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Effect of social influence on a two party election: A Markovian multi-agent model

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***Abstract*—In digital social networks, the filtering algorithms employed by the platform management to sieve the contents shared among the users can alter the social influence intensity. In this paper, a Markov multi-agent model of opinion dynamics is used to analyze possible opinion manipulation under apparently neutral interventions on the influence intensity. We consider a two-party election whose voters, modeled as heterogeneous agents, are connected in a social network with arbitrary topology. The equations describing the variance of the vote share, both in transient and steady state, are derived. The key is the solution of the second-order marginalization problem under the form of a numerically tractable characterization of pairwise joint probabilities of the voters' opinions. In particular, these probabilities are computed by means of a Lyapunov-like matrix differential equation driven by first-order moments. This result is used to answer some important questions, like the possible nonmonotonic effect of the influence intensity on the vote volatility and the interplay of topology and individuals' stubborness to determine the electoral balance between two parties.**

***Index Terms*—Opinion dynamics, Social networks, Multi-agent systems, Markov process.**

I. Introduction

Mathematical models of opinion dynamics have been studied in sociology for more than 50 years, see e.g the pioneering works of [1], [2] and [3]. In the last decade, the advent of digital social networks induced a resurgence of interest in this topic. Among the various aspects investigated, the recent case of Cambridge Analytica [4] is a paradigmatic example of a possible strategy that employs social media and opinion influence in order to affect election results. There is therefore a clear interest for the development of models that describe how the digital platform management may affect opinion formation and evolution in an electoral competition.

The classical opinion dynamics models assume a deterministic real-valued description of the opinions of individuals, interconnected through networks with different topologies. The basic mechanism is simple: each individual opinion is modified by the opinions of her/his neighbors, according to suitable updating rules. For more details, we refer the reader to the surveys [5], [6], [7] and the book [8]. This category of deterministic models has two major shortcomings. Although well suited to treat binary opinions, that can be reduced to a single real-valued variable, it is more difficult to apply these models to multi-dimensional opinion spaces. Moreover, the assumption of a deterministic evolution may be an overly simple description of real-world social phenomena, subject to a variety of random internal and external perturbations.

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In the field of stochastic models of opinion dynamics, the Markovian multi-agent paradigm offers a good compromise between flexibility and analytical tractability. It assumes that individual opinions are random variables, taking values in a finite set, evolving in accordance with a (state-dependent) Markov chain process. Each agent has its own probabilities of changing opinion and these probabilities are affected by the neighbors' current opinions. The overall social network is thus represented as a network of interacting Markov chains. A seminal contribution was the introduction of the so-called 'influence model' [9]. More recently, the Markovian multi-agent framework has been used to investigate the effects of the intensity of social interaction between the users of digital social networks [10], [11]. In particular, the filters devised by the platform managers sieve the contents shared among the users, thus affecting the intensity of interaction on a certain topic. This intensity can be modeled by means of scalar parameters that modulate the 'opinion contagion' between neighbors. Although a social network made by a graph of Markovian agents is itself a Markov chain, it soon becomes analytically intractable for a growing number of agents. A significant achievement of [10], [11] was to show that, under mild assumptions, the joint distributions can be marginalized, making it possible to study the time evolution and the steady-state value of the probabilities of the agents' opinions.

This kind of analysis becomes increasingly difficult when moving from special network topologies and homogeneous agents to general topologies and heterogeneous agents. While results have been made available for the time evolution of single agent's probabilities, much less attention was paid to the time evolution of the joint probabilities of the opinions of pairs of agents. Far from being an abstract issue, the study of the joint probabilities has a direct impact on a very relevant question, such as the effect of the social influence intensity on electoral outcomes. Indeed, the evolution of the marginal probabilities allows one to evaluate the mean value of the vote share, but this information is insufficient to predict the election results without some assessment of the dispersion of the vote share around its mean. Note that, also in the different context of epidemic processes in complex networks, the issue of computing the joint probabilities for pairs of Markov agents is known to play a crucial role in order to go beyond basic mean-field approximations, as pointed out in [12]. Differently from what happens in epidemic models, we will show that the pairwise joint probability can be exactly computed.

The aim of the present paper is to address networks, with arbitrary topology, populated by heterogeneous agents voting in a two-party election. Besides the analytic expression for the time evolution of the expected vote share, we provide also the equations describing the variance, both in transient and steady state. The key is the solution of the second-order marginalization problem under the form of a numerically tractable characterization of pairwise joint probabilities. In particular, these probabilities are computed by means of a Lyapunov-like matrix differential equation driven by first-order moments. This result is used to answer some important questions, like the possible nonmonotonic effect of the influence intensity on the vote volatility and the interplay of topology and individuals' stubborness to determine the electoral balance between two parties.

The paper is organized as follows. After introducing the notation in Section II, the Markovian multi-agent model is briefly reviewed in Section III and the existing results on the propagation of individual probabilities are presented in Section IV. Section V is the core of the paper, as it illustrates how to compute both the transient evolution and the steady-state value of the pairwise joint probabilities. The computation requires the solution of constrained Lyapunov equations, that cannot be easily solved in closed-form. Numerical solutions for such equations, based on different methods, are then presented in Section VI. The expressions for the mean and variance of the vote share are derived in Section VII. Section VIII treats the special case of the so-called Peer Assembly, with identical agents connected through a complete graph. Section IX investigates on the monotonicity of the vote share mean and variance. In particular, sufficient conditions for the lack of monotonicity are derived. Moreover, some considerations are made on the initial sensitivity of mean and variance with respect to the influence intensity parameter. The paper ends with some concluding remarks in Section X.

II. Notation

The vector $\mathbf{1}_N$ is the N-dimensional column vector with all entries equal to 1. The i-th column of the N-th order identity matrix I_N is indicated with $e_i(N)$. A square matrix $A = [a_{ij}]$ is said to be *Metzler* if its off-diagonal entries are nonnegative, namely $a_{ij} \geq 0$ for every $i \neq j$. Let $\sigma(A)$ denote the spectrum of a Metzler matrix A. It is known , see e.g. [13], that $\max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$ is always an eigenvalue of A, called the *Perron-Frobenius eigenvalue* and denoted by λ_F, and that the corresponding eigenspace, when A is irreducible, is generated by a positive eigenvector, v_F, with $\mathbf{1}_n' v_F = 1$, called *Frobenius eigenvector*. For a square matrix A, the spectral radius, i.e. the largest absolute value of its eigenvalues, is denoted by $\rho(A)$. The symbol $\otimes$ stands for the Kronecker product. Given a set of N scalars $v^{[r]}$, indexed by a positive integer $r \in \{1, 2, \ldots, N\}$, the symbol $\mathrm{col}\{v^{[r]}\}$ denotes a vector obtained by stacking the scalars $v^{[r]}$ in a single column vector. Given a set of N square matrices $V^{[r]}$, indexed by a positive integer $r \in \{1, 2, \ldots, N\}$, the symbol $\mathrm{diag}\{V^{[r]}\}$ denotes a block-diagonal matrix with the submatrices $V^{[r]}$, $r = 1, 2, ...N$ on its diagonal. When v is a vector, $V = \mathrm{diag}(v)$ is a diagonal matrix with v on its diagonal. When V is a square matrix, $v = \mathrm{diag}(V)$ is the column vector containing the diagonal entries of V. Hence, the symbol $D = \mathrm{diag}(\mathrm{diag}(V))$ indicates that D is a diagonal matrix with the same diagonal of V. Given a set of N scalars $v^{[r]}$, indexed by a positive integer $r \in \{1, 2, \ldots, N\}$, we will use by short $\mathrm{diag}\{v^{[r]}\}$ to denote the diagonal matrix with the elements $v^{[r]}$ on the diagonal, i.e. $\mathrm{diag}\{v^{[r]}\} = \mathrm{diag}(\mathrm{col}\{v^{[r]}\})$.

The set of probability vectors, i.e. vectors with nonnegative entries that sum up to 1 is indicated as $\mathcal{P}$. Given a discrete set $\mathcal{N}$, the symbol $|\mathcal{N}|$ denotes its cardinality. The symbol $E[v]$ denotes the expectation of the random variable v. Given a random event $\mathcal{A}$, $\mathcal{I}_\mathcal{A}$ represents the indicator function of the event, namely $\mathcal{I}_\mathcal{A} = 1$ if $\mathcal{A}$ occurs, $\mathcal{I}_\mathcal{A} = 0$ otherwise. $\Pr\{\mathcal{A}\}$ will be used to denote the probability of the event $\mathcal{A}$ and $\Pr\{\mathcal{A}|\mathcal{B}\}$ is the conditional probability of $\mathcal{A}$ given the event $\mathcal{B}$

III. The model

The model was first introduced in [10] and further developed in [11], based on a network of interacting Markovian agents. The mutual interaction of agents is described through a weighted undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$, with a finite set of nodes $\mathcal{N} = \{1, 2, \ldots, N\}$ representing the agents, the set of edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ associated to reciprocal influences, and the matrix $W = [w_{rs}] \in \mathbb{R}^{N \times N}$ representing the interpersonal trustiness. For simplicity, we assume uniform trustiness, i.e.

$$w_{rs} = \begin{cases} |\mathcal{N}^{[r]}|^{-1} & , \quad (r, s) \in \mathcal{E} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{1}$$

where $\mathcal{N}^{[r]} = \{s \in \mathcal{N} : (r, s) \in \mathcal{E}\}$ denotes the set of neighbors of agent r.

