\rm sym}} \left|\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right|^q \xi_\delta(|\mathbb{Z}|)\mathrm{d}\mathbb{Z}\right)^{\frac{1}{q}} - \left(\int_{\mathbb{R}^{d\times d}_{\rm sym}} \left|\mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right|^q \xi_\delta(|\mathbb{Z}|)\mathrm{d}\mathbb{Z}\right)^{\frac{1}{q}}\right]^q;
\end{aligned}$$

recalling that for any $\delta > 0$ sufficiently small $\mathrm{supp}\,\xi_\delta \subset K$ with $K \subset \mathbb{R}^{d\times d}_{\rm sym}$ a compact set and that for any $\delta > 0$

$$\int_{\mathbb{R}^{d\times d}_{\rm sym}} \xi_\delta(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} = \frac{1}{\delta^d}\int_{\mathbb{R}^{d\times d}_{\rm sym}} \xi\left(\frac{|\mathbb{Z}|}{\delta}\right)\mathrm{d}\mathbb{Z} = \int_{\mathbb{R}^{d\times d}_{\rm sym}} \xi(|\mathbb{Z}|)\ \mathrm{d}\mathbb{Z} = 1,$$

we obtain the following inequality

$$F_\delta(\mathbb{D}) \ge \mu\left[\left|\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right| - \left(\sup_{\mathbb{Z}\in K}\left|\mathbb{Z} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{Z}]\mathbb{I}\right|^q\right)^{\frac{1}{q}}\right]^q - C_1.$$

Now, for every fixed $q > 1$ and constant $c_1 > 0$, there exist $\alpha = \alpha(q, c_1) \in (0,1)$ and $c_2 = c_2(q, c_1) > 0$ such that

$$(y - c_1)^q \ge \alpha y^q - c_2 \quad \text{for any } y \ge 0;$$

in particular, we get that for all $\mathbb{D} \in \mathbb{R}^{d\times d}_{\rm sym}$

$$F_\delta(\mathbb{D}) \ge \mu\alpha\left|\mathbb{D} - \frac{1}{d}\,\mathrm{Tr}[\mathbb{D}]\mathbb{I}\right|^q - (C_1 + C_2)$$

and thus (4.24) holds choosing $\nu = \mu\alpha$ and $c = C_1 + C_2$. ☐

From (4.22) and (4.23) we can deduce that

$$\|F_\delta(\mathbb{D}_x \mathbf{u}_n)\|_{L^1((0,T)\times\Omega)} \leq c(\overline{E})$$

which, from (4.24), implies

$$\left\| \mathbb{D}_x \mathbf{u}_n - \frac{1}{d}(\operatorname{div}_x \mathbf{u}_n)\mathbb{I} \right\|_{L^q((0,T)\times\Omega;\mathbb{R}^{d\times d})} \leq c(\overline{E}).$$

The previous inequality combined with the L^q-version of the trace-free Korn's inequality, see [4], Theorem 3.1, gives

$$\|\nabla_x \mathbf{u}_n\|_{L^q((0,T)\times\Omega;\mathbb{R}^{d\times d})} \leq c(\overline{E});$$

the standard Poincaré inequality ensures then

$$\mathbf{u} \text{ to be bounded in } L^q(0,T(n);W_0^{1,q}(\Omega;\mathbb{R}^d))$$

by a constant which is independent of n and $T(n) \leq T$. Since all norms are equivalent in X_n, this implies that

$$\mathbf{u} \text{ is bounded in } L^q(0,T(n);W^{1,\infty}(\Omega;\mathbb{R}^d));$$

in particular, by virtue of (4.10) and (4.12), the density $\varrho = \varrho[\mathbf{u}]$ is bounded from below and above by constants independent of $T(n) \leq T$. Since ϱ is bounded from below, one can use (4.22) to easily deduce uniform boundedness in t of $\mathbf{u}$ in the space $L^2(\Omega;\mathbb{R}^d)$. Consequently, the functions $\mathbf{u}(t,\cdot)$ remain bounded in X_n for any t independently of $T(n) \leq T$. Thus we are allowed to iterate the previous local existence result to construct a solution defined on the whole time interval $[0,T]$.

Summarizing, so far we proved the following result.

