

OPTIMALITY CONDITIONS FOR AN EXTENDED TUMOR GROWTH MODEL WITH DOUBLE OBSTACLE POTENTIAL VIA DEEP QUENCH APPROACH

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ABSTRACT. In this work, we investigate a distributed optimal control problem for an extended phase field system of Cahn–Hilliard type which physical context is that of tumor growth dynamics. In a previous contribution, the author has already studied the corresponding problem for the logarithmic potential. Here, we try to extend the analysis by taking into account a non-smooth singular nonlinearity, namely the double obstacle potential. Due to its non-smoothness behavior, the standard procedure to characterize the necessary conditions for the optimality cannot be performed. Therefore, we follow a different strategy which in the literature is known as the “deep quench” approach in order to obtain some optimality conditions that have to be interpreted in a more general framework. We establish the existence of optimal controls and some first-order optimality conditions for the system are derived by employing suitable approximation schemes.

1. Introduction. In recent years there has been an increased focus on the investigation and understanding of tumor growth by the mathematical community (see, e.g., [15]). Unfortunately, the phenomena which occur in real cases is far too complex to be approached by experimental techniques as a whole. Therefore, the mathematical modeling can be a useful instrument to reduce the problem in a manageable one since it is able to isolate some particular mechanisms which can be hopefully sufficient to forecast something helpful for medical treatments. Here, we concentrate on the subclass of models known in the literature as diffuse interface models which are derived by continuum mixture theory. The system we are going to consider in what follows constitutes a variation of the model originally introduced in [30] in order to describe the evolution of a young tumor, before the development of quiescent cells in the presence of a nutrient species (see [14, 29, 31, 38] and also [7, 8, 10, 20]). It consists of a Cahn–Hilliard equation for the phase variable (see, e.g., [33] and the huge references therein), coupled with a reaction-diffusion equation for an unknown species acting as a nutrient, for instance oxygen or glucose. Furthermore, by interpreting the tumor and the healthy cells as inertia-less fluids,

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the contribution of the velocity field can be included in the investigation by assuming a Darcy law or a Stokes-Brinkman equation. In this regards, let us refer to [16, 19, 21, 23, 24, 25, 26, 28], where further mechanisms such as chemotaxis and active transport are also taken into account. Furthermore, we also point out the paper [22], where a non-local model is proposed. Lastly, for different physically meaningful choices of the potentials, we refer to [1] and to the references therein, where some numerical simulations and comparison with clinical data can be found as well. Further investigations and mathematical models related to biology can be also found e.g. in [16] and [21]. Let us begin introducing our problem. First of all, let $\Omega \subset \mathbb{R}^3$ denote some open and bounded domain with smooth boundary denoted by Γ . For convenience, given a fixed final time $T > 0$, we introduce the standard parabolic cylinder and its boundary by setting

$$\begin{aligned} Q_t &:= \Omega \times (0, t), & \Sigma_t &:= \Gamma \times (0, t) \quad \text{for every } t \in (0, T], \\ Q &:= Q_T, & \Sigma &:= \Sigma_T. \end{aligned} \quad (1.1)$$

The problem we are going to deal with consists in minimizing the cost functional

$$\begin{aligned} \mathcal{J}(\varphi, \sigma, u) &:= \frac{b_1}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{b_2}{2} \|\varphi(T) - \varphi_\Omega\|_{L^2(\Omega)}^2 + \frac{b_3}{2} \|\sigma - \sigma_Q\|_{L^2(Q)}^2 \\ &\quad + \frac{b_4}{2} \|\sigma(T) - \sigma_\Omega\|_{L^2(\Omega)}^2 + \frac{b_0}{2} \|u\|_{L^2(Q)}^2 \end{aligned} \quad (1.2)$$

subject to the control constraints

$$u \in \mathcal{U}_{ad} := \{u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q\}, \quad (1.3)$$

and to the state system

$$\alpha \partial_t \mu + \partial_t \varphi - \Delta \mu = P(\varphi)(\sigma - \mu) \quad \text{in } Q \quad (1.4)$$

$$\mu = \beta \partial_t \varphi - \Delta \varphi + \xi + \pi(\varphi) \quad \text{in } Q \quad (1.5)$$

$$\xi \in \partial I_{[-1,1]}(\varphi) \quad \text{in } Q \quad (1.6)$$

$$\partial_t \sigma - \Delta \sigma = -P(\varphi)(\sigma - \mu) + u \quad \text{in } Q \quad (1.7)$$

$$\partial_n \mu = \partial_n \varphi = \partial_n \sigma = 0 \quad \text{on } \Sigma \quad (1.8)$$

$$\mu(0) = \mu_0, \varphi(0) = \varphi_0, \sigma(0) = \sigma_0 \quad \text{in } \Omega. \quad (1.9)$$

At this first stage, we would confine the technicalities as much as possible, trying to describe the general purpose of the article. Anyhow, let us point out that ∂_n stands for the outward normal derivative, and ∂I for the subdifferential of the indicator function of the interval $[-1, 1]$, that is, the function which vanishes there, and takes the value $+\infty$ elsewhere. Indeed, in the interval $[-1, 1]$, we have the following characterization

$$s \in \partial I_{[-1,1]}(r) \quad \text{if and only if} \quad s \begin{cases} \leq 0 & \text{if } r = -1 \\ = 0 & \text{if } -1 < r < 1. \\ \geq 0 & \text{if } r = +1 \end{cases}$$

As far as in the above lines numerous quantities occur, let us sketch the role they cover in our treatment, postponing the complete investigation on the hypotheses we need to require on them to the next section. First we suppose α, β to be strictly positive constant. Secondly we assume that the symbols b_0, b_1, b_2, b_3, b_4 that appear in 1.2 represent nonnegative constants, while $\varphi_Q, \sigma_Q, u_*, u^*$ and $\varphi_\Omega, \sigma_\Omega$ denote some prescribed functions in their respective domains Q and Ω . Furthermore, the set \mathcal{U}_{ad} models the admissible set in which we can choose the control variables.

Let us spend a few words as the physical background of the involved variables are concerned. The variable φ , called phase variable, is devoted to accounting for the presence of the tumor in the evolution process. It represents a rescaled density which ranges in $[-1, 1]$, where the extreme values -1 and 1 model the complete tumorous case and the healthy one, while the values in between denote intermediate concentrations. The second variable μ stands for the chemical potential for φ , while σ is a rescaled density which takes into account the nutrient-rich extracellular water fraction. Furthermore, we assume that σ ranges from 0 to 1 and that when σ is close to zero the nutrient is poor, while when σ is close to one, it is rich. Moreover, the function P stands for a nonlinearity which describes the proliferation of the tumorous cells in the tissue. As regards u , it represents the control variable, which in the application consists of a supply of a nutrient or a drug in chemotherapy. A more complete description of the model, along with some variations, can be found in [7, 8, 10, 20]. Now, let us present some literature concerning some optimal controllability results for these systems. Up to our knowledge, the first control problem for a system very close to our system is [9]. There, the state system slightly differs from 1.4–1.9, since, formally, consists of the same model in the case $\alpha = \beta = 0$. Moreover, the investigation has to be restricted to regular potentials with polynomial growth, so that the double-obstacle one is not allowed. In that regards, we also point out the recent [4], where the authors extend the analysis of [9] employing a time-dependent cost functional which, in addition, penalizes the long-time treatments and the large mass of the tumor at the end of the medication. Next, we point out [34], where the author, adding the two relaxation terms $\alpha\partial_t\mu$, $\beta\partial_t\varphi$ generalizes the optimal control problem [9] extending the analysis to the case of singular and regular double-well potentials. With the current contribution, we aim at showing that [34] can be generalized allowing also non-regular and singular potentials to be considered. Lastly, regarding the models which take into account the velocity field, let us mention the recent contributions [17, 18], where the authors establish the existence of optimal control and provide some necessary and sufficient conditions for optimality (see also [36]).

From a heuristic viewpoint, our problem consists of choosing an admissible control variable u to force the dynamics of the system to evolve towards a fixed final configuration which is considered to be desirable for some practical reason (e.g. surgery).

Summing up, our goal consists in solving the following problem.

- (P_0) Minimize $\mathcal{J}(\varphi, \mu, u)$ subject to the control constraints 1.3 and under the requirement that the variables (φ, σ) yield a solution to 1.4–1.9.

Once the well-posedness of the system 1.4–1.9 has been shown for every admissible u , we can define the corresponding control-to-state mapping \mathcal{S}_0 as the map that assigns to a fixed control the corresponding state, namely $\mathcal{S}_0 : u \mapsto (\mu, \varphi, \sigma)$, where (μ, φ, σ) solves 1.4–1.9. Moreover, let us fix a convention that will be used repeatedly in the paper: with $\mathcal{S}_{0,2}$ we denote the map \mathcal{S}_0 restricted to the second and third components, that is, $\mathcal{S}_{0,2}(u) = (\varphi, \sigma)$. Furthermore, it is possible to introduce, over \mathcal{U}_{ad} , the so-called reduced cost functional as follows

$$\mathcal{J}_{\text{red}}(u) := \mathcal{J}(\mathcal{S}_{0,2}(u), u). \quad (1.10)$$

Thus, since we will assume \mathcal{U}_{ad} to be closed, bounded, convex and nonempty, we are able to formally characterize the necessary condition that every optimal control

\bar{u} has to satisfy by means of the following variational inequality

$$\langle D\mathcal{J}_{\text{red}}(\bar{u}), v - \bar{u} \rangle \geq 0 \quad \text{for every } v \in \mathcal{U}_{ad}, \quad (1.11)$$

where $D\mathcal{J}_{\text{red}}$ denotes the derivative of 1.10 in a proper functional sense. In this direction, it will be necessary to prove some regularity result for \mathcal{S}_0 and we would need $I_{[-1,1]}(\cdot)$ to be differentiable, which is not the case. For this reason, the standard procedure to characterize the optimality conditions, which is essentially based on 1.11, fails and we have to proceed with a different strategy (see, e.g., [32, 37]).

The technique that we are going to exploit is often referred to in the literature as “deep quench” approach and it lies on a suitable approximation and some monotonicity arguments. As far as some recent application of this strategy is concerned, we mention the paper [11] and [6]. The former deals with an optimal control problem for the simpler Allen–Cahn system endowed with dynamic boundary conditions, whereas the latter focuses the attention on the Cahn–Hilliard system combined with dynamic boundary conditions in which the control variable represents the optimal velocity for the system. We also refer the reader to [5, 12, 13] where, with the same technique, other phase separation problems were faced.

Here, let us only sketch the idea behind this approximation scheme. The key idea is the following: the differential inclusion 1.6 is replaced by a function which, at every stage, resembles the logarithmic double-well potential and which, in a proper sense, approximates the double obstacle nonlinearity as the parameter of the approximation goes to zero. This plan will be quite effective since the corresponding approximating problem complies with the framework of [34], and therefore all the results there established are at our disposal. So, we formally substitute the inclusion 1.6 as follows

$$\xi = g(\gamma)h'(\varphi), \quad (1.12)$$

where h is defined by

$$h(s) := \begin{cases} (1-s)\ln(1-s) + (1+s)\ln(1+s) & \text{if } s \in (-1, +1) \\ 2\ln 2 & \text{if } s \in \{-1, +1\}, \end{cases} \quad (1.13)$$

and where g is a positive and regular function acting on the parameter.

