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Density of Complex and Quaternionic Polyanalytic Polynomials in Polyanalytic Fock Spaces

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Abstract

In this paper we consider the polyanalytic Fock spaces both in the complex and in the quaternionic case. In this latter case, the polyanalytic functions are considered in the slice regular case, and we shall treat Fock spaces of the first and of the second kind. In all these spaces we prove quantitative results in the approximation by polyanalytic polynomials. The quantitative approximation results are given in terms of higher order L^p-moduli of smoothness.

Keywords Polyanalytic complex Fock space · Polyanalytic complex functions · Polyanalytic complex polynomials · Polyanalytic quaternionic Fock space of the first kind · Polyanalytic quaternionic Fock space of second kind · Slice quaternionic polyanalytic functions · Slice quaternionic polyanalytic polynomials · Convolution with trigonometric kernels · Quantitative estimates · L^p-moduli of smoothness · Slice regular functions · Best approximation

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This paper is dedicated to John Ryan on the occasion of his retirement.

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This article is part of Topical Collection in Honor of Prof. John Ryan's Retirement.

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1 Introduction

Among the various interesting generalizations of the classical theory of holomorphic functions of a complex variable, there is the theory of polyanalytic functions, which are defined as nullsolutions of higher order powers of the Cauchy–Riemann operator. These functions play an important role since both the complex variable z and its conjugate $\bar{z}$ are involved in their description, and this leads to the consideration of polynomials in the two variables $z, \bar{z}$.

A complete introduction to polyanalytic functions and their basic properties can be found in [10–12]. This class of functions was studied by various authors from different perspectives, see e.g. [2–6, 14, 27] and the references therein. In quantum mechanics polyanalytic functions are also relevant, in fact they are used for the study of the Landau levels associated to Schrödinger operators, see [4]. These functions were used also in [2] to study sampling and interpolation problems on polyanalytic Fock spaces using time frequency analysis techniques such as short-time Fourier transform or Gabor transforms, see, e.g. [17]. For sampling and interpolation in function spaces, see, e.g. [1] where also image and signal processing are considered.

The literature on polyanalytic functions is rather rich, starting with the early paper by Kolossov [43–45] and Pompeiu [55], then Teodorescu's doctoral dissertation, see [56], until the book by Muskhelishvili [50] in the sixties. Other interesting contributions in this field can be found in Pascali's works in the sixties, see [51–54]. A milestone in the theory is the book [10] which also contains further references.

Given $s \in \mathbb{N}$, a complex-valued function f of a complex variable is called *s-analytic* or *polyanalytic of order s* in an open set $G \subset \mathbb{C}$, if $\overline{\partial}^s(f) = 0$ in G, where $\overline{\partial} = \partial/\partial\overline{z}$ is the Cauchy–Riemann operator and $\overline{\partial}^s$ denotes its s-power.

A polyanalytic function f of order s can be written in the form

$$f(z) = f_0(z) + \overline{z} f_1(z) + \cdots + \overline{z}^{s-1} f_{s-1}(z), z \in G, \tag{1}$$

where $f_0, \ldots, f_{s-1}$ are holomorphic in G. If, in particular, all the functions $f_0, \ldots, f_{s-1}$ are polynomials, then f is called an *s-analytic polynomial*.

We define the degree $\deg(f)$ of an s-analytic polynomial f with respect to the variable z as $\max\{\deg(f_j); j = 0, \ldots, s-1\}$. We note that, from now on, we shall omit to specify that the degree is considered with respect to z.

The representation (1) shows that the building blocks of polyanalytic functions are holomorphic functions. Note however that the class of polyanalytic functions shows deep differences with respect to the class of holomorphic functions. See [10] for the basic information on these functions.

There are many directions of research in this field. The focus of this paper is about approximation in the polyanalytic Fock spaces. Questions about approximation have already been considered in the literature. Among them, and with no claim of completeness, we mention: the problem of the uniform approximation by s-analytic polynomials, see, e.g., Fedorovskiy [26–29], Carmona-Fedorovskiy [14, 15], Carmona-Paramonov-Fedorovskiy [16], Baranov-Carmona-Fedorovskiy [12], Mazalov [47, 48], Mazalov-Paramonov-Fedorovskiy [49], Verdera [58].

Concerning approximation by s-analytic polynomials of functions which are s-analytic functions in G and continuous in $\overline{G}$, the above mentioned results are of qualitative type. Quantitative results in approximation of polyanalytic functions by s-analytic polynomials were obtained in the recent paper Gal-Sabadini [39].

Polyanalytic functions have been considered also in the quaternionic and Clifford algebra case by considering powers of suitable generalized Cauchy–Riemann operators, see [13]. In the case one considers the framework of slice regular functions, the polyanalitic case has been recently introduced, see Alpay-Diki-Sabadini [7–9]. The counterpart of the class of holomorphic functions considered in this context is the class of slice regular functions. Several approximation results have been proved in this class of functions in the past decade by Gal-Sabadini, see [33–37], and also the paper by Diki-Gal-Sabadini [24]. In particular, polynomial approximation results in this framework have been obtained in Bergman, Bloch, Besov and also in Fock spaces.

We note that results concerning approximation by polynomials in the classical complex and also quaternionic Bergman spaces can be found in Duren-Schuster [25], Hedenmalm-Korenblum-Zhu [42], Gal [31], and in Gal-Sabadini [35].

More recently we started considering the quaternionic polyanalytic case in the slice regular case, see [38, 39].

In this paper we shall introduce the polyanalytic Fock spaces first in the complex, then in the quaternionic case. In the latter case, we generalize the notion of slice polyanalytic Fock space, already considered in [7]. We also note that in [38] we have only considered the polyanalytic Bergman space of the second kind, whereas in this paper we study the Fock spaces of the first and of the second kind. The difference between the two cases consists in the definition which is done via a 4-dimensional integral in the first kind case, while it is done via a 2-dimensional integral in the second kind space. In both cases, our results are also quantitative, in terms of various L^p-moduli of smoothness and in terms of the best approximation quantity.

To obtain our results, we use the classical method of convolution with various even trigonometric kernels, successfully used by us in the past, see, e.g., Gal [30–32], Gal-Sabadini [35–38], Diki-Gal-Sabadini [24]. The methods work also in this higher dimensional case and in fact most of the computations can be carried out without significant changes. In the paper we repeat some of these computations since it is somewhat necessary to check that they are valid in this case. Where we do so we refer to the sources.

