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Modeling and attitude control of spacecraft with an unbalanced rotating device

Davide Invernizzi[1]

***Abstract*—This paper addresses the problem of controlling the attitude of spacecraft endowed with a rotating device, motivated by recent space applications which will make use of large rotating payloads. Due to the presence of uncertain and potentially large inertial asymmetries in the rotating device, an internal force and a torque can appear at interface between the spacecraft and the rotor, causing performance degradation and even affecting the system stability. To counteract such unbalance effects, active balancing systems, using movable masses mounted on the rotating device, are being considered in the literature. During the balancing phase, it is important for the attitude control system to maintain a stable configuration. After deriving a suitable control-oriented model of the multi-body spacecraft, we propose using a coordinate-free attitude controller that ensures safe balancing operations and desirable pointing performance.**

***Index Terms*—Spacecraft control, geometric control.**

I. Introduction

In recent years there has been an increasing interest in space missions that require the use of spacecraft endowed with rotating devices, such as large antenna reflectors, to achieve high Earth observation capabilities [1], [2]. These systems demand a careful design since the presence of unavoidable inertial asymmetries in the rotating devices give rise to internal forces and torques in the spacecraft, leading to a reduction of pointing accuracy and stability, and undermining the outcome of the mission itself.

While the system considered in this paper falls within the class of asymmetric dual-spin satellites, existing works focus mostly on the study of the torque-free dynamics and typically assume small inertial asymmetries [3], [4], [5]. In order to mitigate the detrimental effects of potentially large inertial asymmetries in the rotating device, two viable approaches can be considered: 1) the development of an attitude control system on the spacecraft base capable of rejecting the interface force and torque; 2) the design of a balancing system made of movable masses mounted on the rotating device to actively cancel the inertial unbalances [6]. The first option is interesting because it would solve the problem just by using suitable control laws. On the other hand, it would be inefficient and risky in case of large unbalances: the attitude control system would waste a lot of power in compensating the unbalance torque and the prolonged application of interface loads could damage the bearings of the motor sustaining the payload rotation. The second option would require carrying additional mass onboard the spacecraft. However, it would also reduce risks associated with the unbalances and potentially lead to better pointing performance. Of course, the attitude control system of the spacecraft should be able to keep a stable configuration while balancing operations are carried out during the first phases of the mission. In this work, we specifically address the latter problem.

First of all, the dynamical model of the multi-body spacecraft is derived leveraging Lagrange's approach developed directly on the configuration manifold with no parametrization. We refer to the center of mass of the overall spacecraft as opposed to the center of mass of the spacecraft base [7]: under reasonable assumptions, this choice allows decoupling the attitude from the position dynamics to obtain a suitable model for attitude control analysis and design purposes. The control design is then carried out using a coordinate-free formulation [8] that avoids singularities or ambiguities associated with parametrization of $\mathrm{SO}(3)$, the attitude configuration manifold. Using Lyapunov methods and results from differential geometry, we show that the considered control law can be tuned to guarantee robust asymptotic stability of the desired attitude configuration and globally uniformly ultimately bounded errors for any inertial unbalance.

Notation. $\mathbb{R}\,(\mathbb{R}_{>0},\mathbb{R}_{\geq 0})$ denotes the set of (positive, nonnegative) real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices. The i-th vector of the canonical basis in $\mathbb{R}^n$ is denoted is denoted as $e_i := [0 \dots 0\ 1\ 0 \dots 0]^\top$ (1 in the i-th entry, 0 elsewhere) for $i \in \{1,\dots,n\}$ and the identity matrix in $\mathbb{R}^{n\times n}$ is $I_n := [e_1 \cdots e_i \cdots e_n]$. Given vectors $x,y \in \mathbb{R}^n$, the standard inner product is defined as $\langle x,y\rangle := x^\top y$. The Euclidean norm of a vector $x\in\mathbb{R}^n$ is $|x| := \sqrt{\langle x,x\rangle}$. The minimum and maximum eigenvalues of $A\in\mathbb{R}^{n\times n}$ are denoted by $\lambda_m(A)$ and $\lambda_M(A)$, respectively. For a matrix $A\in\mathbb{R}^{n\times n}$, $\mathrm{skew}(A) := A - A^\top$ and $\mathrm{sym}(A) := A + A^\top$. The n-dimensional unit sphere is denoted as $\mathbb{S}^n := \{x\in\mathbb{R}^{n+1} : |x| = 1\}$. The set $\mathrm{SO}(3) := \{R\in\mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$ is the 3D Special Orthogonal group. The map $S(\cdot) : \mathbb{R}^3 \to \mathfrak{so}(3) := \{W\in\mathbb{R}^{3\times 3} : W = -W^\top\}$ is defined such that given $x,y\in\mathbb{R}^3$ one has $S(x)y = x\times y$. The inverse of S is denoted S^{-1}. Given a differentiable function $g : \mathbb{R}^{m\times n} \mapsto \mathbb{R}$, the gradient with respect to x is represented by $\nabla_x g(x) : \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{m\times n}$.