We postulate that the function g is positive, such that $g \in C^0(0, 1]$, and that, for every $\gamma \in (0, 1]$, it satisfies the following limit properties

$$\lim_{\gamma \searrow 0} g(\gamma) = 0, \quad \lim_{\gamma \searrow 0} g(\gamma)h(s) = 0 \quad \forall s \in [-1, +1]. \quad (1.14)$$

Moreover, the explicit expression of h , allows us to compute that $h'(s) = \ln\left(\frac{1+s}{1-s}\right)$, and that $h''(s) = \frac{2}{1-s^2} > 0$ for $s \in (-1, 1)$. Hence, combining the requirements 1.14 with these expressions, we also realize that

$$\lim_{\gamma \searrow 0} g(\gamma)h'(s) = 0 \quad \forall s \in (-1, +1), \quad \lim_{\gamma \searrow 0} \left(g(\gamma) \lim_{s \rightarrow \pm 1} h'(s) \right) = \pm\infty.$$

For instance, an admissible choice for the function g could be $g(\gamma) := \gamma^p$, for some $p > 0$. In that framework, it follows that the graph of $g(\gamma)h'(\cdot)$ became closer to the one of $\partial I_{[-1,1]}(\cdot)$ as the parameter γ goes to zero. Starting from this setting, we will show that also the family of control problems generated by this substitution, are close to (P_0) as $\gamma \searrow 0$ in a suitable sense.

We will see that this approximation scheme turns out to be sufficient to prove the existence of an optimal control for the system 1.4–1.9. Anyhow, this approach will also point out some weakness and limitations. In particular, we will realize that nothing can be said as the approximation of optimal control for (P_0) by sequences of optimal controls for the approximating problem is concerned. This fact totally prevents the possibility to recover some necessary conditions for the initial problem by passing to the limit in the necessary conditions of the approximating ones.

However, something can be said if we restrict the analysis to local results. In fact, localizing the problem around a fixed optimal control for (P_0) , we can prove a kind of local density result in terms of approximating optimal controls. Namely, we can show that for every fixed control for (P_0) , say \bar{u} , there exists an approximating sequence which, at every stage, is constituted by an optimal control for a suitable approximating problem. In order to prove such an approximation result, it will be convenient to introduce another cost functional, called adapted, which reads as follows

$$\tilde{\mathcal{J}}(\varphi, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2. \quad (1.15)$$

Let us point out that $\tilde{\mathcal{J}}$ strongly depends on the fixed control \bar{u} and also that if we take (φ, σ) as $(\bar{\varphi}, \bar{\sigma}) = \mathcal{S}(\bar{u})$, the adapted cost functional $\tilde{\mathcal{J}}$ reduces to \mathcal{J} . In this sense $\tilde{\mathcal{J}}$ consists of a local perturbation of \mathcal{J} around the optimum \bar{u} . Next, we will solve the control problem for the approximating system subject to the adapted cost functional and, providing to show some uniform estimates with respect to γ , we will pass to the limit in the corresponding optimality conditions to characterize the one for our initial system.

The plan of the rest of the paper is as follows. In the following section, we provide a precise description of the arguments introduced up to now at a formal level. Moreover, we fix our setting and assumptions, and also state the established results. From the third section on, we focus the attention on the corresponding proofs. Furthermore, Section 3 is totally devoted to the investigation of the approximating problem, while the existence of optimal controls has been proved in Section 4. To conclude, in Section 5 we perform the asymptotic analysis that will allow us to obtain the necessary condition for the problem we are dealing with.

2. Setting and main results. In this section, we state the main results on existence of optimal controls and on the optimality conditions. First of all, let us define some functional spaces that will be extremely useful later on

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\},$$

$$\mathcal{Y} := \left(H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \right)^3, \quad \hat{\mathcal{Y}} := \Pi_2 \circ \mathcal{Y},$$

where Π_2 stands for the projection of the first two components. Furthermore, we endow them with their natural norm to obtain some Banach spaces, and agree that the symbol $\|\cdot\|_X$ denotes the norm associated with a generic Banach space X . Moreover, we denote with $\langle \cdot, \cdot \rangle$ the duality pairing between V and its dual V^* .

Our basic assumptions on the system 1.4–1.9, on \mathcal{U}_{ad} , and on the cost functional 1.2 are as follows

$$b_0, b_1, b_2, b_3, b_4 \text{ are nonnegative constants, but not all zero.} \quad (2.1)$$

$$\varphi_Q, \sigma_Q \in L^2(Q), \varphi_\Omega, \sigma_\Omega \in H^1(\Omega), u_*, u^* \in L^\infty(Q) \text{ with } u_* \leq u^* \text{ a.e. in } Q. \quad (2.2)$$

$$\alpha, \beta > 0. \quad (2.3)$$

$$P \in C^2(\mathbb{R}) \text{ is nonnegative, bounded and Lipschitz continuous.} \quad (2.4)$$

$$\widehat{\pi} \in C^3(\mathbb{R}) \text{ and } \pi := \widehat{\pi}' \text{ is Lipschitz continuous.} \quad (2.5)$$

$$\mu_0, \varphi_0, \sigma_0 \in H^1(\Omega). \quad (2.6)$$

$$|\varphi_0| \leq 1 \text{ a.e. in } \Omega. \quad (2.7)$$

We thus may infer that the choices of u_* and u^* entails that the set \mathcal{U}_{ad} in turn is bounded, closed, convex and nonempty. Moreover, given a positive constant R , we introduce

$\mathcal{U}_R \subset L^2(Q)$ be a nonempty and bounded open set such that it contains \mathcal{U}_{ad} , and $\|u\|_2 \leq R$ for all $u \in \mathcal{U}_R$.

Let us emphasize that the singular, while not regular, double-well potential we are considering is usually referred to as the double-obstacle potential and it reads as

$$F_{2obst} := I_{[-1,1]} + \widehat{\pi}.$$

One has also to keep in mind that $\partial I_{[-1,1]}$ may be multivalued and therefore, we introduce a selection ξ by 1.6 that may not be regular enough to possess a trace.

Now, we can start to list our results.

Theorem 2.1 (Well-posedness). *Suppose that 2.3-2.7 are fulfilled. Then, for every $u \in L^2(Q)$, there exists a unique quadruplet $(\mu, \varphi, \sigma, \xi) \in \mathcal{Y} \times L^2(Q)$ which solves 1.4-1.9.*

Proof. It directly follows as a special case from [7, Thm. 2.2, p. 2426] and [8, Thm. 2.2, p. 97]. \square

The above result allows us to properly introduce the control-to-state mapping $\mathcal{S}_0 : L^2(Q) \rightarrow \mathcal{Y}$, which is the map that assigns to every admissible control u the corresponding solution (μ, φ, σ) to system 1.4-1.9. As the approximating system is concerned, assuming 1.13-1.14, and replacing 1.6 by 1.12, we get the following system which depends on γ and reads as

$$\alpha \partial_t \mu^\gamma + \partial_t \varphi^\gamma - \Delta \mu^\gamma = P(\varphi^\gamma)(\sigma^\gamma - \mu^\gamma) \quad \text{in } Q \quad (2.8)$$

$$\mu^\gamma = \beta \partial_t \varphi^\gamma - \Delta \varphi^\gamma + g(\gamma) h'(\varphi^\gamma) + \pi(\varphi^\gamma) \quad \text{in } Q \quad (2.9)$$

$$\partial_t \sigma^\gamma - \Delta \sigma^\gamma = -P(\varphi^\gamma)(\sigma^\gamma - \mu^\gamma) + u \quad \text{in } Q \quad (2.10)$$

$$\partial_n \mu^\gamma = \partial_n \varphi^\gamma = \partial_n \sigma^\gamma = 0 \quad \text{on } \Sigma \quad (2.11)$$

$$\mu^\gamma(0) = \mu_0^\gamma, \varphi^\gamma(0) = \varphi_0^\gamma, \sigma^\gamma(0) = \sigma_0^\gamma \quad \text{in } \Omega, \quad (2.12)$$

where $\{(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma)\}_\gamma$ denotes a family of initial data. We postulate that such a family fulfills the following requirements

$$(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times W \times H^1(\Omega) \quad \forall \gamma \in (0, 1], \quad (2.13)$$

$$|\varphi_0^\gamma| \leq 1 - \gamma/2 \text{ a.e. in } \Omega \quad \forall \gamma \in (0, 1], \quad (2.14)$$

$$(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma) \rightarrow (\mu_0, \varphi_0, \sigma_0) \text{ strongly in } V \times V \times V \text{ as } \gamma \searrow 0. \quad (2.15)$$

Even though they seem to be reasonable assumptions, the existence of a family that satisfies all the above conditions is not so trivial. For this reason we refer to Appendix 6, where the construction of such a family is shown in detail. Since the approximating problem 2.8-2.12 perfectly fits the framework of [34], the result below follows by a simple application of [34, Thm. 2.1].

Theorem 2.2 (Well-posedness of the approximating system). *Assume that 1.13–1.14, 2.3–2.6, and 2.13–2.15 are in force.*

(i) *For every $\gamma \in (0, 1]$ and every $u \in \mathcal{U}_{ad}$, the system 2.8–2.12 admits a unique solution $(\mu^\gamma, \varphi^\gamma, \sigma^\gamma)$ that possesses the following regularity*

$$\varphi^\gamma \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \subset C^0([0, T]; C^0(\bar{\Omega})) \quad (2.16)$$

$$\mu^\gamma, \sigma^\gamma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V) \quad (2.17)$$

$$\mu^\gamma \in L^\infty(Q), \quad (2.18)$$

and whose second component also satisfies that

$$-1 < \inf \varphi^\gamma \leq \sup \varphi^\gamma < 1 \quad \text{a.e. in } Q.$$

(ii) *For every given $\gamma \in (0, 1]$ there exist constants $\varphi_*(\gamma), \varphi^*(\gamma) \in (-1, 1)$, which depend on γ , on the initial data, and on the data of the system such that*

$$\varphi_*(\gamma) \leq \varphi^\gamma \leq \varphi^*(\gamma) \quad \text{a.e. in } Q, \quad (2.19)$$

where φ^γ is the second component of the unique solution to the approximating system 2.8–2.12 associated to the given u .

Remark 1. Let us point out that, unfortunately, a uniform separation property from $(-1, 1)$ is out of reach. In fact, when $\gamma \searrow 0$, it may occur that

$$\varphi_*(\gamma) \searrow -1 \quad \text{and/or} \quad \varphi^*(\gamma) \nearrow 1.$$

Nonetheless, although we cannot prove a separation result for 1.4–1.9, we will see that in the asymptotic investigation the separation property 2.19 for the approximated system will be using several times. Another ingredient that will be fundamental in the asymptotic analysis is the result below.

Lemma 2.3. *Suppose that $u \in \mathcal{U}_R$, and that assumptions 1.13–1.14, 2.3–2.6, and 2.13–2.15 are fulfilled. Then, whenever $\gamma \in (0, 1]$ and $(\mu^\gamma, \varphi^\gamma, \sigma^\gamma)$ is the corresponding solution to 2.8–2.12, it holds that*

$$\|(\mu^\gamma, \varphi^\gamma, \sigma^\gamma)\|_{\mathcal{Y}} \leq C_2, \quad (2.20)$$

where C_2 is a positive constant which only depends on the data of the system, but it is independent of the parameter γ . Moreover, we have the following estimate

$$\|g(\gamma)h'(\varphi^\gamma)\|_{L^2(Q)} \leq C_3. \quad (2.21)$$

Once the state system 2.8–2.12 has been analyzed, we can address the corresponding control problem and, as above, we set the following minimization problem.

(P_γ) Minimize $\mathcal{J}(\varphi^\gamma, \sigma^\gamma, u)$ subject to the control constraints 1.3 and under the requirement that the variables $(\varphi^\gamma, \sigma^\gamma)$ are the components of the solution to 2.8–2.12.