The Laplacian matrix associated with the graph $\mathcal{G}$ is $L = I_N - W$. Note that $L\mathbf{1}_N = 0$. In the sequel, we will assume that the graph $\mathcal{G}$ is connected.

Each individual's opinion on a given issue at a given time may assume a value in the finite set $\mathcal{M} = \{1, 2, \ldots, M\}$. Herein, we will mainly consider the special case of binary opinions, i.e. $M = 2$.

The opinion of agent r at time $t \in \mathbb{R}$ is denoted by $\sigma^{[r]}(t) \in \mathcal{M}$, and its evolution in time is governed by a finite-state continuous-time Markov chain with transition rate matrix

$$\tilde{Q}^{[r]}(t) = Q^{[r]} + A^{[r]}(t)\,. \tag{2}$$

The matrix $Q^{[r]} \in \mathbb{R}^{M \times M}$ is the transition rate matrix of agent r when isolated. The off-diagonal entries $q_{ij}^{[r]} \geq$

$0, i \neq j$ represent the transition rates between opinions, i.e.

$$\Pr\{\sigma^{[r]}(t+dt)=j|\sigma^{[r]}(t)=i\} = q_{ij}^{[r]}dt + o(dt), \quad i \neq j\,. \tag{3}$$

The diagonal entries are defined as $q_{ii}^{[r]} = -\sum_{j=1,j\neq i}^{M} q_{ij}^{[r]}$, so that $Q^{[r]}$ is a Metzler matrix satisfying $Q^{[r]}\mathbf{1}_M = 0$. To ensure ergodicity of the Markov process, it is assumed that $Q^{[r]}$ is irreducible. For instance, this happens if $q_{ij}^{[r]} > 0, i \neq j$.

When $M = 2$ it is convenient to use the so-called (α, β)-parametrization introduced in [11]. Precisely, define

$$\alpha^{[r]} = q_{12}^{[r]} + q_{21}^{[r]} \quad , \quad \beta^{[r]} = \frac{q_{21}^{[r]}}{q_{12}^{[r]} + q_{21}^{[r]}}$$

so that

$$Q^{[r]} = \alpha^{[r]} \begin{bmatrix} -(1-\beta^{[r]}) & 1-\beta^{[r]} \\ \beta^{[r]} & -\beta^{[r]} \end{bmatrix}.$$

The stationary probability distribution associated with $Q^{[r]}$ is $\bar{\pi}^{[r]} = [\ \beta^{[r]} \quad 1-\beta^{[r]}\ ]'$. Hence, $\beta^{[r]} \in (0,1)$ can be interpreted as a bias parameter. As $\beta^{[r]}$ approaches 1, the agent opinion becomes more biased towards opinion $\sigma^{[r]} = 1$, and viceversa when $\beta^{[r]}$ approaches 0. The second parameter $\alpha^{[r]} > 0$ can be seen as a time-scale parameter measuring volatility (high values of $\alpha^{[r]}$ imply frequent opinion changes).

For a fixed t, the second term $A^{[r]}(t) \in \mathbb{R}^{M\times M}$ in (2) is a random matrix and takes into account the influence of the neighbors at time t. For $i \neq j$,

$$a_{ij}^{[r]}(t) = \lambda\eta^{[r]} \sum_{s\in\mathcal{N}^{[r]}} w_{rs}\mathcal{I}_{\sigma^{[s]}(t)=j} \tag{4}$$

$$\mathcal{I}_{\sigma^{[s]}(t)=j} = \begin{cases} 1, & \sigma^{[s]}(t)=j \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

and the elements $a_{ii}^{[r]}(t)$ are such that $A^{[r]}(t)\mathbf{1}_M = 0$. According to this interaction model, the instantaneous transition rates to opinion j increase proportionally to the weighted number of neighbors that share opinion j.

The nonnegative parameter λ reflects the interaction intensity among agents on the considered issue, possibly manipulated by the social platform through content filtering algorithms. The nonnegative parameter $\eta^{[r]}$ represents the individual influenceability of agent r. The extreme value $\eta^{[r]} = 0$ is associated with an agent which is not influenced by the others. Increasing values of $\eta^{[r]}$ indicate that agent r is more and more heavily influenced by the opinion of its neighbors. The influence mechanism is illustrated with an example in Figure 1.

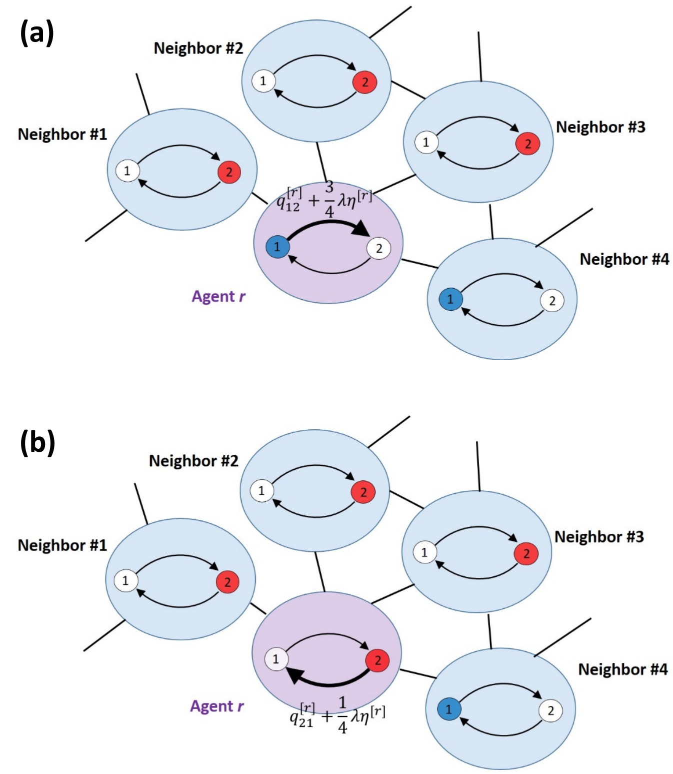

Figure 1. The influence mechanism of the Markovian multi-agent model. Panel (a): If agent r is in state 1 and three neighbors out of four are in state 2, its transition rate towards state 2 is $q_{12}^{[r]} + \frac{3}{4}\lambda\eta^{[r]}$. Panel (b): If agent r is in state 2 and one neighbor out of four is in state 1, its transition rate towards state 1 is $q_{21}^{[r]} + \frac{1}{4}\lambda\eta^{[r]}$.

IV. Propagation of individual probabilities

Let $z_r(t)$ denote the probability that agent r has opinion 1 at time t. It was shown in [11] that the vector $z(t) = [z_1(t)\, z_2(t) \ldots z_N(t)]'$ describing the probabilities of all agents obeys the following differential equation:

$$\dot{z}(t) = -(F + \lambda HL)\, z(t) + g \tag{6}$$

where $F = \text{diag}\{\alpha^{[r]}\}$, $g = \text{col}\{\alpha^{[r]}\beta^{[r]}\}$ and $H = \text{diag}\{\eta^{[r]}\}$ is the influenceability matrix. Note that $\hat{F}(\lambda) = -(F + \lambda HL)$ is a Metzler matrix.

In view of ergodicity, the solution $z(t)$ converges asymptotically to an equilibrium $\bar{z}$, which is a function of λ and will therefore referred to as $\bar{z}(\lambda)$. Of course, it results that

$$\bar{z}(\lambda) = (F + \lambda HL)^{-1}\, g\,. \tag{7}$$

Moreover, it can be shown that:

$$\bar{z}(\lambda) = \left(I_N + \lambda F^{-1}HL\right)^{-1} \bar{z}(0) \tag{8}$$

where $\bar{z}(0) = \text{col}\{\beta^{[r]}\}$ is the vector of isolated probabilities.

It was also shown in [11] that, when λ tends to ∞, all agents in the network reach a probabilistic consensus, i.e.

$$\lim_{\lambda\to\infty} \bar{z}(\lambda) = \gamma\mathbf{1}_N\,. \tag{9}$$

The common probability γ can be obtained from the equation $\left[\gamma \quad 1-\gamma\right]\hat{Q} = 0$, where $\hat{Q}$ is the transition rate matrix of a weighted average agent. More precisely $\hat{Q} = \sum_{r=1}^{N} \varphi_r Q^{[r]}$ with φ_r being the entries of the unit-sum left eigenvector φ of matrix HL, i.e. $\varphi' HL = 0, \varphi'\mathbf{1}_N = 1$.

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It turns out, see [11, Section VI], that

$$\gamma = \frac{\sum_{r=1}^{N} \varphi_r \alpha^{[r]} \beta^{[r]}}{\sum_{r=1}^{N} \varphi_r \alpha^{[r]}} \,. \tag{10}$$

The agents synchronize and tend to act as a single agent, whose opinion is a Bernoulli random variable with parameter γ.