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II. Mathematical modeling

A. Multi-body system configuration

The spacecraft that we consider in this work is a multi-body system made by a base and a payload which is rotating about a fixed axis in the base frame through a revolute joint constraint. The main difference with respect to the convectional "gyrostat" model [3] lies in the fact that the rotating device is neither assumed to be axis-symmetric nor to have its Center of Mass (CoM) lying on the axis of rotation (see Figure 1). To describe the system configuration we introduce several Cartesian reference frames:

- an inertial frame $\mathcal{F}_i := (O_i, \{i_1, i_2, i_3\})$ fixed at center of the Earth;
- a base-fixed frame $\mathcal{F}_b := (O_b, \{b_1, b_2, b_3\})$, with O_b being the CoM of the base. The difference vector between O_b and O_i resolved in $\mathcal{F}_i$ is denoted by $x_b \in \mathbb{R}^3$.
- a base-fixed frame $\mathcal{F}_a := (O_a, \{a_1, a_2, a_3\})$, with O_a being the attachment point between the base and the payload and $a_3 \in \mathbb{S}^2$ identifying the axis of rotation. Without loss of generality, we assume $a_i \equiv b_i \ \forall i \in \{1,2,3\}$, namely, $\mathcal{F}_a$ is aligned with $\mathcal{F}_b$. The difference vector between O_a and O_b resolved in $\mathcal{F}_b$ is denoted by $h_a \in \mathbb{R}^3$.
- a payload-fixed frame $\mathcal{F}_p := (O_p, \{p_1, p_2, p_3\})$, with $O_p \equiv O_a$. The difference vector between O_p and O_i resolved in $\mathcal{F}_i$ is denoted by $x_p \in \mathbb{R}^3$. The difference vector between the payload CoM and O_p resolved in $\mathcal{F}_p$ is denoted by $\bar{x}_p \in \mathbb{R}^3$.

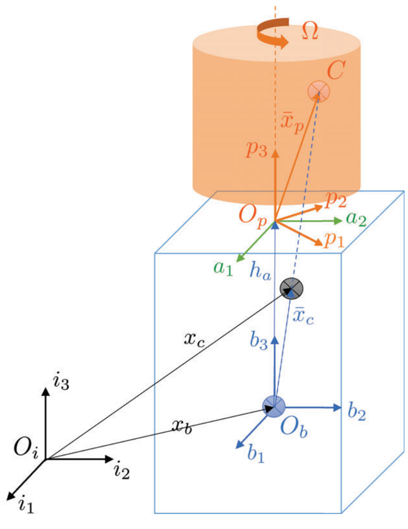

Fig. 1. Multibody spacecraft configuration.

The inertial properties of the system are defined as follows: $m_b \in \mathbb{R}_{>0}$ and $m_p \in \mathbb{R}_{>0}$ denote the mass of the base and of the payload, respectively; $J_b = J_b^\top \in \mathbb{R}^{3\times 3}_{>0}$ and $J_p = J_p^\top \in \mathbb{R}^{3\times 3}_{>0}$ denote the inertia matrices of the base and of the payload, resolved in $\mathcal{F}_b$ and $\mathcal{F}_p$, respectively.

The attitude of the spacecraft is identified by the rotation matrix $R := [\, b_1 \; b_2 \; b_3 \,] \in \mathrm{SO}(3)$, with each axis $b_i \in \mathcal{S}^2$ resolved in $\mathcal{F}_i$. The payload attitude with respect to the inertial frame is given by the composition of rotations

$$R_p := RQ, \tag{1}$$

where Q is the (planar) rotation matrix between $\mathcal{F}_p$ and $\mathcal{F}_a$. For what concerns the position kinematics, the main point of departure with respect to other modeling approaches [7] is that we do not refer to a fixed point in the base frame (such as O_b) to derive the position dynamics but rather refer to the CoM of the overall spacecraft, denoted hereafter as C. The difference vector between C and O_i, resolved in $\mathcal{F}_i$, is denoted x_c and is related to the base and payload CoM as follows:

$$x_c = \frac{m_b x_b + m_p x_p}{m_b + m_p}. \tag{2}$$

The configuration of the spacecraft can be uniquely identified by the tuple (R, x_c, Q), which is an element of the nonlinear seven-dimensional manifold $\mathrm{SO}(3) \times \mathbb{R}^3 \times \mathrm{SO}(3)$.

Before proceeding, we derive some useful expressions relating the position of the base and of the payload CoM to the configuration variables. To this aim, let us introduce the difference vector between C and O_b, resolved in $\mathcal{F}_b$:

$$\begin{aligned} \bar{x}_c &:= R^\top (x_c - x_b) = R^\top \left(\frac{m_b x_b + m_p (x_b + R(h_a + Q\bar{x}_p))}{m_b + m_p} - x_b \right) \\ &= R^\top \left(\frac{m_p (R(h_a + Q\bar{x}_p))}{m_b + m_p} \right) = \mu_p (h_a + Q\bar{x}_p) = \bar{x}_c(Q) \end{aligned} \tag{3}$$

where $\mu_p := \frac{m_p}{m_p + m_b} \in \mathbb{R}_{>0}$, and where we used equation (2) and $x_p = x_b + R(h_a + Q\bar{x}_p)$ in the first line and the identity $R^\top R = I_3$ in the second line. Equation (3) shows the dependency of location of the CoM in the body frame on the orientation of the payload Q. Indeed, the only case in which this dependency is removed occurs when $\bar{x}_p$ lies on the axis of rotation, *i.e.*, when $\bar{x}_p = [\, 0 \; 0 \; \bar{x}_{p_3} \,]^\top$, so that one would have $Q\bar{x}_p = \bar{x}_{p_3} Q e_3 = x_{p_3} e_3 = \bar{x}_p$. Based on (3), the inertial position of the base CoM can be written in terms of x_c as follows:

$$x_b = x_c - R\bar{x}_c = x_c - \mu_p R(h_a + Q\bar{x}_p). \tag{4}$$

Using geometric arguments and (4), the inertial position of the payload CoM can be computed as a function of x_c as follows:

$$x_p = x_b + R(h_a + Q\bar{x}_p) = x_c + \mu_b R(h_a + Q\bar{x}_p) \tag{5}$$

where $\mu_b := \frac{m_b}{m_p + m_b}$.

