AAS 14-226

ASYMPTOTIC SOLUTION FOR THE TWO BODY PROBLEM WITH RADIAL PERTURBING ACCELERATION

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In this article, an approximate analytical solution for the two body problem perturbed by a radial, low acceleration is obtained, using a regularized formulation of the orbital motion and the method of multiple scales. The results reveal that the physics of the problem evolve in two fundamental scales of the true anomaly. The first one drives the oscillations of the orbital parameters along each orbit. The second one is responsible of the long-term variations in the amplitude and mean values of these oscillations. A good agreement is found with high precision numerical solutions.

INTRODUCTION

There is a great theoretical and practical interest in obtaining analytical solutions for the two body problem subjected to perturbing forces. The motion of a spacecraft around a primary under continuous thrust is studied in the classic astrodynamics book by Battin,[1] for radial and tangential acceleration: in the first case, an exact solution for initially circular orbits is reached in terms of elliptic integrals; whereas for the second one, an approximate solution for low thrust acceleration is developed. More recently, an asymptotic solution for low tangential acceleration was obtained by Bombardelli et al.,[2] using Dromo orbital formulation[4] and a regular expansion in the non-dimensional thrust. These approximate solutions can be applied to study several practical problems, such as the effects of solar radiation pressure or comet outgassing[3] in the radial case.

In this article, an approximate analytical solution for the two body problem perturbed by a small radial acceleration is obtained, using the regularized formulation of the orbital motion known as Dromo, and the method of multiple scales. Compared to the results by Battin,[1] it has the main advantage of not being restricted to initially circular orbits, requiring only small initial eccentricity.

It should be pointed out that an exact solution for the radial problem in the general case has been recently obtained by Izzo et al.,[10] in terms of a fictitious time introduced with a Sundmann transformation and the Weierstrass elliptic functions. However, our asymptotic solution, despite being less precise, has the advantage of being expressed in terms of much simpler functions, and provides more insight about the underlying physics of the low thrust case.

First of all, the equations of motion for the two body problem perturbed by a small radial acceleration of constant magnitude are posed, using Dromo orbital formulation. Dromo was initially introduced by the Grupo de Dinámica de Tethers (now Space Dynamics Group-UPM), and has been under active development.[4,5,6] It has proven to be an excellent propagation tool, and its suitability for the formulation of low thrust optimal control problems has been recently studied.[7]

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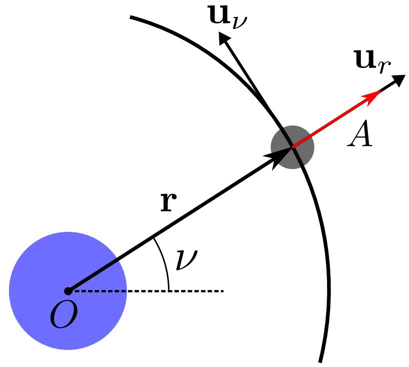

Figure 1. Schematic representation of the radial thrust problem

In the next section, an asymptotic solution for the problem is obtained using a regular expansion in the small perturbing acceleration. It is checked that this solution breaks for large values of the independent variable, suggesting the use of more complex perturbation techniques.

The third section deals with the solution of the problem using the method of multiple scales. The results obtained reveal that the physics of the problem evolve in two fundamental scales of the independent variable. The first one drives the oscillations of the orbital parameters along each orbit. The second one, scaled by the magnitude of the thrust, is responsible for the long-term variations in the amplitude and mean values of these oscillations. Moreover, it is verified that the behavior is periodic in both scales, with different periods; this shows that, provided the escape is not reached, the orbital parameters of the perturbed orbit oscillate between certain limit values indefinitely.

Finally, the multiple scales solution is compared with high precision numerical propagations for several cases, finding a good agreement between them. The asymptotic solution obtained through the regular expansion is also included in these comparisons, highlighting the great gains in accuracy and validity range of the asymptotic solution achieved through the method of multiple scales.

EQUATIONS OF MOTION

Let us consider a particle of mass m orbiting around a primary of gravitational constant μ. The only forces acting on the particle are the attraction of the primary and a small radial acceleration of constant magnitude A. Since this perturbation force is always contained in the osculating plane of the orbit, the resulting motion will be *planar*. Moreover, given that all the forces are central, *the angular momentum remains constant*; these properties will allow us to simplify the formulation for the equations of motion. The initial distance between the particle and the center of the primary is denoted as R_0, while ν_0 is the initial value of the true anomaly. All equations and variables considered hereafter are expressed in non-dimensional form: to this end, three characteristic magnitudes are introduced for length (R_0), mass (m) and time (n_0^{-1}), where n_0 is the angular frequency of the circular orbit of radius R_0 around the primary, $n_0 = (\mu/R_0)^{1/2}$.

To describe the motion of the body, the Dromo orbital formulation developed by Peláez et al.[4,6,5] is used. In this formulation, a fictitious time θ is introduced through a change of independent variable given by a Sundman transformation. Then, the variation of parameters technique is applied to obtain seven generalized orbital parameters $\mathbf{q}$ which, along the non-dimensional time t, describe the state of the particle. These orbital parameters are constant in the unperturbed problem, but evolve in presence of perturbing forces; this property is very convenient for the mathematical developments in this study. Moreover, it is important to note that three of these parameters describe the geometry of the orbit in its plane, while the other four are related to the orientation of said plane. Therefore, the later are constant for the planar case, and the motion of the particle can be described by a 4-dimensional state vector:

$$(t, q_1, q_2, q_3) \tag{1}$$

whose evolution is given by the following system of four differential equations:

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{q_3 s^2} \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{q_3 s^3}\begin{bmatrix} s\sin\theta & (s+q_3)\cos\theta \\ -s\cos\theta & (s+q_3)\sin\theta \\ 0 & -q_3 \end{bmatrix}\begin{bmatrix} a_r \\ a_\theta \end{bmatrix} \tag{3}$$

with

$$s = q_3 + q_1\cos\theta + q_2\sin\theta \tag{4}$$

$$r = \frac{1}{q_3 s} \tag{5}$$

where a_r and a_θ are the non-dimensional perturbing forces along the radial and transversal directions, and r is the non-dimensional orbital distance. Since Equations (2) and (3) are uncoupled, the problem can be solved for (q_1, q_2, q_3) without calculating t.

It is possible to establish several relations between the generalized orbital parameters and the classical orbital elements:[6,5]

$$q_1 = \frac{e}{h}\cos\gamma, \qquad q_2 = \frac{e}{h}\sin\gamma, \qquad q_3 = \frac{1}{h} \tag{6}$$

$$e = \frac{\sqrt{q_1^2+q_2^2}}{q_3}, \quad \gamma = \tan^{-1}\left(\frac{q_2}{q_1}\right), \quad h = \frac{1}{q_3}, \quad a = \frac{1}{q_3^2-q_1^2-q_2^2}, \quad E = \frac{q_1^2+q_2^2-q_3^2}{2} \tag{7}$$

where e is the eccentricity, h is the non-dimensional angular momentum, a is the non-dimensional semimayor axis, E is the non-dimensional total energy, and angle γ is the difference between the variations of the fictitious time and the true anomaly[5]

$$\Delta\theta = \Delta\nu + \gamma \tag{8}$$

In the planar case, γ coincides with the angle between the eccentricity vectors of the initial and of the osculating orbit, and θ becomes the angular position of the particle measured from the eccentricity vector of the initial orbit.

For the radial thrust problem, the non-dimensional perturbing acceleration takes the form:

$$\mathbf{a} = \begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \quad , \qquad \varepsilon = \frac{A}{R_0 n_0^2} = \frac{A}{\mu/R_0^2} \tag{9}$$

where ε is the non-dimensional acceleration parameter. Introducing these values into Equations (3) yields:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{\varepsilon}{q_3 s^2}\begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \tag{10}$$

The third equation shows that q_3 is always constant in the radial case; this result is equivalent to the conservation of the angular momentum, since $q_3 = 1/h$. Regarding the other two components, approximate solutions for the low thrust case can be searched for using perturbation techniques.