V. Propagation of joint probabilities

The next theorem, whose proof can be found in the Appendix, provides a way to compute the time evolution of the joint probability that two distinct agents (r, s) have opinions (j, i). Notably, this result is valid in general when the number $M \geq 2$ of opinions is arbitrary, i.e $\mathcal{M} = \{1, 2, \dots M\}$.

Theorem 1. *The joint probability* $\pi_{ji}^{rs}(t) = E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)], t \geq 0, r \neq s$ *satisfies the following differential equation:*

$$\begin{aligned}
\dot{\pi}_{ji}^{rs}(t) &= \sum_{k\in\mathcal{M}} q_{kj}^{[r]} \pi_{ki}^{rs}(t) + \sum_{k\in\mathcal{M}} q_{ki}^{[s]} \pi_{jk}^{rs}(t) \\
&+ \lambda\Bigg(-(\eta^{[r]} + \eta^{[s]})\pi_{ji}^{rs}(t) \\
&+ \frac{\eta^{[r]}}{|\mathcal{N}^r|} \sum_{l\in\mathcal{N}^r} \pi_{ji}^{ls}(t) + \frac{\eta^{[s]}}{|\mathcal{N}^s|} \sum_{l\in\mathcal{N}^s} \pi_{ji}^{rl}(t) \Bigg).
\end{aligned} \tag{11}$$

Note that the number of variables in (11) is $M^2 N(N-1)/2$, since the number of unordered agent pairs is $N(N-1)/2$ and the number of opinion pairs is M^2. Hence, the computation soon becomes intractable as N and M increase.

However, in the case of binary opinions ($M = 2$) the computation of the correlation between agents can be made much simpler. Assume $M = 2$ and define $v^r(t) = \mathcal{I}_1^r(t)$. Let $V(t) = E[v(t)v(t)']$ denote the correlation matrix of vector $v(t) = \text{col}\{v^r(t)\}$. The diagonal entry $V_{rr}(t)$ represents $E[v^r(t)]$, i.e. the probability that agent r has opinion 1 at time t. Hence $V_{rr}(t) = z_r(t)$, where $z_r(t)$ was defined in Section IV. On the contrary, the off-diagonal entries $V_{rs}(t)$ represent $E[v^r(t)v^s(t)]$, i.e. the probability that the agents r and s share opinion 1 at time t.

Notice that the matrix $V(t)$ fully characterizes the second-order properties of the process, because all the joint probabilities $\pi_{ij}^{rs}(t)$ can be obtained from $V(t)$ as follows:

$$\pi_{11}^{rs}(t) = V_{rs}(t) \tag{12a}$$
$$\pi_{12}^{rs}(t) = V_{rr}(t) - V_{rs}(t) \tag{12b}$$
$$\pi_{21}^{rs}(t) = V_{ss}(t) - V_{rs}(t) \tag{12c}$$
$$\pi_{22}^{rs}(t) = 1 - V_{rr}(t) - V_{ss}(t) + V_{rs}(t)\,. \tag{12d}$$

The time evolution of the correlation matrix $V(t)$ is described in the following theorem.

Theorem 2. *The entries of the symmetric matrix* $V(t), t \geq 0$, *satisfy the following differential equations:*

$$\begin{aligned}
\dot{V}_{rs}(t) &= (q_{11}^{[r]} - q_{21}^{[r]})V_{rs}(t) + (q_{11}^{[s]} - q_{21}^{[s]})V_{rs}(t) \\
&+ q_{21}^{[r]} V_{ss}(t) + q_{21}^{[s]} V_{rr}(t) \\
&+ \lambda\Bigg(-(\eta^{[r]} + \eta^{[s]})V_{rs}(t) + \frac{\eta^{[r]}}{|\mathcal{N}^r|} \sum_{l\in\mathcal{N}^r} V_{ls}(t) \\
&\quad + \frac{\eta^{[s]}}{|\mathcal{N}^s|} \sum_{l\in\mathcal{N}^s} V_{rl}(t) \Bigg), \quad r \neq s
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{V}_{rr}(t) &= (q_{11}^{[r]} - q_{21}^{[r]})V_{rr}(t) + q_{21}^{[r]} \\
&+ \lambda\eta^{[r]}\Bigg(-V_{rr}(t) + \frac{1}{|\mathcal{N}^r|} \sum_{l\in\mathcal{N}^r} V_{ll}(t) \Bigg).
\end{aligned} \tag{14}$$

Proof. The proof of equation (13) is immediate by recalling Theorem 1 and using (12). Equation (14) corresponds to the time evolution of the probability $z_r(t) = V_{rr}(t)$ and is consistent with (6). □

According to (13) there are $N(N-1)/2$ correlation functions for all pairs of agents, while (14) contains N probabilities of being in opinion 1 for each single agent.

The notation can be made more compact using (6) and the definitions of matrices F, H, L and vector g introduced in Section IV. More precisely, letting $z(t)$ be the solution of (6) and $\hat{F}(\lambda) = -(F + \lambda HL)$, the correlation function $V(t)$ is the solution of the following linear differential Lyapunov-like equation:

$$\dot{V}(t) = \hat{F}(\lambda)V(t) + V(t)\hat{F}(\lambda)' + gz(t)' + z(t)g' + D(V(t)) \tag{15}$$

where the diagonal matrix $D(V(t))$ is introduced so as to enforce that $\text{diag}(V(t)) = z(t), \forall t$. More precisely,

$$\begin{aligned}
D(V) &= \text{diag}(\hat{F}(\lambda)\text{diag}(V) + g) \\
&\quad -\text{diag}\Big(\text{diag}(\hat{F}(\lambda)V + V\hat{F}(\lambda) \\
&\quad +g(\text{diag}(V))' + \text{diag}(V)g')\Big)\,.
\end{aligned}$$

The off-diagonal terms of the Lyapunov equation obtained from (15) when $D(V(t)) = 0$ are consistent with (13). The presence of $D(V(t))$ accounts for the different time evolution of the diagonal entries, that obeys (14).

The steady-state correlation matrix $\bar{V}(\lambda)$ is the solution of the algebraic Lyapunov-like equation:

$$0 = \hat{F}(\lambda)\bar{V}(\lambda) + \bar{V}(\lambda)\hat{F}(\lambda)' + g\bar{z}(\lambda)' + \bar{z}(\lambda)g' + \bar{D}(\lambda) \tag{16}$$

where $\bar{D}(\lambda) = D(\bar{V}(\lambda))$.

More details on the computational tools available to solve these constrained Lyapunov equations will be provided in Section VII.

It is interesting to consider the steady-state correlation matrix $\bar{V}(\infty)$ corresponding to the limit case when $\lambda \to \infty$. In such a case, all random processes $v^r(t)$ converge to the same Bernoulli random variable with parameter γ, so that $E[v^r(t)] = E[v^r(t)v^s(t)] = \gamma, \forall r, s$, and

$$\lim_{\lambda\to\infty} \bar{V}(\lambda) = \gamma \mathbf{1}_N \mathbf{1}_N' \tag{17}$$

where γ is the common probability defined in Section IV. A formal proof of this result is provided in the Appendix. Note that all entries of the steady-state correlation matrix coincide with the entries of the steady-state probability $\bar{z}(\infty)$. In other words, irrespective of the topology and the agents' parameters (bias, volatility and influenceability), the correlation of the agents' opinions is so high that the probability of any pair of agents r and s having different opinions tends to zero. In the social network literature, this kind of probabilistic synchronization is known as *herding behaviour*.

VI. Computational issues

The computation of the correlation matrix $V(t)$, both in the transient and at steady-state, is not direct as the solutions of the Lyapunov equations (15) and (16) are constrained on the diagonal. In this Section, we provide numerical methods to evaluate them efficiently.

A. Transient behaviour of the correlation matrix

A simple way to approximate the time-behaviour of $V(t)$ is to rely on Euler discretization of (15) combined with diagonal reset. The numerical procedure is as follows.

First compute $z(t)$ from (6), for $t \in [0, T]$. Then, given a sufficiently small integration step h, the solution $V(t), t \in [0, T]$, of (15) can be approximated by means of the following recursion:

$$\begin{aligned} V^*(t+h) &= V(t) + h(\hat{F}(\lambda)V(t) + V(t)\hat{F}(\lambda)' \\ &\quad + gz(t)' + z(t)g') && (18) \\ V(t+h) &= V^*(t+h) - \mathrm{diag}(\mathrm{diag}(V^*(t+h))) \\ &\quad + \mathrm{diag}(z(t+h)) \,. && (19) \end{aligned}$$

B. Steady-state correlation matrix

In this subsection, three different methods to calculate $\bar{V}(\lambda)$ from eq. (16) are illustrated.

1) Asymptotic: The steady-state correlation $\bar{V}(\lambda)$ is obtained by calculating the transient behaviour of $V(t), t \in [0, T]$, with T sufficiently large.