B. Kinematics

The attitude kinematics of the base-fixed frame is given by the matrix differential equation

$$\dot{R} = RS(\omega) \tag{6}$$

where $\omega \in \mathbb{R}^3$ is the body angular velocity of the base. The relative kinematics of the payload-fixed frame is a planar rotation (due to the revolute joint constraint) described by

$$\dot{Q} = QS(\Omega e_3) = QS(e_3)\Omega, \tag{7}$$

where $\Omega \in \mathbb{R}$ is the angular rate of the payload with respect to the base about the spin axis. The attitude kinematics of the payload is obtained by differentiating equation (1):

$$\begin{aligned} \dot{R}_p &= \dot{R}Q + R\dot{Q} = RS(\omega)Q + RQS(e_3)\Omega \\ &= RQ(Q^\top S(\omega) Q + S(e_3)\Omega) = R_p S(Q^\top \omega + e_3 \omega) \end{aligned} \tag{8}$$

where we exploited the linearity property of the $S^{-1}(\cdot)$ map and the property $S^{-1}(R^\top S(\omega) R) = R^\top \omega \ \forall (R, \omega) \in \mathrm{SO}(3) \times$

$\mathbb{R}^3$. From equation (8), one can define the payload angular velocity resolved in $\mathcal{F}_p$ as

$$\omega_p = Q^\top \omega + \Omega e_3, \tag{9}$$

which gives the compact expression $\dot{R}_p = R_p S(\omega_p)$.

For what concerns the position kinematics, the motion of the CoM of the overall system is given by

$$\dot{x}_c = v_c, \tag{10}$$

with $v_c \in \mathbb{R}^3$ representing the CoM inertial velocity. With the aim of computing the inertial velocity of the base CoM as a function of v_c, one has to differentiate equation (4) as follows:

$$\dot{x}_b = v_c - \mu_p R\left(S(\omega)(h_a + Q\bar{x}_p) + QS(e_3)\bar{x}_p\Omega\right) =: v_b \tag{11}$$

where (6) and (7) are used. Similarly, the inertial velocity of the payload CoM is obtained by differentiation of (5):

$$\begin{aligned} \dot{x}_p &= v_c + \mu_b \dot{R}(h_a + Q\bar{x}_p) + \mu_b R\dot{Q}\bar{x}_p \\ &= v_c + \mu_b R\left(S(\omega)(h_a + Q\bar{x}_p) + QS(e_3)\bar{x}_p\Omega\right) =: v_p \end{aligned} \tag{12}$$

The kinematics of the system is fully characterized by the tuple $(\omega, v_c, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, which is consistent with the dimension of the configuration manifold.

C. Dynamics and equations of motion

The equations of motion are derived in this section using Euler-Lagrange equations for systems evolving in the product manifold $\mathrm{SO}(3) \times \mathbb{R}^3$, which deserves some care as SO(3) is a nonlinear manifold. The Lagrangian of the system described in Section II-A is given by $\mathcal{L} := T - U$, where

$$T := \tfrac{1}{2}m_b|v_b|^2 + \tfrac{1}{2}\omega^\top J_b\omega + \tfrac{1}{2}m_p|v_p|^2 + \tfrac{1}{2}\omega_p^\top J_p\omega_p, \tag{13}$$

$$U := -\int_{\mathcal{B}_b} \tfrac{\mu}{|x_b + R\rho|}\, dm(\rho) - \int_{\mathcal{B}_p} \tfrac{\mu}{|x_p + R_p\rho|}\, dm(\rho) \tag{14}$$

are the kinetic and the potential energy associated with the gravity field, respectively, $\mu := 398600.5\,\mathrm{km}^3\mathrm{s}^{-2}$ is the (earth) gravitational constant, $\mathcal{B}_b$ and $\mathcal{B}_p$ are all the base and payload material points, respectively, $dm(\rho)$ is the infinitesimal mass element at a given location $\rho \in \mathbb{R}^3$ such that $\int_{\mathcal{B}_i} dm(\rho) = m_i$.

Let us focus on the kinetic energy (13). Expanding the expressions for v_b in (11) and v_p in (12), the translational part of the kinetic energy can be written as follows:

$$\begin{aligned} \tfrac{1}{2}m_b|v_b|^2 + \tfrac{1}{2}m_p|v_p|^2 &= \tfrac{1}{2}(m_b + m_p)|v_c|^2 \\ &+ \tfrac{1}{2}m_s\left|S(h_a + Q\bar{x}_p)^\top\omega + QS(\bar{x}_p)^\top e_3\Omega\right|^2 \end{aligned} \tag{15}$$

where $m_s := \frac{m_b m_p}{m_b + m_p}$ and where we exploited that the cross-term

$$(m_b\mu_p - m_p\mu_b)v_c^\top R\left(S(\omega)(h_a + Q\bar{x}_p) + QS(e_3)\bar{x}_p\Omega\right)$$

vanishes because $m_b\mu_p - m_p\mu_b = m_b\frac{m_p}{m_b+m_p} - m_p\frac{m_b}{m_b+m_p} = 0$. Hence, the kinetic energy of the spacecraft is given by

$$\begin{aligned} T &= \tfrac{1}{2}m_s|v_c|^2 + \tfrac{1}{2}\omega^\top J_s(Q)\omega \\ &+ \omega^\top\left(Q\bar{J}_p + S(h_a)^\top QS(\bar{I}_p)\right)e_3\Omega + \tfrac{1}{2}e_3^\top\bar{J}_p e_3\Omega^2, \end{aligned} \tag{16}$$

where

$$J_s(Q) := \bar{J}_b + Q\bar{J}_pQ^\top + \mathrm{sym}(S(h_a)^\top S(Q\bar{I}_p)) \tag{17}$$
$$\bar{J}_b := J_b + m_sS(h_a)S(h_a)^\top \tag{18}$$
$$\bar{J}_p := J_p + m_sS(\bar{x}_p)S(\bar{x}_p)^\top \tag{19}$$
$$\bar{I}_p := m_s\bar{x}_p. \tag{20}$$