To close the mathematical formulation of the dynamics, the initial conditions are obtained from Equations (6), taking into account that $\theta_0 = \nu_0$, $\gamma_0 = 0$:

$$q_3(\theta_0) = 1/h_0 = q_{3i}\ , \quad q_1(\theta_0) = e_0/h_0 = e_0 q_{3i} = q_{1i}\ , \quad q_2(\theta_0) = 0 \tag{11}$$

with $h_0 = \sqrt{1 + e_0 \cos\nu_0}$. For simplicity and clarity sake, in the following developments the initial value of the independent variable is assumed to be zero, $\theta_0 = 0$. The results can be generalized for an arbitrary value of θ_0 by introducing the corresponding integration constants.

REGULAR EXPANSION

An asymptotic solution for the low-thrust two body problem defined by Equations (10) and (11) is now searched for, in the form of a regular expansion in the non-dimensional thrust parameter ε. To this end, the state is expanded in power series of $\varepsilon \ll 1$ as follows:

$$q_k(\theta;\varepsilon) = q_{k0}(\theta) + \varepsilon q_{k1}(\theta) + \varepsilon^2 q_{k2}(\theta) + \mathcal{O}(\varepsilon^3) \qquad k = 1, 2 \tag{12}$$

Expanding also the initial conditions, with $\theta_0 = 0$, and identifying terms of equal power of ε:

$$q_{10}(0) = \frac{e_0}{h_0} = q_{1i}, \qquad q_{1l}(0) = 0 \quad l \geq 1\,, \qquad q_{2l}(0) = 0 \quad l \geq 0 \tag{13}$$

Introducing the expansion of the state into the first two Equations of (10), expanding in Taylor series of ε and retaining terms of $\mathcal{O}(1)$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} q_{10}(\theta) \\ q_{20}(\theta) \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} q_{10}(\theta) \\ q_{20}(\theta) \end{bmatrix} = \begin{bmatrix} q_{1i} \\ 0 \end{bmatrix}$$

That is, the zeroth order terms of the asymptotic solution* are constant and equal to their initial values; this result was expected, since the limit $\varepsilon = 0$ corresponds to the unperturbed orbit, for which Dromo orbital parameters remain constant.

To retain the effect of the thrust, it is necessary to consider the first order terms of the asymptotic solution. The differential equations which describe their evolution with θ are obtained canceling terms of $\mathcal{O}(\varepsilon)$ in the expansion of Equations (10):

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} q_{11}(\theta) \\ q_{21}(\theta) \end{bmatrix} = \frac{1}{q_{3i}\left(q_{3i} + q_{10}\cos\theta + q_{20}\sin\theta\right)^2}\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} \tag{14}$$

Since the previous equations are uncoupled, $q_{11}(\theta)$ and $q_{21}(\theta)$ can be obtained independently as quadratures. Introducing the known results for q_{10} and q_{20}, and taking into account the initial conditions given by Equations (13), the solution for $q_{11}(\theta)$ is straightforward:

$$q_{11}(\theta) = \frac{1 - \cos\theta}{q_{3i}(q_{3i} + q_{1i})(q_{3i} + q_{1i}\cos\theta)} \tag{15}$$

*The order of each term in the asymptotic solution is denoted hereafter through the corresponding exponent of ε.

The solution for $q_{21}(\theta)$ is more complex. Assuming $q_{3i} > q_{1i}$* and integrating yields:

$$q_{21}(\theta) = \frac{-\sin\theta}{(q_{3i} + q_{1i}\cos\theta)\left(q_{3i}^2 - q_{1i}^2\right)} + \frac{2q_{1i}}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)^{3/2}}\left(\frac{\theta}{2} + \arctan\mathcal{K}(\theta)\right) \tag{16}$$

with

$$\mathcal{K}(\theta) = -\frac{\sin\theta\left(-q_{3i} + q_{1i} + \sqrt{q_{3i}^2 - q_{1i}^2}\right)}{(1+\cos\theta)\sqrt{q_{3i}^2 - q_{1i}^2} + (1-\cos\theta)(q_{3i} - q_{1i})} \tag{17}$$

The second term of the expression for $q_{21}(\theta)$ introduces a secular behavior in θ; as a result, $\varepsilon q_{21}(\theta)$ becomes of $\mathcal{O}(1)$ for $\theta \sim \varepsilon^{-1}$, and the regular asymptotic expansion breaks. This posses a clear limitation to the applicability of this solution, and suggests the convenience of resorting to more complex formulations.

It is not straightforward to give physical interpretations of Dromo generalized parameters. Therefore, the previous results are now expressed in terms of more familiar orbital elements. Using the first of Equations (7), introducing the asymptotic solutions for $q_1(\theta;\varepsilon)$ and $q_2(\theta;\varepsilon)$, expanding in Taylor series of ε and retaining terms up to $\mathcal{O}(\varepsilon)$, the eccentricity e can be expressed as:

$$e(\theta;\varepsilon) = e_0 + \varepsilon e_1(\theta) + \mathcal{O}(\varepsilon^2) \qquad \text{with } \; e_1(\theta) = \frac{1-\cos\theta}{q_{3i}^4(1+e_0)(1+e_0\cos\theta)} \tag{18}$$

As a first approximation, eccentricity oscillates between e_0 and $e_0 + 2\varepsilon q_{30}^{-4}(1-e_0^2)^{-1}$, with a period of 2π in the pseudo time θ. Note that, for this expansion to be valid, $\varepsilon e_1(\theta)$ must be a small correction of e_0; specifically, it fails for initially circular or quasi-circular orbits. This limitation in e_0 to the applicability of Equation (18) is not inherent to the asymptotic solution of the problem, but rather associated to the additional expansion performed for $e(\theta;\varepsilon)$; in all cases, eccentricity can be calculated using the first of Equations (7).

Proceeding in a similar manner with the non-dimensional orbital radius yields:

$$r(\theta;\varepsilon) = \frac{1}{q_{3i}\left(q_{3i} + q_{1i}\cos\theta\right)} + \varepsilon r_1(\theta) + \mathcal{O}(\varepsilon^2) \tag{19}$$

$$r_1(\theta) = \frac{-q_{1i}\sin\theta\left(\theta + 2\arctan\mathcal{K}(\theta)\right) + (1-\cos\theta)\sqrt{q_{3i}^2 - q_{1i}^2}}{q_{3i}^2(q_{3i}^2 - q_{1i}^2)^{3/2}(q_{3i} + q_{1i}\cos\theta)^2}$$

This expression is valid regardless the value of e_0, but, same as $q_2(\theta;\varepsilon)$, breaks for $\theta \sim \varepsilon^{-1}$. From a physical point of view, the presence of the secular term in θ would imply that the orbital radius is unbounded for any value of e_0 and ε, which is in contradiction with the results given by Battin.[1]

Finally, it is interesting to consider the evolution of the true anomaly ν with the fictitious time θ. According to Equations (8) and (7), both variables are related through an angular drift γ whose regular expansion up to $\mathcal{O}(\varepsilon)$ can be given as:

$$\gamma(\theta;\varepsilon) = -\varepsilon\left(\frac{\sin\theta}{q_{1i}(q_{3i}^2 - q_{1i}^2)(q_{3i} + q_{1i}\cos\theta)} - \frac{\theta + 2\arctan\mathcal{K}(\theta)}{q_{3i}(q_{3i}^2 - q_{1i}^2)^{3/2}}\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{20}$$

Same as with the eccentricity, the previous expansion is not valid for the case of initially circular or quasi-circular orbits; in those cases, the general expression in Equations (7) must be used.

*This assumption is valid as long as $e_0 < 1$, since $q_{1i} = e_0 q_{3i}$.

Initially circular orbit

For the particular case of an orbit with $e_0 = 0$, the regular asymptotic solution presented in this section takes a simpler form:

$$q_1(\theta;\varepsilon) = \varepsilon\frac{1-\cos\theta}{q_{3i}^3} + \mathcal{O}(\varepsilon^2) \quad , \quad q_2(\theta;\varepsilon) = \varepsilon\frac{-\sin\theta}{q_{3i}^3} + \mathcal{O}(\varepsilon^2) \tag{21}$$

and from it:

$$e(\theta;\varepsilon) = \varepsilon\frac{\sqrt{2}}{q_{3i}^4}\sqrt{1-\cos\theta} + \mathcal{O}(\varepsilon^2) \quad , \quad r(\theta;\varepsilon) = \frac{1}{q_{3i}^2} + \varepsilon\frac{1-\cos\theta}{q_{3i}^6} + \mathcal{O}(\varepsilon^2)$$
$$\gamma(\theta;\varepsilon) = \frac{\theta-\pi}{2} + \mathcal{O}(\varepsilon^2)$$

The secular term in θ has vanished from the solutions, but this does not imply a good behavior for large values of the independent variable θ. Certainly, the numerical results displayed in later sections show that the approximation is still bad for $\theta \sim \varepsilon^{-1}$.