We note that one could consider also the case of the so-called true Fock spaces (for their definition see [57]) however this is outside the scopes of this work.

The plan of the paper is as follows. In Sect. 2 we introduce the complex polyanalytic Fock spaces and we obtain quantitative approximation results by polyanalytic polynomials. Section 3 contains the notions of quaternionic polyanalytic Fock spaces of first and second kind. Section 4 contains qualitative and quantitative approximation results in these Fock spaces of first kind and, similarly, Sect. 5 contains the analogous results in Fock spaces of the second kind.

2 Polyanalytic Complex Fock Spaces and Approximation by Polyanalytic Polynomials

Fock spaces in the complex setting are well known and widely studied, especially for their importance in quantum mechanics. We recall below the definition and some basic facts and we refer the reader to, e.g., [59], for more information.

Definition 2.1 (see, e.g., [59, p. 36]) Let $0 < p < \infty$ and $\alpha > 0$. The Fock space $F^p_\alpha(\mathbb{C})$ is defined as the space of all entire functions in $\mathbb{C}$ with the property that $\frac{\alpha p}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z) < +\infty$, where $dA(z) = dxdy = rdrd\theta$, $z = x + iy = re^{i\theta}$, is the area measure in the complex plane.

Remark 2.2 Let us define

$$\|f\|_{p,\alpha} = \left(\frac{\alpha p}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z)\right)^{1/p}.$$

It is well-known (see, e.g., [59, p. 36]) that $F^p_\alpha(\mathbb{C})$ endowed with $\|f\|_{p,\alpha}$ is a Banach space for $1 \le p < \infty$, and a complete metric space for $\|\cdot\|^p_{p,\alpha}$ with $0 < p < 1$. Moreover, when $p = +\infty$, the space $F^\infty_\alpha(\mathbb{C})$ endowed with $\|f\|_{\infty,\alpha} = \text{ess sup}\{|f(z)|e^{-\alpha|z|^2/2}; z \in \mathbb{C}\}$ turns out to be a Banach space.

We now introduce the polyanalytic complex Fock spaces as follows (see, e.g., Abreu-Gröchenig [2]):

Definition 2.3 Let $0 < p < +\infty$. The s-analytic Fock space denoted by $\mathcal{F}^p_{\alpha,s}(\mathbb{C})$ consists of all polyanalytic functions of order s in $\mathbb{C}$, such that

$$\|f\|_{p,\alpha} := \left(\frac{\alpha p}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z)\right)^{1/p} < +\infty.$$

Remark 2.4 In the case $p = 2$ and $\alpha = 2$, the space $\mathcal{F}^p_{\alpha,s}(\mathbb{C})$ was introduced and studied in, e.g., Abreu-Feichtinger [4, p. 6], Balk [10, p. 170], Vasilevski [57].

As we mention in the introduction, there are results in the literature concerning the approximation by polynomials in Fock spaces, but all of them are of qualitative type and no quantitative estimates were obtained. For example, for any $0 < p < \infty$, and $f \in F^p_\alpha$, there exists a polynomial sequence $(P_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|f - P_n\|_{p,\alpha} = 0$ (see, e.g., Proposition 2.9, p. 38 in [59]). The proof of the result is not constructive and consists in two steps: at step 1, one approximates $f(z)$ by its dilations $f(rz)$ with $r \to 1^-$ and at step 2 one approximates each f_r by its attached Taylor polynomials. If $1 < p < \infty$, then one can construct P_n as the Taylor polynomials attached to f (see, e.g., Exercise 5, p. 89 in [59]) but if $0 < p \le 1$, then there exists $f \in F^p_\alpha$ which cannot be approximated by its associated Taylor polynomials (see, e.g., Exercise 6, p. 89 in [59]). However, if $f \in F^\infty_\alpha$ is such that $\lim_{z\to\infty} f(z)e^{\alpha|z|^2/2} = 0$, then f can be approximated by polynomials in the norm $\|\cdot\|_{\infty,\alpha}$ (see, e.g., Exercise 8, p. 89 in [59]).

In addition, by using the convolution method, we showed in the recent paper [39] that quantitative results in terms of various moduli of smoothness for approximation of polyanalytic functions by polyanalytic polynomials in the complex unit disc can be obtained.

In the spirit of the above results, in this section we consider the approximation by polyanalytic polynomials in the polyanalytic complex Fock spaces $\mathcal{F}^p_{\alpha,s}(\mathbb{C})$.

The novelty is that, for $1 \le p < +\infty$ we present a constructive proof for the density result with quantitative estimates in terms of higher order moduli of smoothness and in terms of the best approximation quantity.

To this end, for the sake of clarity and completeness, we need a number of definition and notations.

We denote by $\mathcal{P}_{n,s}$ denotes the set of all s-analytic polynomials of degree $\le n$.

Definition 2.5 Let $0 < p < +\infty$ and $f \in \mathcal{F}^p_{\alpha,s}(\mathbb{C})$.
The higher order L^p-moduli of smoothness of k-th order is defined by

$$\begin{aligned}\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} &= \sup_{0\le|h|\le\delta}\left\{\int_{\mathbb{C}}|\Delta_h^k f(z)|^p\cdot[e^{-\alpha|z|^2/2}]^p dA(z)\right\}^{1/p}\\ &= \sup_{0\le|h|\le\delta}\|w_\alpha\Delta_h^k f\|_{L^p(\mathbb{C})},\end{aligned}$$

where $k \in \mathbb{N}$, $w_\alpha(z) = e^{-\alpha|z|^2/2}$,

$$\Delta_h^k f(z) = \sum_{s=0}^{k}(-1)^{k+s}\binom{k}{s}f(ze^{ish}) \text{ and } \|f\|_{L^p(\mathbb{C})} = \left(\int_{\mathbb{C}}|f(z)|^p dA(z)\right)^{1/p}.$$

In other words, $\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} = \omega_k(f;\delta)_{w_\alpha,L^p(\mathbb{C})}$ is a weighted modulus of smoothness with the weight $w_\alpha(z) = e^{-\alpha|z|^2/2}$.