Remark 1: According to the proposed selection of configuration variables, the kinetic energy can be decomposed as the sum of a translational and a rotational term as follows:

$$T = T_{pos}(v_c) + T_{rot}(Q, \Omega, \omega) \tag{21}$$

where $T_{pos}(v_c) := \frac{1}{2}m_s|v_c|^2$ and $T_{rot}(Q,\Omega,\omega) := \frac{1}{2}\omega^\top J_s(Q)\omega + \omega^\top\left(Q\bar{J}_p + S(h_a)^\top QS(\bar{I}_p)\right)e_3\Omega + \frac{1}{2}e_3^\top\bar{J}_pe_3\Omega^2$. Moreover, the rotational kinetic energy is invariant to changes in the base attitude. Combined with suitable assumptions, these important properties will be exploited in the control design.

At this point, assume the payload relative rotation Q be regulated at a constant rate in such a way that $\dot{\Omega} = 0$. According to this assumption, Q is not any more a free variable. When considering that the base of the spacecraft has an actuation mechanism capable of delivering a desired torque $\tau_c \in \mathbb{R}^3$, the Euler-Lagrange equations on $\mathrm{SO}(3) \times \mathbb{R}^3$ read

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\nabla_{v_c}\mathcal{L} - \nabla_{x_c}\mathcal{L} = 0 \tag{22}$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\nabla_\omega\mathcal{L} + S(\omega)\nabla_\omega\mathcal{L} - S^{-1}(\mathrm{skew}(R^\top\nabla_R\mathcal{L})) = \tau_c, \tag{23}$$

which fully describe the dynamical model of the overall system together with the kinematic equations (6), (7), (10). By substituting the Lagrangian with (21) into (22)-(23), the full set of equations is given by (6), (7), (10), and

$$(m_p + m_b)\dot{v}_c = f_u(R, x_c, Q) \tag{24}$$

$$\begin{aligned} J_s(Q)\dot{\omega} &= \tau_u(R, x_c, Q) - \dot{J}_s(Q)\omega + \\ &- S(\omega)h_g(Q,\Omega,\omega) + \tau_e(Q,\Omega) + \tau_c, \end{aligned} \tag{25}$$

$f_u = \nabla_{x_c}U$, $\tau_u = S^{-1}(\mathrm{skew}(R^\top\nabla_R U))$, $\dot{J}_s = \Omega(Q(S(e_3)\bar{J}_p) + \bar{J}_pS(e_3))Q^\top + \mathrm{sym}(S(h_a)^\top S(QS(e_3)\bar{I}_p))$ and

$$h_g := J_s(Q)\omega + \left(Q\bar{J}_p + S(h_a)^\top QS(\bar{I}_p)\right)e_3\Omega \tag{26}$$

$$\tau_e := \left(QS(e_3)^\top\bar{J}_p + S(h_a)QS(e_3)S(\bar{I}_p)\right)e_3\Omega^2. \tag{27}$$

The following Lemma provides a useful characterization of the dependence of h_g and τ_e on the inertial unbalances.

Lemma 1: Let $d := \begin{bmatrix}\bar{I}_{p_1} & \bar{I}_{p_2} & \bar{J}_{p_{13}} & \bar{J}_{p_{23}}\end{bmatrix}^\top$, then the generalized angular momentum h_g in (26) and the perturbing torque τ_e in (27) can be written as affine functions with respect to d as follows:

$$h_g = J_s(Q)\omega + \bar{J}_{p_{33}}\Omega e_3 + \Omega\left(QW_J + S(h_a)^\top QW_I\right)d \tag{28}$$

$$\tau_e = \Omega^2\left(QS(e_3)^\top W_J + S(h_a)QS(e_3)W_I\right)d. \tag{29}$$

where $W_J := \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$, $W_I := \begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\end{bmatrix}$.

Proof: The expressions (28)-(29) can be obtained by rewriting the terms $Q\bar{J}_pe_3$ and $S(\bar{I}_p)e_3$ which appear in both (26) and (27). Noting that $Q\bar{J}_pe_3 = Q(I_3 \pm e_3e_3^\top)\bar{J}_pe_3 = \bar{J}_{p_{33}}Qe_3 + Q(I_3 - e_3e_3^\top)\bar{J}_pe_3$, the first term in the sum simplifies as $\bar{J}_{p_{33}}Qe_3 = \bar{J}_{p_{33}}e_3$ since Q is a planar rotation about e_3. As for the second term of the sum, one has $(I_3 - e_3 e_3^\top)\bar{J}_p e_3 = [\bar{J}_{p13}\ \bar{J}_{p23}\ 0]^\top = W_J d$. Given $S(\bar{I}_p)e_3 = -S(e_3)\bar{I}_p = W_I d$, the Lemma is proven with straightforward substitutions. ■

It is worth noting that d includes the static unbalances of the payload CoM only in the plane perpendicular to the payload rotation axis $(\bar{I}_{p_1}, \bar{I}_{p_2})$ and that d does not include the dynamic unbalance on the same plane, *i.e.*, $\bar{J}_{p_{12}}$. The torque $\tau_e(Q,\Omega)$ depends on the payload rotation Q, is null whenever $\Omega = 0$ or $d = 0$ and is periodically time-varying $\forall \Omega \neq 0$ when seeing $Q = Q(t)$ as the solution to (7) for some initial condition $Q(t_0)$.