MULTIPLE SCALES SOLUTION

The breakdown of the regular expansion for $\theta \sim 1/\varepsilon$ suggests the existence of a slow 'time' scale in the independent variable θ. This hypothesis is further supported by the perturbation model for the tangential case presented by Bombardelli et al.,[2] and the solutions in terms of elliptic equations given by Battin[1] for the radial problem. All those cases are characterized by a fast, periodic evolution associated to the orbital period, and a slow, secular behavior whose characteristic period depends on the magnitude of the thrust. Based on this, the following two 'time' scales are proposed:

$$\tau = \theta$$
$$T = \varepsilon\,\theta$$

The derivative operator can be rewritten in terms of the new independent variables as:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \partial_\tau\frac{\mathrm{d}\tau}{\mathrm{d}\theta} + \partial_T\frac{\mathrm{d}T}{\mathrm{d}\theta} = \partial_\tau + \varepsilon\partial_T$$

and introducing it into Equations (10) yields:

$$\partial_\tau\begin{bmatrix}q_1(\tau,T)\\ q_2(\tau,T)\end{bmatrix} + \varepsilon\partial_T\begin{bmatrix}q_1(\tau,T)\\ q_2(\tau,T)\end{bmatrix} = \frac{\varepsilon}{q_3 s^2}\begin{bmatrix}\sin\tau\\ -\cos\tau\end{bmatrix} \tag{22}$$

The series expansions for $q_1(\tau,T)$ and $q_2(\tau,T)$ in $\varepsilon \ll 1$ up to first order terms are now of the form:

$$q_k(\tau,T;\varepsilon) = q_{k0}(\tau,T) + \varepsilon q_{k1}(\tau,T) + \mathcal{O}(\varepsilon^2) \qquad k = 1,2 \tag{23}$$

$$q_{10}(0,0) = \frac{e_0}{h_0} = q_{1i}, \qquad q_{1l}(0,0) = 0 \quad l \geq 1\,, \qquad q_{2l}(0,0) = 0 \quad l \geq 0 \tag{24}$$

Substituting Equations (23) into Equations (22), expanding in Taylor series of the small parameter ε and retaining terms of $\mathcal{O}(1)$ yields:

$$\partial_\tau q_{k0} = 0 \quad \Rightarrow \quad q_{k0}(\tau,T) = q_{k0}(T) \qquad \text{for } k = 1,2$$
$$\text{with} \quad q_{10}(0) = q_{1i}\,,\ \ q_{20}(0) = 0$$

Consequently, q_{10} and q_{20} are no longer constants, but rather functions of the slow scale T. Moreover, $q_{20}(T) \neq 0$ in general, and the corresponding simplifications introduced in the derivation of the regular solution cannot be made.

The zeroth order equations do not provide enough information to fully determine $q_{10}(T)$ and $q_{20}(T)$. This is the expected behavior when applying the method of multiple scales; as shown in many classic perturbation texts,[8,9] the equations to close the zeroth order solution will be obtained from the cancellation of the secular terms in the next order solution (also known as *secularity condition*). The equations for $\mathcal{O}(\varepsilon)$ are:

$$\partial_\tau \begin{bmatrix} q_{11}(\tau,T) \\ q_{21}(\tau,T) \end{bmatrix} + \partial_T \begin{bmatrix} q_{10}(T) \\ q_{20}(T) \end{bmatrix} = \frac{1}{q_{3i}\left(q_{3i} + q_{10}(T)\cos\tau + q_{20}(T)\sin\tau\right)^2} \begin{bmatrix} \sin\tau \\ -\cos\tau \end{bmatrix}$$

Rearranging terms and integrating in τ, the first order solutions $q_{11}(\tau,T)$ and $q_{21}(\tau,T)$ are obtained in terms of the unknown functions of the slow scale $q_{10}(T)$, $q_{20}(T)$, $g_{11}(T)$ and $g_{21}(T)$:

$$q_{k1}(\tau,T) = \mathcal{P}^*_{k1}(\tau,T) + \mathcal{S}^*_{k1}(T)\left(\frac{\tau}{2} + \arctan\mathcal{K}^*(\tau,T)\right) - \tau\partial_T q_{k0}(T) + g_{k1}(T) \quad k = 1,2$$

Since the expressions for $\mathcal{P}^*_{k1}(\tau,T)$, $\mathcal{S}^*_{k1}(T)$ and $\mathcal{K}^*(\tau,T)$ are rather long, they have been compiled in the Appendix. Suffice to say that their dependence with τ is trigonometric, in $\sin\tau$ and $\cos\tau$, while the slow time scale T only appears through the zeroth order terms $q_{10}(T)$ and $q_{20}(T)$. There are two main differences between these solutions and those obtained for the regular expansion. On the one hand, the coefficients of $q_{20}(T)$ are retained, causing a secular term to appear for $q_{11}(\tau,T)$. On the other hand, the amplitudes of the oscillations are no longer constant, but vary in the slow scale with $q_{10}(T)$ and $q_{20}(T)$.

Imposing the cancellation of the secular terms in $q_{11}(\tau,T)$ and $q_{21}(\tau,T)$, the following ODE system for determining $q_{10}(T)$ and $q_{20}(T)$ is reached:

$$\frac{\mathrm{d}q_{10}(T)}{\mathrm{d}T} = \frac{1}{2}\mathcal{S}^*_{11}(T) = -\frac{q_{20}(T)}{q_{3i}\left(q_{3i}^2 - q_{10}^2(T) - q_{20}^2(T)\right)^{3/2}} \tag{25}$$

$$\frac{\mathrm{d}q_{20}(T)}{\mathrm{d}T} = \frac{1}{2}\mathcal{S}^*_{21}(T) = \frac{q_{10}(T)}{q_{3i}\left(q_{3i}^2 - q_{10}^2(T) - q_{20}^2(T)\right)^{3/2}} \tag{26}$$

$$q_{10}(0) = q_{1i} \quad , \quad q_{20}(0) = 0 \tag{27}$$

Dividing Equation (25) by Equation (26) and integrating, a first integral is found in the form:

$$q_{10}^2(T) + q_{20}^2(T) = q_{1i}^2 \tag{28}$$

This result, combined with Equations (7), implies that the zeroth order terms of the asymptotic solutions for the eccentricity, the semimayor axis and the energy remain constant and equal to the unperturbed problem. In other words, the variations in e, a and E due to a small radial perturbing acceleration ε are of $\mathcal{O}(\varepsilon)$.

Introducing the first integral into Equations (25)-(26), they can be integrated to reach:

$$q_{10}(T) = q_{1i}\cos\frac{T}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)^{3/2}} \quad , \quad q_{20}(T) = q_{1i}\sin\frac{T}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)^{3/2}} \tag{29}$$

These expressions represent the variations of the zeroth order terms due to the accumulation of small effects during long times. Comparing them with the constant and secular terms in the regular expansion, it is possible to identify the later as the approximation of $q_{10}(T)$ and $q_{20}(T)$ for $T \ll 1$; consequently, the multiple scales formulation improves the solution by retaining more accurate information about the physics of the problem. Regarding the frequency of the solutions, it is important to point out that it evolves from 1 for $e_0 = 0$, since $q_{3i}(e_0 = 0) = 1$, $q_{1i}(e_0 = 0) = 0$, to ∞ for $e_0 = 1$, since $q_{3i}(e_0 = 1) = q_{1i}(e_0 = 1)$. As a consequence, the multiple scales solution will behave badly for near escape orbit, where a singularity is found.