The best approximation quantity is defined by

$$E_{n,s}(f;\mathbb{C})_{p,\alpha} = \inf\{\|f-P\|_{p,\alpha};\ P \in \mathcal{P}_{n,s}\}.$$

As in the case of the L^p-moduli of smoothness for functions of real variable (see, e.g., [23, pp. 44–45]), it can be proved that

$$\lim_{\delta\to 0}\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} = 0, \tag{2}$$

$$\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} \le (\lambda+1)^k\cdot\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})},\ \text{if } 1\le p<+\infty \tag{3}$$

and

$$[\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})}]^p \le (\lambda+1)^k\cdot[\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})}]^p,\ \text{if } 0<p<1. \tag{4}$$

It is in fact a standard argument the fact that, setting (for fixed z) $g(x) = f(ze^{ix})$, we get $\Delta_h^k f(z) = \overline{\Delta}_h^k g(0)$, where $\overline{\Delta}_h^k g(x_0) = \sum_{s=0}^{k} (-1)^{s+k} \binom{k}{s} g(x_0 + sh)$.

Also next definitions are standard and we already used them in various cases, see e.g. [36].

For any $1 \le p < +\infty$ and f an s-analytic function in $\mathbb{C}$, we define the convolution operators

$$L_n(f)(z) = \int_{-\pi}^{\pi} f(ze^{it}) \cdot K_n(t)dt,\ z \in \mathbb{C}.$$

Here $K_n(t)$ is a positive and even trigonometric polynomial satisfying the property $\int_{-\pi}^{\pi} K_n(t)dt = 1$.

In the case of the Fejér kernel $K_n(t) = \frac{1}{2\pi n} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$, we shall denote $L_n(f)(z)$ by $F_n(f)(z)$.

For $K_{n,r}(t) = \frac{1}{\lambda_{n',r}} \cdot \left(\frac{\sin(n't/2)}{\sin(t/2)}\right)^{2r}$, $n' = [n/r] + 1$, where r is the smallest integer with $r \ge \frac{m+3}{2}$, $m \in \mathbb{N}$ and the constants $\lambda_{n',r}$ are chosen such that $\int_{-\pi}^{\pi} K_{n,r}(t)dt = 1$, let us define the polyanalytic polynomials

$$I_{n,m,r}(f)(z) = -\int_{-\pi}^{\pi} K_{n,r}(t) \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} f(ze^{ikt})dt,\ z \in \mathbb{C}.$$

Then $K_{n,r}$ is a trigonometric polynomial of degree n, see [46, p. 57], and consequently $I_{n,m,r}(f)(z)$ is an s-analytic polynomial of degree $n+s-1$ (see, e.g., Theorem 2.2 in [39]).

Finally, we set $V_n(f)(z) = 2F_{2n}(f)(z) - F_n(f)(z)$, $z \in \mathbb{C}$, and since $F_n(f)(z)$ is a trigonometric polynomial of degree $n+s-1$, it follows that $V_n(f)(z)$ are s-analytic polynomials of degree $\le 2n + s - 1$ (see, e.g., Theorem 2.2 in [39]).

Next result was proved in [38] in the case of polyanalytic Bergman spaces in the unit ball. The calculations in the proof are performed in the same way in the case of Fock spaces in $\mathbb{C}$. Note however that here the domain of the functions is unbounded so, in principle, it is necessary to verify the validity of the various arguments. We provide the main lines of the proof and we refer the reader to [38] for more details.

Theorem 2.6 *Let* $1 \le p < +\infty$, $0 < \alpha$, $s \in \mathbb{N}$, $m \in \mathbb{N} \bigcup \{0\}$ *and* $f \in \mathcal{F}^p_{\alpha,s}(\mathbb{C})$ *be arbitrary fixed.*

(i) $I_{n,m,r}(f)(z)$ *is* s*-analytic polynomial of degree* $n + s - 1$, *which satisfies the estimate*

$$\|I_{n,m,r}(f) - f\|_{p,\alpha} \le C_{p,m,r,\alpha} \cdot \omega_{m+1}\left(f; \frac{1}{n}\right)_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})},\ n \in \mathbb{N},$$

where $m \in \mathbb{N}$, r *is the smallest integer with* $r \ge \frac{p(m+1)+2}{2}$ *and* $C_{p,m,r,\alpha} > 0$ *is a constant independent of* f *and* n.

(ii) $V_n(f)(z)$ *is a s-analytic polynomials of degree* $\leq 2n+s-1$, *satisfying the estimate*

$$\|V_n(f) - f\|_{p,\alpha} \leq [2^{(p-1)/p} \cdot (2^p+1)^{1/p} + 1] \cdot E_{n,s}(f;\mathbb{C})_{p,\alpha} \; n \in \mathbb{N}.$$

Proof We first prove point (i) reasoning as in [31] (see also [38], Theorem 3.2 in the case of the Bergman space in the unit disc). To this end we apply a well-known Jensen type inequality for integrals applied to the convex function $\varphi(t) = t^p$, $1 \leq p < \infty$, and we get

$$\begin{aligned} |f(z) - I_{n,m,r}(f)(z)|^p &= \left| \int_{-\pi}^{\pi} \Delta_t^{m+1} f(z) K_{n,r}(t) dt \right|^p \\ &\leq \int_{-\pi}^{\pi} |\Delta_t^{m+1} f(z)|^p K_{n,r}(t) dt. \end{aligned}$$

Now we multiply the expression above by $e^{-\alpha p|z|^2/2}$ and we integrate on $\mathbb{C}$ with respect to $dA(z)$. Using Fubini's theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{C}} |I_{n,m,r}(f)(z) - f(z)|^p e^{-\alpha p|z|^2/2} dA(z) \\ &\leq \int_{-\pi}^{\pi} \left[\int_{\mathbb{C}} |\Delta_t^{m+1} f(z)|^p e^{-\alpha p|z|^2/2} dA(z) \right] K_{n,r}(t) dt \\ &\leq \int_{-\pi}^{\pi} \omega_{m+1}(f;|t|)^p_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} \cdot K_{n,r}(t) dt \\ &\leq \int_{-\pi}^{\pi} \omega_{m+1}(f;1/n)^p_{\mathcal{F}^p_{\alpha,s}(\mathbb{C})} (n|t|+1)^{(m+1)p} K_{n,r}(t) dt. \end{aligned}$$

Using (5) in [46, p. 57], for $r \in \mathbb{N}$ with $r \geq \frac{p(m+1)+2}{2}$, we obtain

$$\int_{-\pi}^{\pi} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t) dt \leq C_{p,m,r} < +\infty, \tag{5}$$

from which (i) follows.