Lemma 4.3. *For every fixed $\delta > 0$, $\varepsilon > 0$, $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right] \mathrm{d}x \leq \overline{E},$$

where the constant $\overline{E}$ is independent of n, there exist

$$\begin{aligned} \varrho = \varrho_{\delta,\varepsilon,n} &\in L^2((0,T);W^{1,2}(\Omega)) \cap C([0,T];L^2(\Omega)), \\ \mathbf{u} = \mathbf{u}_{\delta,\varepsilon,n} &\in C([0,T];X_n), \end{aligned}$$

such that

(i) the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho\cdot\nabla_x\varphi)\ \mathrm{d}x$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_{0,n}$;

(ii) the integral identity

$$\begin{aligned} \left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + a\varrho\operatorname{div}_x\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \partial F_\delta(\mathbb{D}_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t \end{aligned}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T];X_n)$, with $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$;

(iii) the integral equality

$$\int_\Omega \left[\frac{1}{2}\varrho|\boldsymbol{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x + \int_0^\tau\int_\Omega \partial F_\delta(\mathbb{D}_x\boldsymbol{u}) : \nabla_x\boldsymbol{u}\ \mathrm{d}x\mathrm{d}t + \varepsilon\int_0^\tau\int_\Omega P''(\varrho)|\nabla_x\varrho|^2\mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right]\mathrm{d}x$$

holds for any time $\tau\in[0,T]$.

4.3 Limit $\delta\to 0$

Let now $\varepsilon>0$ and $n\in\mathbb{N}$ be fixed, and let $\{\varrho_\delta,\mathbf{u}_\delta\}_{\delta>0}$ be the family of weak solutions to problem (4.1)–(4.8) as in Lemma 4.3. Proceeding as before, we can deduce that

$$\{\mathbf{u}_\delta\}_{\delta>0}\ \text{is unifrmly bounded in}\ L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^d)).$$

As n is fixed and all norms are equivalent on the finite-dimensional space X_n, we get that

$$\{\nabla_x\mathbf{u}_\delta\}_{\delta>0}\ \text{is unifrmly bounded in}\ L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}),$$

and therefore, we are ready to perform the limit $\delta\to 0$. Accordingly, we obtain the following result.

Lemma 4.4. *For every fixed $\varepsilon>0$, $n\in\mathbb{N}$, and any $\varrho_{0,n}\in C(\overline{\Omega})$ such that*

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right]\mathrm{d}x \le \overline{E},$$

where the constant $\overline{E}$ is independent of n, there exist

$$\varrho = \varrho_{\varepsilon,n}\in L^2((0,T);W^{1,2}(\Omega))\cap C([0,T];L^2(\Omega)),$$
$$\mathbf{u} = \mathbf{u}_{\varepsilon,n}\in C([0,T];X_n),$$

such that

(i) the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho\cdot\nabla_x\varphi)\ \mathrm{d}x \tag{4.25}$$

holds for any $\tau\in[0,T]$ and any $\varphi\in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot)=\varrho_{0,n}$;

(ii) there exists

$$\mathbb{S} = \mathbb{S}_{\varepsilon,n}\in L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

such that the integral identity

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega\nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t \tag{4.26}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C^1([0,T];X_n)$, with $(\varrho\mathbf{u})(0,\cdot)=\mathbf{m}_0$;

(iii) the integral inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\boldsymbol{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\ \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^\tau \int_\Omega P''(\varrho)|\nabla_x\varrho|^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right]\mathrm{d}x \tag{4.27}$$

holds for a.e. $\tau \in (0,T)$.

4.4 Limit $\varepsilon \to 0$

In order to perform the limit $\varepsilon \to 0$, we need the following result.