Taking the first integral into account, and canceling the secular parts, the equations for $q_{11}(\tau, T)$ and $q_{21}(\tau, T)$ assume simpler forms:

$$q_{k1}(\tau, T) = \mathcal{P}_{k1}(\tau, T) + \mathcal{S}_{k1}(T) \arctan \mathcal{K}(\tau, T) + g_{k1}(T) \qquad k = 1, 2 \tag{30}$$

with $\mathcal{P}_{k1}(\tau, T)$, $\mathcal{S}_{k1}(T)$ and $\mathcal{K}(\tau, T)$ given by Equations (39)-(43) in the Appendix. Since the expressions for $q_{10}(T)$ and $q_{20}(T)$ are known, it only remains to determine $g_{11}(T)$ and $g_{12}(T)$ to close the first order solution; same as before, this is done by imposing the secularity condition to the next order solution. Introducing the expansions for $q_1(\tau, T; \varepsilon)$ and $q_2(\tau, T; \varepsilon)$ into Equations (22), and retaining terms of $\mathcal{O}(\varepsilon^2)$:

$$\partial_\tau \begin{bmatrix} q_{12}(\tau, T) \\ q_{22}(\tau, T) \end{bmatrix} + \partial_T \begin{bmatrix} q_{11}(\tau, T) \\ q_{21}(\tau, T) \end{bmatrix} = \frac{2\left(q_{11}(\tau, T)\cos\tau + q_{21}(\tau, T)\sin\tau\right)}{q_3\left(q_3 + q_{10}(T)\cos\tau + q_{20}(T)\sin\tau\right)^3} \begin{bmatrix} -\sin\tau \\ \cos\tau \end{bmatrix} \tag{31}$$

Due to the complexity of the expressions for $q_{11}(\tau, T)$ and $q_{21}(\tau, T)$, it is not feasible to identify all the components in Equations (31) which give raise to secular behaviors; to address this problem, an approximate solution for the case of small initial eccentricity e_0 is searched for instead. Writing $q_{1i} = e_0 q_{3i}$, expanding in Taylor series of e_0 up to the leading order terms, and forcing the cancellation of the parts which introduce secular components in the quadrature, the following system of ODEs is reached:

$$\frac{\mathrm{d}g_{11}}{\mathrm{d}T} = -\frac{g_{21}(T)}{q_{3i}^4} \tag{32}$$

$$\frac{\mathrm{d}g_{21}}{\mathrm{d}T} = \frac{g_{11}(T)}{q_{3i}^4} - \frac{1}{q_{3i}^7} \tag{33}$$

The initial conditions can be obtained imposing $q_{11}(0,0) = q_{21}(0,0) = 0$:

$$g_{11}(0) = \frac{2}{q_{3i}^3\left(1 - e_0^2\right)} = \frac{2}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)} \quad , \qquad g_{21}(0) = 0$$

Then, the solution for Equations (32)-(33) is:

$$g_{11}(T) = \frac{1}{q_{3i}^3} + \frac{q_{3i}^2 + q_{1i}^2}{q_{3i}^3(q_{3i}^2 - q_{1i}^2)} \cos\left(\frac{T}{q_{3i}^4}\right) \tag{34}$$

$$g_{21}(T) = \frac{q_{3i}^2 + q_{1i}^2}{q_{3i}^3(q_{3i}^2 - q_{1i}^2)} \sin\left(\frac{T}{q_{3i}^4}\right) \tag{35}$$

Note that q_{1i} has been deliberately retained in the initial condition, despite being of $\mathcal{O}(e_0)$, to improve the behavior of the results. This approximate solution is only accurate for initially circular orbits, and its precision degrades rapidly with e_0. Nevertheless, the numerical results in Figures 5 and 6 show that it is good enough for moderate values of the initial eccentricity e_0 and the non-dimensional thrust ε.

When applying the method of multiple scales, it is common to obtain a succession of increasingly slower time scales, with the number of said scales depending on the order of the approximation. Some of this scales may be associated with underlying physical properties of the problem, while others are just mathematical corrections for the previous orders. In the solution developed so far, since $\tau \sim \mathcal{O}(1)$ and q_k has been expanded up to $\mathcal{O}(\varepsilon)$, the presence of a third, super slow scale would be expected.[9] This is also consistent with the numerical results in the following section, in which a small angular drift between the reference and the asymptotic solutions can be observed as θ grows. The absence of this scale in the previous developments stems from the approximation made when determining $g_{k1}(T)$, since it would be introduced by applying the secularity condition to the exact system of ODEs for those functions. Nevertheless, although this third scale remains unknown, the mathematical structure of the problem shows that any new slower scale would act as a correction of the slow time scale, so the problem only shows *two fundamental frequencies*, and their combinations. This is in correspondence with the recent developments by Izzo el al.,[10] where an exact solution for r is obtained in terms of a fictitious time introduced with a Sundmann transformation and the doubly periodic Weierstrass $\wp$ function. Moreover, the results by Izzo et al. also include terms in the acceleration parameter up to order 2, which suggests that this term still has a physical meaning. Comparing the formulation presented in this article with the one by Izzo et al., the later has the advantage of being exact and valid for any value of ε and e_0, while the former can be expressed in terms of simpler functions and provides a greater insight on the underlying physics of the problem.

Same as for the regular expansion, it is interesting to express the previous results in terms of the eccentricity, the non-dimensional radial distance and the true anomaly. Regarding the eccentricity, it is noteworthy that the terms involving $\mathcal{K}(\tau, T)$ cancel each other in the expansion up to $\mathcal{O}(\varepsilon)$, yielding:

$$e(\tau, T; \varepsilon) = e_0 + \varepsilon e_1(\tau, T) + \mathcal{O}\left(\varepsilon^2\right) \tag{36}$$

$$e_1(\tau, T) = \frac{1}{q_{1i} q_{3i}} \left[q_{10}(T)\left(\mathcal{P}_{11}(\tau, T) + g_{11}(T)\right) + q_{20}(T)\left(\mathcal{P}_{21}(\tau, T) + g_{21}(T)\right)\right]$$

Note that this expansion is only valid if $\varepsilon e_1(\tau, T)$ is a small correction to $e(\tau, T)$, that is, if the initial eccentricity is sufficiently large. Otherwise, the first of Equations (7) must be used.

The expression for the non-dimensional orbital radius takes the form:

$$r(\tau, T; \varepsilon) = \frac{1}{q_{3i}\left(q_{3i} + q_{10}(T)\cos\tau + q_{20}(T)\sin\tau\right)^2} + \varepsilon r_1(\tau, T) + \mathcal{O}(\varepsilon^2) \tag{37}$$

$$r_1(\tau, T) = -\frac{2}{q_{3i}\left(q_{3i} + q_{10}(T)\cos\tau + q_{20}(T)\sin\tau\right)^3}\left(q_{11}(\tau, T)\cos\tau + q_{21}(\tau, T)\sin\tau\right)$$

Unlike the results obtained using the regular expansion, this expression for r is bounded, as it should.[1] It is also noteworthy that the terms $q_{10}(T)\cos\tau + q_{20}(T)\sin\tau$ and $q_{11}(\tau, T)\cos\tau + q_{21}(\tau, T)\sin\tau$ introduce new frequencies in the problem, as combinations of the two fundamental ones already obtained. This behavior is more clearly seen in the solutions for the initially circular orbit, given bellow.

Regarding the angular drift γ between the true anomaly ν and the fictitious time θ, expanding the second of Equations (7) for the multiple scales solution yields:

$$\gamma(\tau, T; \varepsilon) = \frac{T}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)^{3/2}} + \varepsilon\gamma_1(\tau, T) + \mathcal{O}(\varepsilon^2) \tag{38}$$

$$\gamma_1(\tau, T) = \frac{1}{q_{1i}^2}\left[q_{10}(T)\left(\mathcal{P}_{21} + g_{21}\right) - q_{20}(T)\left(\mathcal{P}_{11} + g_{11}\right) + \frac{2q_{1i}^2}{q_{3i}\left(q_{3i}^2 - q_{1i}^2\right)^{3/2}} \arctan \mathcal{K}\right]$$

The leading order term, which was absent from the regular expansion, now varies in the slow scale. This can be compared with the regular expansion having a secular component in θ inside its first order term. On the other hand, the first order term $\gamma_1(\tau, T)$ is now periodic and bounded.

Initially circular orbit

The particular case of initially circular orbit is specially interesting for this multiple scales solution, since the expressions given for $g_{11}(T)$ and $g_{21}(T)$ are then exact (the third scale, however, will still be missing). Particularizing the previous results for $e_0 = 0$ yields:

$$q_1(\tau, T; \varepsilon) = \frac{\varepsilon}{q_{3i}^3}\left(\cos\frac{T}{q_{3i}^4} - \cos\tau\right) + \mathcal{O}(\varepsilon^2)\ , \quad q_2(\tau, T; \varepsilon) = \frac{\varepsilon}{q_{3i}^3}\left(\sin\frac{T}{q_{3i}^4} - \sin\tau\right) + \mathcal{O}(\varepsilon^2)$$

Note that, although the zeroth order terms no longer appear, the variation with the slow scale is retained through $g_{11}(T)$ and $g_{21}(T)$. Comparing these expressions with those obtained for the regular expansion, it is straightforward to check that the later coincide with the former particularized for $T = 0$.