Let us recall that in Theorem 2.2 the well-posedness of system 2.8–2.12 has been already shown. Therefore, we are also in a position to define, for every $\gamma \in (0, 1]$, the well-posed map \mathcal{S}_γ consisting of the control-to-state mapping associated to the system 2.8–2.12, and the corresponding restriction $\mathcal{S}_{\gamma,2}$. In addition, since (P_γ) complies with the framework of [34], a simple application of [34, Thm. 2.6] leads to the following lemma.

Lemma 2.4. *Assume that 1.13–1.14, 2.1–2.7, and 2.13–2.15 are in force. Then, whenever $\gamma \in (0, 1]$ is given, the problem (P_γ) admits at least an optimal control.*

Now, let us present the existence result, whose proof will be structured this way: first, we approximate in a suitable sense (P_0) as $\gamma \searrow 0$ by (P_γ) and then, accounting for some compactness and monotonicity arguments, we pass to the limit.

Theorem 2.5 (Existence of optimal controls). *Suppose that 2.1-2.7 are fulfilled. Then (P_0) admits at least a solution.*

As already mentioned, the existence result combined with some asymptotic techniques turns out to be insufficient to properly characterize the optimality conditions we are looking for. In particular, the fact that every optimal control for (P_0) can be found as a limit, in a proper topology, of some approximating sequences of optimal controls for (P_{γ_n}) cannot be proven, whenever $\{\gamma_n\} \subset (0, 1]$ denotes a sequence which goes to zero as $n \nearrow \infty$. This gives no hope to recover some necessary conditions for (P_0) by the mere investigation of the ones of (P_{γ_n}) , since the passage to the limit at that stage will be meaningless.

However, even though we are not able to prove such a global result, a partial one can be stated localizing the problem. The key idea, which was introduced by Barbu in [2], consists of investigating the same approximating problem 2.8-2.12, but focusing the attention on the control problem corresponding to the adapted cost functional. Hence, the so-called adapted control problem reads as follows:

(\tilde{P}_γ) Minimize $\tilde{\mathcal{J}}(\varphi^\gamma, \sigma^\gamma, u)$ subject to the control constraints 1.3 and under the requirement that the variables $(\varphi^\gamma, \sigma^\gamma)$ yield a solution to 2.8-2.12,

where let us remind that $\tilde{\mathcal{J}}$ is defined, once that \bar{u} has been fixed, by

$$\tilde{\mathcal{J}}(\varphi, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2.$$

Again, we can easily prove the following lemma which straightforwardly follows as an application of [34, Thm. 2.6].

Lemma 2.6. *Under the assumptions 1.13-1.14, 2.1-2.6, and 2.13-2.15, whenever $\bar{u} \in \mathcal{U}_{ad}$ and $\gamma \in (0, 1]$ are given, the optimal control problem (\tilde{P}_γ) possesses at least a solution.*

The key result which motivate the interest toward the adapted optimal control problem is formalized in the next theorem where we will show that every fixed optimal control for (P_0) can be obtained as a limit of a sequence of optimal controls for (\tilde{P}_γ) which is of great importance for the forthcoming asymptotic analysis. Indeed, we have:

Theorem 2.7. *Assume that 1.13-1.14, 2.1-2.7, and 2.13-2.15 are in force. Moreover, let $(\bar{\varphi}, \bar{\sigma}, \bar{u}) \in \hat{\mathcal{Y}} \times \mathcal{U}_{ad}$ be an optimal choice for (P_0) . Then, for every sequence $\{\gamma_n\}_n$ which goes to zero as $n \nearrow \infty$, and for every $n \in \mathbb{N}$, there exists an approximating optimal triple for (\tilde{P}_{γ_n}) , namely $(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n})$, whose components satisfy the following convergences as $n \nearrow \infty$*

$$\bar{u}^{\gamma_n} \rightarrow \bar{u} \text{ strongly in } L^2(Q) \tag{2.22}$$

$$\bar{\varphi}^{\gamma_n} \rightarrow \bar{\varphi} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \tag{2.23}$$

$$\bar{\sigma}^{\gamma_n} \rightarrow \bar{\sigma} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \tag{2.24}$$

$$\tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) \rightarrow \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}). \tag{2.25}$$

Let us emphasize that this is the correct way in which the assertion “ (P_γ) approximates (P_0) as $\gamma \searrow 0$ ” has to be interpreted. By virtue of the above result, it is somehow reasonable that some optimality conditions can be earned exploiting this result. Indeed, we will establish the first optimality conditions for (P_0) by passing to the limit as $n \nearrow \infty$ in the corresponding optimality conditions for the approximating problem (\tilde{P}_{γ_n}) .

In Section 5, we will present the adjoint system for 2.8–2.12 which was originally treated in [34]. That system is formulated in terms of the variables $q^\gamma, p^\gamma, r^\gamma$ and it represents a core argument in order to pass to the limit to characterize the optimality conditions for 1.4–1.9. At this stage, let us only disclose that it admits existence and uniqueness of a solution and also that we are able to show the regularity that its solution enjoys (cf. Theorem 5.3). Next, we investigate the properties of this system in order to let $\gamma \searrow 0$ to characterize the optimality conditions we are looking for. To precisely state the asymptotic result we have established, let us first introduce further spaces that will naturally come out from the mathematical analysis. We set

$$\begin{aligned} \mathcal{Z} := & (L^\infty(0, T; H) \cap L^2(0, T; V)) \times (H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)) \\ & \times (H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)), \end{aligned} \quad (2.26)$$

$$\mathcal{W}(0, T) := H^1(0, T; V^*) \cap L^2(0, T; V) \subset C^0([0, T]; H), \quad (2.27)$$

$$\mathcal{W}_0(0, T) := \{v \in \mathcal{W}(0, T) : v(0) = 0\}, \quad (2.28)$$

and endow these spaces with their natural norm to get three Banach spaces. To avoid an heavy notation, we will denote the norm of $\mathcal{W}(0, T)$ and $\mathcal{W}_0(0, T)$ by $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{W}_0}$, respectively. Furthermore, we convey to use $\mathcal{W}_0^*(0, T) \langle \cdot, \cdot \rangle_{\mathcal{W}_0(0, T)}$ for the duality product between the dual of $\mathcal{W}_0(0, T)$, $\mathcal{W}_0(0, T)^*$ and $\mathcal{W}_0(0, T)$ itself. Moreover, it is worth noting that the space $L^2(0, T; V^*)$ is embedded into $\mathcal{W}_0(0, T)^*$. In fact, if z belongs to $L^2(0, T; V^*)$, for every v in $\mathcal{W}_0(0, T)$, we have that

$$\mathcal{W}_0(0, T)^* \langle z, v \rangle_{\mathcal{W}_0(0, T)} = \int_0^T V^* \langle z(t), v(t) \rangle_V dt.$$

In conclusion, we present the results we are able to prove which specify how the new optimality conditions have to be read.

Theorem 2.8. *Suppose that 1.13–1.14, 2.1–2.7, and 2.13–2.15 are satisfied, and let us define*

$$\lambda^{\gamma_n} := g(\gamma_n) h''(\bar{\varphi}^{\gamma_n}) \bar{q}^{\gamma_n} \quad \text{for every } n \in \mathbb{N}, \quad (2.29)$$

where h is the function introduced by 1.13 and where $\{\gamma_n\} \subset (0, 1]$ is a sequence which goes to zero as $n \nearrow \infty$. Then, there exists a positive constant C_4 such that, for every $n \in \mathbb{N}$, the following holds true

$$\|(\bar{q}^{\gamma_n}, \bar{p}^{\gamma_n}, \bar{r}^{\gamma_n})\|_{\mathcal{Z}} + \|\partial_t \bar{q}^{\gamma_n}\|_{\mathcal{W}_0(0, T)^*} + \|\lambda^{\gamma_n}\|_{\mathcal{W}_0(0, T)^*} \leq C_4, \quad (2.30)$$

where the variables $\bar{q}^{\gamma_n}, \bar{p}^{\gamma_n}, \bar{r}^{\gamma_n}$ denote the unique solutions to the adjoint problem for 2.8–2.12 considered for the element of the sequence γ_n instead of for γ , and where the constant C_4 may depend on the data of the system, but it is independent of n . In addition, up to a subsequence, we deduce the following convergences

$$\bar{q}^{\gamma_n} \rightharpoonup q \quad \text{weakly star in } L^\infty(0, T; H) \cap L^2(0, T; V) \quad (2.31)$$

$$\bar{p}^{\gamma^n} \rightarrow p \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.32)$$

$$\bar{r}^{\gamma^n} \rightarrow r \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.33)$$

$$\lambda^\gamma \rightarrow \lambda \text{ weakly in } \mathcal{W}_0^*. \quad (2.34)$$

This will allow us to let $\gamma \searrow 0$ and prove the following optimality conditions.

Theorem 2.9. *Let the assumptions 1.13–1.14, 2.1–2.7, and 2.13–2.15 be fulfilled. Furthermore, let $(\bar{\varphi}, \bar{\sigma}, \bar{u}) \in \hat{\mathcal{Y}} \times \mathcal{U}_{ad}$ be an optimal choice for (P_0) . Then the following properties hold true.*

(i) *Whenever a sequence $\{\gamma_n\} \subset (0, 1]$ which goes to zero as $n \nearrow \infty$ is fixed, we have that for every $n \in \mathbb{N}$ there exists an approximating optimal triple $(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n})$ to $(\bar{\varphi}, \bar{\sigma}, \bar{u})$, namely a triplet which solves the adapted control problem (\tilde{P}_{γ_n}) and which, as $n \nearrow \infty$, satisfies the convergences pointed out by 2.31–2.34.*

(ii) *Moreover, under the same assumptions we have that for every subsequence $\{n_k\}_k$ of \mathbb{N} , there exists a subsequence $\{n_{kj}\}_j$, a triple $(p, q, r) \in \mathcal{Z}$, and a functional $\lambda \in \mathcal{W}_0(0, T)^*$ such that the variational inequality which characterizes the optimality*

$$\int_Q (r + b_0 \bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \quad (2.35)$$

is satisfied. Furthermore, for a.a. $t \in (0, T)$, the triplet (q, p, r) solves the adjoint system consisting of the following variational equation

$$\begin{aligned} & \int_0^T \langle \partial_t v, p - \beta q \rangle - \mathcal{W}_0^* \langle \lambda, v \rangle_{\mathcal{W}_0} + \int_Q \nabla q \cdot \nabla v - \int_Q \pi''(\bar{\varphi}) q v \\ & + \int_Q P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) v = \int_Q b_1(\bar{\varphi} - \varphi_Q) v + \int_\Omega b_2(\bar{\varphi}(T) - \varphi_\Omega) v(T), \end{aligned} \quad (2.36)$$

which holds for every $v \in \mathcal{W}_0(0, T)$, combined with

$$q - \alpha \partial_t p - \Delta p + P(\bar{\varphi})(p - r) = 0 \quad \text{in } Q \quad (2.37)$$

$$-\partial_t r - \Delta r + P(\bar{\varphi})(r - p) = b_3(\bar{\sigma} - \sigma_Q) \quad \text{in } Q \quad (2.38)$$

$$\partial_n p = \partial_n r = 0 \quad \text{on } \Sigma \quad (2.39)$$

$$\alpha p(T) = 0, \quad r(T) = b_4(\bar{\sigma}(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (2.40)$$

In addition, we can show that

$$\liminf_{n \rightarrow \infty} \int_Q \lambda^{\gamma_n} \bar{q}^{\gamma_n} \geq 0, \quad (2.41)$$

and also a complementary slackness condition

$$\lim_{n \rightarrow \infty} \int_Q \lambda^{\gamma_n} (1 - (\bar{\varphi}^{\gamma_n})^2) \Phi = 2 \lim_{n \rightarrow \infty} \int_Q g(\gamma_n) \bar{q}^{\gamma_n} \Phi = 0, \quad (2.42)$$

are satisfied, where Φ is a general smooth function which vanishes at zero. Note that the above limits should be considered, in principle, for the subsequence $\{n_{kj}\}_j$ as $j \nearrow \infty$.