To prove (ii) we consider $f, g \in \mathcal{F}^p_{\alpha,s}(\mathbb{C})$ and $1 \leq p < +\infty$. Standard arguments show that for all $z \in \mathbb{C}$ we have

$$\begin{aligned} |V_n(f)(z) - V_n(g)(z)| &\leq 2|F_{2n}(f)(z) - F_{2n}(g)(z)| + |F_n(f)(z) - F_n(g)(z)| \\ &\leq 2\int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})| \cdot K_{2n}(t) dt \\ &\quad + \int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})| \cdot K_n(t) dt \end{aligned}$$

and

$$|V_n(f)(z) - V_n(g)(z)|^p \leq 2^{p-1}\left[\left(2\int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})| \cdot K_{2n}(t)dt\right)^p + \left(\int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})| \cdot K_n(t)dt\right)^p\right]$$
$$\leq 2^{p-1}\left[2^p \int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})|^p \cdot K_{2n}(t)dt + \int_{-\pi}^{\pi} |f(ze^{it}) - g(ze^{it})|^p \cdot K_n(t)dt\right].$$

Following the standard procedure, we multiply the expression above by $\frac{\alpha p}{2\pi} e^{-\alpha p|z|^2/2}$ and we integrate on $\mathbb{C}$. We get

$$\|V_n(f) - V_n(g)\|_{p,\alpha}^p \leq 2^{p-1}\left[2^p \int_{-\pi}^{\pi} \frac{\alpha p}{2\pi} \left(\int_{\mathbb{C}} |f(ze^{it}) - g(ze^{it})|^p e^{-\alpha p|z|^2/2} dA(z)\right) K_{2n}(t)dt + \int_{-\pi}^{\pi} \frac{\alpha p}{2\pi} \left(\int_{\mathbb{C}} |f(ze^{it}) - g(ze^{it})|^p e^{-\alpha p|z|^2/2} dA(z)\right) K_n(t)dt\right].$$

Setting $F(z) = |f(z) - g(z)|^p e^{-\alpha p|z|^2/2}$, $z \in \mathbb{C}$, and $z = r\cos(\theta) + ir\sin(\theta)$, we get

$$\int_{\mathbb{C}} |F(ze^{it})|^p dA(z) = \int_{\mathbb{C}} |F(z)|^p dA(z), \text{ for all } t,$$

which replaced in the above gives, after some calculations

$$\|V_n(f) - V_n(g)\|_{p,\alpha} \leq 2^{(p-1)/p} \cdot (2^p + 1)^{1/p} \|f - g\|_{p,\alpha}.$$

Let $P_{n,s}^*$ be a polynomial of best approximation by elements in $\mathcal{P}_{n,s}$ in the norm in $\|\cdot\|_{p,\alpha}$, that is

$$E_{n,s}(f; \mathbb{C})_{p,\alpha} = \|f - P_{n,s}^*\|_{p,\alpha}.$$

Since $\dim(\mathcal{P}_{n,s}) = n + s - 1$ such a polynomial $P_{n,s}^*$ exists.

We prove that $V_n(P_{n,s}^*)(z) = P_{n,s}^*(z)$, for all $z \in \mathbb{C}$.

Indeed, let us set

$$P_{n,s}^*(z) = \sum_{j=0}^{s-1} \overline{z}^j p_{n,j}^*(z) = \sum_{j=0}^{s-1} \overline{z}^j \left[\sum_{k=0}^{n} a_{n,k}^{(j)} z^k\right].$$

Denoting $a_{n,k}^{(j)} = r_{n,k}^{(j)} e^{i\theta_{n,k}^{(j)}}$ and $z = re^{ix}$, we obtain

$$P_{n,s}^*(z) = \sum_{j=0}^{s-1} re^{-ixj} \left[r_{n,k}^{(j)} e^{i\theta_{n,k}^{(j)}} \cdot re^{ixk} \right]$$
$$= \sum_{j=0}^{s-1} \sum_{k=0}^{n} r^2 r_{n,k}^{(j)} \left[\cos(\theta_{n,k}^{(j)} + (k-j)x) + i \sin(\theta_{n,k}^{(j)} + (k-j)x) \right].$$

Using standard formulas in trigonometry, we deduce that

$$P_{n,s}^*(z) = P_n(x) + i Q_n(x),$$

where P_n and Q_n are trigonometric polynomials of degree $\leq n$ with real coefficients.

We then have

$$V_n(P_{n,s}^*)(z) = V_n(P_n)(x) + i V_n(Q_n)(x), \qquad z = re^{ix},$$

which by Lorentz [46, p. 93], immediately implies

$$V_n(P_{n,s}^*)(z) = P_n(x) + i Q_n(x) = P_{n,s}^*(z).$$

We conclude that

$$\|f - V_n(f)\|_{p,\alpha} \leq \|f - P_{n,s}^*\|_{p,\alpha} + \|V_n(P_{n,s}^*) - V_n(f)\|_{p,\alpha}$$
$$\leq E_{n,s}(f; \mathbb{C})_{p,\alpha} + 2^{(p-1)/p} \cdot (2^p + 1)^{1/p} \|P_{n,s}^* - f\|_{p,\alpha},$$

which proves (ii). □

3 Preliminaries on Polyanalytic Quaternionic Fock Spaces

We shall denote by $\mathbb{H}$, in honor of Hamilton who introduced it, the noncommutative field of quaternions. It consists of elements of the form

$$q = x_0 + x_1 i + x_2 j + x_3 k, \quad x_i \in \mathbb{R}, \;\; i = 0, 1, 2, 3,$$

where the imaginary units i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \; ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

The real number x_0 is called real part of q while $x_1 i + x_2 j + x_3 k$ is called imaginary or vector part of q. The conjugate of q is $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$, while the norm of q is defined as $|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

By $\mathbb{S}$ we denote the unit sphere of purely imaginary quaternion, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$ and so for any fixed $I \in \mathbb{S}$ the subset $\mathbb{C}_I := \{x + Iy;\ |\ x, y \in \mathbb{R}\}$ can be identified with a complex plane.

As it is well known and easily seen, any non real quaternion q is uniquely associated to the element $I_q \in \mathbb{S}$ defined by $I_q := (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3|$ and q belongs to the complex plane $\mathbb{C}_{I_q}$.

The real axis, obtained setting $x_1 = x_2 = x_3 = 0$, belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated to any imaginary unit I. Moreover, we have $\mathbb{H} = \cup_{I\in\mathbb{S}}\mathbb{C}_I$.