From this solution, it is possible to derive:

$$\begin{aligned}
e(\tau, T; \varepsilon) &= \frac{\varepsilon\sqrt{2}}{q_{3i}^3}\left[1 - \cos\left(\tau - \frac{T}{q_{3i}^4}\right)\right]^{1/2} + \mathcal{O}(\varepsilon^2) \\
r(\tau, T; \varepsilon) &= \frac{1}{q_{3i}^3} + \varepsilon\frac{2}{q_{3i}^7}\left[1 - \cos\left(\tau - \frac{T}{q_{3i}^4}\right)\right] + \mathcal{O}(\varepsilon^2) \\
\gamma(\tau, T; \varepsilon) &= \arctan\frac{\sin(T/q_{3i}^4) - \sin\tau}{\cos(T/q_{3i}^4) - \cos\tau} + \mathcal{O}(\varepsilon)
\end{aligned}$$

Note that e and r now present a new period, resulting from the combination of those for the fast and slow scales.

NUMERICAL EVALUATION OF THE RESULTS

In this section, the quality of the asymptotic solutions obtained so far is evaluated by comparing them with reference numerical solutions, computed using a high-precision integrator. It will be seen that the use of multiple scales not only improves the quality of the results, but also provides interesting information about the physics of the problem.

Figures 2-4 show the evolution of Dromo parameters (q_1, q_2), eccentricity e, non-dimensional orbital radius r, angular displacement of the eccentricity vector γ and non-dimensional semimayor axis a for several values of the initial eccentricity e_0 and the non-dimensional thrust acceleration parameter ε. The values of e, r, γ and a have been calculated using Equations (7) and (5). The first

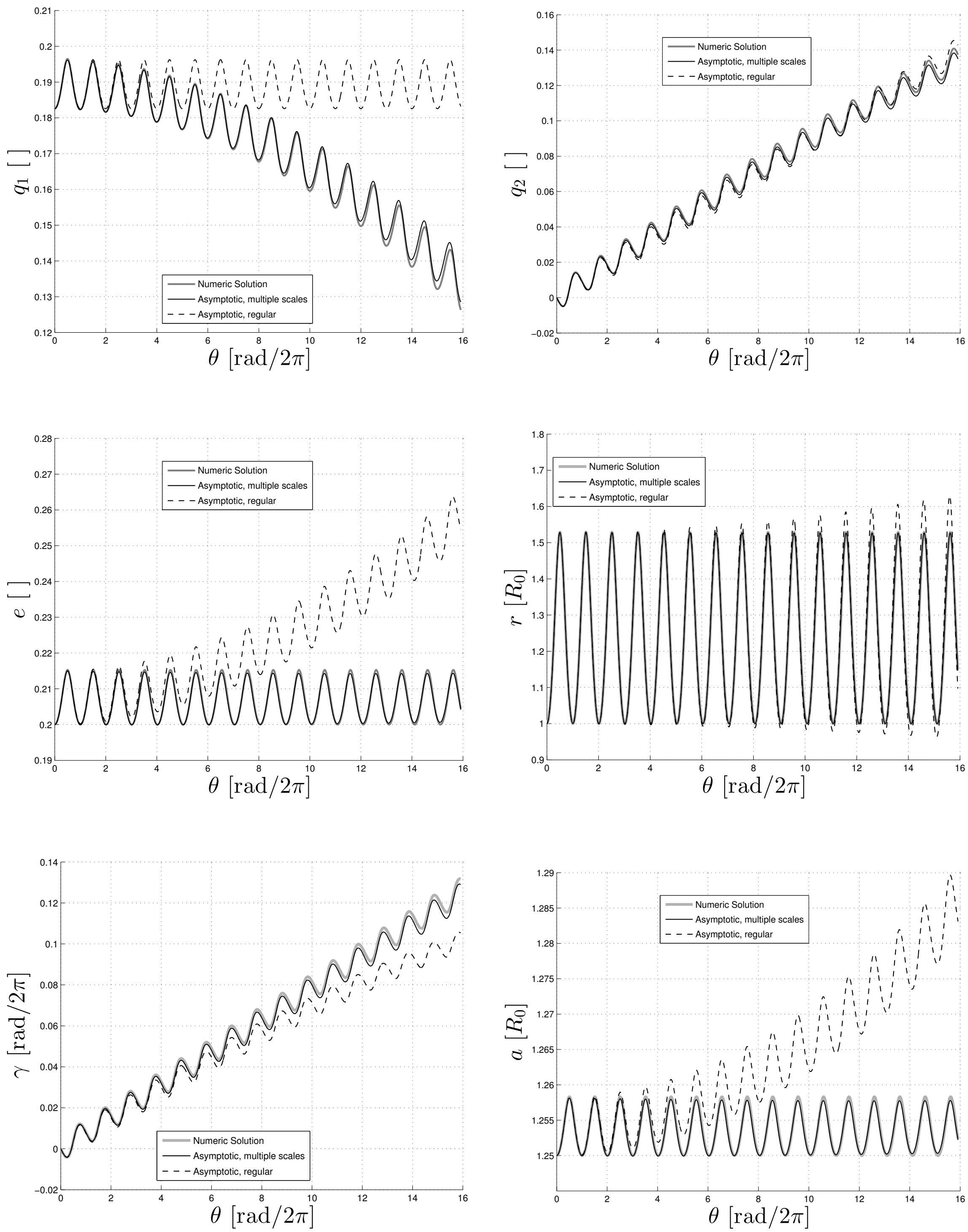

Figure 2. Comparison of the solution obtained for q_1, q_2, e, r, γ and a in the radial thrust case, for $e_0 = 0.2$ and $\varepsilon = 0.005$