Remark 2. It can possibly occur that the limit triple (q, p, r) , and/or λ may be not uniquely determined. Indeed, it should happen that for different subsequences the corresponding limit q, p, r , and λ may change. Anyhow, a proper kind of uniqueness can be stated in term of a suitable projection. In fact, whenever $b_0 > 0$, it follows from the variational inequality 2.35 that

$$\bar{u} = \mathbb{P}_{\mathcal{U}_{ad}}(-b_0^{-1}r),$$

where $\mathbb{P}_{\mathcal{U}_{ad}}$ stands for the orthogonal projection onto \mathcal{U}_{ad} with respect to the standard inner product of $L^2(Q)$, so that r is uniquely determined in terms of \bar{u} .

In the remainder, we recollect some useful inequalities and properties that are applied several times in the paper. First, we often owe to the well-known Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \quad (2.43)$$

Furthermore, since the evolution set Ω is a bounded subset of \mathbb{R}^3 and possesses regular boundary, we can account for the Sobolev continuous embedding

$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{which is satisfied for every } q \in [1, 6], \quad (2.44)$$

i.e. we have the existence of suitable constant for which the following inequality holds true

$$\|v\|_{L^q(\Omega)} \leq C_q \|v\|_V \quad \text{for every } v \in V \text{ and } q \in [1, 6].$$

Let us conclude the section explaining a convention that we are going to employ as far as constants are concerned. Since in the following we have to deal with several estimates, we agree to use the symbol c for every constant which depends only on the final time T , on Ω , the shape of the nonlinearities, on the norms of the involved functions, and possibly on α and β , but it has to be independent of γ . Therefore, the meaning of c might change from line to line and even in the same chain of inequalities. On the other hand, the capital letters are devoted to denoting precise constants which we eventually will refer to.

3. Approximating system. From this section on, we start with the proofs of the stated results. In this section, we deal with the approximating system [2.8–2.12](#). Since a lot of properties immediately follows from [\[34\]](#) no repetition is required here. As a matter of fact, we only need to prove [Lemma 2.3](#), namely check that the constant C_2 involved in the lemma turns out to be independent of γ . This property will be fundamental later on to let $\gamma \searrow 0$ in order to prove the existence of an optimal control for (P_0) .

Proof of Lemma 2.3. We employ similar estimates to the ones performed in [\[34, Proof of Thm. 2.1\]](#) while referred to the approximating problem [2.8–2.12](#). Furthermore, we will have the care to show that all the constants that will appear do not depend on γ .

First estimate: We add to both sides of [2.9](#) the term φ^γ , multiply [2.8](#) by μ^γ , this new second equation by $-\partial_t \varphi^\gamma$ and [2.10](#) by σ^γ , then we add the resulting equations and integrate over Q_t and by parts to obtain

$$\begin{aligned} & \frac{\alpha}{2} \int_{\Omega} |\mu^\gamma(t)|^2 + \int_{Q_t} |\nabla \mu^\gamma|^2 + \beta \int_{Q_t} |\partial_t \varphi^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\varphi^\gamma(t)|^2 + \frac{1}{2} \int_{\Omega} |\nabla \varphi^\gamma(t)|^2 \\ & \quad + g(\gamma) \int_{\Omega} h(\varphi^\gamma(t)) + \frac{1}{2} \int_{\Omega} |\sigma^\gamma(t)|^2 + \int_{Q_t} |\nabla \sigma^\gamma|^2 + \int_{Q_t} P(\varphi^\gamma)(\sigma^\gamma - \mu^\gamma)^2 \\ & \leq \frac{\alpha}{2} \int_{\Omega} |\mu_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\varphi_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\nabla \varphi_0^\gamma|^2 + g(\gamma) \int_{\Omega} h(\varphi_0^\gamma) \\ & \quad + \frac{1}{2} \int_{\Omega} |\sigma_0^\gamma|^2 + \int_{Q_t} u \sigma^\gamma + \int_{Q_t} \varphi^\gamma \partial_t \varphi^\gamma + \int_{Q_t} \pi(\varphi^\gamma) \partial_t \varphi^\gamma, \end{aligned}$$

where almost all the integrals on the left-hand side are nonnegative since they are squares and P is so by 2.4. In fact, the only term that needs further manipulations is the sixth, the one in which g and h appear. First of all, let us recall that g is assumed to be positive, that h is defined by 1.13 and that it remains bounded in the interval $(-1, 1)$. Moreover, as a solution, φ^γ possesses the regularity stated by 2.16 and also enjoys the separation result 2.19. Therefore, both $\varphi^\gamma(t)$ and φ_0^γ range in the inner part of the interval $(-1, 1)$, and from this property, along with the fact that h is bounded from below in $[-1, 1]$, we infer that $h(\varphi^\gamma(t))$ is bounded from below as well. Hence, we neglect that term and we are reduced to control the integrals on the right-hand side which we denote by I_1, \dots, I_8 , in this order. Due to the requirements 2.13–2.15 and the above observation we immediately deduce that

$$\sum_{i=1}^5 |I_i| \leq c.$$

As the other terms are concerned, we invoke the Young inequality to show that

$$|I_6| + |I_7| + |I_8| \leq \frac{1}{2} \int_{Q_t} |u|^2 + \frac{1}{2} \int_{Q_t} |\sigma^\gamma|^2 + 2\delta \int_{Q_t} |\partial_t \varphi^\gamma|^2 + c_\delta \int_{Q_t} (|\varphi^\gamma|^2 + 1),$$

for a positive δ yet to be determined. Hence, we fix $0 < \delta < \beta/2$ and a Gronwall argument yields

$$\|\mu^\gamma\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\varphi^\gamma\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\sigma^\gamma\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c.$$

Second estimate: Now, we multiply 2.8 by $\partial_t \mu^\gamma$ and 2.10 by $\partial_t \sigma^\gamma$, add the resulting equalities and integrate over Q_t . Owing to the above estimate we easily conclude that

$$\|\mu^\gamma\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\sigma^\gamma\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c.$$

Third estimate: Equations 2.8 and 2.10 show a parabolic structure with respect to the variables μ^γ and σ^γ , respectively. In turn, it follows from the above estimates that their forcing terms both belong to $L^2(0, T; H)$. Hence, since the initial data 2.12 are regular due to 2.13, the elliptic regularity theory for homogeneous Neumann boundary problems produces

$$\|\mu^\gamma\|_{L^2(0,T;W)} + \|\sigma^\gamma\|_{L^2(0,T;W)} \leq c.$$

Fourth estimate: Next, we aim at improving the regularity of φ^γ arguing in a similar way via a comparison argument in 2.9. So, let us rearrange 2.9 as follows

$$-\Delta \varphi^\gamma + g(\gamma)h'(\varphi^\gamma) = f, \quad \text{where } f := \mu^\gamma - \beta \partial_t \varphi^\gamma - \pi(\varphi^\gamma), \quad (3.1)$$

and we realize that the above estimates entail that $f \in L^2(0, T; H)$. We then test 3.1 by $-\Delta \varphi^\gamma$ and integrate over Ω to get, for a.a. $t \in (0, T)$, the following inequality

$$\int_{\Omega} |\Delta \varphi^\gamma(t)|^2 + g(\gamma) \int_{\Omega} h''(\varphi^\gamma(t)) |\nabla \varphi^\gamma(t)|^2 \leq \frac{1}{2} \int_{\Omega} |\Delta \varphi^\gamma(t)|^2 + \frac{1}{2} \int_{\Omega} |f(t)|^2,$$

where we have applied Young's inequality on the right-hand side. The second term of the left-hand side turns out to be positive since $\varphi^\gamma(t)$ satisfies 2.19 and h'' is nonnegative in such an interval. Hence, we realize that

$$\frac{1}{2} \int_{\Omega} |\Delta \varphi^\gamma(t)|^2 \leq \frac{1}{2} \int_{\Omega} |f(t)|^2,$$

and the elliptic regularity theory for homogeneous Neumann problem yields that

$$\|\varphi^\gamma\|_{L^2(0,T;W)} \leq c.$$

Combining all the above estimates, it is straightforward to realize that 2.20 has been proved.

Let us conclude by proving the second part of the lemma.

Fifth estimate: On account of the above estimates, a comparison in 2.9 directly gives that

$$\|g(\gamma)h'(\varphi^\gamma)\|_{L^2(Q)} \leq c, \tag{3.2}$$

that is the estimate we are looking for. \square

4. Existence and approximation of optimal controls. Here, we essentially aim to show the validity of Theorems 2.5 and 2.7.

Proof of Theorem 2.5. Let us pick an arbitrary sequence $\{\gamma_n\} \subset (0, 1]$ which goes to zero as $n \nearrow \infty$. In view of Lemma 2.4, we can take an optimal triple for (P_{γ_n}) associated with that sequence. Namely, for every $n \in \mathbb{N}$, we consider the following triple

$$(\varphi^{\gamma_n}, \sigma^{\gamma_n}, u^{\gamma_n}) \in \widehat{\mathcal{Y}} \times \mathcal{U}_{ad}, \tag{4.1}$$

where $(\varphi^{\gamma_n}, \sigma^{\gamma_n}) = \mathcal{S}_{\gamma_n, 2}(u^{\gamma_n})$. Moreover, from Lemma 2.3 and Theorem 2.2 it follows that

$$|\varphi^{\gamma_n}| < 1 \text{ a.e. in } Q, \text{ and } \|(\varphi^{\gamma_n}, \sigma^{\gamma_n})\|_{\widehat{\mathcal{Y}}} \leq C_2 \text{ for every } n \in \mathbb{N}, \tag{4.2}$$

where the constant C_2 is independent of γ . Therefore, thanks to well-known weak star compactness, it is a standard matter to show that there exist some $\bar{u} \in \mathcal{U}_{ad}$, and a triple $(\bar{\mu}, \bar{\varphi}, \bar{\sigma}) \in \mathcal{Y}$, such that the following convergences

$$\begin{aligned} u^{\gamma_n} &\rightharpoonup \bar{u} \text{ weakly star in } L^\infty(Q) \\ \varphi^{\gamma_n} &\rightharpoonup \bar{\varphi} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \sigma^{\gamma_n} &\rightharpoonup \bar{\sigma} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \mu^{\gamma_n} &\rightharpoonup \bar{\mu} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \end{aligned}$$

hold as $n \nearrow \infty$. Furthermore, due to the continuity of the embedding

$$H^1(0, T; H) \cap L^2(0, T; W) \subset C^0([0, T]; V),$$

we infer that $\bar{\varphi}, \bar{\sigma} \in C^0([0, T]; V)$. In addition, also some strong convergences can be recovered invoking the Aubin-Lions lemma (see, e.g., [35, Sect. 8, Cor. 4]). Indeed, one can show that

$$\varphi^{\gamma_n} \rightarrow \bar{\varphi} \text{ strongly in } C^0([0, T]; H) \cap L^2(0, T; V) \tag{4.3}$$

$$\sigma^{\gamma_n} \rightarrow \bar{\sigma} \text{ strongly in } C^0([0, T]; H) \cap L^2(0, T; V) \tag{4.4}$$

$$\mu^{\gamma_n} \rightarrow \bar{\mu} \text{ strongly in } C^0([0, T]; H) \cap L^2(0, T; V). \tag{4.5}$$