III. Control model and problem formulation

By inspecting equations (10), (24) and (6), (25) one sees that the position and attitude dynamics are only coupled through the force and torque associated with gravity, *i.e.*, through f_u and τ_u. This work addresses the scenario in which the residual disturbance torque τ_e due to inertial unbalances is much larger than environmental torques. In this condition, it is reasonable to neglect τ_u for control design purposes.[1]As a byproduct of this assumption, the dynamical system takes a cascade structure, wherein the attitude subsystem (6), (25) perturbs the position subsystem (10), (24). The control design for the attitude dynamics, being the upper subsystem in the cascade, can be carried out independently. The model for control is derived based on the following modeling assumptions.

Assumption 1: 1) The base and the rotating device are rigid bodies; 2) the angular velocity Ω of the device relative to the base is constant; 3) the environmental torque (τ_u in (25)) is much smaller in magnitude than the internal torque associated with the inertial unbalances of the rotating device (τ_e in (29)).

Based on the above reasoning, we are concerned with the design of a control torque τ_c to stabilize a desired attitude $R_d \in \mathrm{SO}(3)$ for the system comprising the base kinematics (6) and the angular velocity dynamics (25) with $\tau_u = 0$, *i.e.*,

$$J_s(Q)\dot{\omega} = -\dot{J}_s(Q)\omega - S(\omega)h_g(Q,\Omega,\omega) + \tau_e(Q,\Omega) + \tau_c. \tag{30}$$

We can now formalize the control problem that we are addressing in this work as follows.

Problem 1: Consider the dynamics of the multi-body spacecraft in (6), (7), (30). Given any desired attitude $R_d \in \mathrm{SO}(3)$, design a controller delivering a control torque $\tau_c \in \mathbb{R}^3$ such that the point $(R,\omega) = (R_d, 0)$ is locally ISS with respect to d and the solutions of the closed-loop system are Globally Uniformly Ultimately Bounded (GUUB) $\forall d \in \mathbb{R}^4$.
Any controller solving Problem 1 is a good candidate to ensure safe balancing operations because it would guarantee ultimate boundedness of the errors for any inertial unbalance and for any initial condition (globally) while guaranteeing the nice properties associated with local ISS[2], in particular, the closed-loop system would be robustly asymptotically stable.

[1]For the kind of space missions relevant to this work, the gravity gradient torque is in the order of 10^{-5}Nm whereas admissible residual unbalances are in the order of 10^{-1}Nm [9]. Hence, gravity gradient and environmental perturbations can be considered as exogenous disturbances affecting the steady-state performance of the control system.

[2]Asking for local ISS is not restrictive when referring to continuous time-invariant control laws on SO(3) due to the unavoidable presence of multiple equilibria in the closed-loop dynamics (see also Footnote 3).

IV. Control law design and stability analysis

A. Control architecture and closed-loop error dynamics

To solve Problem 1, define the tracking errors

$$R_e := R_d^\top R, \qquad \omega = 0 \tag{31}$$

and consider the following control torque:

$$\tau_c := -\gamma_R(R_e) - K_\omega \omega, \tag{32}$$

where

$$\gamma_R(R_e) := \tfrac{1}{2} S^{-1}(\mathrm{skew}(K_R R_e)), \tag{33}$$

with $K_\omega \in \mathbb{R}^{3\times 3}$ being a positive definite matrix and $K_R \in \mathbb{R}^{3\times 3}$ being a symmetric matrix such that $\mathrm{trace}(K_R)I_3 - K_R$ is positive definite. The use of a control law directly developed on $\mathrm{SO}(3)\times\mathbb{R}^3$ allows avoiding singularity issues associated with mininimal parametrizations or ambiguities associated with quaternions [8]. When using any minimal parametrization of SO(3), it can be shown that γ_R acts as a proportional controller for small attitude errors: given small angles $\theta_e \in \mathbb{R}^3$ such that $R_e \approx I_3 + S(\theta_e)$, one has $\gamma_R(R_e(\theta_e)) \approx K_R \theta_e$.

Remark 2: The control law (32) is just one representative candidate that solves Problem 1 despite being designed for the dynamics of a single rigid body (which is much simpler that the dynamics considered here). The use of more advanced attitude controllers developed in recent years could be considered as well [10], [11], [12], [13]. Nonetheless, the control law (32) stands out for its simplicity and is therefore appealing for space applications, being independent of the system inertial parameters[3] and requiring only (standard) attitude and velocity measurements of the spacecraft base.

The resulting closed-loop error dynamics is presented in the following proposition.

Proposition 1: Consider the dynamics (6), (7) (30) and the controller (32). Using the stabilization error in (31), the closed-loop error dynamics is an autonomous system described by the payload kinematics (7) and by

$$\dot{R}_e = R_e S(\omega) \tag{34}$$

$$\begin{aligned} J_s(Q)\dot{\omega} = &-(K_\omega + \dot{J}_s(Q))\omega \\ &- S(\omega)h_g(Q,\Omega,\omega) + \tau_e(Q,\Omega) - \gamma_R(R_e). \end{aligned} \tag{35}$$

Proof: The only equation requiring some effort obtaining is (34), which can be derived as follows: $\dot{R}_e = \dot{R}_d^\top R + R_d^\top \dot{R} = R_d^\top R S(\omega) = R_e S(\omega)$, where we exploited (6) and $\dot{R}_d = 0$. ■

B. Stability analysis and main results

We note that for null inertial unbalances, *i.e.*, for $d = 0$, the closed-loop error system (34), (35) has an equilibrium set given by $\{R_e, \omega : \gamma_R(R_e) = 0, \omega = 0\}$, which contains the desired attitude ($R_e = I_3$) plus additional equilibria.[4] The existence of multiple equilibria cannot be avoided when using a time-invariant continuous stabilizer on SO(3) and requires a careful stability analysis [8]. While the control law in (32)

[3]As shown next, in Theorem 1, one needs to know only (conservative) bounds on the inertial parameters for stability reasons.