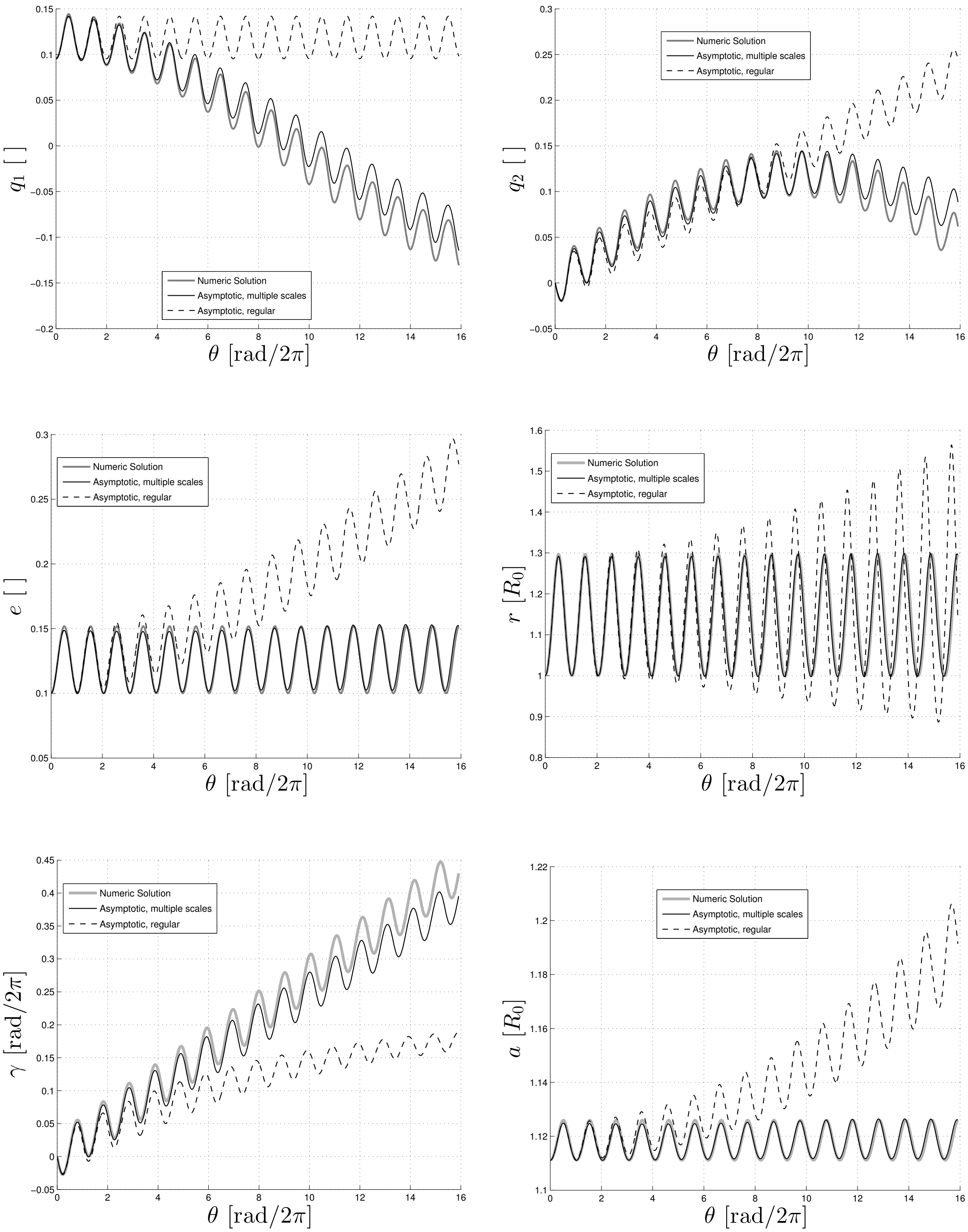

Figure 3. Comparison of the solution obtained for q_1, q_2, e, r, γ and a in the radial thrust case, for $e_0 = 0.1$ and $\varepsilon = 0.02$

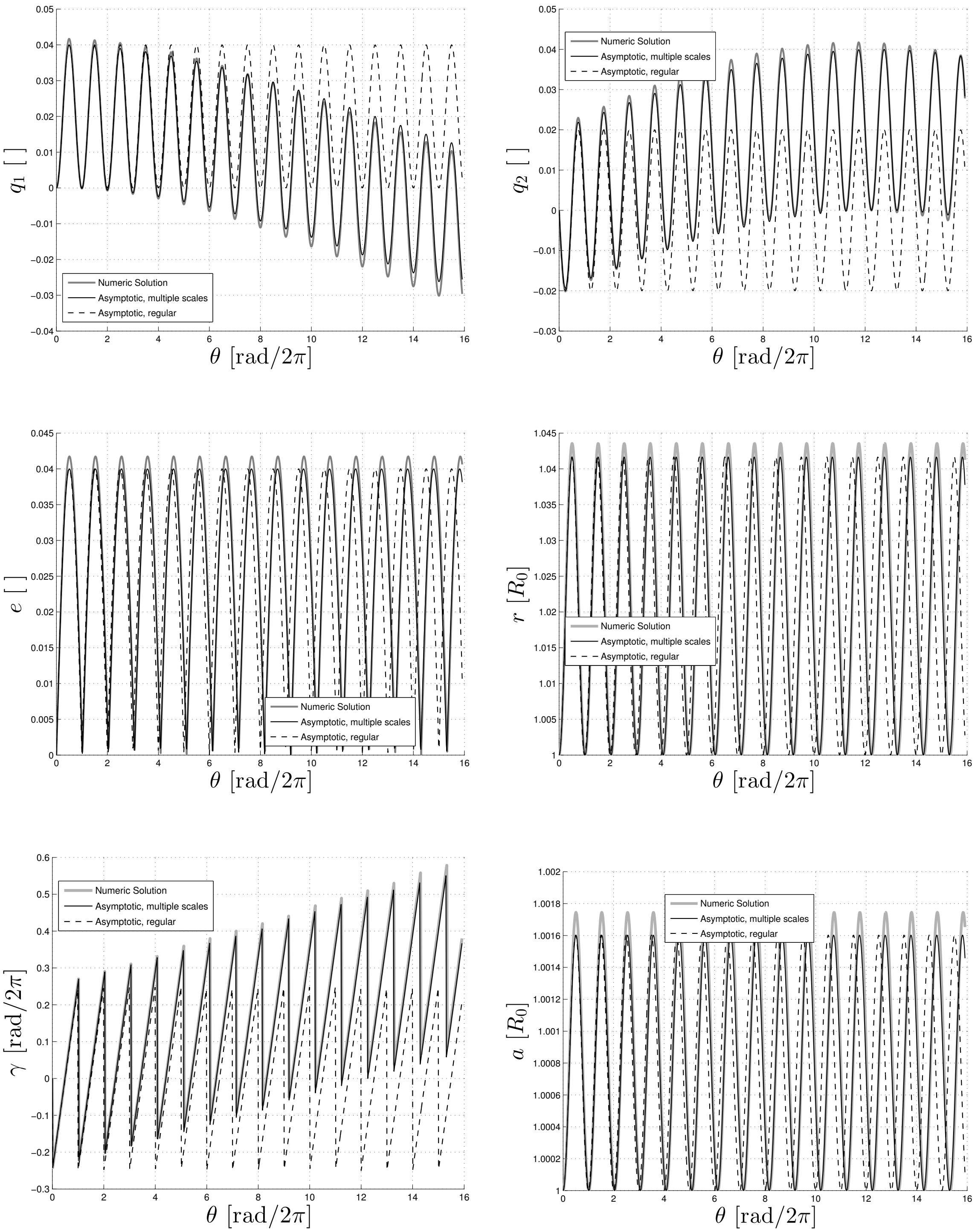

Figure 4. Comparison of the solution obtained for q_1, q_2, e, r, γ and a in the radial thrust case, for $e_0 = 0$ and $\varepsilon = 0.02$