Lemma 4.5. *Let* $n \in \mathbb{N}$ *be fixed and let* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbb{S}_\varepsilon\}_{\varepsilon>0}$ *be as in Lemma 4.4. Moreover, let*

$$\boldsymbol{f}(\varrho_\varepsilon) := \sqrt{\varepsilon}\ \nabla_x\varrho_\varepsilon$$
$$\boldsymbol{g}(\varrho_\varepsilon, \mathbf{u}_\varepsilon) := \sqrt{\varepsilon}\ \nabla_x\varrho_\varepsilon \cdot \nabla_x\mathbf{u}_\varepsilon.$$

Then, passing to a suitable subsequences as the case may be, the following convergences hold as $\varepsilon \to 0$.

$$\varrho_\varepsilon \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty((0,T)\times\Omega), \tag{4.28}$$
$$\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)), \tag{4.29}$$
$$\varrho_\varepsilon\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \varrho\mathbf{u} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d), \tag{4.30}$$
$$\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}), \tag{4.31}$$
$$\mathbb{S}_\varepsilon \rightharpoonup \mathbb{S} \quad \text{in } L^1((0,T)\times\Omega;\mathbb{R}^{d\times d}), \tag{4.32}$$
$$\boldsymbol{f}(\varrho_\varepsilon) \rightharpoonup \overline{\boldsymbol{f}(\varrho)} \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d), \tag{4.33}$$
$$\boldsymbol{g}(\varrho_\varepsilon,\mathbf{u}_\varepsilon) \rightharpoonup \overline{\boldsymbol{g}(\varrho,\mathbf{u})} \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d). \tag{4.34}$$

Proof. From (4.27) it is easy to deduce the following uniform bounds

$$\|F(\mathbb{D}_x\mathbf{u}_\varepsilon)\|_{L^1((0,\infty)\times\Omega)} \leq c(\overline{E}), \tag{4.35}$$
$$\|F^*(\mathbb{S}_\varepsilon)\|_{L^1((0,\infty)\times\Omega)} \leq c(\overline{E}). \tag{4.36}$$

Similarly to the previous section, from (4.35) we obtain

$$\|\mathbf{u}_\varepsilon\|_{L^q(0,T;W^{1,q}(\Omega;\mathbb{R}^d))} \leq c_1$$

for some $q > 1$ and a positive constant c_1 independent of $\varepsilon > 0$, yielding, in view of Lemmas 4.1, conditions (ii) and (iii),

$$e^{-c_1T}\underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq e^{c_1T}\overline{\varrho}, \quad \text{for all } (t,x)\in[0,T]\times\overline{\Omega}. \tag{4.37}$$

We recover convergence (4.28). From the energy inequality (4.27), it is easy to deduce

$$\sup_{t\in[0,T]} \|\mathbf{u}_\varepsilon(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \leq c_2, \tag{4.38}$$

from which convergence (4.29) follows. Combining (4.37) and (4.38), we can recover

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d).$$

Now, notice that (4.28) can be strengthened to

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}([0,T];L^p(\Omega)) \quad \text{for all } 1 < p < \infty$$

as $\varepsilon \to 0$, so that, relaying on the compact Sobolev embedding

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,1}(\Omega) \quad \text{for all } p \geq 1,$$

we obtain

$$\varrho_\varepsilon \to \varrho \quad \text{in } C([0,T];W^{-1,1}(\Omega))$$

as $\varepsilon \to 0$. The last convergence combined with (4.29), implies

$$\mathbf{m} = \varrho \mathbf{u} \quad \text{a.e. in } (0,T) \times \Omega,$$

and thus, we get (4.30). Similarly, from (4.29) and (4.30) we can deduce (4.31). Convergence (4.32) can be deduced from (4.36) using the superlinearity of F^* (2.10) combined with the De la Vallée–Poussin criterion and the Dunford–Pettis theorem. Finally, from (4.37) we have in particular that

$$\frac{e^{c_1 T}\overline{\varrho}}{\varrho(t,x)} \geq 1, \quad \text{for all } (t,x) \in [0,T] \times \overline{\Omega},$$

and thus, from the energy inequality (4.27),

$$\varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho|^2 \, \mathrm{d}x\mathrm{d}t \leq \varepsilon \ e^{c_1 T}\overline{\varrho} \int_0^\tau \int_\Omega P''(\varrho)|\nabla_x \varrho|^2 \, \mathrm{d}x\mathrm{d}t \leq c(\overline{\varrho}, T).$$

In this way we get (4.33) and, in view of (4.38), (4.34). □

Remark 4.6. It is worth noticing that the limit density ϱ admits the same upper and lower bounds as in (4.37):

$$e^{-c_1 T}\underline{\varrho} \leq \varrho(t,x) \leq e^{c_1 T}\overline{\varrho}, \quad \text{for all } (t,x) \in [0,T] \times \overline{\Omega}.$$

We are now ready to let $\varepsilon \to 0$ in the weak formulations (4.25), (4.26); notice in particular that, in view of (4.34), for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T];X_n)$

$$\varepsilon \int_0^\tau \int_\Omega \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = \sqrt{\varepsilon} \int_0^\tau \int_\Omega \sqrt{\varepsilon} \ \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to 0$$

as $\varepsilon \to 0$.