[4]For instance, when K_R is diagonal with distinct eigenvalues, the undesired equilibria are $R_i = R_d \exp(\pi e_i)$, $i \in \{1,2,3\}$, *i.e.*, 180deg rotations about the axes of the desired attitude. See [8] for further details.

is an implementation of the one suggested in [8] for attitude tracking, the complex dynamics of the multi-body spacecraft that we consider in this work requires additional effort to prove that it solves Problem 1, as claimed by the following theorem.

Theorem 1: The controller (32) with K_ω satisfying $\lambda_m(K_\omega) > c_m := |\Omega| \left(\sqrt{\bar{J}_{p_{12}}^2 + \frac{\bar{J}_{p_{11}} - \bar{J}_{p_{22}}}{4}} + |h_a| \sqrt{\bar{I}_{p_1}^2 + \bar{I}_{p_2}^2} \right)$ solves Problem 1.

Proof: We start the proof by showing local ISS of the desired equilibrium point. To this end, consider the Lyapunov candidate

$$V(R_e, Q, \omega) := \tfrac{1}{2}|R_e|_{K_R}^2 + \tfrac{1}{2}\omega^\top J_s(Q)\omega + c\omega^\top J_s(Q)\gamma_R(R_e), \tag{36}$$

where $|R_e|_{K_R}^2 := \frac{1}{2}\text{tr}\left((I_3 - R_e)^\top K_R (I_3 - R_e)\right) = \text{tr}(K_R(I_3 - R_e))$ and $c > 0$ is an arbitrary small positive scalar. Consider now the following two lemmas.

Lemma 2: Let $x_e := \begin{bmatrix} |\omega| & |R_e|_{K_R} \end{bmatrix}^\top \in \mathbb{R}_{\geq 0}^2$ and $\ell_R := \lambda_m(\text{tr}(K_R)I_3 - K_R) > 0$. Then, $\forall 0 < \ell < \ell_R$ there exist positive constants a_1, a_2, c such that V in (36) is a quadratic function with respect to x_e in the set $\Omega_\ell := \left\{(R, Q, \omega) \in \text{SO}(3) \times \text{SO}(3) \times \mathbb{R}^3 : |R|_{K_R} \leq \ell\right\}$, namely:

$$x_e^\top W_1 x_e \leq V(R_e, Q, \omega) \leq x_e^\top W_2 x_e, \quad \forall (R_e, Q, \omega) \in \Omega_\ell \tag{37}$$

where $W_1 := \frac{1}{2}\begin{bmatrix} b_1 & -ca_2b_2 \\ -ca_2b_2 & 1 \end{bmatrix}$ and $W_2 := \frac{1}{2}\begin{bmatrix} b_2 & ca_2b_2 \\ ca_2b_2 & 1 \end{bmatrix}$, $b_1 := \min_{Q \in \text{SO}(3)} \lambda_m(J_s(Q))$, $b_2 := \max_{Q \in \text{SO}(3)} \lambda_M(J_s(Q))$.

The proof of the Lemma hinges on [8, Lemma 12], according to which there exist two positive constants a_1, a_2 such that $a_2^{-1}|\gamma_R| \leq |R_e|_{K_R} \leq a_1^{-1}|\gamma_R|$ $\forall R_e \in \{R \in \text{SO}(3) : |R|_{K_R} \leq \ell\}$, $\forall 0 < \ell < \ell_R$. Constants b_1 and b_2 are finite thanks to the continuity of the eigenvalue function and the compactness of SO(3). Matrices W_1, W_2 are positive definite $\forall c < \sqrt{b_1}a_2^{-2}b_2^{-2}$.

Lemma 3: There exists a constant $c > 0$ such that the Lie derivative of V in (36) along the closed-loop system (34)-(35) satisfies the following dissipation inequality:

$$\dot{V} \leq -c_4|x_e|^2 + c_5|x_e||d| \leq -c_6|x_e| \quad \forall |x_e| \geq \tfrac{c_5}{c_4\lambda}|d|, \tag{38}$$

$\forall (R_e, Q, \omega) \in \Omega_\ell$, $\forall 0 < \ell < \ell_R$, $\forall |d| \leq \bar{d} < \frac{c_4\lambda\sqrt{\ell}}{c_5\sqrt{\lambda_M(W_2)}}$, $\forall \lambda \in (0,1)$, $c_6 := c_4(1-\lambda)$, for some constants $c_4, c_5 > 0$.

The proof of Lemma 3 is reported in the Appendix. Based on Lemma 2 and Lemma 3, local ISS of $x_e = 0$, *i.e.*, $R = R_d$, $\omega = 0$, with respect to d then follows by resorting to the local version of [14, Theorem 4.19].

To finally conclude that the controller (32) solves Problem 1, we need to show that the closed-loop solutions are GUUB for any inertial unbalance d. To this aim, we refer to the Lyapunov function $V_1 := \frac{1}{2}|R_e|_{K_R}^2 + \frac{1}{2}\omega^\top J_s(Q)\omega$ and compute its Lie derivative along the closed-loop system (7), (34), (35), which can be bounded as follows:

$$\dot{V}_1 \leq -c_7|\omega|^2 + \omega^\top \tau_e \leq -\tfrac{c_7}{2}|\omega|^2 + \tfrac{2}{c_7}|\tau_e|^2, \tag{39}$$