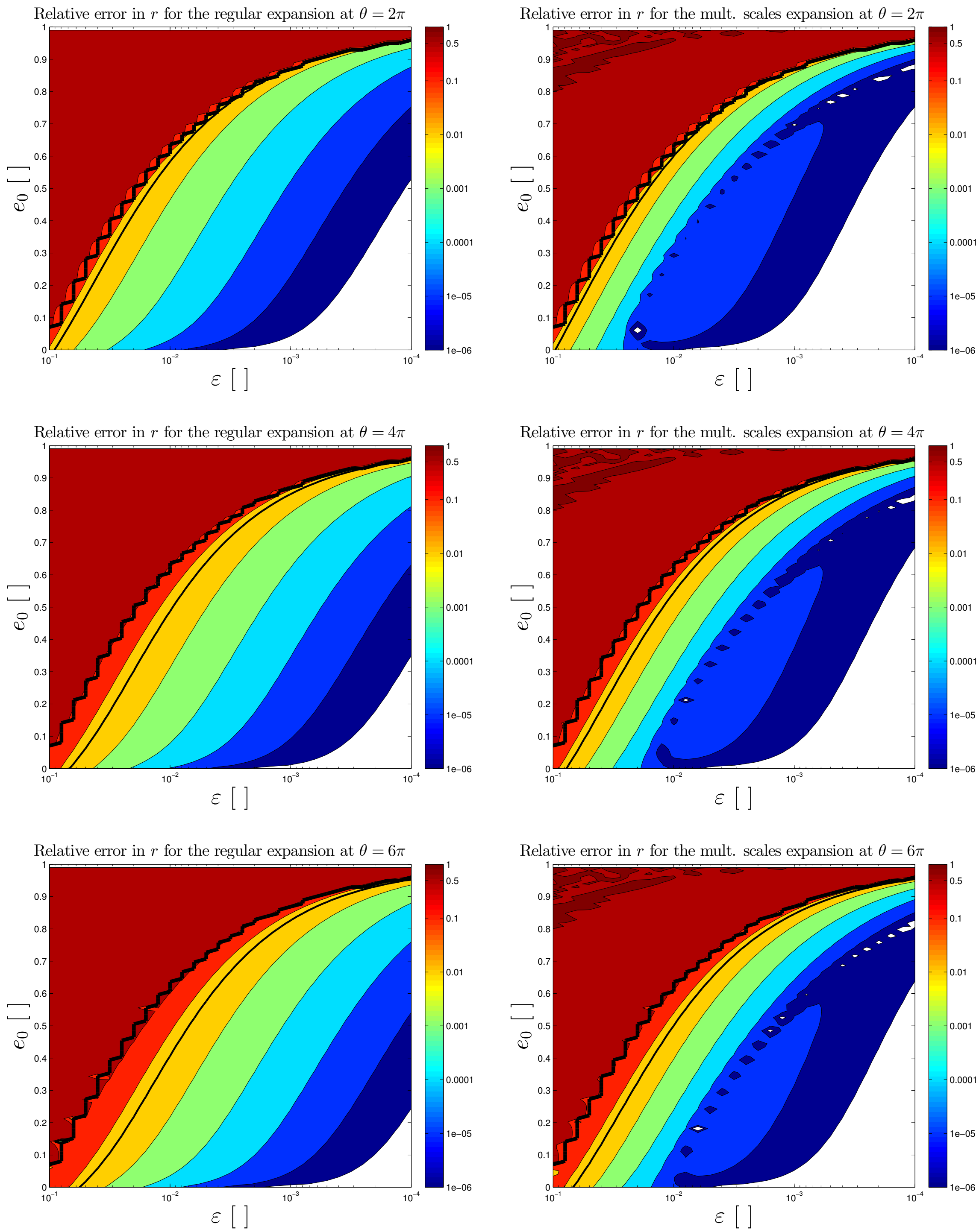

Figure 5. Relative errors in the non-dimensional orbital distance r of the regular and multiple scales asymptotic solutions, as a function of e_0 and ε. The thin black line marks the 5% error. The region to the left of the thick black line corresponds to orbits which escape before completing one revolution. For easier comparison, the color scale is the same in all the figures; the white areas correspond to relative errors lower than 10^{-6}.

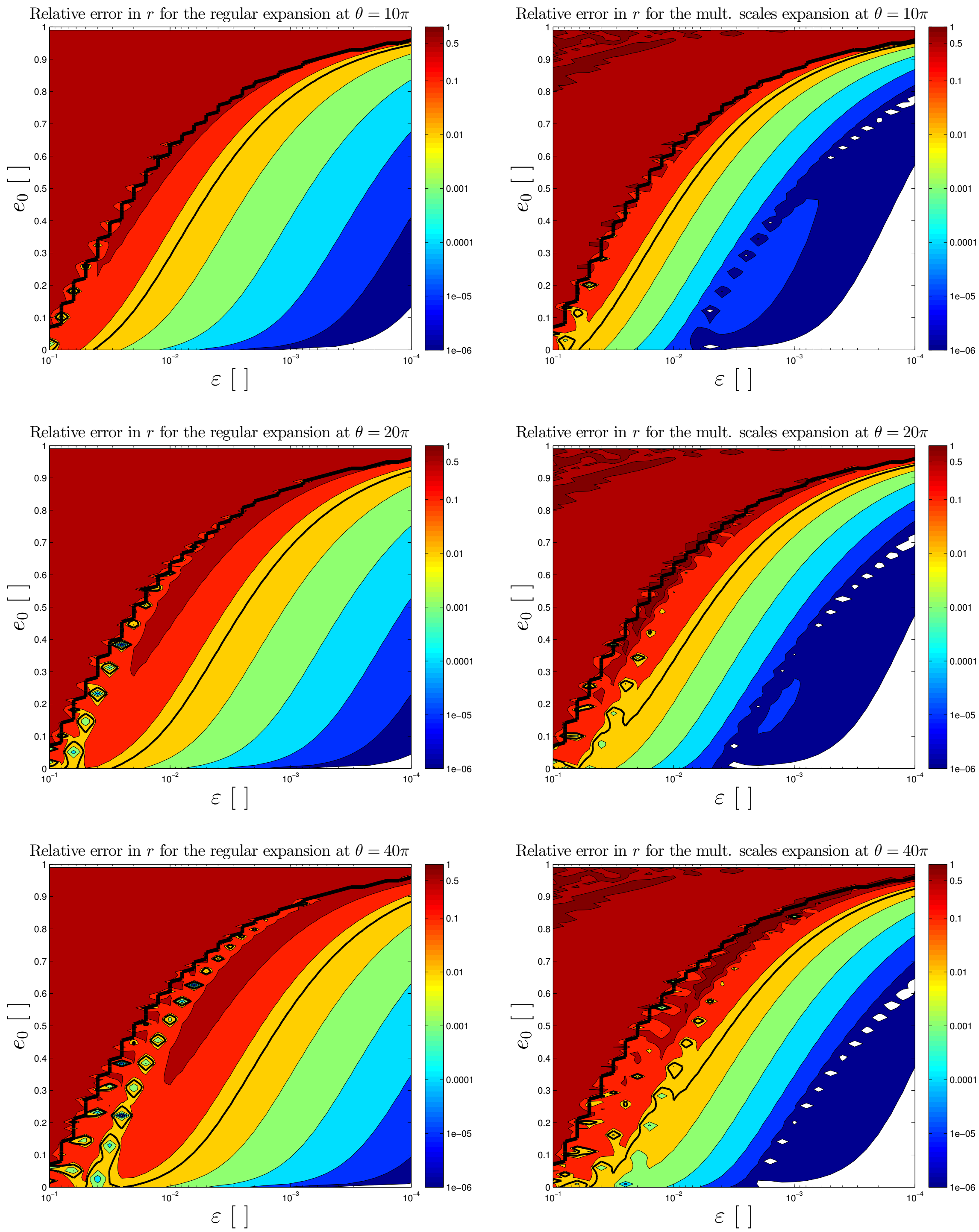

Figure 6. Relative errors in the non-dimensional orbital distance r of the regular and multiple scales asymptotic solutions, as a function of e_0 and ε. The thin black line marks the 5% error. The region to the left of the thick black line corresponds to orbits which escape before completing one revolution. For easier comparison, the color scale is the same in all the figures; the white areas correspond to relative errors lower than 10^{-6}.

set of results, Figures 2, corresponds to the case of $\varepsilon = 0.005$ and $e_0 = 0.2$; using Equations (11), the initial values of the Dromo variables associated to this e_0 are $q_{1i} = 0.1826$ and $q_{3i} = 0.9129$. The first conclusion is that the regular expansion fails very soon for $q_1(\theta)$; the mean value remains constant, so it cannot reproduce the evolution in the slow scale. Its behavior is better for $q_2(\theta)$, since it contains a secular term that approximately reproduces the sinusoidal slow scale evolution for small values of θ. It is interesting to remember that the secular components in the regular solution correspond to the first terms of the Taylor expansions of the expressions for $q_{10}(T)$ and $q_{20}(T)$ given by the multiple scales method. The multiple scales solution turns out to be remarkably good, slowly separating from the real one as θ grows. The results for e, a and r inherit the properties from $q_1(\theta)$ and $q_2(\theta)$; since the reference solutions for e, a and r oscillate between fixed values, the evolutions in the slow scale of $q_1(\theta)$ and $q_2(\theta)$ must compensate each other to obtain a good approximation. This is not possible for the regular expansion, which only contains a secular term in $q_2(\theta)$. As a consequence, a spurious secular evolution appears for e and a, separating them from the reference solution very soon; while the amplitude of r increases with θ instead of remaining constant. On the other hand, the multiple scales formulation faithfully represents the real solution for the range of θ shown in the figures. Finally, a good agreement is observed between the reference and multiple scales solutions for γ, while the regular expansion slowly diverges from them.

Figures 3 correspond to an orbital propagation with $\varepsilon = 0.02$ and $e_0 = 0.1$ ($q_{1i} = 0.09350$, $q_{3i} = 0.9535$). Most of the comments made for the previous case still hold, only now the separation between the reference and the multiple scales solutions grows faster with θ. It is checked that the greater errors for the multiple scales solution come from the evolution of the mean values in the slow scale, not the amplitude or period of the oscillations in the fast scale. Therefore, the agreement with the reference solution is still very good for e, r and a, since in those cases the secular evolutions of the mean values cancel. On the other hand, the values of γ and a given by the regular expansion not only separate very soon from the reference solution, but are also incapable of reproducing the amplitude of the oscillations. The figure for $q_2(\theta)$ is particularly interesting, clearly showing the sinusoidal evolution of the mean value this parameter in the slow scale.