At this point, from the assumptions on the initial data 2.13–2.15, combined with the above strong convergences, we infer that $\bar{\varphi}(0) = \varphi_0$ and $\bar{\sigma}(0) = \sigma_0$. By the same token, we can handle both the nonlinearities P and π , that are Lipschitz continuous by 2.4 and 2.6, respectively. In fact, as $n \nearrow \infty$, we also realize that

$$P(\varphi^{\gamma_n}) \rightarrow P(\bar{\varphi}) \text{ strongly in } L^2(0, T; H)$$

$$\pi(\varphi^{\gamma_n}) \rightarrow \pi(\bar{\varphi}) \text{ strongly in } L^2(0, T; H).$$

Moreover, estimate 3.2 also leads to the following weak convergence

$$g(\gamma_n)h'(\varphi^{\gamma_n}) \rightharpoonup \xi \text{ weakly in } L^2(0, T; H). \tag{4.6}$$

Now, we claim that its limit, that we have denoted by ξ , has the same meaning of the variable ξ introduced in 1.5, namely that $\xi \in \partial I_{[-1,1]}(\bar{\varphi})$ a.e. in Q . For this purpose, we account for the convexity property of h defined by 1.13. It yields that, for every $n \in \mathbb{N}$, the following inequality is satisfied

$$\int_Q g(\gamma_n) h(\varphi^{\gamma_n}) + \int_Q g(\gamma_n) h'(\varphi^{\gamma_n})(w - \varphi^{\gamma_n}) \leq \int_Q g(\gamma_n) h(w) \quad \text{for all } w \in B_1,$$

where $B_1 := \{v \in L^2(Q) : |v| \leq 1 \text{ a.e. in } Q\}$. Then, we owe to 1.14 and deduce that in the above inequality the former and the latter terms go to zero as $n \nearrow \infty$. Therefore, we combine the strong convergence 4.3 with the weak one 4.6 to find in the limit

$$\int_Q \xi(\bar{\varphi} - w) \geq 0 \quad \text{for every } w \in B_1, \quad (4.7)$$

which means exactly that ξ is an element of the subdifferential of the extension of $I_{[-1,1]}(\cdot)$ to $L^2(Q)$, or equivalently (cf. [3, Ex. 2.3.3., p. 25]) that $\xi \in \partial I_{[-1,1]}(\bar{\varphi})$, as we claimed. Hence, all the above convergences ensure us the possibility to pass to the limit as $n \nearrow \infty$ and realize that $(\bar{\varphi}, \bar{\sigma}, \bar{u})$ solves 1.4-1.9 and also that it is an admissible choice for (P_0) , i.e. that $(\bar{\varphi}, \bar{\sigma}) = \mathcal{S}_{0,2}(\bar{u})$.

To conclude, it remains to show that $(\bar{\varphi}, \bar{\sigma}, \bar{u})$ is in fact not only admissible, but also optimal and we will manage this problem by accounting for monotonicity arguments. In fact, recalling that 1.2 is lower semicontinuous, it turns out that the following

$$\begin{aligned} \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) &= \mathcal{J}(\mathcal{S}_{0,2}(\bar{u}), \bar{u}) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_{\gamma_n,2}(u^{\gamma_n}), u^{\gamma_n}) \\ &\leq \liminf_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_{\gamma_n,2}(v), v) \leq \lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_{\gamma_n,2}(v), v) = \mathcal{J}(\mathcal{S}_{0,2}(v), v) \end{aligned}$$

is satisfied for every fixed $v \in \mathcal{U}_{ad}$, where in the last inequality we exploit the continuity of the cost functional with respect to the first component. \square

As a consequence of the above proof, we also realize the following corollary.

Corollary 1. *Assume 1.13-1.14, 2.1-2.7, 2.13-2.15, and let $\{\gamma_n\} \subset (0, 1]$ be a sequence which goes to zero as $n \nearrow \infty$. Then, whenever a sequence $\{u^{\gamma_n}\}$ such that $u^{\gamma_n} \rightarrow \bar{u}$ weakly star in $L^\infty(Q)$ is given, we have that*

$$\mathcal{S}_{\gamma_n,2}(u^{\gamma_n}) \rightarrow \mathcal{S}_{0,2}(\bar{u}) \quad \text{weakly star in } \hat{\mathcal{Y}}.$$

Moreover, for every $v \in \mathcal{U}_R$, we have that

$$\mathcal{S}_{\gamma_n,2}(v) \rightarrow \mathcal{S}_{0,2}(v) \quad \text{strongly in } L^2(0, T; V),$$

and, due to the continuity of the cost functional with respect to the first component, also that

$$\lim_{n \rightarrow \infty} \mathcal{J}(\mathcal{S}_{\gamma_n,2}(v), v) = \mathcal{J}(\mathcal{S}_{0,2}(v), v) \quad \text{for every } v \in \mathcal{U}_R. \quad (4.8)$$

Below, we prove Theorem 2.7, which is the best we can say as far as the approximation of optimal controls for (P_0) is concerned. Monotonicity and compactness arguments will be the key arguments to show this result.

Proof of Theorem 2.7. Let $\gamma \in (0, 1]$ be fixed, and let $(\bar{\varphi}^\gamma, \bar{\sigma}^\gamma, \bar{u})$ be an optimal triple for (\tilde{P}_γ) which exists by virtue of Lemma 2.6. The boundedness of \mathcal{U}_{ad} and

Lemma 2.3 yield that there exist φ, σ, u , and a sequence $\{\gamma_n\}$, which goes to zero as $n \nearrow \infty$, for which the convergences

$$\bar{u}^{\gamma_n} \rightarrow u \text{ weakly star in } L^\infty(Q) \quad (4.9)$$

$$\bar{\varphi}^{\gamma_n} \rightarrow \varphi \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (4.10)$$

$$\bar{\sigma}^{\gamma_n} \rightarrow \sigma \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (4.11)$$

are satisfied. Moreover, in view of Corollary 1, we also realize that (φ, σ, u) is an admissible triple for (P_0) , i.e. that $(\varphi, \sigma) = \mathcal{S}_{0,2}(u)$.

As a matter of fact, monotonicity arguments will allow us to say more. Indeed, we are able to show that the limit u coincides with \bar{u} , and from the well-posedness of \mathcal{S}_0 also that $(\varphi, \sigma) = \mathcal{S}_{0,2}(u) = \mathcal{S}_{0,2}(\bar{u}) = (\bar{\varphi}, \bar{\sigma})$. As far as $\tilde{\mathcal{J}}$ is lower semicontinuous and $(\bar{\varphi}, \bar{\sigma}, \bar{u})$ is optimal for (P_0) , we deduce that the following inequality holds true

$$\begin{aligned} \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) &\geq \tilde{\mathcal{J}}(\varphi, \sigma, u) = \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 \\ &\geq \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2, \end{aligned} \quad (4.12)$$

where we also exploit the definition of the reduced cost functional $\tilde{\mathcal{J}}$ introduced by 1.15. In addition, owing to the optimality of $(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n})$ for (\tilde{P}_{γ_n}) , we realize that

$$\tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) = \tilde{\mathcal{J}}(\mathcal{S}_{\gamma_n,2}(\bar{u}^{\gamma_n}), \bar{u}^{\gamma_n}) \leq \tilde{\mathcal{J}}(\mathcal{S}_{\gamma_n,2}(\bar{u}), \bar{u}) \quad \text{for every } n \in \mathbb{N}.$$

Therefore, we pass to the superior limit in both sides of the above inequality to obtain that

$$\limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) \leq \tilde{\mathcal{J}}(\mathcal{S}_{0,2}(\bar{u}), \bar{u}) = \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}), \quad (4.13)$$

where the last equality has been treated invoking 4.8 and the fact that $\tilde{\mathcal{J}}$ reduces to \mathcal{J} whenever it is considered to act on an optimal triple for (P_0) . Thus, combining inequality 4.12 and 4.13 imply that

$$\frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 = 0, \quad (4.14)$$

which consists of the first convergence 2.22. Moreover, this also establishes that $\bar{u} = u$ and $(\bar{\varphi}, \bar{\sigma}) = (\varphi, \sigma)$ which prove 2.23 and 2.24 as well. Finally, due to the above estimates, we also find that

$$\begin{aligned} \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) &= \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) \\ &= \limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}) = \lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}^{\gamma_n}, \bar{\sigma}^{\gamma_n}, \bar{u}^{\gamma_n}), \end{aligned}$$

so that the desired convergence 2.25 has been shown. \square

5. Optimality results. This final section is devoted to the check Theorems 2.8 and 2.9. We aim to characterize the optimality conditions for the initial system 1.4–1.9 by passing to the limit in the first-order necessary conditions of the approximating problem. As already underlined, the mathematical analysis that comes out will be quite involved since a sort of double approximation has to be considered. To begin with, let us start to investigate the optimality results for the approximating system 2.8–2.12 which perfectly fits the framework of [34]. As far as numerous problems were already investigated there, no repetition for the proofs is needed in what follows.

5.1. The linearized system of the approximating problem. In the remainder, we require that $\bar{u} \in \mathcal{U}_R$ is given, and we denote $(\bar{\mu}^\gamma, \bar{\varphi}^\gamma, \bar{\sigma}^\gamma)$ the corresponding solution to 2.8–2.12. At this stage is not so important if \bar{u} is an optimal control or not, it only matters that \bar{u} is fixed.

For a given $\gamma \in (0, 1]$ and for every $\psi \in L^2(Q)$, the linearized system corresponding to 2.8–2.12 reads as follows (c.f. [34, Sec. 4.2])

$$\alpha \partial_t \eta^\gamma + \partial_t \vartheta^\gamma - \Delta \eta^\gamma = P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma) \vartheta^\gamma + P(\bar{\varphi}^\gamma)(\rho^\gamma - \eta^\gamma) \quad \text{in } Q \quad (5.1)$$

$$\eta^\gamma = \beta \partial_t \vartheta^\gamma - \Delta \vartheta^\gamma + g(\gamma) h''(\bar{\varphi}^\gamma) \vartheta^\gamma + \pi'(\bar{\varphi}^\gamma) \vartheta^\gamma \quad \text{in } Q \quad (5.2)$$

$$\partial_t \rho^\gamma - \Delta \rho^\gamma = -P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma) \vartheta^\gamma - P(\bar{\varphi}^\gamma)(\rho^\gamma - \eta^\gamma) + \psi \quad \text{in } Q \quad (5.3)$$

$$\partial_n \rho^\gamma = \partial_n \vartheta^\gamma = \partial_n \eta^\gamma = 0 \quad \text{on } \Sigma \quad (5.4)$$

$$\rho^\gamma(0) = \vartheta^\gamma(0) = \eta^\gamma(0) = 0 \quad \text{in } \Omega. \quad (5.5)$$

An application of [34, Thm. 2.4] leads to the result below. Let us only point out that the symbol h covers a different role there.

Lemma 5.1. *Assume 1.13–1.14, 2.1–2.7, and 2.13–2.15. Then, whenever $\psi \in L^2(Q)$ is given, the system 5.1–5.5 admits a unique solution $(\eta^\gamma, \vartheta^\gamma, \rho^\gamma)$ that belongs to \mathcal{Y} .*

5.2. Fréchet differentiability of \mathcal{S}_γ . Our next goal is concerned with the Fréchet differentiability of the control-to-state mapping \mathcal{S}_γ . Again, since the system 2.8–2.12 was already investigated, let us recall the obtained result (c.f. [34, Thm. 2.5]).