In the sequel we will need to introduce convolution operators of a quaternion variable, and so we recall the notion of exponential function of quaternion variable. For any arbitrary, but fixed, $I \in \mathbb{S}$, we define, following [41]: $e^{It} = \cos(t) + I\sin(t),\ \ t \in \mathbb{R}$. With this definition we have an Euler's kind formula: $(\cos(t)+I\sin(t))^k = \cos(kt)+I\sin(kt)$, and therefore we can write $(e^{It})^k = e^{Ikt}$.

A class of functions of a quaternionic variable which has been widely studied in the past fifteen years is that one of the so called slice regular (or slice hyperholomorphic) functions. For more information on these functions and for their various applications, we refer the reader to [20, 21, 40] and the references therein. Various approximation results can be proved in this class of functions in this framework and a summary can be found in [36].

Definition 3.1 Let U be an open set $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is said to be (left) slice analytic (regular) if, for every $I \in \mathbb{S}$, its restriction f_I of f to the complex plane $\mathbb{C}_I$ satisfies

$$\overline{\partial}_I f(x + Iy) = \frac{1}{2}\Big(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\Big) f_I(x + Iy) = 0.$$

The set of (left) slice analytic (regular) functions on U will be denoted by $\mathcal{SR}(U)$.

Remark 3.2 Since we are in a non commutative setting, we could give the previous definition as

$$\overline{\partial}_I f(x + Iy) = \frac{1}{2}\Big(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\Big) = 0.$$

The class thus obtained is that one of right slice regular functions. The two classes of functions contain different elements but the function theory is basically equivalent. Thus we shall consider only the case of left slice regular functions and we shall not specify anymore "left". We note that $\mathcal{SR}(U)$ is a right linear space over $\mathbb{H}$.

Despite other function theories over the quaternions, the class of slice regular functions contains converging power series in the variable q, thus also polynomials, and with quaternionic coefficients written on the right, i.e.

$$\sum_{n=0}^{\infty} q^n a_n, \qquad a_n \in \mathbb{H}.$$

Various function spaces can be defined within the class of slice regular functions, for example: two kinds of Bergman spaces, see, e.g., [18, 19] and Bloch, Besov and Dirichlet spaces, see, e.g., [17].

As a further generalization, we now define the class of slice polyanalytic functions of order s.

Let $s \in \mathbb{N}$ and let $\mathcal{C}^s(U)$ be the set of continuously differentiable functions with all their derivatives up to order s on an open set $U \subseteq \mathbb{H}$. Assume that U is axially symmetric, i.e. when $x + Jy \in U$ then $x + Iy \in U$ for all $I \in \mathbb{S}$. We set $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \;\; x + Iy \in U\}$.

Definition 3.3 Let $f : U \to \mathbb{H}$ belong to $\mathcal{C}^s(U)$, $s \in \mathbb{N}$ and let f be of the form

$$f(q) = \alpha(x, y) + I\beta(x, y) \qquad \text{for } q = x + Iy \in U \tag{6}$$

with $\alpha, \beta : \mathcal{U} \to \mathbb{H}$ satisfying the compatibility conditions $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$. If

$$(\partial_x + I\partial_y)^s f(x + Iy) = 0, \quad \text{for all } \; I \in \mathbb{S} \tag{7}$$

then f is called slice polyanalytic function of order $s \in \mathbb{N}$, or s-analytic for short, on U.

On $\mathbb{H}$, slice polyanalytic functions of order s are of the form (see Proposition 3.6 in [7] and also [8])

$$f(q) = f_0(q) + \overline{q} f_1(q) + \cdots + \overline{q}^{p-1} f_{s-1}(q), \qquad q \in \mathbb{H}, \tag{8}$$

where $f_j(q) = \sum_{l=0}^{+\infty} q^l c_l^{(j)}$, $c_l^{(j)} \in \mathbb{H}$, $j = 0, \ldots, s-1$, $l = 0, 1, \ldots,$ and the series is convergent in $\mathbb{H}$, i.e., $f_j(q)$ is a slice analytic function in $\mathbb{H}$. In particular, $f_j(q)$ can be a polynomial and if $f_j(q)$ is a polynomial for all $j = 0, \ldots, p-1$ we say that f is a slice s-analytic polynomial in $\mathbb{H}$, whose degree $\deg(f)$ is defined as the maximum degree of the f_j's.

We can now introduce the quaternionic counterpart of slice polyanalytic Fock spaces of order s, beginning with the following definition that, in this generality, has not been previously considered in the literature.

Definition 3.4 Let $0 < p < +\infty$, $s \in \mathbb{N}$ and $0 < \alpha < +\infty$. The polyanalytic quaternionic Fock space of the first kind, denoted by $\mathcal{F}^p_{\alpha,s}(\mathbb{H})$, is defined as the space of functions f slice polyanalytic of order s on $\mathbb{H}$ and are such that

$$\|f\|_{p,\alpha} := \left(\frac{\alpha p}{2\pi} \int_{\mathbb{H}} |f(q)|^p (e^{-\alpha|q|^2/2})^p dm(q) \right)^{1/p} < +\infty,$$

where $dm(q)$ represents the Lebesgue volume element in $\mathbb{R}^4$.

Remark 3.5 Using standard techniques, like in the complex case, one can prove that for $1 \le p < +\infty$, $\|\cdot\|_{p,\alpha}$ is a norm, while for $0 < p < 1$, $\|f-g\|_{p,\alpha}^p$ is a quasi-norm.

Remark 3.6 Any slice s-analytic quaternionic polynomial in $\mathbb{H}$, $P(q) = p_0(q) + \overline{q}p_1(q) + \cdots + \overline{q}^{p-1}p_{s-1}(q)$ with all $p_j(q)$ quaternionic polynomials, belongs to the space $\mathcal{F}^p_{\alpha,s}(\mathbb{H})$.

Indeed, this is immediate from the obvious inequality

$$\int_{\mathbb{H}} |q^k|^p (e^{-\alpha|q|^2/2})^p dm(q) < +\infty, \text{ for all } k \in \mathbb{N} \cup \{0\}.$$

Before to define the Fock spaces of the second kind, originally introduced in [24], we give the following:

Definition 3.7 For $I \in \mathbb{S}$, $0 < \alpha < +\infty$ and $0 < p < +\infty$, let us denote

$$\|f\|_{p,\alpha,I} = \left(\frac{\alpha p}{2\pi}\int_{\mathbb{C}_I} |f(q)|^p (e^{-\alpha|q|^2/2})^p dm_I(q)\right)^{1/p},$$

with $dm_I(q)$ representing the area measure on $\mathbb{C}_I$.