Lemma 4.7. *For every fixed $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \mathrm{d}x \leq \overline{E},$$

where the constant $\overline{E}$ is independent of n, there exist

$$\varrho = \varrho_n \in L^\infty((0,T) \times \Omega),$$
$$\mathbf{u} = \mathbf{u}_n \in C([0,T];X_n),$$

with

$$e^{-cT}\underline{\varrho} \leq \varrho(t,x) \leq e^{cT}\overline{\varrho}, \quad \textit{for all } (t,x) \in [0,T] \times \overline{\Omega},$$

for a positive constant c, such that *(i) the integral identity*

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\ \mathrm{d}x \tag{4.39}$$

holds for any $\tau \in [0,T]$ *and any* $\varphi \in C^1([0,T]\times\overline{\Omega})$, *with* $\varrho(0,\cdot) = \varrho_{0,n}$*;*

(ii) there exists

$$\mathbb{S} = \mathbb{S}_n \in L^1((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

such that the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.40}$$

holds for any $\tau \in [0,T]$ *and any* $\boldsymbol{\varphi} \in C^1([0,T];X_n)$, *with* $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$*;*

(iii) the integral inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\boldsymbol{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x + \int_0^\tau\int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\ \mathrm{d}x\mathrm{d}t \le \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n})\right]\mathrm{d}x \tag{4.41}$$

holds for a.e. $\tau \in (0,T)$.

Remark 4.8. In the energy inequality (4.41) we used the lower semi-continuity of the function

$$[\varrho,\mathbf{m}] \mapsto \begin{cases} 0 & \text{if } \mathbf{m} = 0, \\ \frac{|\mathbf{m}|^2}{\varrho} & \text{if } \varrho > 0, \\ \infty & \text{otherwise,}\end{cases}$$

and the weak lower semi-continuity in L^1 of the functions F and F^*, and thus for a.e. $\tau > 0$

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x &\le \liminf_{\varepsilon\to\infty} \int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + P(\varrho)\right](\tau,\cdot)\ \mathrm{d}x, \\ \int_0^\tau\int_\Omega [F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})]\ \mathrm{d}x\mathrm{d}t &\le \liminf_{\varepsilon\to 0} \int_0^\tau\int_\Omega [F(\mathbb{D}_x\mathbf{u}_\varepsilon) + F^*(\mathbb{S}_\varepsilon)]\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

4.5 Limit $n \to \infty$

Let $\{\varrho_n, \mathbf{m}_n = \varrho_n\mathbf{u}_n\}_{n\in\mathbb{N}}$ be the family of approximate solutions obtained in Lemma 4.7, with correspondent viscous stress tensor $\mathbb{S}_n$. At this stage, as the initial energies are uniformly bounded by a constant independent of n, we can perform the same procedure done in [2], Section 5.1 with $\gamma = 1$, to get the following family of convergences as $n \to \infty$, passing to suitable subsequences as the case may be:

$$\varrho_n \to \varrho \quad \text{in } C_{\mathrm{weak}}([0,T];L^1(\Omega)), \tag{4.42}$$

$$\mathbf{m}_n \to \mathbf{m} \quad \text{in } C_{\mathrm{weak}}([0,T];L^1(\Omega;\mathbb{R}^d)), \tag{4.43}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^q(0,T;W_0^{1,q}(\Omega;\mathbb{R}^d)) \tag{4.44}$$

$$\mathbb{S}_n \rightharpoonup \mathbb{S} \quad \text{in } L^1(0,T;L^1(\Omega;\mathbb{R}^{d\times d})), \tag{4.45}$$

$$\mathbb{1}_{\varrho_n>0}\frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n} \overset{*}{\rightharpoonup} \overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{4.46}$$

with

$$\mathbf{m} = \varrho\mathbf{u} \quad \text{a.e. in } (0,T)\times\Omega,$$

as a consequence of Lemma 5.2 in [2].