2) Closed-form: Consider the algebraic Lyapunov equation (16), where the unknowns are the off-diagonal entries of $\bar{V}(\lambda)$ and the diagonal entries of $\bar{D}(\lambda)$, while the diagonal of $\bar{V}(\lambda)$ is constrained to be equal to $\bar{z}(\lambda)$. Since $\bar{D}(\lambda)$ is diagonal, it can be written as $\bar{D}(\lambda) = \sum_{i=1}^N d_i(\lambda)e_i(N)e_i(N)'$, where the scalars $d_i(\lambda)$ are the diagonal entries of $\bar{D}(\lambda)$. By superposition, the solution $\bar{V}(\lambda)$ of (16) can be expressed as

$$\bar{V}(\lambda) = P(\lambda) + \sum_{i=1}^{N} d_i(\lambda)X^{[i]}(\lambda) \tag{20}$$

where $P(\lambda)$ is the solution of:

$$0 = \hat{F}(\lambda)P(\lambda) + P(\lambda)\hat{F}(\lambda)' + g\bar{z}(\lambda)' + \bar{z}(\lambda)g'$$

and $X^{[i]}(\lambda), i \in \mathcal{N}$, is the solution of:

$$0 = \hat{F}(\lambda)X^{[i]}(\lambda) + X^{[i]}(\lambda)\hat{F}(\lambda)' + e_i(N)e_i(N)' \,.$$

Notice that, in view of (7), $P(\lambda) = \bar{z}(\lambda)\bar{z}(\lambda)'$. Now, define $Y(\lambda) \in \mathbb{R}^{N\times N}$ as

$$Y(\lambda) = [\mathrm{diag}(X^{[1]}(\lambda)) \;\; \mathrm{diag}(X^{[2]}(\lambda)) \;\; \cdots \;\; \mathrm{diag}(X^{[N]}(\lambda))] \,. \tag{21}$$

By considering (20) and letting $p(\lambda) = \mathrm{diag}(P(\lambda))$, the constraint on the diagonal entries of $\bar{V}(\lambda)$ entails that:

$$\begin{aligned} \mathrm{diag}(\bar{V}(\lambda)) &= p(\lambda) + \sum_{i=1}^{N} d_i(\lambda)\mathrm{diag}(X^{[i]}(\lambda)) \\ &= p(\lambda) + Y(\lambda)\mathrm{diag}(\bar{D}(\lambda)) = \bar{z}(\lambda) \,. \end{aligned}$$

Therefore, $d(\lambda) = \mathrm{diag}(\bar{D}(\lambda))$ is given by $d(\lambda) = Y(\lambda)^{-1}(\bar{z}(\lambda) - p(\lambda))$. Observe that this vector can be obtained from $\bar{z}(\lambda)$ and $Y(\lambda)$. Then the steady-state correlation matrix $\bar{V}(\lambda)$ is computed in closed-form as the unique solution of the Lyapunov equation (16) with $\bar{D}(\lambda) = \mathrm{diag}(d(\lambda))$.

This method requires the solution of N Lyapunov equations with unknowns of size $N \times N$, in order to compute the matrices $X^{[i]}(\lambda)$, the inversion of matrix $Y(\lambda)$ of size $N \times N$ to obtain $\bar{D}(\lambda)$ and finally the solution of one more Lyapunov equation with unknown $\bar{V}(\lambda)$ of size $N \times N$.

3) Iterative: The following algorithm provides an iterative method to find $\bar{V}(\lambda)$ and $\bar{D}(\lambda)$ satisfying (16) with the constraint $\mathrm{diag}(\bar{V}(\lambda)) = \bar{z}(\lambda)$. In the sequel, we use the notation $\tilde{F}(\lambda) = \mathrm{diag}(\mathrm{diag}(\hat{F}(\lambda)))$.

1. Select a sufficiently small step length $h > 0$ and a precision parameter $\epsilon > 0$.
2. Start from an initial guess D^0 of the unknown diagonal matrix $\bar{D}(\lambda)$. Set the iteration counter to $k = 0$.
3. Compute the solution V^k of the Lyapunov equation:
$$0 = \hat{F}(\lambda)V^k + V^k\hat{F}(\lambda)' + g\bar{z}(\lambda)' + \bar{z}(\lambda)g' + D^k \,.$$
4. Update D^k according to the iteration:
$$D^{k+1} = D^k + 2h\tilde{F}(\lambda)\left(\mathrm{diag}(\mathrm{diag}(V^k)) - \mathrm{diag}(\bar{z}(\lambda))\right) .$$
5. If $\|D^{k+1} - D^k\| < \epsilon$ quit, otherwise set $k = k+1$ and go to step 3.

The matrices V^k and D^k returned by the algorithm in the last step are estimates of $\bar{V}(\lambda)$ and $\bar{D}(\lambda)$, respectively.

The convergence of the algorithm is guaranteed if and only if

$$h < h^* = \rho^{-1}(-\tilde{F}(\lambda)Y(\lambda)) \tag{22}$$

where, since the matrix $-\tilde{F}(\lambda)Y(\lambda)$ is positive, the spectral radius ρ coincides with its Perron-Frobenius eigenvalue. As a matter of fact, by letting $d^k = \mathrm{diag}(D^k)$, it can be shown that the iteration in Step 4 is equivalent to:

$$d^{k+1} = \left(I_N + 2h\tilde{F}(\lambda)Y(\lambda)\right)d^k - 2h\tilde{F}(\lambda)\left(\bar{z}(\lambda) - p\right) .$$

Such a vector iteration converges to an equilibrium if and only if the matrix $I_N + 2h\tilde{F}(\lambda)Y(\lambda)$ is Schur stable. In view of positivity of $-h\tilde{F}(\lambda)Y(\lambda)$, this condition is equivalent to Schur stability of $-h\tilde{F}(\lambda)Y(\lambda)$, and the condition (22) directly follows.

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It is interesting to notice that, when $\lambda \to \infty$, the bound h^* on the iteration step length tends to zero, so preventing convergence.

C. Computing time comparison

The three different methods to calculate $\bar{V}(\lambda)$ are now compared in terms of required computing time. Precisely, the algorithms have been implemented in Matlab and run on networks of different size N with different values of the interaction intensity parameter λ. All networks are described by a complete graph and the agents' parameters $\alpha^{[r]}$, $\beta^{[r]}$, $\eta^{[r]}$, are randomly extracted in each experiment from a uniform distribution in the interval $[0, 1]$.

For the asymptotic method 1, the integration step h is appropriately adapted as a nonincreasing function of λ ($h = \min(0.1, 0.1/\lambda)$) and the algorithm is terminated when the norm of $V(t+h) - V(t)$ is below the accuracy threshold $\epsilon = 0.001$. In the iterative method 3, the step length and the precision parameter have been set to $h = 0.1$ and $\epsilon = 0.01$, respectively.

The results of the experiments are reported in Figure 2. It is apparent that the performance of both methods 2 and 3 are relatively insensitive to the value of λ, while the required cpu-time increases rapidly with the size of the network. Both these methods are based on the solution of a sequence of Lyapunov equations with a matrix unknown of size $N \times N$. It is then expected that the computational burden is proportionial to N^3. The cpu-time of method 2 is generally longer than method 3 since the construction of the $Y(\lambda)$ matrix in the closed-form method always calls for the solution of N Lyapunov equations, whereas method 3 solves a single Lyapunov equation at each iteration, and the required number of iterations to achieve convergence is typically smaller than the number of agents N.

On the contrary, method 1, which is based on a time-discretization is obviously affected by the value of λ. Larger values of the interaction intensity speed up the time-dynamics and require smaller integration steps. Note that the transient behaviour of $V(t)$ is essentially governed by the eigenvalues of matrix $\hat{F}(\lambda) = -(F + \lambda HL)$, which, for very small values of λ, slightly depart from those of matrix F, that represents the isolated agents dynamics. This explains why for moderate λ (see top panels of Figure 2) the performance curve is quite irregular, reflecting the randomness of the volatility parameters $\alpha^{[r]}$.

VII. Vote share mean and variance

We are now in a position to exploit the results on joint probabilities in order to characterize the distribution of the vote share. Precisely, define $n_1(t)$ as the random process of the number of agents sharing opinion 1 at time t, and the vote share $s_1(t) = n_1(t)/N = \mathbf{1}'_N v(t)/N$ as the corresponding fraction. Since $E[v(t)] = z(t)$, the mean vote share at time t is given by

$$\mu_{s_1}(t) = E[s_1(t)] = \frac{1}{N}\mathbf{1}'_N z(t)\,. \qquad (23)$$

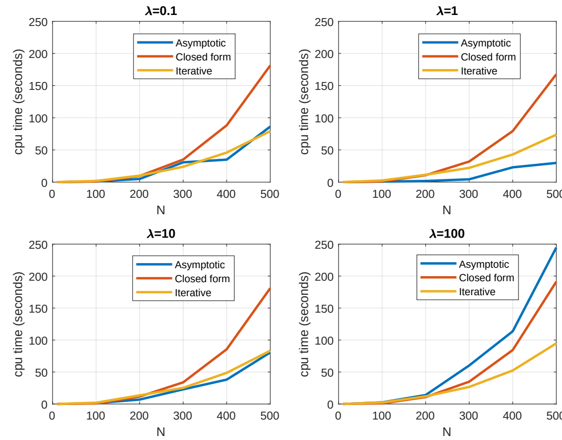

Figure 2. Computing times required to calculate $\bar{V}(\lambda)$ with different values of N and λ and different methods.