The space of slice s-analytic functions on $\mathbb{H}$ satisfying $\|f\|_{p,\alpha,I} < +\infty$ will be denoted with $\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)$.

Standard techniques in slice analysis based on the Representation formula (see e.g. Proposition 4.1 in [24]) show that for any $I, J \in \mathbb{S}$ the spaces $\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)$ and $\mathcal{F}^p_{\alpha,s,J}(\mathbb{C}_J)$ contains the same elements and have equivalent norms. Thus we give the following definition:

Definition 3.8 Let $0 < p < +\infty$, $s \in \mathbb{N}$ and $0 < \alpha < +\infty$. The polyanalytic quaternionic Fock space of the second kind, denoted by $\mathcal{F}^{(2),p}_{\alpha,s}(\mathbb{H})$, is defined as the space of functions f slice polyanalytic of order s on $\mathbb{H}$ such that $f \in \mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)$ for some $I \in \mathbb{S}$.

Notice that for $p = \alpha = 2$, the second kind Fock space of slice s-analytic functions have been introduced in [7], Sect. 4.

4 Density in Polyanalytic Quaternionic Fock Spaces of the First Kind

In this section we introduce the slice polyanalytic quaternionic Fock spaces of the first kind. To this end, we need a number of notions that we list keeping the notations in Sect. 3.

Definition 4.1 Let $0 < p < +\infty$ and $f \in \mathcal{F}^p_{\alpha,s}(\mathbb{H})$. The higher order L^p-moduli of smoothness of k-th order is defined by

$$\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} = \sup_{0\le|h|\le\delta}\left\{\int_{\mathbb{H}} |\Delta_h^k f(q)|^p \cdot [e^{-\alpha|q|^2/2}]^p dm(q)\right\}^{1/p}$$

$$= \sup_{0\le|h|\le\delta} \|w_\alpha \Delta_h^k f\|_{L^p(\mathbb{H})},$$

where $k \in \mathbb{N}$, $w_\alpha(q) = e^{-\alpha|q|^2/2}$,

$$\Delta_h^k f(q) = \sum_{s=0}^{k}(-1)^{k+s}\binom{k}{s} f(qe^{I_q sh}) \text{ and } \|f\|_{L^p(\mathbb{H})} = \left(\int_{\mathbb{H}} |f(q)|^p dm(q)\right)^{1/p}.$$

In other words, $\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} = \omega_k(f;\delta)_{w_\alpha, L^p(\mathbb{H})}$ is a weighted modulus of smoothness with the weight $w_\alpha(q) = e^{-\alpha|q|^2/2}$.

The best approximation quantity is defined by

$$E_{n,s}(f;\mathbb{H})_{p,\alpha} = \inf\{\|f - P\|_{p,\alpha};\ P \in \mathcal{P}_{n,s}\},$$

where $\mathcal{P}_{n,s}$ denotes the set of all slice s-analytic polynomials on $\mathbb{H}$ of degree $\le n$.

Note that also in this case we obtain the counterparts of (2)-(4), namely:

$$\lim_{\delta\to 0} \omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} = 0,$$
$$\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} \le (\lambda+1)^k \cdot \omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})}, \text{ if } 1 \le p < +\infty \tag{9}$$

and

$$[\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})}]^p \le (\lambda+1)^k \cdot [\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})}]^p, \text{ if } 0 < p < 1. \tag{10}$$

Now, for any $1 \le p < +\infty$ and f a slice polyanalytic function of order s in $\mathbb{H}$, we define the convolution operators

$$L_n(f)(q) = \int_{-\pi}^{\pi} f(qe^{It}) \cdot K_n(t)dt,\ q \in \mathbb{H},$$

where $K_n(t)$ is a positive and even trigonometric polynomial such that $\int_{-\pi}^{\pi} K_n(t)dt = 1$.

As in the complex case, we can choose $K_{n,r}(t) = \frac{1}{\lambda_{n',r}} \cdot \left(\frac{\sin(n't/2)}{\sin(t/2)}\right)^{2r}, n' = [n/r]+1$, where r is the smallest integer with $r \ge \frac{m+3}{2}$, $m \in \mathbb{N}$ and the constants $\lambda_{n',r}$ are such that $\int_{-\pi}^{\pi} K_{n,r}(t)dt = 1$.

For any $1 \le p < +\infty$ and f a slice polyanalytic function of order s in $\mathbb{H}$, we define the convolution operators

$$I_{n,m,r}(f)(q) = -\int_{-\pi}^{\pi} K_{n,r}(t) \sum_{k=1}^{m+1}(-1)^k \binom{m+1}{k} f(qe^{I_q kt})dt,\ q \in \mathbb{H}.$$

Note that $K_{n,r}$ is a trigonometric polynomial of degree n and consequently $I_{n,m,r}(f)(q)$ is a slice polyanalytic polynomial of order s of degree $n+s-1$ (see, e.g., Theorem 2.2 in [39]).

Also in the quaternionic case we can prove the following result:

Theorem 4.2 *Let $1 \le p < +\infty$, $0 < \alpha$, $s \in \mathbb{N}$, $m \in \mathbb{N}\bigcup\{0\}$ and $f \in \mathcal{F}^p_{\alpha,s}(\mathbb{H})$ be arbitrarily fixed.*

$I_{n,m,r}(f)(q)$ is a slice polyanalytic polynomial of degree $n+s-1$, which satisfies the estimate

$$\|I_{n,m,r}(f) - f\|_{p,\alpha} \le C_{p,m,r,\alpha} \cdot \omega_{m+1}\left(f; \frac{1}{n}\right)_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})}, n \in \mathbb{N},$$

where $m \in \mathbb{N}$, r is the smallest integer with $r \ge \frac{p(m+1)+2}{2}$ and $C_{p,m,r,\alpha} > 0$ is a constant independent of f and n.

Proof As it was mentioned before the statement, the convolution operators $I_{n,m,r}(f)(z)$ are slice s-analytic polynomials of the mentioned degrees.