Lemma 5.2 (Fréchet differentiability of \mathcal{S}_γ). *Assume that 1.13–1.14, 2.3–2.5, and 2.13–2.15 are fulfilled. Then the control-to-state mapping \mathcal{S}_γ , viewed as a mapping from \mathcal{U}_R into the state space \mathcal{Y} , is Fréchet differentiable at \bar{u} . Moreover, for any $\bar{u} \in \mathcal{U}_R$ the Fréchet derivative $D\mathcal{S}_\gamma(\bar{u})$ is a linear and continuous operator from $L^2(Q)$ to \mathcal{Y} . Furthermore, for every $\psi \in L^2(Q)$ we have that $D\mathcal{S}_\gamma(\bar{u})\psi = (\eta^\gamma, \vartheta^\gamma, \rho^\gamma)$, where $(\eta^\gamma, \vartheta^\gamma, \rho^\gamma)$ is the unique solution to 5.1–5.5 associated with ψ .*

5.3. Optimality conditions for the adapted problem. In the following, we deal with the optimality conditions for (\tilde{P}_γ) that will turn out to be extremely fruitful in view of the forthcoming asymptotic analysis. At a formal stage, we can assert that wherever \bar{u}^γ represents an optimal control for (\tilde{P}_γ) , which exists by virtue of Lemma 2.6, the variational inequality, which characterizes the optimality conditions we are looking for, reads

$$\langle D\tilde{\mathcal{J}}_{\text{red},\gamma}(\bar{u}^\gamma), v - \bar{u}^\gamma \rangle \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \quad (5.6)$$

where $D\tilde{\mathcal{J}}_{\text{red},\gamma}$ denotes the Fréchet derivative of $\tilde{\mathcal{J}}_{\text{red},\gamma}$ and where this latter stands for the reduced cost functional corresponding to $\tilde{\mathcal{J}}$ and it can be defined as made in 1.10, while referred to $\tilde{\mathcal{J}}$ instead of \mathcal{J} . Moreover, we can appeal to the chain rule to infer that, for every fixed $\gamma \in (0, 1]$, we have

$$D\tilde{\mathcal{J}}_{\text{red},\gamma}(\bar{u}) = D_{(\bar{\varphi}, \bar{\sigma})} \tilde{\mathcal{J}}(\mathcal{S}_{\gamma,2}(\bar{u}), \bar{u}) \circ D\mathcal{S}_{\gamma,2}(\bar{u}) + D_{\bar{u}} \tilde{\mathcal{J}}(\mathcal{S}_{\gamma,2}(\bar{u}), \bar{u}). \quad (5.7)$$

Having Lemma 5.2 at disposal, a simple application of [34, Cor. 2.7] yields:

Corollary 2 (First necessary condition). *Suppose that the assumptions 1.13–1.14, 2.1–2.5, and 2.13–2.15 are fulfilled. Let $\gamma \in (0, 1]$ be given, and let $\bar{u}^\gamma \in \mathcal{U}_{ad}$ be an optimal control for (\tilde{P}_γ) with its corresponding state $(\bar{\mu}^\gamma, \bar{\varphi}^\gamma, \bar{\sigma}^\gamma)$. Then, the*

necessary condition for the optimality reads as follows

$$\begin{aligned} & b_1 \int_Q (\bar{\varphi}^\gamma - \varphi_Q) \vartheta^\gamma + b_2 \int_\Omega (\bar{\varphi}^\gamma(T) - \varphi_\Omega) \vartheta^\gamma(T) + b_3 \int_Q (\bar{\sigma}^\gamma - \sigma_Q) \rho^\gamma \\ & + b_4 \int_\Omega (\bar{\sigma}^\gamma(T) - \sigma_\Omega) \rho^\gamma(T) + \int_Q (b_0 \bar{u}^\gamma + (\bar{u}^\gamma - \bar{u}))(v - \bar{u}^\gamma) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \end{aligned} \quad (5.8)$$

where ϑ^γ and ρ^γ are the second and third components of the unique solution $(\eta^\gamma, \vartheta^\gamma, \rho^\gamma)$ to the linearized system 5.1–5.5 associated with $\psi = v - \bar{u}^\gamma$.

Anyhow, the presence of the linearized variables ϑ^γ and ρ^γ in the above inequality is rather unpleasant, thus we try to eliminate them by solving the so-called adjoint system that consists of the problem below

$$\begin{aligned} & \beta \partial_t q^\gamma - \partial_t p^\gamma + \Delta q^\gamma - g(\gamma) h''(\bar{\varphi}^\gamma) q^\gamma - \pi''(\bar{\varphi}^\gamma) q^\gamma \\ & + P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma)(r^\gamma - p^\gamma) = b_1(\bar{\varphi}^\gamma - \varphi_Q) \quad \text{in } Q \end{aligned} \quad (5.9)$$

$$q^\gamma - \alpha \partial_t p^\gamma - \Delta p^\gamma + P(\bar{\varphi}^\gamma)(p^\gamma - r^\gamma) = 0 \quad \text{in } Q \quad (5.10)$$

$$- \partial_t r^\gamma - \Delta r^\gamma + P(\bar{\varphi}^\gamma)(r^\gamma - p^\gamma) = b_3(\bar{\sigma}^\gamma - \sigma_Q) \quad \text{in } Q \quad (5.11)$$

$$\partial_n q^\gamma = \partial_n p^\gamma = \partial_n r^\gamma = 0 \quad \text{on } \Sigma \quad (5.12)$$

$$p^\gamma(T) - \beta q^\gamma(T) = b_2(\bar{\varphi}^\gamma(T) - \varphi_\Omega), \quad \alpha p^\gamma(T) = 0, \quad r^\gamma(T) = b_4(\bar{\sigma}^\gamma(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (5.13)$$

The well-posedness of the above system has been discussed in [34] where the following result was proved.

Theorem 5.3. *Under the assumptions 1.13–1.14, 2.1–2.6, 2.13–2.15 and for every fixed $\gamma \in (0, 1]$, the system 5.9–5.13 has a unique solution $(\bar{q}^\gamma, \bar{p}^\gamma, \bar{r}^\gamma)$ that in turn satisfies the following regularity requirements*

$$\bar{q}^\gamma, \bar{p}^\gamma, \bar{r}^\gamma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V). \quad (5.14)$$

This will allow us to obtain the corresponding necessary condition for optimality which reads as follows.

Theorem 5.4 (Well-posedness and necessary condition). *Under the assumptions 1.13–1.14, 2.1–2.6, and 2.13–2.15, whenever \bar{u}^γ represents an optimal control for (\tilde{P}_γ) , there holds*

$$\int_Q (\bar{r}^\gamma + (b_0 \bar{u}^\gamma + (\bar{u}^\gamma - \bar{u}))(v - \bar{u}^\gamma)) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \quad (5.15)$$

where \bar{r}^γ is the unique solution to the adjoint problem introduced by Theorem 5.3.

At this stage, we would be tempted to pass to the limit as $\gamma \searrow 0$ in the above inequality to characterize the necessary conditions for (P_0) , but, unfortunately, the corresponding mathematical analysis turns out to be more delicate and the precise description is the purpose of the next paragraph.

5.4. First-order necessary condition for (P_0) . As sketched in the above lines, we try to recover some optimality conditions for system 1.4–1.9 from inequality 5.15 showing that, in a proper sense, we can pass to the limit as $\gamma \searrow 0$. In this direction, some compactness properties for the solution to the adjoint problem 5.9–5.13 need

to be shown. Before moving on, let us introduce a further notation: in addition to 1.1, it will be useful to set the backward-in-time cylinder by setting

$$Q_t^T := \Omega \times [t, T], \quad \text{for } 0 \leq t < T.$$

Proof of Theorem 2.8. First, let us show some uniform estimates, with respect to γ , that will allow us to justify the passage to the limit as the parameter goes to zero.

First estimate: To begin with, let us add to both the members of 5.10 the term \bar{p}^γ . Then, let us test 5.9 by $-\bar{q}^\gamma$, this new second equation by $-\partial_t \bar{p}^\gamma$, and 5.11 by \bar{r}^γ . Finally, we add these equalities and integrate over Q_t^T and by parts to find that

$$\begin{aligned} & \frac{\beta}{2} \int_{\Omega} |\bar{q}^\gamma(t)|^2 + \int_{Q_t^T} \partial_t \bar{p}^\gamma \bar{q}^\gamma + \int_{Q_t^T} |\nabla \bar{q}^\gamma|^2 + g(\gamma) \int_{Q_t^T} h''(\bar{\varphi}^\gamma) |\bar{q}^\gamma|^2 - \int_{Q_t^T} \partial_t \bar{p}^\gamma \bar{q}^\gamma \\ & + \alpha \int_{Q_t^T} |\partial_t \bar{p}^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\nabla \bar{p}^\gamma(t)|^2 + \frac{1}{2} \int_{\Omega} |\bar{p}^\gamma(t)|^2 + \frac{1}{2} \int_{\Omega} |\bar{r}^\gamma(t)|^2 + \int_{Q_t^T} |\nabla \bar{r}^\gamma|^2 \\ & = \frac{\beta}{2} \int_{\Omega} |\bar{q}^\gamma(T)|^2 + \frac{1}{2} \int_{\Omega} |\nabla \bar{p}^\gamma(T)|^2 + \frac{1}{2} \int_{\Omega} |\bar{p}^\gamma(T)|^2 + \frac{1}{2} \int_{\Omega} |\bar{r}^\gamma(T)|^2 \\ & - \int_{Q_t^T} \pi''(\bar{\varphi}^\gamma) \bar{q}^\gamma{}^2 + \int_{Q_t^T} P'(\bar{\varphi}^\gamma) (\bar{\sigma}^\gamma - \bar{\mu}^\gamma) (\bar{r}^\gamma - \bar{p}^\gamma) \bar{q}^\gamma \\ & - \int_{Q_t^T} b_1(\bar{\varphi}^\gamma - \varphi_Q) \bar{q}^\gamma + \int_{Q_t^T} P(\bar{\varphi}^\gamma) (\bar{r}^\gamma - \bar{p}^\gamma) \partial_t \bar{p}^\gamma - \int_{Q_t^T} \bar{p}^\gamma \partial_t \bar{p}^\gamma \\ & + \int_{Q_t^T} b_3(\bar{\sigma}^\gamma - \sigma_Q) \bar{r}^\gamma - \int_{Q_t^T} P(\bar{\varphi}^\gamma) (\bar{r}^\gamma - \bar{p}^\gamma) \bar{r}^\gamma, \end{aligned}$$

where we denote by I_1, \dots, I_{11} the integrals of the right-hand side, in that order. As regards the left-hand side, we point out that the second and the fifth integrals cancel out. Moreover, all the other terms on that side are nonnegative since we also have that $h''(\bar{\varphi}^\gamma)$ is so, due to the separation result 2.19 for φ^γ and owing to the explicit form of h'' . Therefore, it remains to control the right-hand side. The assumptions on the final conditions allow us to straightforwardly establish that

$$\sum_{i=1}^4 |I_i| = \frac{b_2^2}{2\beta} \int_{\Omega} |\bar{\varphi}^\gamma(T) - \varphi_\Omega|^2 + \frac{b_4^2}{2} \int_{\Omega} |\bar{\sigma}^\gamma(T) - \sigma_\Omega|^2 \leq c.$$