$c_7 := \lambda_m(K_\omega) - c_m$ (positive by assumption), where we used the property $\omega^\top S(\omega)h_g = 0$ and Young's inequality. By further noting that $|\omega|^2 \geq \frac{1}{b_2}\omega^\top J_s(Q)\omega$ and that $\omega^\top J_s(Q)\omega = 2V_1 - |R_e|_{K_R}$, the following inequality is obtained from (39):

$$\dot{V}_1 \leq -\tfrac{c_7}{b_2}\left(V_1 - \tfrac{1}{2}|R_e|_{K_R}\right) + \tfrac{2}{c_7}|\tau_e|^2 \leq -c_8V_1 + \delta \tag{40}$$

where $c_8 := \frac{c_7}{b_2} > 0$, and $\delta := \frac{2}{c_7}\max_{Q \in \text{SO}(3)}|\tau_e(Q, \Omega)| + \frac{c_7}{2b_2}\lambda_M(\text{tr}(K_R)I_3 - K_R))$ is a positive constant which exists finite because d is bounded and $\tau_e(Q, \Omega)$ is continuous in both Q and Ω, with SO(3) compact and Ω finite. By leveraging the Comparison Lemma [14, Lemma 3.4], $V_1(t)$ satisfies the inequality $V_1(t) \leq \exp(-c_8(t - t_0))V_1(t_0) + \frac{\delta}{c_8}(1 - \exp(-c_8(t - t_0)))$ $\forall t \geq t_0$, which gives GUUB of the closed-loop solutions. ■

V. Numerical results

We report a numerical example to show the effectiveness of the proposed control design in a mission scenario in which the attitude of the multi-body spacecraft must be stabilized starting from perturbed initial conditions. The simulation data are inspired by the CIMR mission [1]: the spacecraft operates along an almost polar, slightly elliptical orbit ($i = 98.7$deg, $e = 0.0011$), with an altitude of 824.6km, a corresponding orbital period of $T = 6074.7$s and an orbital rate $\omega_o = 0.001$rad/s. The payload is rotating at $\Omega = 1$ rad/s. The following inertial and geometric parameters are used, $m_b = 1000$kg, $J_b = \text{diag}(1000, 850, 500)\text{kgm}^2$, $m_p = 500$kg, $J_p = \begin{bmatrix} 5000 & -5 & -1 \\ -5 & 5200 & -2 \\ -1 & -2 & 1000 \end{bmatrix}\text{kgm}^2$, $h_a = \begin{bmatrix} 0 & 0 & 1.25 \end{bmatrix}^\top$m, $\bar{x}_p = \begin{bmatrix} 0.02 & 0.03 & 2 \end{bmatrix}^\top$m. The corresponding inertial unbalance is $d = \begin{bmatrix} 6.67\text{kgm} & 10\text{kgm} & -14.33\text{kgm}^2 & -22\text{kgm}^2 \end{bmatrix}^\top$. The initial conditions are $\omega(0) = \begin{bmatrix} 0.03 & 0.015 & 0.02 \end{bmatrix}^\top$ rad/s and $\phi(0) = 60$deg, $\theta(0) = 50$deg, $\psi(0) = -25$deg, where $\phi, \theta, \psi \in \mathbb{R}$ represent the roll, pitch and yaw angle, respectively.

The control goal is to stabilize the spacecraft at $(R, \omega) = (I_3, 0)$. The gains of the controller (32) are tuned to have an approximately critically damped behavior in the proximity of the desired attitude for $d = 0$. Specifically, we select $K_\omega = 2\omega_c\bar{J}_s(I_3)$ and $K_R = \bar{J}_s(I_3)\omega_c^2$, $\omega_c = 50\omega_o$, for which $\lambda_m(K_\omega) = 150$ satisfies the condition in Theorem 1, with $c_m \approx 115$.

In the simulation, we assume the presence of an ideal balancing system on the rotating payload that starts operating at 1000s and is capable of reducing the inertial unbalances in nine steps below 10% of the initial unbalances (behaving as a first order system with time constant 0.05s). Such residual unbalances are kept constant for 2000s and then completely removed starting from 4000s (see Figure 2, top plot). The considered behavior of the balancing system is inspired by [6], where a discrete-time harmonic controller is proposed, and is just one representative candidate to show that the controller (32) works as established in Theorem 1, covering both the case of partial balancing ($t < 4000$s) and the one of complete balancing ($t \geq 4000$s). Nonetheless, different balancing mechanisms could be considered as well. Assuming a mechanism that compensates for the inertial unbalances in small steps and with slow variations of the inertial properties is reasonable given the criticality of the balancing operations and at the same time keeps approximately valid the model (30), which has been developed assuming static unbalances.

The attitude tracking performance obtained by the proposed controller along one orbit is illustrated in Figure 2 (middle) in terms of roll, pitch and yaw angle errors. As expected from Theorem 1, the closed-loop solutions are ultimately bounded when the payload is not actively balanced (d constant), *i.e.*, for

$t < 1000$s. The amplitude of the residual oscillations is then reduced down to a small ultimate bound ($\approx 0.02\,$deg) when the residual unbalances are below 10% of the initial unbalance $3000\text{s} < t < 4000\text{s}$. In the ideal case of full compensation ($t \geq 4000$s), the oscillations induced by the unbalances disappear and only a small residual attitude error (associated with gravity gradient) is present, confirming the ISS property proven in Thereom 1. By inspecting Figure 2 (bottom), the amplitude of the control torque decreases accordingly with the evolution of the balancing operation.

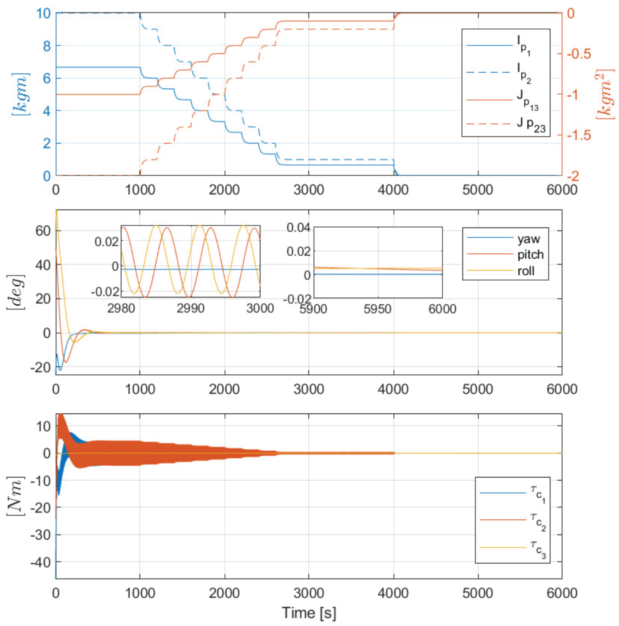

Fig. 2. Assumed inertial balancing d (top), attitude tracking performance (middle) and control torque τ_c (bottom).