In what follows we will reason as in [31] and we shall apply the Jensen type inequality as we did in the proof of Theorem 2.6, by choosing $\varphi(t) = t^p$, $1 \le p < \infty$. We get

$$\begin{aligned}|f(q) - I_{n,m,r}(f)(q)|^p &= \left|\int_{-\pi}^{\pi} \Delta_t^{m+1} f(q) K_{n,r}(t)dt\right|^p \\ &\le \int_{-\pi}^{\pi} |\Delta_t^{m+1} f(q)|^p K_{n,r}(t)dt.\end{aligned}$$

Now we multiply the expression above by $e^{-\alpha p|q|^2/2}$ and we integrate on $\mathbb{H}$ with respect to $dm(q)$. Taking into account the Fubini's theorem, we obtain

$$\begin{aligned}&\int_{\mathbb{H}} |I_{n,m,r}(f)(q) - f(q)|^p e^{-\alpha p|q|^2/2} dm(q) \\ &\le \int_{-\pi}^{\pi} \left[\int_{\mathbb{H}} |\Delta_t^{m+1} f(q)|^p e^{-\alpha p|q|^2/2} dm(q)\right] \\ &K_{n,r}(t)dt \le \int_{-\pi}^{\pi} \omega_{m+1}(f; |t|)^p_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} \cdot K_{n,r}(t)dt \\ &\le \int_{-\pi}^{\pi} \omega_{m+1}(f; 1/n)^p_{\mathcal{F}^p_{\alpha,s}(\mathbb{H})} (n|t|+1)^{(m+1)p} K_{n,r}(t)dt.\end{aligned}$$

Using relation (5) in [46, p. 57], for $r \in \mathbb{N}$ with $r \ge \frac{p(m+1)+2}{2}$, we obtain

$$\int_{-\pi}^{\pi} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t)dt \le C_{p,m,r} < +\infty, \tag{11}$$

and thus we get the required estimate. □

5 Density in Polyanalytic Quaternionic Fock Spaces of the Second Kind

In this section we prove the density of polyanalitic polynomials in slice polyanalytic Fock spaces of second kind. The result provides quantitative estimates in terms of higher order L^p-moduli of smoothness and in terms of the best approximation quantity.

For $1 \leq p < +\infty$ we present a constructive proof for the density result with quantitative estimates in terms of higher order L^p-moduli of smoothness and in terms of the best approximation quantity. For that purpose, we need some more terminology.

Definition 5.1 Let $0 < p < +\infty$, $0 < \alpha$, $I \in \mathbb{S}$ and $f \in \mathcal{F}^p_{\alpha,s,I}(\mathbb{H})$.
The higher order L^p-moduli of smoothness of k-th order is defined by

$$\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)} = \sup_{0\leq|h|\leq\delta} \left\{ \int_{\mathbb{C}_I} |\Delta_h^k f(q)|^p [e^{-\alpha|q|^2/2}]^p dm_I(q) \right\}^{1/p},$$

where $k \in \mathbb{N}$ and $\Delta_h^k f(q) = \sum_{s=0}^k (-1)^{k+s} \binom{k}{s} f(qe^{Ish})$.
The best approximation quantity is defined by

$$E_{n,s,I}(f;\mathbb{C}_I)_{p,\alpha,s} = \inf\{\|f - P\|_{p,\alpha,I};\, P \in \mathcal{P}_{n,s,I}\},$$

where $\mathcal{P}_{n,s,I}$ denotes the set of all quaternionic slice s-analytic polynomials of degree $\leq n$ in $\mathbb{C}_I$.

Also in this case we have the analogs of (2)–(4):

$$\lim_{\delta\to 0} \omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)} = 0,$$
$$\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)} \leq (\lambda+1)^k \cdot \omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)},\ \text{if } 1 \leq p < +\infty \quad (12)$$

and

$$[\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)}]^p \leq (\lambda+1)^k \cdot [\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)}]^p,\ \text{if } 0 < p < 1. \quad (13)$$

Indeed, this is immediate from the fact that setting (for fixed q and I) $g(x) = f(qe^{Ix})$, we get $\Delta_h^k f(q) = \overline{\Delta}_h^k g(0)$, where $\overline{\Delta}_h^k g(x_0) = \sum_{s=0}^k (-1)^{s+k} \binom{k}{s} g(x_0 + sh)$.

Now, for any $1 \leq p < +\infty$ and $f \in \mathcal{F}^p_{\alpha,s,I}(\mathbb{C}_I)$, we define the convolution operators

$$L_n(f)(q) = \int_{-\pi}^{\pi} f(qe^{I_q t}) \cdot K_n(t)dt,\ q \in \mathbb{C}_I.$$

As in the previous section, $K_n(t)$ is a positive and even trigonometric polynomial with the property $\int_{-\pi}^{\pi} K_n(t)dt = 1$.

In particular, we take the Fejér kernel $K_n(t) = \frac{1}{2\pi n} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$, and in this case we denote $L_n(f)(q)$ by $F_n(f)(q)$.

For $K_{n,r}(t) = \frac{1}{\lambda_{n',r}} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2r}$, where $n' \geq [n/r] + 1$, $r \geq \frac{p(m+1)+2}{2}$, $m \in \mathbb{N}$ and the constants $\lambda_{n',r}$ are chosen such that $\int_{-\pi}^{\pi} K_{n,r}(t)dt = 1$. We define

$$G_{n,m,r}(f)(q) = -\int_{-\pi}^{\pi} K_{n,r}(t) \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} f(qe^{I_q kt})dt, \; q \in \mathbb{C}_I.$$

Moreover, we set

$$V_n(f)(q) = 2F_{2n}(f)(q) - F_n(f)(q), \qquad q \in \mathbb{C}_I.$$

According to the reasonings in Sect. 2, for fixed $I \in \mathbb{S}$, if $q \in \mathbb{C}_I$ then $L_n(f)(q)$, $G_{n,m,r}(f)(q)$ and $V_n(f)(q)$ are slice polyanalytic polynomials of order s in q on $\mathbb{C}_I$.

The main result of this section is the following whose proof closely follows the proof of Theorem 2.6 in Sect. 2 and for this reason some details will be omitted.

Theorem 5.2 *Let* $1 \leq p < +\infty$, $0 < \alpha$, $m \in \mathbb{N} \bigcup \{0\}$ *and* $f \in \mathcal{F}^p_{\alpha,s,I}(\mathbb{H})$ *be arbitrary fixed.*

(i) $G_{n,m,r}(f)(q)$ *is an* s*-regular quaternionic polynomial of degree* $\leq n + s + 1$, *which for any* $I \in \mathbb{S}$ *satisfies the estimate*

$$\|G_{n,m,r}(f) - f\|_{p,\alpha,I} \leq C_{p,m,r,\alpha} \cdot \omega_{m+1}\left(f; \frac{1}{n}\right)_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{H})}, n \in \mathbb{N},$$

where $m \in \mathbb{N}$, r *is the smallest integer with* $r \geq \frac{p(m+1)+2}{2}$ *and* $C(p, m, r, \alpha) > 0$ *is a constant independent of* f, n *and* I.