A case of initially circular orbit is considered in Figures 4, for a non-dimensional perturbing acceleration of $\varepsilon = 10^{-2}$. This example is of special interest for the study of the multiple scales solution, since the expressions used for $g_{11}(T)$ and $g_{21}(T)$ are now exact. It is observed that the amplitudes of the oscillations in both the regular and multiple scales solutions are slightly smaller than the amplitudes for the reference solution; this error is due to the order of the asymptotic approximation, and could be reduced by retaining terms of greater order in the expansion. Regarding the period of the oscillations, the results for initially circular orbits developed in the previous section suggested that said period is a combination of the characteristic periods for the fast and slow scales. This behavior is confirmed by the excellent agreement between the oscillation periods of both the reference and the multiple scales solutions, with a small drift driven by the slower time scales not included in the formulation. Meanwhile, the regular expansion, which does not take into account the slow time scale, shows a slightly shorter period than the order two

The particular cases considered so far have shown that the multiple scales solution behaves much better than the regular expansion, giving a more accurate description of the physics of the problem. Nevertheless, a systematic evaluation of the error of both methods is advisable. To this end, the relative error in the non-dimensional orbital position, $|r^{\text{ref}} - r|/r^{\text{ref}}$, has been calculated for a significant range in both the initial eccentricity and the non-dimensional thrust parameter.[2] The results are displayed in Figures 5 and 6, for several values of the independent variable θ. For those orbits

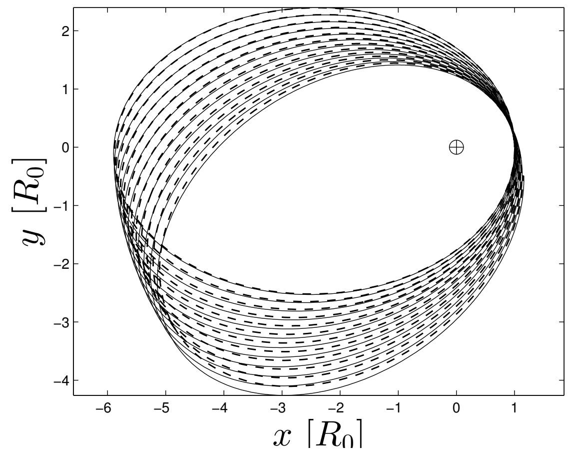

Figure 7. Evolution of an orbit with $e_0 = 0.7$ and $\varepsilon = 0.001$. The solid line corresponds to the reference numerical solution, while the dashed orbit is computed using the multiple scales solution.

which reach the escape before completing one revolution, the error is calculated at the last point of the reference numerical solution; a thick black line delimits the area corresponding to this group of orbits, which is associated with high values of e_0 and ε. In general, the errors are notably smaller for the multiple scales solution, and the difference between both asymptotic solutions increase with θ. However, the multiple scales solution presents large areas of very bad behavior for high values of e_0; this is due to the approximations taken for the calculation of $g_{11}(T)$ and $g_{21}(T)$, which are only valid for low values of the initial eccentricity, and the singularity at $e_0 = 1$. It is also noteworthy the apparition of 'high precision islands' in the regular expansion, for large values of θ. This areas do not really correspond to a good behavior of the regular expansion, but are due to the accumulation of the drift in θ between the reference and the asymptotic solutions; for long times, this drift may cause two different minimums of r to coincide, yielding a false low reading of the error.

Finally, Figure (7) shows an orbit with $e_0 = 0.7$ and $\varepsilon = 0.001$, for both the reference and the multiple scales solutions. The asymptotic solution remains close to the exact solution during the first orbital revolutions, but slowly separates for larger values of θ. The rotation of the eccentricity vector, given by γ, can be clearly appreciated.

CONCLUSIONS

Two asymptotic solutions for the two body problem perturbed by a small, constant acceleration oriented along the radial direction have been obtained, using regular expansions and the method of multiple scales. The equations of motion have been defined using Dromo orbital formulation, which has proven very adequate for this purpose. Some important conclusions are reached from studying the structure and behavior of these solutions:

- The regular expansion fails for $\theta \sim 1/\varepsilon$, where θ is Dromo independent variable, related with the true anomaly, and ε is the non-dimensional perturbing acceleration. Consequently, it cannot be used to propagate orbits for long periods of time, unless a reinitialization process like the one proposed by Bombardelli et al.[2] is included.

- The method of multiple scales reveals that the problem has *two fundamental scales*. The first one is responsible of the 2π-periodic oscillations along each orbit; while the second one, with a period depending on ε and e_0, drives the long term variations of the mean values and the amplitudes of the oscillations. New periods appear in some cases as the combination of the two fundamental ones. As expected, this solution behaves noticeably better than the regular expansion in most cases.

Finally, to close the multiple scales solution for the terms of $\mathcal{O}(\varepsilon)$, an additional expansion in the initial eccentricity e_0 has been introduced. Since this degrades both the quality of the solution for high initial eccentricities and the accuracy of the expression for the slow time scale, an interesting future work would be to derive a better approximation for these elements.

APPENDIX: COMPONENTS OF THE FIRST ORDER TERMS OF THE MULTIPLE SCALES SOLUTION

The first order terms of the multiple scales solution for the two-body, low radial constant thrust problem are given as a combination of the following functions:

$$\mathcal{P}^*_{11}(\tau,T) = -\frac{(q_{10}(T)+q_{3i})\,(1+\cos\tau)+q_{20}(T)\sin\tau}{q_{3i}\left(q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)\right)(q_{3i}+q_{10}(T)\cos\tau+q_{20}(T)\sin\tau)}$$

$$\mathcal{S}^*_{11}(T) = -\frac{2q_{20}(T)}{q_{3i}\left(q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)\right)^{3/2}}$$

$$\mathcal{P}^*_{21}(\tau,T) = \frac{q_{10}(T)q_{20}(T)\,(1+\cos\tau)+\left(-q_{3i}^2+q_{20}^2(T)+q_{3i}q_{10}(T)\right)\sin\tau}{q_{3i}\,(q_{3i}-q_{10}(T))\left(q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)\right)(q_{3i}+q_{10}(T)\cos\tau+q_{20}(T)\sin\tau)}$$

$$\mathcal{S}^*_{21}(T) = \frac{2q_{10}(T)}{q_{3i}\left(q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)\right)^{3/2}}$$

$$\mathcal{K}^*(\tau,T) = -\frac{\left(\sqrt{q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)}-q_{3i}+q_{10}(T)\right)\sin\tau-q_{20}(T)(1+\cos\tau)}{(q_{3i}-q_{10}(T))(1-\cos\tau)+q_{20}(T)\sin\tau+(1+\cos\tau)\sqrt{q_{3i}^2-q_{10}^2(T)-q_{20}^2(T)}}$$

Introducing the first integral $q_{10}^2(T)+q_{20}^2(T)=q_{1i}^2$, these expressions take simpler forms

$$\mathcal{P}_{11}(\tau,T) = -\frac{(q_{10}(T)+q_{3i})\,(1+\cos\tau)+q_{20}(T)\sin\tau}{q_{3i}\left(q_{3i}^2-q_{1i}^2\right)(q_{3i}+q_{10}(T)\cos\tau+q_{20}(T)\sin\tau)} \tag{39}$$

$$\mathcal{S}_{11}(T) = -\frac{2q_{20}(T)}{q_{3i}\left(q_{3i}^2-q_{1i}^2\right)^{3/2}} \tag{40}$$

$$\mathcal{P}_{21}(\tau,T) = \frac{q_{10}(T)q_{20}(T)\,(1+\cos\tau)+\left(-q_{3i}^2+q_{20}^2(T)+q_{3i}q_{10}(T)\right)\sin\tau}{q_{3i}\,(q_{3i}-q_{10}(T))\left(q_{3i}^2-q_{1i}^2\right)(q_{3i}+q_{10}(T)\cos\tau+q_{20}(T)\sin\tau)} \tag{41}$$

$$\mathcal{S}_{21}(\tau,T) = \frac{2q_{10}(T)}{q_{3i}\left(q_{3i}^2-q_{1i}^2\right)^{3/2}} \tag{42}$$

$$\mathcal{K}(\tau,T) = -\frac{\left(\sqrt{q_{3i}^2-q_{1i}^2}-q_{3i}+q_{10}(T)\right)\sin\tau-q_{20}(T)(1+\cos\tau)}{(q_{3i}-q_{10}(T))(1-\cos\tau)+q_{20}(T)\sin\tau+(1+\cos\tau)\sqrt{q_{3i}^2-q_{1i}^2}} \tag{43}$$

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