The second-order moment of the vote share is

$$\begin{aligned} E[s_1^2(t)] &= \frac{1}{N^2}E[\mathbf{1}'_N v(t)v(t)'\mathbf{1}_N] \\ &= \frac{1}{N^2}\mathbf{1}'_N V(t)\mathbf{1}_N = \frac{1}{N^2}\sum_r\sum_s V_{rs}(t)\,. \end{aligned} \qquad (24)$$

The variance of the vote share is

$$\begin{aligned} \sigma^2_{s_1}(t) &= Var[s_1(t)] = E[s_1^2(t)] - \mu^2_{s_1} \\ &= \frac{1}{N^2}\sum_r\sum_s (V_{rs}(t) - V_{rr}(t)V_{ss}(t))\,. \end{aligned} \qquad (25)$$

Since the process $s_1(t)$ is asymptotically stationary and ergodic, its steady-state mean and variance only depend on the interaction intensity λ and can be evaluated as:

$$\bar{\mu}_{s_1}(\lambda) = \frac{1}{N}\mathbf{1}'_N \bar{z}(\lambda) \qquad (26)$$

$$\bar{\sigma}^2_{s_1}(\lambda) = \frac{1}{N^2}\sum_r\sum_s (\bar{V}_{rs}(\lambda) - \bar{V}_{rr}(\lambda)\bar{V}_{ss}(\lambda))\,. \qquad (27)$$

Note that, in view of (9), (17), when $\lambda \to \infty$, it results that $\lim_{\lambda\to\infty} \bar{\mu}_{s_1}(\lambda) = \gamma$, $\lim_{\lambda\to\infty} \bar{\sigma}^2_{s_1}(\lambda) = \gamma(1-\gamma)$, where the parameter γ is defined in (10). In other words, the entire network asymptotically behaves like a single Markovian agent and $s_1(t)$ is a Bernoulli process with parameter γ.

VIII. Peer Assembly

A special case of the model presented in Section III is the so-called *Peer Assembly* (PA) model, already discussed in [10]. In a PA model, the social network is composed by identical individuals, with binary opinions, sharing the same isolated transition-rate matrix Q and connected by a complete graph. Due to the inherent indistinguishability of the agents, the network opinion dynamics can be analyzed by means of a classical birth-death Markov process. Hence, closed-form results for the vote share mean and variance can be worked out. In this Section, we aim to compare the theoretical results reported in [10] with those based on the methods developed in the present paper.

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The main results derived in [10] about the PA model with unbiased influence (i.e. λ is opinion-independent, as assumed in the present paper) are as follows:
(i) The time-evolution (and the steady-state value) of the marginal probability distribution of each single agent does coincide with the time-evolution (and the steady-state value) of the probability distribution of the isolated agent. In turn, the mean vote share $s(t)$ is not affected by the interaction between agents (see Proposition 3 of [10]).
(ii) The bivariate distribution of any couple of agents is obtained from the solution of an affine differential equation (see Theorem 3 of [10]).
(iii) The steady-state variance of the vote share is an increasing function of the interaction intensity parameter λ (see Theorem 4 of [10]).

The results (ii) and (iii) can be revisited and complemented in the light of the theory developed in the previous Sections. In a PA model, with unitary influenceability, it results that

$$Q^{[r]} = Q = \alpha \begin{bmatrix} -(1-\beta) & 1-\beta \\ \beta & -\beta \end{bmatrix}, \quad \eta^{[r]} = \eta = 1, \forall r\,.$$

Since all agents in a PA network are not distinguishable, they all share the same probability $z_r(t)$ of a generic agent r. Moreover, all pairs of agents are not distinguishabe as well. So, the correlation $V_{rs}(t), r \neq s$, of a generic pair is identical for all pairs. Finally recall that $V_{rr}(t) = z_r(t)$. Then, for any pair of agents (r,s), (13) becomes:

$$\begin{aligned} \dot{V}_{rs}(t) &= -2\alpha V_{rs}(t) + 2\alpha\beta z_r(t) - 2\lambda V_{rs}(t) \\ &\quad + \frac{\lambda}{N-1}\left(2(N-2)V_{rs}(t) + 2z_r(t)\right) \\ &= -V_{rs}(t)\left(2\alpha + \frac{2\lambda}{N-1}\right) + 2z_r(t)\left(\alpha\beta + \frac{\lambda}{N-1}\right)\,. \end{aligned} \tag{28}$$

By recalling that the joint probabilities are given by equations (12), it is then easy to show that (28) is in perfect agreement with the formula for the time evolution of joint probabilities provided in Theorem 3 of [10].

We are now in a position to derive a simple formula for the time evolution of the vote share variance in a PA model. This result is new and is based on (6), (28), and the observation that, in a PA model,

$$\begin{aligned} \sigma^2_{s_1}(t) &= \frac{1}{N^2} E\left[\left(\sum_{r=1}^{N} \mathcal{I}_{\sigma^{[r]}(t)=1}\right)^2\right] - \mu^2_{s_1}(t) \\ &= \frac{1}{N} z_r(t) + \frac{N-1}{N} V_{rs}(t) - z_r^2(t)\,. \end{aligned}$$

Theorem 3. *In a PA network of size N, the time evolution of $\sigma^2_{s_1}(t)$ can be computed as the output of the following dynamical system:*

$$\begin{aligned} \begin{bmatrix} \dot{V}_{rs}(t) \\ \dot{z}_r(t) \end{bmatrix} &= \begin{bmatrix} -2\left(\alpha + \frac{\lambda}{N-1}\right) & 2\left(\alpha\beta + \frac{\lambda}{N-1}\right) \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} V_{rs}(t) \\ z_r(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \alpha\beta \end{bmatrix} \end{aligned} \tag{29}$$

$$\sigma^2_{s_1}(t) = \frac{N-1}{N} V_{rs}(t) + z_r(t)\left(\frac{1}{N} - z_r(t)\right)\,. \tag{30}$$

The equilibrium state of system (29), (30) is:

$$z_r = \beta \quad , \quad V_{rs} = \frac{\beta\left(\lambda + \alpha\beta(N-1)\right)}{\lambda + \alpha(N-1)}$$

so that the steady-state variance, after tedious but simple manipulation, can be expressed as:

$$\bar{\sigma}^2_{s_1} = \frac{1}{N}\beta(1-\beta)\left(1 + \frac{\lambda(N-1)}{\lambda + \alpha(N-1)}\right) \tag{31}$$

which is consistent with the formula derived in Theorem 4 of [10] by means of a different rationale. It is remarkable that such a variance is an increasing function of the interaction intensity λ and it varies from the variance $\beta(1-\beta)/N$ of the non-interacting case ($\lambda = 0$) to the value $\beta(1-\beta)$ associated to the variance of a single isolated agent (when $\lambda \to \infty$). The latter case corresponds to a herding phenomenon, with all agents moving unanimously from opinion 1 to opinion 2 and viceversa.

IX. Monotonicity of mean and variance

It is obviously of great interest to be able to predict how changes of the social influence are going to affect the probability distribution of the vote share. Indeed, knowledge of such a distribution makes it possible to compute the probability of winning the elections. In previous sections the effects of the social influence parameter λ on both mean and variance have been investigated and quantitatively characterized. While interventions on λ that influence the mean have a simple and direct effect on the election outcome, it is worth observing that there is also scope for manipulation of the variance. For instance, the majority party (according to intentions of vote) takes advantage from a variance reduction that increases the probability of an outcome close to the mean value. On the contrary, the minority party could benefit from a variance increase that implies a larger volatility of the votes. Hence the interest for investigating the sensitivity of mean and variance with respect to λ.

A. Monotonicity of the mean

Concerning the monotonicity of the mean vote share $\bar{\mu}_{s1}(\lambda)$ with respect to λ, in our previous work [11] it was shown that a nonmonotonic behavior may occur even in simple situations. In order to better understand this phenomenon, it is convenient to study the derivative with respect to λ of the function $\bar{\mu}_{s1}(\lambda)$ given in (26). In particular, the following Proposition provides a closed-form expression, based on model data only, for the derivative at $\lambda = 0$, thus quantifying the initial sensitivity to the social influence parameter λ.

Proposition 1. *Let*

$$\tilde{\beta}^{[r]} = \frac{\eta^{[r]}}{\alpha^{[r]}} \sum_{k=1}^{N} L_{rk}\beta^{[k]}\,. \tag{32}$$

It holds that

$$\frac{d\bar{\mu}_{s1}(\lambda)}{d\lambda}|_{\lambda=0} = -\frac{1}{N}\sum_{r=1}^{N} \tilde{\beta}^{[r]}\,. \tag{33}$$

The above result easily follows by observing that $\bar{z}_r(0) = \beta^{[r]}$, $\frac{dz_r}{d\lambda}|_{\lambda=0} = -\tilde{\beta}^{[r]}$. Here we provide a sufficient condition for the lack of monotonicity of the mean, based on the initial value $\bar{\mu}_{s1}(0)$, the asymptotic value $\bar{\mu}_{s1}(\infty) = \gamma$, see (10), and the initial derivative.

Theorem 4. *If*

$$\sum_{r=1}^{N}(\gamma - \beta^{[r]}) \sum_{r=1}^{N} \tilde{\beta}^{[r]} > 0$$

then $\bar{\mu}_{s_1}(\lambda)$ is not monotonic.