Moreover, we have that

$$|I_5| + |I_7| + |I_{10}| \leq c + c \int_{Q_t^T} |\bar{q}^\gamma|^2 + c \int_{Q_t^T} |\bar{r}^\gamma|^2,$$

appealing to the Young inequality, to 2.2, and to 2.5. As the remaining integrals are concerned, we compute

$$\begin{aligned} |I_8| + |I_9| + |I_{11}| & \leq 2\delta \int_{Q_t^T} |\partial_t \bar{p}^\gamma|^2 + c_\delta \int_{Q_t^T} |P(\bar{\varphi}^\gamma) (\bar{r}^\gamma - \bar{p}^\gamma)|^2 \\ & + c_\delta \int_{Q_t^T} |\bar{p}^\gamma|^2 + c \int_{Q_t^T} |\bar{r}^\gamma|^2 \\ & \leq 2\delta \int_{Q_t^T} |\partial_t \bar{p}^\gamma|^2 + c_\delta \int_{Q_t^T} |\bar{p}^\gamma|^2 + c_\delta \int_{Q_t^T} |\bar{r}^\gamma|^2, \end{aligned}$$

where we have applied 2.4 and 2.43. Moreover, we obtain from 2.4, 2.5, 2.17, 2.19, the Sobolev embedding 2.44, Young's and Hölder's inequality that

$$\begin{aligned}
|I_6| &\leq c \int_{Q_t^T} |P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma)\bar{q}^\gamma|^2 + c \int_{Q_t^T} |\bar{r}^\gamma - \bar{p}^\gamma|^2 \\
&\leq c \int_{Q_t^T} (|\bar{\sigma}^\gamma|^2 + |\bar{\mu}^\gamma|^2)|\bar{q}^\gamma||\bar{q}^\gamma| + c \int_{Q_t^T} |\bar{r}^\gamma|^2 + c \int_{Q_t^T} |\bar{p}^\gamma|^2 \\
&\leq c \int_t^T (\|\bar{\sigma}^\gamma\|_6^2 + \|\bar{\mu}^\gamma\|_6^2)\|\bar{q}^\gamma\|_6\|\bar{q}^\gamma\|_2 + c \int_{Q_t^T} |\bar{r}^\gamma|^2 + c \int_{Q_t^T} |\bar{p}^\gamma|^2 \\
&\leq c \int_t^T (\|\bar{\sigma}^\gamma\|_V^2 + \|\bar{\mu}^\gamma\|_V^2)\|\bar{q}^\gamma\|_V\|\bar{q}^\gamma\|_H + c \int_{Q_t^T} |\bar{r}^\gamma|^2 + c \int_{Q_t^T} |\bar{p}^\gamma|^2 \\
&\leq \frac{1}{2} \int_{Q_t^T} (|\bar{q}^\gamma|^2 + |\nabla \bar{q}^\gamma|^2) + c \int_t^T (\|\bar{\sigma}^\gamma\|_V^4 + \|\bar{\mu}^\gamma\|_V^4)\|\bar{q}^\gamma\|_H^2 \\
&\quad + c \int_{Q_t^T} |\bar{r}^\gamma|^2 + c \int_{Q_t^T} |\bar{p}^\gamma|^2 \\
&\leq \frac{1}{2} \int_{Q_t^T} |\nabla \bar{q}^\gamma|^2 + c \int_{Q_t^T} |\bar{r}^\gamma|^2 + c \int_{Q_t^T} |\bar{p}^\gamma|^2 + c \int_{Q_t^T} |\bar{q}^\gamma|^2.
\end{aligned}$$

Then, upon collecting the above estimates, we fix $0 < \delta < \alpha/2$ and apply the backward-in-time Gronwall lemma to infer that

$$\begin{aligned}
&\|\bar{q}^\gamma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\bar{p}^\gamma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\bar{r}^\gamma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\quad + g(\gamma) \int_Q h''(\bar{\varphi}^\gamma)|\bar{q}^\gamma|^2 \leq c.
\end{aligned} \tag{5.16}$$

Second estimate: We proceed multiplying 5.10 by $\Delta \bar{p}^\gamma$. Using the Young inequality and the above estimate, we deduce that

$$\|\Delta \bar{p}^\gamma\|_{L^2(0,T;H)} \leq c,$$

and, accounting for the elliptic regularity theory for homogeneous Neumann boundary problems, also that

$$\|\bar{p}^\gamma\|_{L^2(0,T;W)} \leq c. \tag{5.17}$$

Third estimate: Next, let us rewrite the equation 5.11 as follows:

$$-\partial_t \bar{r}^\gamma - \Delta \bar{r}^\gamma = b_3(\bar{\sigma}^\gamma - \sigma_Q) - P(\bar{\varphi}^\gamma)(\bar{r}^\gamma - \bar{p}^\gamma) =: f. \tag{5.18}$$

Owing to the above estimates, 2.4, and using the fact that $\bar{\varphi}^\gamma$, as solution to 1.4–1.9, satisfies 2.16, it is easy to realize that the forcing term $f \in L^2(0,T;H)$. Hence, since it reads as a backward-in-time parabolic equation, on account for the boundary condition 5.12 and for the regularity of the final datum 5.13, we obtain that

$$\|\bar{r}^\gamma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c. \tag{5.19}$$

Fourth estimate: Now, let us point out that whenever $v \in \mathcal{W}_0(0,T)$ is given, a simple application of the integration by parts formula and the last of 5.13 lead to

$$\beta \mathcal{W}_0^* \langle \partial_t \bar{q}^\gamma, v \rangle_{\mathcal{W}_0} = -\beta \int_0^T \langle \partial_t v, \bar{q}^\gamma \rangle - b_2 \int_\Omega (\bar{\varphi}^\gamma(T) - \varphi_\Omega)v(T),$$

which, in turn, implies that

$$\begin{aligned} \left| \mathcal{W}_0^* \langle \partial_t \bar{q}^\gamma, v \rangle_{\mathcal{W}_0} \right| &\leq \beta \|\partial_t v\|_{L^2(0,T;V^*)} \|\bar{q}^\gamma\|_{L^2(0,T;V)} + \|b_2(\bar{\varphi}^\gamma(T) - \varphi_\Omega)\|_H \|v(T)\|_H \\ &\leq c \|v\|_{\mathcal{W}_0}, \end{aligned}$$

where the previous estimates and the continuous embedding $\mathcal{W}_0(0,T) \subset C^0([0,T];H)$ are taken into account. Dividing both sides by $\|v\|_{\mathcal{W}_0}$ and passing to the supremum, we conclude that there exists a positive constant c such that

$$\|\partial_t \bar{q}^\gamma\|_{\mathcal{W}_0^*} \leq c. \quad (5.20)$$

Fifth estimate: We are left with the task to control λ^γ which was introduced by 2.29. First, let us rewrite the equation 5.9 as

$$\begin{aligned} \lambda^\gamma &= g(\gamma) h''(\bar{\varphi}^\gamma) \bar{q}^\gamma = \beta \partial_t \bar{q}^\gamma - \partial_t \bar{p}^\gamma + \Delta \bar{q}^\gamma - \pi''(\bar{\varphi}^\gamma) \bar{q}^\gamma \\ &\quad + P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma)(\bar{r}^\gamma - \bar{p}^\gamma) - b_1(\bar{\varphi}^\gamma - \varphi_Q). \end{aligned}$$

Then, we consider the duality pairing between λ^γ and an arbitrary function $v \in \mathcal{W}_0(0,T)$ to get

$$\begin{aligned} \mathcal{W}_0^* \langle \lambda^\gamma, v \rangle_{\mathcal{W}_0} &= \beta \mathcal{W}_0^* \langle \partial_t \bar{q}^\gamma, v \rangle_{\mathcal{W}_0^*} - \int_Q \partial_t \bar{p}^\gamma v - \int_Q \nabla \bar{q}^\gamma \cdot \nabla v - \int_Q \pi''(\bar{\varphi}^\gamma) \bar{q}^\gamma v \\ &\quad + \int_Q P'(\bar{\varphi}^\gamma)(\bar{\sigma}^\gamma - \bar{\mu}^\gamma)(\bar{r}^\gamma - \bar{p}^\gamma) v - \int_Q b_1(\bar{\varphi}^\gamma - \varphi_Q) v, \end{aligned} \quad (5.21)$$

where also 5.12 and 5.13 are taken into account. Hence, due to 2.2, 2.5 and to the previous estimates, we claim that the above inequality implies that

$$\left| \mathcal{W}_0^* \langle \lambda^\gamma, v \rangle_{\mathcal{W}_0} \right| \leq c \|v\|_{\mathcal{W}_0}, \quad (5.22)$$

for a positive constant c . Indeed, bearing in mind 5.16–5.17, and 5.19–5.20, it is clear that the only term which deserves further investigation is the fifth product on the right-hand side of 5.21 that we denote by \tilde{I} . Then, let us separately prove how it can be controlled. By virtue of the Hölder inequality, the boundedness of P' and the Sobolev embedding 2.44, we conclude that

$$\begin{aligned} |\tilde{I}| &\leq c \int_Q |\bar{\sigma}^\gamma - \bar{\mu}^\gamma| |\bar{r}^\gamma - \bar{p}^\gamma| |v| \leq c \int_0^T \|\bar{\sigma}^\gamma - \bar{\mu}^\gamma\|_4 \|\bar{r}^\gamma - \bar{p}^\gamma\|_2 \|v\|_4 \\ &\leq c \int_0^T \|\bar{\sigma}^\gamma - \bar{\mu}^\gamma\|_V \|\bar{r}^\gamma - \bar{p}^\gamma\|_H \|v\|_V \leq c \|\bar{r}^\gamma - \bar{p}^\gamma\|_{L^2(0,T;H)} \|v\|_{L^2(0,T;V)}, \end{aligned} \quad (5.23)$$

where in the last inequality we also appeal to the fact that $\bar{\sigma}^\gamma$ and $\bar{\mu}^\gamma$, as solutions to 2.8–2.12, satisfy 2.16 and 2.17 and therefore they both belong to $L^\infty(0,T;V)$. So that 5.22 is shown. Lastly, dividing both sides of 5.22 by $\|v\|_{\mathcal{W}_0}$ and passing to the supremum, we obtain that there exists a positive constant c such that

$$\|\lambda^\gamma\|_{\mathcal{W}_0^*} \leq c. \quad (5.24)$$

To conclude, the application of the aforementioned estimates 5.16–5.17, 5.19–5.20, and 5.24, lead to infer the existence of some q, p, r and λ such that, as $n \nearrow \infty$, 2.31–2.34, are verified. \square

We will see that the above result will be sufficient to pass to the limit as $\gamma \searrow 0$, at least in a suitable weak framework, as rigorously described in Theorem 2.9.

Proof of Theorem 2.9. Owing to the uniform estimates pointed out in Theorem 2.8, we realize that there exists some subsequence, which is again indexed by n , for which we can pass to the limit in the inequality 5.15 to obtain that, for some limit r , 2.35 is satisfied.