VI. Conclusions

In this paper we considered the problem of controlling the attitude of a dual-spin satellite in which the rotating part is characterized by inertial asymmetries that can have a huge impact on the stability and attitude performance of the system. After deriving a suitable control-oriented model, the stabilizing property of a coordinate-free control design were analyzed leveraging results from differential geometry and nonlinear control theory. Important properties such as global uniform ultimate boundedness of the closed-loop solutions and local ISS with respect to the inertial unbalances were proven.

Appendix: proof of Lemma 3

The Lie derivative of V along (34)-(35) reads:

$$\begin{aligned}\dot{V} &= \gamma_R^\top \omega + \tfrac{1}{2}\omega^\top \dot{J}_s \omega + \omega^\top J_s(Q)\dot{\omega} \\ &\quad + c(\dot{\omega}^\top J_s(Q) + \omega^\top \dot{J}_s)^\top \gamma_R + c\omega^\top J_s(Q)\dot{\gamma}_R \\ &= -\omega^\top \left(\tfrac{1}{2}\dot{J}_s + K_\omega\right)\omega + \omega^\top \tau_e + c\gamma_R^\top \left(S(\omega)h_g + \tau_e\right) \\ &\quad - \gamma_R - K_\omega \omega)) + c\tfrac{1}{2}\omega^\top J_s(Q)(\mathrm{tr}(K_R R_e)I_3 - K_R R_e)\omega\end{aligned} \tag{41}$$

where the expression $\dot{\gamma}_R = \frac{1}{2}S^{-1}(\mathrm{skew}(K_R\dot{R}_e)) = \frac{1}{2}(\mathrm{tr}(R_e^\top K_R)I_3 - R_e^\top K_R)\omega$ has been used ([11]) as well as $\omega^\top S(\omega(h_g)) = 0$. Using the chain of inequalities $|\dot{J}_s(Q)| \leq |\Omega||Q(S(e_3)\bar{J}_p + \bar{J}_p S(e_3))Q^\top| + 2|\Omega||S(h_a)||QS(e_3)\bar{I}_p)| \leq |\Omega||S(e_3)\bar{J}_p + \bar{J}_p S(e_3)| + 2|\Omega||h_a||S(e_3)\bar{I}_p)| \leq 2c_m$, the Lie derivative (41) can be bounded in Ω_ℓ as follows

$$\begin{aligned}\dot{V} \leq -c_1|\omega|^2 + \tfrac{c|J_s|}{\sqrt{2}}\mathrm{tr}(K_R)|\omega|^2 + c|\gamma_R|\left(|S(\omega)|(|J_s||\omega| + \bar{J}_{33}|\Omega|\right. \\ \left. +|W_g||d|) - |\gamma_R| + |W_e||d| + \lambda_M(K_\omega)|\omega|\right) + |x_e||W_e||d|\end{aligned} \tag{42}$$

where we used $\frac{1}{2}|(\mathrm{tr}(R_e K_R)I_3 - R_e K_R)| \leq \frac{1}{\sqrt{2}}\mathrm{tr}(K_R)$ and defined $c_1 := \lambda_m(K_\omega) - c_m$ (positive by assumption) and $W_g(Q) := \Omega\left(QW_J + S(h_a)^\top QW_I\right)$, $W_e(Q) := \Omega^2\left(QS(e_3)^\top W_J + S(h_a)QS(e_3)W_I\right)$, which are both uniformly bounded together with $\bar{\gamma}_R := \max_{R_e \in \mathrm{SO}(3)}|\gamma_R(R_e)|$ (continuous function on a compact set) and therefore $c_5 := \max_{Q\in \mathrm{SO}(3)}(c\bar{\gamma}_R|W_g(Q)| + (c+1)|W_e(Q)|) > 0$ is finite as well. Then, one obtains $\dot{V} \leq -c_4|x_e|^2 + c_5|x_e||d|$, in which $c_4 := \lambda_m\left(\begin{bmatrix} c_1 - cc_2 & -cc_3 \\ -cc_3 & ca_1^2\end{bmatrix}\right)$, with $c_2 := \left(\frac{\mathrm{tr}(K_R)}{\sqrt{2}} + \bar{\gamma}_R\right)b_2 > 0$, $2c_3 := \bar{J}_{33}|\Omega| + \lambda_M(K_\omega) > 0$. Choosing $c < \min(\sqrt{b_1}a_2^{-2}b_2^{-2}, c_1 a_1(c_2 a_1^2 + c_3^2)^{-1})$ ensures $c_4 > 0$. Thus, $\dot{V}$ satisfies the dissipation inequality in (38) $\forall (R_e, Q, \Omega) \in \Omega_\ell$, $|d| \leq \bar{d} < \frac{c_4 \lambda \sqrt{\ell}}{c_5\sqrt{\lambda_M(W_2)}}$, $\lambda \in (0,1)$. The magnitude of d had to be restricted to a value $\bar{d} > 0$ such that the set $\left\{(R_e, \Omega, Q) : |x_e| \leq \frac{c_5}{c_4\lambda}\bar{d}\right\}$ is strictly contained in Ω_ℓ.

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