Proof. The condition is equivalent to

$$(\bar{\mu}_{s_1}(\infty) - \bar{\mu}_{s_1}(0))\frac{d\bar{\mu}_{s_1}(\lambda)}{d\lambda}|_{\lambda=0} < 0$$

i.e. the variation of the differentiable function $\bar{\mu}_{s_1}(\lambda)$ over $[0, \infty)$ and its initial derivative have opposite sign. This is sufficient to prove that $\bar{\mu}_{s_1}(\lambda)$ is not monotonic. □

A simple 4-agent example satisfying the condition of Theorem 4 was presented in Example 1 of [11].

B. Monotonicity of the variance

The next issue is to investigate on the possible monotonicity property of the variance. As recalled in the previous section, a closed-form expression (31) for the variance is available for the special case of the Peer Assembly, showing that the variance is a monotonically increasing function of λ and, for $\lambda \to \infty$, tends to the variance of the herd's vote. Since the herding behaviour is not restricted to the Peer Assembly, but is a general feature for $\lambda \to \infty$, one would be tempted to conjecture that monotonicity of the variance may hold in general. For instance, consider the following two examples.

Example 1. *The social network is composed by $N = 100$ individuals, with $\alpha^{[r]} = 1, \forall r$, $\beta^{[r]}$ randomly extracted from a uniform distribution with support $[0.4, 1]$ and $\eta^{[r]}$ randomly extracted from a uniform distribution with support $[0, 1]$. The interaction graph is a Watts-Strogatz small world network, see [14], with parameters $k = 10$ and $p = 0.2$. The theoretical steady-state values of the mean $\bar{\mu}_{s1}$ and the variance $\bar{\sigma}^2_{s1}$ as functions of the influence intensity λ being varied from 0 to 500 are shown in Figure 3. From the plots, it is observed that the mutual increased interaction reduces the bias towards opinion 1, starting from $\bar{\mu}_{s1}(0) = 0.6993$. The initial decrease of the mean is in accordance with the formula (33) of Proposition 1, that, in this case, predicts a negative initial derivative. Afterwards, the mean is monotonic decreasing towards the asymptotic value $\bar{\mu}_{s1}(\infty) = \gamma = 0.6644$. As for the variance, the increase in the cross-correlation generates a higher variance in the vote share, with a monotonic pattern going from the initial value $\bar{\sigma}^2_{s1}(0) = 0.00182$ to the asymptotic value $\bar{\sigma}^2_{s1}(\infty) = \gamma(1-\gamma) = 0.2230$.*

Repeated experiments with the same network model, but with different samples of the random parameters, always lead to a monotonically increasing curve for the variance, while the behaviour of the mean highly depended on the asymptotic value γ being smaller or larger than the initial value.

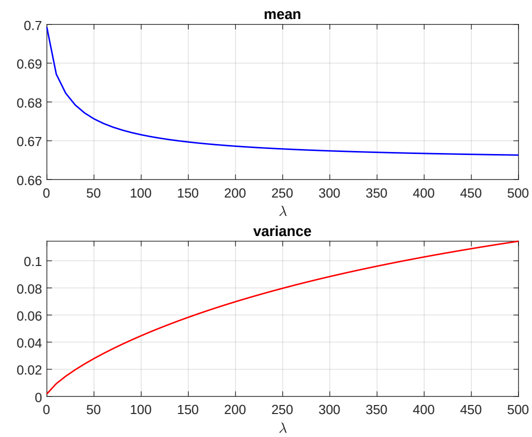

Figure 3. Mean and variance of the vote share as functions of λ in Example 1.

Example 2. *Consider a star network composed by $N = 30$ agents, all sharing the same value of $\alpha^{[r]} = 1, \forall r$. The hub node $r = 1$ is highly biased towards opinion 1 with $\beta^{[1]} = 0.99$ and strongly stubborn with $\eta^{[1]} = 1e-6$, while the peripheral nodes are neutral, $\beta^{[r]} = 0.5, r = 2, 3, \dots, N$, and more influenceable $\eta^{[r]} = 1$. The effect of varying λ on the steady-state mean and variance is displayed in Figure 4. While the mean is steadily increasing with λ, due to the growing driving effect of the hub node on the peripheral nodes, the variance exhibits an oscillating pattern. The curve starts from $\bar{\sigma}^2_{s1}(0) = 8.1e{-}3$ and tends asymptotically to $\bar{\sigma}^2_{s1}(\infty) = \gamma(1-\gamma) = 9.9e-3$. For small values of λ, the oscillation can be explained as the effect of the interplay between two contrasting forces. On one side, when the correlation between agents is negligible and the influence of the hub node on the rest of the network is small, the variance tends to decrease as the mean vote share is deviating from the initial value, close to 0.5 (note that, for a population of independent identical Bernoulli agents, the variance would be maximized in correspondence of a mean value equal to 0.5). On the other hand, the increase of the correlation for larger values of the social influence produces a significant growth of the variance.*

Having shown that the variance may be not monotonic with respect to λ, we can further inquire the issue by studying its derivative at $\lambda = 0$, i.e. the initial sensitivity of the variance with respect to λ.

Proposition 2. *Let $\tilde{\beta}^{[r]}$ be defined as in (32). Moreover, let*

$$\tilde{v}^{[r]} = \eta^{[r]} \sum_{k=1, k\neq r}^{N} L_{rk} \frac{\beta^{[k]}(1-\beta^{[k]})}{\alpha^{[r]} + \alpha^{[k]}} . \tag{34}$$

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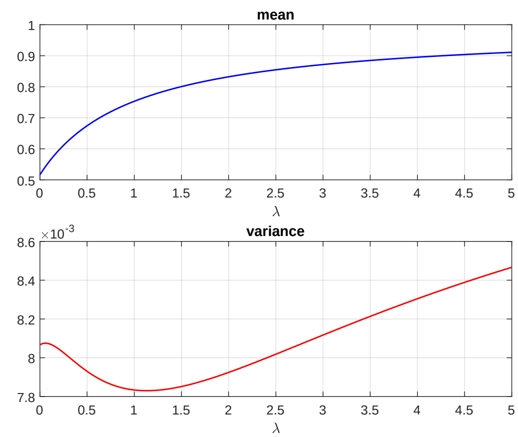

Figure 4. Mean and variance of the vote share as functions of λ in Example 2.

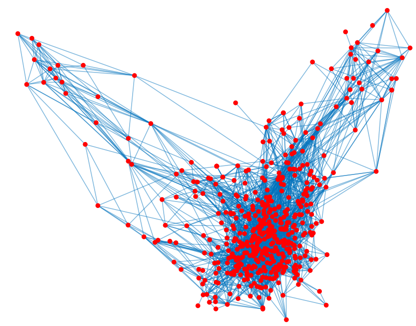

Figure 5. Graph of the social network considered in Example 3.

It holds that

$$\frac{d\bar{\sigma}_{s1}^2}{d\lambda}|_{\lambda=0} = -\frac{1}{N^2}\sum_{r=1}^{N}\tilde{\beta}^{[r]}(1-2\beta^{[r]}+2\tilde{v}^{[r]})\,. \tag{35}$$

Proof. A simple yet cumbersome computation (detailed in the Appendix) shows that

$$\frac{d\bar{V}_{rs}}{d\lambda}|_{\lambda=0} = -(\tilde{\beta}^{[r]}\beta^{[s]}+\tilde{\beta}^{[s]}\beta^{[r]}),\ r\neq s \tag{36a}$$

$$\frac{d\bar{V}_{rr}}{d\lambda}|_{\lambda=0} = -\tilde{\beta}^{[r]}\,. \tag{36b}$$

Then, by exploiting (27) the conclusion follows. □

We now provide a sufficient condition for the lack of monotonicity of the variance. To this purpose, recall that $\bar{\sigma}_{s_1}^2(0)=\frac{1}{N^2}\sum_r\beta^{[r]}(1-\beta^{[r]})$ and $\bar{\sigma}^2(\infty)=\gamma(1-\gamma)$.

Theorem 5. *If*

$$(\gamma(1-\gamma)-\frac{1}{N^2}\sum_{r=1}^{N}\beta^{[r]}(1-\beta^{[r]}))\sum_{r=1}^{N}\tilde{\beta}^{[r]}(1-2\beta^{[r]}+2\tilde{v}^{[r]})>0$$

then $\bar{\sigma}_{s_1}^2(\lambda)$ is not monotonic.

Proof. The condition is equivalent to

$$(\bar{\sigma}_{s_1}^2(\infty)-\bar{\sigma}_{s_1}^2(0))\frac{d\bar{\sigma}_{s_1}^2(\lambda)}{d\lambda}|_{\lambda=0}<0$$

so implying that $\bar{\sigma}_{s_1}^2(\lambda)$ is not monotonic, by repeating the same argument as in the proof of Theorem 4. □

Example 3. *In this example, we consider a population of 500 individuals, out of which* 40% *supporting Party 1 and* 60% *supporting Party 2, connected in a network artificially generated with the algorithm of [15]. The artificial network is representative of a real-world social network, that is typically assortative, i.e., highly connected nodes tend to connect to other highly connected nodes, and has broad degree distributions. The graph of the network is shown in Figure 5*

Supporters of Party 1 have parameters $\alpha=1$, $\beta=0.9$, $\eta=0.1$ (strong bias in favour of opinion 1 and low influenceability). Supporters of Party 2 have parameters $\alpha=1$, $\beta=0.1$, $\eta=1$ (strong bias in favour of opinion 2 and high influenceability).