We are now ready to let $n\to\infty$ in the weak formulation of the continuity equation (4.39) and the balance of momentum (4.40), obtaining that

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\,\mathrm{d}x\mathrm{d}t$$

holds for any $\tau\in[0,T]$ and any $\varphi\in C^1([0,T]\times\overline{\Omega})$, with $\varrho(0,\cdot)=\varrho_0$, and

$$\begin{aligned}\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}:\nabla_x\boldsymbol{\varphi} + a\varrho\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t\\ &\quad -\int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}}\nabla_x\boldsymbol{\varphi}:\mathrm{d}\mathfrak{R}\,\mathrm{d}t\end{aligned} \tag{4.47}$$

holds for any $\tau\in[0,T]$ and any $\boldsymbol{\varphi}\in C^1([0,T];X_n)$, with n arbitrary. As clearly explained by Abbatiello, Feireisl and Novotný [1], Section 3.4, by a density argument it is possible to extend the validity of the integral identity (4.47) for any $\boldsymbol{\varphi}\in C^1([0,T]\times\overline{\Omega})$, $\boldsymbol{\varphi}|_{\partial\Omega}=0$. Finally, notice that from the energy inequality (4.41) we have the following uniform bounds

$$\begin{aligned}\left\|\frac{\mathbf{m}_n}{\sqrt{\varrho_n}}\right\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} &\le c(\overline{E}),\\ \|P(\varrho_n)\|_{L^\infty(0,T;L^1(\Omega))} &\le c(\overline{E}),\end{aligned}$$

from which it is possible to deduce that

$$\begin{aligned}\frac{|\mathbf{m}_n|^2}{\varrho_n} &\overset{*}{\rightharpoonup} \overline{\frac{|\mathbf{m}|^2}{\varrho}} \quad \text{in } L^\infty(0,\infty;\mathcal{M}(\overline{\Omega}))\\ P(\varrho_n) &\overset{*}{\rightharpoonup} \overline{P(\varrho)} \quad \text{in } L^\infty(0,\infty;\mathcal{M}(\overline{\Omega}))\end{aligned}$$

as $n\to\infty$. Thus,

$$\mathfrak{R}\in L^\infty_{\text{weak}}(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}}))$$

appearing in (4.47) has been chosen in such a way that

$$\mathrm{d}\mathfrak{R} = \left(\overline{\mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}} - \mathbb{1}_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)\mathrm{d}x + \psi(t)\mathbb{I},$$

where the time-dependent function ψ is chosen in such a way to guarantee

$$\frac{1}{\lambda}\,\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}] = \frac{1}{2}\left(\overline{\frac{|\mathbf{m}|^2}{\varrho}} - \frac{|\mathbf{m}|^2}{\varrho}\right)\mathrm{d}x + \left(\overline{P(\varrho)} - P(\varrho)\right)\mathrm{d}x$$

for a.e. $\tau\in(0,T)$; see [2], Section 5.4 for further details.

We proved the following result.

Theorem 4.9. *For every fixed initial data*

$$[\varrho_0,\mathbf{m}_0]\in L^1(\Omega)\times L^1(\Omega;\mathbb{R}^d),$$

with

$$\int_\Omega\left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0\log\varrho_0\right]\mathrm{d}x < \infty, \tag{4.48}$$

problem (2.1)–(2.9) *admits a dissipative solution in the sense of Definition 3.1.*

5 Existence of weak solutions

Choosing $q > d$ in (2.5), we get the existence of weak solutions to models describing a general viscous compressible fluid (2.1)–(2.9), or equivalently, the Reynold stress $\mathfrak{R}$ appearing in Definition 3.1 is identically zero. In particular, we improve the work by Matušů-Nečasová and Novotný [13], where existence was achieved in the framework of measure-valued solutions.

We can repeat the same procedure performed in the previous section until we get to Lemma 4.7. We can now prove the following crucial result.