Thus, we are led to show that such a limit solves a suitable adjoint problem for 1.4–1.9, and eventually prove some additional features on this system. To do that, let us multiply the equations 5.9 by an arbitrary $v \in \mathcal{W}_0(0, T)$ and integrate over Q . Accounting for the boundary conditions 5.12, and for the final ones 5.13, the application of the integration by parts, leads to the following problem consisting of a variational formulation

$$\begin{aligned} & \int_0^T \langle \partial_t v, \bar{p}^{\gamma_n} - \beta \bar{q}^{\gamma_n} \rangle - \mathcal{W}_0^* \langle \lambda^{\gamma_n}, v \rangle_{\mathcal{W}_0} + \int_Q \nabla \bar{q}^{\gamma_n} \cdot \nabla v \\ & - \int_Q \pi''(\bar{\varphi}^{\gamma_n}) \bar{q}^{\gamma_n} v + \int_Q P'(\bar{\varphi}^{\gamma_n})(\bar{\sigma}^{\gamma_n} - \bar{\mu}^{\gamma_n})(\bar{r}^{\gamma_n} - \bar{p}^{\gamma_n}) v \\ & = \int_Q b_1(\bar{\varphi}^{\gamma_n} - \varphi_Q) v + \int_\Omega b_2(\bar{\varphi}^{\gamma_n}(T) - \varphi_\Omega) v(T) \quad \text{for every } v \in \mathcal{W}_0(0, T), \end{aligned} \quad (5.25)$$

combined with the system

$$\bar{q}^{\gamma_n} - \alpha \partial_t \bar{p}^{\gamma_n} - \Delta \bar{p}^{\gamma_n} + P(\bar{\varphi}^{\gamma_n})(\bar{p}^{\gamma_n} - \bar{r}^{\gamma_n}) = 0 \quad \text{in } Q \quad (5.26)$$

$$-\partial_t \bar{r}^{\gamma_n} - \Delta \bar{r}^{\gamma_n} + P(\bar{\varphi}^{\gamma_n})(\bar{r}^{\gamma_n} - \bar{p}^{\gamma_n}) = b_3(\bar{\sigma}^\gamma - \sigma_Q) \quad \text{in } Q \quad (5.27)$$

$$\partial_n \bar{p}^{\gamma_n} = \partial_n \bar{r}^{\gamma_n} = 0 \quad \text{on } \Sigma \quad (5.28)$$

$$\alpha \bar{p}^{\gamma_n}(T) = 0, \quad \bar{r}^{\gamma_n}(T) = b_4(\bar{\sigma}^\gamma(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (5.29)$$

With the convergences 2.31–2.34 at our disposal, we would like to let $n \nearrow \infty$ to show that in the limit we get 2.36–2.40. Let us recall that, as pointed out by 4.3–4.4, we have that

$$\bar{\varphi}^{\gamma_n} \rightarrow \bar{\varphi}, \quad \bar{\sigma}^{\gamma_n} \rightarrow \bar{\sigma}, \quad \bar{\mu}^{\gamma_n} \rightarrow \bar{\mu} \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V).$$

In a similar fashion, we immediately deduce from 2.31–2.34 that, up to not relabeled subsequence, there holds

$$\bar{p}^{\gamma_n} \rightarrow p \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V)$$

$$\bar{r}^{\gamma_n} \rightarrow r \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V).$$

Therefore, even though some terms in 5.25–5.29 possess a strongly nonlinear behavior, recalling that π'' , P and P' are continuous, we have that the above strong convergences allow us to let $n \nearrow \infty$ to obtain that in the limit 2.36–2.40 is satisfied. This is the sense in which we can state that the limit (q, p, r) and λ enjoy an adjoint system corresponding to 1.4–1.9.

To conclude, let us provide some additional properties which characterize such a limit. First, let us multiply λ^{γ_n} , which is defined by 2.29, by \bar{q}^{γ_n} and integrate over Q to realize that

$$\int_Q \lambda^{\gamma_n} \bar{q}^{\gamma_n} = \int_Q g(\gamma_n) h''(\bar{\varphi}^{\gamma_n}) |\bar{q}^{\gamma_n}|^2 \geq 0 \quad \text{for every } n \in \mathbb{N},$$

where we used that g is nonnegative, that $h''(s) = \frac{2}{1-s^2}$, and the fact that $\bar{\varphi}^{\gamma_n}$, as a solution to 2.8–2.12, verifies the separation result 2.19 from which we infer that $h''(\bar{\varphi}^{\gamma_n}) > 0$. Then, by passing to the inferior limit, as $n \nearrow \infty$, we conclude the

first condition 2.41. Lastly, let us show a limit behavior that should suggest the fact that, in the limit, λ^{γ_n} tends to concentrate in the sets where $|\bar{\varphi}| = 1$. We now multiply λ^{γ_n} by $(1 - (\bar{\varphi}^{\gamma_n})^2)\Phi$, for an arbitrary regular function Φ which vanishes at zero, integrate over Q , and then pass to the limit as $n \nearrow \infty$ to find

$$L_\Phi := \lim_{n \rightarrow \infty} \int_Q \lambda^{\gamma_n} (1 - (\bar{\varphi}^{\gamma_n})^2) \Phi.$$

Moreover, exploiting the definition of λ^{γ_n} , the explicit expression of h'' , and accounting for the asymptotic properties 1.13–1.14 we are assuming, we realize that

$$L_\Phi = 2 \lim_{n \rightarrow \infty} \int_Q g(\gamma_n) \bar{q}^{\gamma_n} \Phi = 0,$$

which is exactly the condition 2.42 we are going to prove. □

6. Appendix. Here, we focus on showing a possible way to construct an approximating family of data which fulfills requirements 2.13–2.15.

Lemma 6.1. *Let $(\mu_0, \varphi_0, \sigma_0)$ be a triplet belonging to $V \times V \times V$. Then, there exists an approximating family $\{(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma)\}_\gamma$ which satisfies all the following properties*

$$\begin{aligned} &(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times W \times H^1(\Omega) \quad \forall \gamma \in (0, 1], \\ &|\varphi_0^\gamma| \leq 1 - \gamma/2 \quad \text{a.e. in } \Omega \quad \forall \gamma \in (0, 1], \\ &(\mu_0^\gamma, \varphi_0^\gamma, \sigma_0^\gamma) \rightarrow (\mu_0, \varphi_0, \sigma_0) \quad \text{strongly in } V \times V \times V \quad \text{as } \gamma \searrow 0. \end{aligned}$$

Proof. Obviously, since only existence is stated above, we only show one possible way to proceed. As the first and third variables are concerned, the choices are quite natural. As σ_0^γ we can straightforwardly take, for every γ , σ_0 itself. Secondly, it is natural to choose as μ_0^γ a suitable truncation of μ_0 , since we want μ_0^γ to remain bounded in V and to be uniformly bounded as well. So, we take μ_0^γ as the truncation at level $1/\gamma$ of μ_0 , namely the function defined by

$$\mu_0^\gamma := \begin{cases} 1/\gamma & \text{if } \mu_0 > 1/\gamma \\ \mu_0 & \text{if } |\mu_0| \leq 1/\gamma. \\ -1/\gamma & \text{if } \mu_0 < -1/\gamma \end{cases}$$

It is now a standard argument to check that $\mu_0^\gamma \rightarrow \mu_0$ strongly in V . To conclude, let us face the remaining term which will require more attention. To recover the zero normal derivative condition, we are tempted to choose as φ_0^γ the solution to the following homogeneous Neumann boundary problem

$$\begin{cases} \varphi_0^\gamma - \gamma \Delta \varphi_0^\gamma = \varphi_0 & \text{in } \Omega \\ \partial_n \varphi_0^\gamma = 0 & \text{on } \Gamma, \end{cases}$$

in which φ_0 appears as forcing term. Owing to the regularity of φ_0 , for every $\gamma \in (0, 1]$, it follows from standard results that there exists a unique solution φ_0^γ which belongs to W . Let us mention that in this framework one can also verify that $\varphi_0^\gamma \rightarrow \varphi_0$ strongly in V . However, we also care to control the absolute value of φ_0^γ by $1 - \gamma/2$, which is not guaranteed if we proceed this way.

Thus, we will follow a similar technique, but first we need to truncate the data in order to properly control the absolute value of the approximating variable. For

convenience, let us denote as $\tilde{\varphi}_0^\gamma$ the truncation of φ_0 at level $1 - \gamma/2$. Then, we consider the following problem which strictly resembles the above one

$$\begin{cases} \varphi_0^\gamma - \gamma \Delta \varphi_0^\gamma = \tilde{\varphi}_0^\gamma & \text{in } \Omega \\ \partial_n \varphi_0^\gamma = 0 & \text{on } \Gamma. \end{cases} \quad (6.1)$$

Similarly, it is easy deduce that, for every $\gamma \in (0, 1]$, the unique solution φ_0^γ to **6.1** belongs to W . Moreover, by adding to both sides of the first equation of **6.1** the term $1 - \gamma/2$, we arrive at the identity

$$(\varphi_0^\gamma + 1 - \gamma/2) - \gamma \Delta(\varphi_0^\gamma + 1 - \gamma/2) = \tilde{\varphi}_0^\gamma + 1 - \gamma/2,$$

where the right-hand side is positive by construction of $\tilde{\varphi}_0^\gamma$. Hence, the maximum principle yields that

$$\varphi_0^\gamma + 1 - \gamma/2 \geq 0, \text{ which implies } \varphi_0^\gamma \geq -1 + \gamma/2.$$

By repeating the same strategy adding to both sides the term $-1 + \gamma/2$, it is easy to conclude that $|\varphi_0^\gamma| \leq 1 - \gamma/2$. Finally, we are left with the task of showing that $\varphi_0^\gamma \rightarrow \varphi_0$ strongly in V . In this direction, we multiply the first equation of **6.1** by φ_0^γ and integrate over Ω . Using the boundary condition and estimating the right-hand side by mean of the Young inequality, we discover that

$$\int_{\Omega} |\varphi_0^\gamma|^2 + \gamma \int_{\Omega} |\nabla \varphi_0^\gamma|^2 \leq \frac{1}{2} \int_{\Omega} |\varphi_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\tilde{\varphi}_0^\gamma|^2,$$

and rearranging the terms, we obtain the boundedness of $\{\varphi_0^\gamma\}_\gamma$ in V . Thus, from weak compactness arguments, we immediately realize that $\varphi_0^\gamma \rightarrow \varphi_0$ at least weakly in V . Now, we multiply the first equation by $-\Delta \varphi_0^\gamma$ and integrate by parts to get

$$\int_{\Omega} |\nabla \varphi_0^\gamma|^2 + \gamma \int_{\Omega} |\Delta \varphi_0^\gamma|^2 = \int_{\Omega} \nabla \tilde{\varphi}_0^\gamma \cdot \nabla \varphi_0^\gamma,$$

where we also account for the homogeneous boundary condition for φ_0^γ . Due to the Young inequality we control the right-hand side as follows

$$\int_{\Omega} |\nabla \varphi_0^\gamma|^2 + \gamma \int_{\Omega} |\Delta \varphi_0^\gamma|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla \varphi_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\nabla \tilde{\varphi}_0^\gamma|^2,$$

and, adding this latter with the above estimate and rearranging the terms, we obtain that

$$\frac{1}{2} \int_{\Omega} |\varphi_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\nabla \varphi_0^\gamma|^2 + \gamma \int_{\Omega} |\nabla \varphi_0^\gamma|^2 + \gamma \int_{\Omega} |\Delta \varphi_0^\gamma|^2 \leq \frac{1}{2} \int_{\Omega} |\tilde{\varphi}_0^\gamma|^2 + \frac{1}{2} \int_{\Omega} |\nabla \tilde{\varphi}_0^\gamma|^2,$$

from which we realize that $\|\varphi_0^\gamma\|_V^2 \leq \|\tilde{\varphi}_0^\gamma\|_V^2$, and since that inequality continue to hold if we pass to the superior limit, we realize that actually $\varphi_0^\gamma \rightarrow \varphi_0$ strongly in V , as $\gamma \searrow 0$, which is the desired conclusion. \square

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