Three experiments were performed. In the first one, the agents were randomly assigned to the network nodes. In the second (respectively third) experiment, the agents of Party 1 (respectively 2) were placed in the key nodes of the network, i.e. the nodes with higher degree.

Figure 6, left and center panel, shows the dependence on λ of the mean and variance of the vote share of Party 1. Note the monotonically increasing behaviour of the mean in all experiments. This means that an increase of λ always plays in favour of the less influenceable Party 1. In experiments 1 and 2 (blue and green line, respectively), Party 1 reaches the majority for moderate values of λ, while in experiment 3 the occupation of the key nodes by supporters of Party 2, makes the climb much harder. The pattern of the variance in the center panel is increasing as well, though less predictable. Finally, the third panel describes the effect of λ on the probability that Party 1 wins the election, obtained by assuming a Gaussian distribution of the vote share. The different curves in the three experiments behave as intuitively expected.

In experiment 3 with $\lambda=1$, we also derived a sample estimate of the vote share distribution by taking samples from a single long Monte Carlo realization of the vote share, discarding the initial transient. Such a distribution is shown in Figure 7 compared with the Gaussian distribution based on theoretical mean and variance. A good agreement is evident. The probability of win computed by means of the sample realization is 0.0566 against the value 0.0549 predicted by the Gaussian distribution for $\lambda=1$ (see Figure 6, red curve). Figure 8 displays the boxplots of the sample vote share for different values of λ in experiment 3. The results are in good accordance with the theoretical values of Figure 6. In particular, the theoretical mean, drawn in red, provides a good approximation of the sample median value.

X. Concluding remarks

In our previous paper [10], the Markovian agent model was used to investigate the effect of a biased management

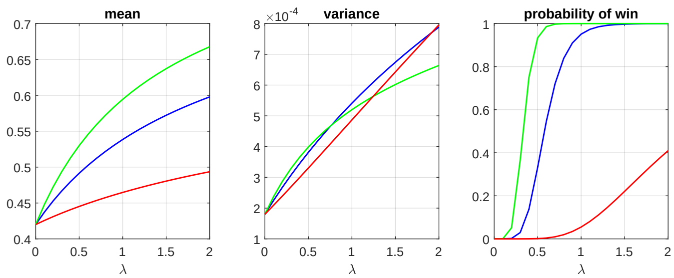

Figure 6. Mean and variance of the vote share, and probability of win as functions of λ in Example 3.Blue: random assignments; Green: key nodes assigned to Party 1; Red: key nodes assigned to Party 2

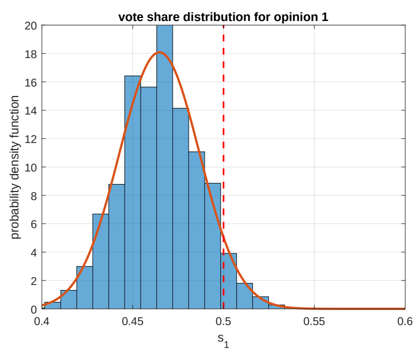

Figure 7. Sample probability distribution of the vote share $s_1(t)$ with $\lambda = 1$ in Example 3, with key nodes assigned to Party 2 (azure histogram). The orange curve represents the Gaussian distribution based on theoretical mean and variance. The dashed red line indicates the threshold for Party 1 to win the election.

of the social influence intensity, that obviously favours one opinion against the other. In the unbiased case, the platform management modifies the influence intensity irrespective of the content of the shared messages, so that an apparently neutral action is applied. Nevertheless, as already observed in [11], neutrality is only apparent because, in the presence of uneven stubborness, changes of the influence intensity can alone unbalance the expected opinion prevalence in the social network.

In this paper we have gone further, providing a more complete characterization of the effect of modulating the interaction strength between users. In particular, attention was focused on possible manipulation of the vote share in a two-party election. The main results are the derivation of pairwise correlations, that are instrumental to the exact calculation of the vote share variance. Based on the knowledge of mean and variance a first estimation of the win probability for each party can be attempted.

A side conclusion is that, apart from extremal scenarios (no interaction or herding behavior), it is difficult to establish general monotonicity results of mean and variance. This means that intentional manipulation of the interaction intensity by the platform management in order to favour one party might somehow backfire, possibly leading to unintended outcomes. It is conjectured that monotonicity of the variance holds in the case of homogeneous networks with arbitrary topology, but this

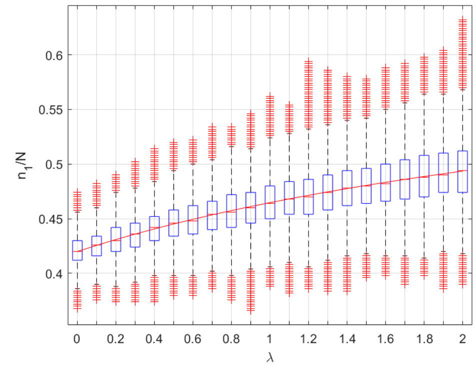

Figure 8. Boxplots of sample vote share for different values of λ in Example 3, with key nodes assigned to Party 2. The red solid line represents the theoretical mean of the vote share.

is still to be proven.

A couple of topics deserve further investigation. The present analysis assumes the knowledge of the agents' parameters. A first development regards the derivation of average properties of a given social network based only on the knowledge of the probability distributions of these parameters. A second issue is the extension of the analysis so as to include the autocovariance and spectral properties of the vote share process.

Acknowledgements

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Appendix

A. Proof of Theorem 1

Proof. Let $\Sigma(t)$ be the state of the network at time t. Then, considering an infinitesimal interval dt, it results that:

$$\begin{aligned}
&E[\mathcal{I}_j^r(t+dt)\mathcal{I}_i^s(t)dt)|\Sigma(t)]\\
&=\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\left(1-dt\Phi_j^r(t)\right)\left(1-dt\Phi_i^s(t)\right)\\
&+(1-\mathcal{I}_j^r(t))\mathcal{I}_i^s(t)dt\Gamma_j^r(t)\left(1-dt\Phi_i^s(t)\right)\\
&+\mathcal{I}_j^r(t)(1-\mathcal{I}_i^s(t))dt\Gamma_i^s(t)\left(1-dt\Phi_j^r(t)\right)+o(dt)
\end{aligned}$$

where $o(dt)$ denotes an infinitesimal term ultimately smaller than dt, i.e. $\lim_{dt\to 0}\frac{o(dt)}{dt}=0$, and

$$\begin{aligned}
\Psi_k^r(t) &= \frac{\lambda\eta^{[r]}}{|\mathcal{N}^r|}\sum_{l\in\mathcal{N}^r}\mathcal{I}_k^l(t)\\
\Phi_j^r(t) &= \sum_{k\neq j}\left(q_{jk}^{[r]}+\Psi_k^r(t)\right)\\
\Gamma_j^r(t) &= \sum_{k\neq j}q_{kj}^{[r]}\mathcal{I}_k^r(t)+\Psi_j^r(t)\,.
\end{aligned}$$

By reordering the terms and defining

$$\Theta_j^r(t)=\sum_k\Psi_k^r(t)+\sum_{k\neq j}q_{jk}^{[r]}$$

we obtain that:

$$\begin{aligned}&E[\mathcal{I}_j^r(t+dt)\mathcal{I}_i^s(t)]dt)|\Sigma(t)] = \mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\\&-dt\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\Big(\Theta_j^r(t)+\Theta_i^s(t)\\&+\sum_{k\neq j} q_{kj}^{[r]}\mathcal{I}_k^r(t)+\sum_{k\neq i} q_{ki}^{[s]}\mathcal{I}_k^s(t)\Big)\\&+dt\Big(\mathcal{I}_i^s(t)\Gamma_j^r(t)+\mathcal{I}_j^r(t)\Gamma_i^s(t)\Big)+o(dt)\,. \qquad (37)\end{aligned}$$

By observing that $\sum_i \mathcal{I}_i^k(t)=1, \forall k, \mathcal{I}_j^r(t)\mathcal{I}_i^r(t)=0, \forall r, i\neq j$, the following identities hold:

$$E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^r(t)] = \lambda\eta^{[r]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] \qquad (38a)$$

$$E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^s(t)] = \lambda\eta^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] \qquad (38b)$$

$$\mathcal{I}_j^r(t)\sum_{k\neq j} q_{kj}^{[r]}\mathcal{I}_k^r(t) = \mathcal{I}_i^s(t)\sum_{k\neq i} q_{ki}^{[s]}\mathcal{I}_k^s(t) = 0 \qquad (38c)$$

$$E[\mathcal{I}_j^r(t)\Psi_i^s(t)] = \frac{\lambda\eta^{[s]}}{|\mathcal{N}^s|}\sum_{l\in\mathcal{N}^s} E[\mathcal{I}_j^r(t)\mathcal{I}_i^l(t)] \qquad (38d)$$

$$E[\mathcal{I}_i^s(t)\Psi_j^r(t)] = \frac{\lambda\eta^{[r]}}{|\mathcal{N}^r|}\sum_{l\in\mathcal{N}^r} E[\mathcal{I}_i^s(t)\mathcal{I}_j^l(t)]\,. \qquad (38e)$$