Lemma 5.1. *Let $q > d$ in (2.5) and let $\{\varrho_n, \mathbf{m}_n = \varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$ be the family of approximate solutions obtained in Lemma 4.7. Then, passing to a suitable subsequence as the case may be,*

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{in } L^1((0,T)\times\Omega;\mathbb{R}^{d\times d}) \tag{5.1}$$

as $n \to \infty$.

Proof. Proceeding as in [2], Sections 5.1 and 5.2, we have

$$\begin{aligned} \varrho_n &\to \varrho \quad \text{in } C_{\text{weak}}([0,T]; L^1(\Omega)), \\ \varrho_n \mathbf{u}_n &\to \varrho \mathbf{u} \quad \text{in } C_{\text{weak}}([0,T]; L^1(\Omega;\mathbb{R}^d)) \end{aligned}$$

as $n \to \infty$, where the sequence $\{\varrho_n \mathbf{u}_n(t,\cdot)\}_{n\in\mathbb{N}}$ is equi-integrable in $L^1(\Omega;\mathbb{R}^d)$ for a.e. $t \in (0,T)$. Thanks to the slightly modified De la Vallée–Poussin criterion, which we report in the Appendix, Theorem A.2, there exists a Young function Ψ satisfies the Δ_2-condition (A.1) such that

$$\varrho_n \mathbf{u}_n \overset{*}{\rightharpoonup} \varrho \mathbf{u} \quad \text{in } L^\infty(0,T; L_\Psi(\Omega;\mathbb{R}^d)),$$

Moreover, due to the compact Sobolev embedding

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,q'}(\Omega) \quad \text{for any } p \geq 1,$$

which is true since $q > d$ from our hypothesis, we can prove that the sequence $\{\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n\}_{n\in\mathbb{N}}$ is equi-integrable in $L^1((0,T)\times\Omega;\mathbb{R}^{d\times d})$. Indeed, let $\varepsilon > 0$ be fixed and let the constant $c > 0$ be such that

$$\|\mathbf{u}_n\|_{L^q(0,T;W^{1,q}(\Omega;\mathbb{R}^d))} \leq c,$$

uniformly in n. Let $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\varepsilon) > 0$ be chosen in such a way that

$$\widetilde{\varepsilon} < \left(c\, T^{\frac{1}{q'}}\right)^{-1} \varepsilon.$$

From the equi-integrability of the sequence $\{\varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$, there exists $\delta = \delta(\widetilde{\varepsilon}) > 0$ such that

$$\int_M |\varrho_n \mathbf{u}_n|(t)\, \mathrm{d}x < \widetilde{\varepsilon}, \quad \text{for every } M \subset \Omega \text{ s.t. } |M| < \delta,$$

for every $n \in \mathbb{N}$. Let $(t_1,t_2)\times M \subset [0,T]\times\Omega$ such that

$$|(t_1,t_2)\times M| < \delta.$$

Then, for every $n \in \mathbb{N}$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_M |\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n| \, \mathrm{d}x\mathrm{d}t &\leq \int_0^T\int_M |\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n| \, \mathrm{d}x\mathrm{d}t \\ &\leq \|\varrho_n \mathbf{u}_n\|_{L^{q'}(0,T;L^1(M))} \|\mathbf{u}_n\|_{L^q(0,T;W^{1,q}(M))} \\ &\leq c \left[\int_0^T \left(\int_M |\varrho_n \mathbf{u}_n|(t) \, \mathrm{d}x\right)^{q'} \mathrm{d}t\right]^{\frac{1}{q'}} \\ &\leq c\, \widetilde{\varepsilon}\, T^{\frac{1}{q'}} \\ &< \varepsilon. \end{aligned}$$

Consequently, we can adapt Lemma 5.2 in [2] replacing the sequence of densities $\{\varrho_n\}_{n\in\mathbb{N}}$ with the sequence of momenta $\{\varrho_n \mathbf{u}_n\}_{n\in\mathbb{N}}$ to obtain (5.1). □

Letting $n \to \infty$ in the weak formulation of the continuity equation (4.39) and the balance of momentum (4.40), we obtain the following result.

Theorem 5.2. *Let $q > d$ in (2.5). For every fixed initial data*

$$[\varrho_0, \mathbf{m}_0] \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^d),$$

with

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0 \log \varrho_0\right] \mathrm{d}x < \infty, \tag{5.2}$$

problem (2.1)–(2.9) admits a weak solution

$$[\varrho, \varrho\mathbf{u}] \in C_{\mathrm{weak}}([0,T]; L^1(\Omega)) \times C_{\mathrm{weak}}([0,T]; L^1(\Omega; \mathbb{R}^d)),$$

meaning that the following holds.

(i) $\varrho \geq 0$ *in* $(0,T) \times \Omega$.

(i) The integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot) \, \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1_c([0,T] \times \overline{\Omega})$, with $\varrho(0,\cdot) = \varrho_0$.

(iii) There exists

$$\mathbb{S} \in L^1(0,T; L^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

such that the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u} \cdot \boldsymbol{\varphi}(t,\cdot) \, \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + a\varrho \operatorname{div}_x \boldsymbol{\varphi}] \, \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t\end{aligned}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1_c([0,T] \times \overline{\Omega}; \mathbb{R}^d)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, with $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$.

(iv) the energy inequality

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + a\varrho \log \varrho\right](\tau,\cdot) \, \mathrm{d}x + \int_0^\tau \int_\Omega [F(\mathbb{D}\boldsymbol{u}) + F^*(\mathbb{S})] \, \mathrm{d}x\mathrm{d}t \\ \leq \int_\Omega \left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_0} + a\varrho_0 \log \varrho_0\right] \mathrm{d}x\end{aligned}$$

holds for a.e. $\tau \in (0,T)$.

A De la Vallée–Poussin criterion

In this section, we prove a slightly modified version of the De la Vallée–Poussin criterion as we require the stronger condition, with respect to the standard formulation, that the Young function satisfies the Δ_2-condition. We first recall the definitions of Young function and Δ_2-condition.

Definition A.1. (i) We say that Φ is a *Young function* generated by φ if

$$\Phi(t) = \int_0^t \varphi(s)\ \mathrm{d}s \quad \text{for any } t \geq 0,$$

where the real-valued function φ defined on $[0, \infty)$ is non-negative, non-decreasing, left-continuous and such that

$$\varphi(0) = 0, \quad \lim_{s\to\infty} \varphi(s) = \infty.$$

(ii) A Young function Φ is said to satisfy the Δ_2-*condition* if there exist a positive constant K and $t_0 \leq 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for any } t \geq t_0. \tag{A.1}$$

Theorem A.2. *Let $Q \subset \mathbb{R}^d$ be a bounded measurable set and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(Q)$. Then, the following statements are equivalent.*

(i) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-integrable, meaning that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_M |f_n(y)|\ \mathrm{d}y < \varepsilon \quad \textit{for any } M \subset Q \textit{ such that } |M| < \delta,$$

independently of n.

(ii) There exists a Young function Φ satisfying the Δ_2-condition (A.1) *such that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the Orlicz space $L_\Phi(Q)$.*

Proof. (ii) $\Rightarrow$ (i) See Pedregal [14], Chapter 6, Lemma 6.4.

(i) $\Rightarrow$ (ii) For $n \in \mathbb{N}$ and $j \geq 1$ fixed, let

$$\mu_j(f_n) := |\{y \in Q : \ |f_n(y)| > j\}|.$$

As the sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-integrable, from the Dunford-Pettis theorem there exists a strictly increasing sequence of positive integers $\{C_m\}_{m\in\mathbb{N}}$ such that for each m

$$\sup_{n\in\mathbb{N}} \int_{\{|f_n|>C_m\}} |f_n(y)|\ \mathrm{d}y \leq \frac{1}{2^m}.$$

For $n \in \mathbb{N}$ and $m \geq 1$ fixed

$$\int_{\{|f_n|>C_m\}} |f_n(y)|\ \mathrm{d}y = \sum_{j=C_m}^{\infty} \int_{\{j<|f_n|\leq j+1\}} |f_n(y)|\ \mathrm{d}y \geq \sum_{j=C_m}^{\infty} j\ [\mu_j(f_n) - \mu_{j+1}(f_n)] \geq \sum_{j=C_m}^{\infty} \mu_j(f_n).$$