By taking the expectation with respect to the conditioning event $\Sigma(t)$ on both sides of (37) and using the definitions of $\Theta_j^r(t)$ and $\Gamma_j^r(t)$, one obtains:

$$\begin{aligned}&E[\mathcal{I}_j^r(t+dt)\mathcal{I}_i^s(t)] = E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&-dtE[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\Theta_j^r(t)] - dtE[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\Theta_i^s(t)]\\&-dt\sum_{k\neq j} q_{kj}^{[r]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]\\&-dt\sum_{k\neq i} q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\mathcal{I}_k^s(t)]\\&+dtE[\mathcal{I}_i^s(t)\Gamma_j^r(t)]+dtE[\mathcal{I}_j^r(t)\Gamma_i^s(t)]+o(dt)\\&= E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&-dtE[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^r(t)] - dt\sum_{k\neq j} q_{jk}^{[r]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&-dtE[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^s(t)] - dt\sum_{k\neq i} q_{ik}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&-dt\sum_{k\neq j} q_{kj}^{[r]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]\\&-dt\sum_{k\neq i} q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\mathcal{I}_k^s(t)]\\&+dt\sum_{k\neq j} q_{kj}^{[r]}E[\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]+dtE[\mathcal{I}_i^s(t)\Psi_j^r(t)]\\&+dt\sum_{k\neq i} q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_k^s(t)]+dtE[\mathcal{I}_j^r(t)\Psi_i^s(t)]+o(dt)\,.\end{aligned}$$

Note that, in view of (38c), the two terms in the third and fourth last lines are null. Then, dividing by dt and letting $dt\to 0$, we get:

$$\begin{aligned}&\frac{d}{dt}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] = -E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^r(t)]\\&-\sum_{k\neq j} q_{jk}^{[r]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] - E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)\sum_k \Psi_k^s(t)]\\&-dt\sum_{k\neq i} q_{ik}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&+\sum_{k\neq j} q_{kj}^{[r]}E[\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]+E[\mathcal{I}_i^s(t)\Psi_j^r(t)]\\&+\sum_{k\neq i} q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_k^s(t)]+E[\mathcal{I}_j^r(t)\Psi_i^s(t)]\,.\end{aligned}$$

Now, using the identities (38a), (38b), (38d) and (38e), we obtain:

$$\begin{aligned}&\frac{d}{dt}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] = -E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)](\sum_{k\neq j} q_{jk}^{[r]}+\sum_{k\neq i} q_{ik}^{[s]})\\&+\sum_{k\neq j} q_{kj}^{[r]}E[\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]+\sum_{k\neq i} q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_k^s(t)]\\&-\lambda(\eta^{[r]}+\eta^{[s]})E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&+\frac{\lambda\eta^{[r]}}{|\mathcal{N}^r|}\sum_{l\in\mathcal{N}^r} E[\mathcal{I}_i^s(t)\mathcal{I}_j^l(t)]+\frac{\lambda\eta^{[s]}}{|\mathcal{N}^s|}\sum_{l\in\mathcal{N}^s} E[\mathcal{I}_j^r(t)\mathcal{I}_i^l(t)]\,.\end{aligned}$$

Finally, recalling that $\sum_{k\neq j} q_{jk}^{[r]} = -q_{jj}^{[r]}$, this equation becomes:

$$\begin{aligned}&\frac{d}{dt}E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)] = \sum_k q_{kj}^{[r]}E[\mathcal{I}_i^s(t)\mathcal{I}_k^r(t)]\\&+\sum_k q_{ki}^{[s]}E[\mathcal{I}_j^r(t)\mathcal{I}_k^s(t)]\\&+\lambda\Big(-(\eta^{[r]}+\eta^{[s]})E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]\\&+\frac{\lambda\eta^{[r]}}{|\mathcal{N}^r|}\sum_{l\in\mathcal{N}^r} E[\mathcal{I}_i^s(t)\mathcal{I}_j^l(t)]+\frac{\lambda\eta^{[s]}}{|\mathcal{N}^s|}\sum_{l\in\mathcal{N}^s} E[\mathcal{I}_j^r(t)\mathcal{I}_i^l(t)]\Big)\end{aligned}$$

which coincides with (11) by replacing the notation $E[\mathcal{I}_j^r(t)\mathcal{I}_i^s(t)]$ with π_{ji}^{rs}. □

B. *Proof of equation* (17)

Proof. Consider the algebraic Lyapunov equation (16), with $F(\lambda) = -(\text{diag}\{\alpha^{[r]}\}+\lambda HL)$. Notice that, using the notation of Section VI,

$$\bar{V}(\lambda) = \bar{z}(\lambda)\bar{z}(\lambda)' + \sum_{i=1}^N d_i(\lambda)X^{[i]}(\lambda)$$

with

$$\hat{F}(\lambda)\bar{X}^{[i]}(\lambda)+\bar{X}^{[i]}(\lambda)\hat{F}(\lambda)'+e_i(N)e_i(N)'=0\,.$$

Since $L\mathbf{1}=0$, we have that

$$\lim_{\lambda\to\infty}\bar{X}^{[i]}(\lambda) = \mathbf{1}_N\mathbf{1}_N'\xi_i\,.$$

Taking φ such that $\varphi' HL=0$, $\varphi'\mathbf{1}_N=1$, it can be seen that

$$\xi_i = \frac{\varphi_i^2}{2\sum_{i=1}^N \varphi_i\alpha_i}\,.$$

Hence,

$$\lim_{\lambda\to\infty}\bar{V}(\lambda)=\mathbf{1}_N\mathbf{1}_N'(\gamma^2+\lim_{\lambda\to\infty}\sum_i\xi_i\lim_{\lambda\to\infty}d_i(\lambda))\,.$$

Finally, since $\bar{V}_{rr}(\lambda)=\bar{z}_r(\lambda),\forall r$ and $\lim_{\lambda\to\infty}\bar{z}_r(\lambda)=\gamma),\forall r$, we conclude that

$$\lim_{\lambda\to\infty}\bar{V}(\lambda)=\gamma\mathbf{1}_N\mathbf{1}_N'\,.$$

□

C. Proof of equations (36a), (36b)

Proof. Consider the matrix $\bar{V}(\lambda)$ solution of the Lyapunov equation (15), expressed in the form (20). For $\lambda=0$, we have that

$$\begin{aligned}
\bar{z}_i(0) &= \beta_i \quad , \quad X^{[i]}(0)=\frac{1}{2\alpha_i}e_i(N)e_i(N)' \\
Y(0) &= \frac{1}{2}(\text{diag}\{\alpha^{[r]}\})^{-1} \\
d(0) &= 2\text{diag}\{\alpha^{[r]}\}\text{diag}\{\beta^{[r]}\}(I_N-\text{diag}\{\beta^{[r]}\}) \\
\bar{V}_{rs}(0) &= \beta^{[r]}\beta^{[s]},\, r\neq s \quad , \quad \bar{V}_{rr}(0)=\beta^{[r]}\,.
\end{aligned}$$

Letting $\tilde{\beta}^{[r]}$ be defined as in (32), the derivatives at $\lambda=0$ of the involved variables are computed as follows:

$$\begin{aligned}
\frac{d\bar{z}_i}{d\lambda}|_{\lambda=0} &= -\tilde{\beta}_i \quad , \quad \frac{d\bar{z}_i^2}{d\lambda}|_{\lambda=0}=-2\beta^{[i]}\tilde{\beta}^{[i]} \\
\frac{dX_{kk}^{[i]}}{d\lambda}|_{\lambda=0} &= 0,\, i\neq k \quad , \quad \frac{dX_{kk}^{[k]}}{d\lambda}|_{\lambda=0}=-\frac{\eta^{[k]}}{2(\alpha^{[k]})^2} \\
\frac{dY}{d\lambda}|_{\lambda=0} &= -\frac{1}{2}\text{diag}\{\eta^{[r]}\}(\text{diag}\{\alpha^{[r]}\})^{-2} \\
\frac{d}{d\lambda}d|_{\lambda=0} &= -2\text{diag}\{\eta^{[r]}\}\text{diag}\{\beta^{[r]}\}(I_N-\text{diag}\{\beta^{[r]}\}) \\
&\quad -2\text{diag}\{\alpha^{[r]}\}\text{diag}\{\tilde{\beta}^{[r]}\}(I_N-2\text{diag}\{\beta^{[r]}\}) \\
\frac{d\bar{V}_{rs}}{d\lambda}|_{\lambda=0} &= -(\tilde{\beta}^{[r]}\beta^{[s]}+\tilde{\beta}^{[s]}\beta^{[r]}),\, r\neq s \\
\frac{d\bar{V}_{rr}}{d\lambda}|_{\lambda=0} &= -\tilde{\beta}^{[r]}\,.
\end{aligned}$$

□

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