In particular, we obtain

$$\sum_{m=1}^{\infty} \sum_{j=C_m}^{\infty} \mu_j(f_n) \leq \sum_{m=1}^{\infty} \int_{\{|f_n|>C_m\}} |f_n(y)|\ \mathrm{d}y \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

For $m \geq 0$, we define

$$\alpha_m = \begin{cases} 0 & \text{if } m < C_1, \\ \max\{k : \ C_k \leq m\} & \text{if } m \geq C_1. \end{cases}$$

Notice that

$$\alpha_m \geq j \quad \Leftrightarrow \quad C_j \leq m. \tag{A.2}$$

It is straightforward that $\alpha_m \to \infty$ as $m \to \infty$. We define a step function φ on $[0, \infty)$ by

$$\varphi(s) = \sum_{m=0}^{\infty} \alpha_m \chi_{(m,m+1]}(s) \quad \text{for any } 0 \leq s < \infty.$$

It is clear that φ is non-negative, non-decreasing, left-continuous and such that $\varphi(0) = 0$, $\lim_{s\to\infty} \varphi(s) = \infty$. Then, we can define the Young function Φ generated by φ as

$$\Phi(t) = \int_0^t \varphi(s) \ \mathrm{d}s, \quad \text{for any } 0 \leq t < \infty.$$

At this point, notice that we have the freedom to take the constants C_j, $j \geq 1$, as large as we want and consequently, the constants α_m, $m \geq 1$, will be as small as we want. More precisely, we may find a positive constant c such that

$$\alpha_{2m} \leq c \ \alpha_m \quad \text{for any } m \geq 1.$$

We then obtain, for all $s \in [0, \infty)$,

$$\varphi(2s) = \sum_{m=0}^{\infty} \alpha_m \chi_{\left(\frac{m}{2}, \frac{m+1}{2}\right)}(s) = \sum_{k=0}^{\infty} \alpha_{2k} \chi_{\left(k, k+\frac{1}{2}\right)}(s) \leq c \sum_{k=0}^{\infty} \alpha_k \chi_{\left(k, k+\frac{1}{2}\right)}(s) \leq c \ \varphi(s);$$

consequently, for all $t \in [0, \infty)$,

$$\Phi(2t) = \int_0^{2t} \varphi(s) \ \mathrm{d}s = 2 \int_0^t \varphi(2z) \ \mathrm{d}z \leq 2c \int_0^t \varphi(z) \ \mathrm{d}z = 2c \ \Phi(t),$$

and thus we get that the Young function Φ satisfies the Δ_2-condition (A.1).

Finally, for $n \in \mathbb{N}$ fixed, using the fact that $\Phi(0) = \Phi(1) = 0$ and for $j \geq 1$, noticing that $\alpha_0 = 0$,

$$\Phi(j+1) = \int_0^{j+1} \varphi(s) \ \mathrm{d}s = \sum_{m=0}^{j} \int_m^{m+1} \varphi(s) \ \mathrm{d}s \leq \sum_{m=0}^{j} \varphi(m+1) = \sum_{m=0}^{j} \alpha_m = \sum_{m=1}^{j} \alpha_m,$$

we get

$$\begin{aligned}
\int_Q \Phi(|f_n(y)|) \ \mathrm{d}y &= \int_{\{|f_n|=0\}} \Phi(|f_n(y)|) \ \mathrm{d}y + \sum_{j=0}^{\infty} \int_{\{j<|f_n|\leq j+1\}} \Phi(|f_n(y)|) \ \mathrm{d}y \\
&\leq \sum_{j=1}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)] \ \Phi(j+1) \\
&\leq \sum_{j=1}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)] \sum_{m=1}^{j} \alpha_m \\
&= \sum_{m=1}^{\infty} \alpha_m \sum_{j=m}^{\infty} [\mu_j(f_n) - \mu_{j+1}(f_n)] \\
&= \sum_{m=1}^{\infty} \alpha_m \mu_m(f_n) = \sum_{m=1}^{\infty} \mu_m(f_n) \sum_{j=1}^{\alpha_m} 1 = \sum_{j=1}^{\infty} \sum_{m=C_j}^{\infty} \mu_m(f_n) \leq 1
\end{aligned}$$

where we used (A.2) in the last line. In particular, we obtain that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the Orlicz space $L_\Phi(Q)$. □

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