(ii) $V_n(f)(z)$ *is a quaternionic polynomials of degree* $\leq 2n + s - 1$, *satisfying for any* $I \in \mathbb{S}$ *the estimate*

$$\|V_n(f) - f\|_{p,\alpha,I} \leq [2^{(p-1)/p} \cdot (2^p + 1)^{1/p} + 1] \cdot E_{n,s,I}(f; \mathbb{C}_I)_{p,\alpha,s}, \; n \in \mathbb{N}.$$

Proof For the fact that the convolution operators $G_{n,m,r}(f)(q)$ and $V_n(f)(q)$ are slice polyanalytic quaternionic polynomials of order s of the corresponding degrees, one uses the arguments in Sect. 2.

(i) Below we apply the well-known Jensen type inequality for integrals already mentioned in Sect. 2 and we follow the reasoning as in the complex case by choosing $\varphi(t) = t^p$, $1 \leq p < \infty$. We obtain

$$\begin{aligned} |f(q) - G_{n,m,r}(f)(q)|^p &= \left| \int_{-\pi}^{\pi} \Delta_t^{m+1} f(q) K_{n,r}(t)dt \right|^p \\ &\leq \int_{-\pi}^{\pi} |\Delta_t^{m+1} f(q)|^p K_{n,r}(t)dt. \end{aligned}$$

Multiplying above by $e^{-\alpha p|q|^2/2}$, integrating on $\mathbb{C}_I$ with respect to $dm_I(q)$ and using the Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{C}_I} |G_{n,m,r}(f)(q) - f(q)|^p e^{-\alpha p|q|^2/2} dm_I(q) \\
&\leq \int_{-\pi}^{\pi} \left[\int_{\mathbb{C}_I} |\Delta_t^{m+1} f(q)|^p e^{-\alpha p|q|^2/2} dm_I(q) \right] K_{n,r}(t)dt \\
&\leq \int_{-\pi}^{\pi} \omega_{m+1}(f; |t|)^p_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{H})} \cdot K_{n,r}(t)dt \\
&\leq \int_{-\pi}^{\pi} \omega_{m+1}(f; 1/n)^p_{\mathcal{F}^p_{\alpha,s,I}(\mathbb{H})} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t)dt.
\end{aligned}$$

Relation (5), p. 57 in [46] implies that, for $r \in \mathbb{N}$ with $r \geq \frac{p(m+1)+2}{2}$, it holds that

$$\int_{-\pi}^{\pi} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t)dt \leq C_{p,m,r} < +\infty, \tag{14}$$

which proves the estimate in (i).

(ii) Let $f, g \in \mathcal{F}^p_{\alpha,s,I}(\mathbb{H})$ and $1 \leq p < +\infty$. From the obvious inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, valid for all $a, b \geq 0$, we deduce that for all $q \in \mathbb{C}_I$ we have

$$\begin{aligned}
&|V_n(f)(q) - V_n(g)(q)| \leq 2|F_{2n}(f)(q) - F_{2n}(g)(q)| + |F_n(f)(q) - F_n(g)(q)| \\
&\leq 2\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_{2n}(t)dt + \int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_n(t)dt
\end{aligned}$$

and

$$\begin{aligned}
|V_n(f)(q) - V_n(g)(q)|^p &\leq 2^{p-1}\left[\left(2\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_{2n}(t)dt \right)^p \right. \\
&\quad \left. + \left(\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_n(t)dt \right)^p \right] \\
&\leq 2^{p-1}\left[2^p \int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})|^p \cdot K_{2n}(t)dt \right. \\
&\quad \left. + \int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})|^p \cdot K_n(t)dt \right].
\end{aligned}$$

Again, following the standard procedure, namely if we multiply the expression above by $\frac{\alpha p}{2\pi} e^{-\alpha p|q|^2/2}$ and we integrate on $\mathbb{C}_I$ with respect to $dm_I(q)$, taking into account the Fubini's theorem too, we obtain

$$\begin{aligned}
&\|V_n(f) - V_n(g)\|^p_{p,\alpha,I} \\
&\leq 2^{p-1}\left[2^p \int_{-\pi}^{\pi} \frac{\alpha p}{2\pi} \left(\int_{\mathbb{C}_I} |f(qe^{It}) - g(qe^{It})|^p e^{-\alpha p|q|^2/2} dm_I(q) \right) \right.
\end{aligned}$$

$$K_{2n}(t)dt + \int_{-\pi}^{\pi} \frac{\alpha p}{2\pi} \left(\int_{\mathbb{C}_I} |f(qe^{It}) - g(qe^{It})|^p e^{-\alpha p|q|^2/2} dm_I(q) \right) K_n(t)dt \Bigg].$$

But denoting $F(q) = |f(q) - g(q)|^p e^{-\alpha p|q|^2/2}$, $q \in \mathbb{C}_I$, writing $q = r\cos(\theta) + Ir\sin(\theta)$ in polar coordinates and taking into account that

$$dm_I(q) = rdrd\theta, q \in \mathbb{C}_I,$$

simple calculations (made exactly as in the complex case) lead to the equality

$$\int_{\mathbb{C}_I} |F(qe^{It})|^p dm_I(q) = \int_{\mathbb{C}_I} |F(q)|^p dA_I(q), \text{ for all } t,$$

which replaced in the above inequality immediately implies

$$\begin{aligned} \|V_n(f) - V_n(g)\|_{p,\alpha,I}^p &\leq 2^{p-1}[2^p \|f - g\|_{p,\alpha,I}^p + \|f - g\|_{p,\alpha,I}^p] \\ &= 2^{p-1}(2^p + 1)\|f - g\|_{p,\alpha,I}^p, \end{aligned}$$

that is

$$\|V_n(f) - V_n(g)\|_{p,\alpha,I} \leq 2^{(p-1)/p} \cdot (2^p + 1)^{1/p} \|f - g\|_{p,\alpha,I}.$$

Now, let us denote by $P_{n,s,I}^*$ a polynomial of best approximation by elements in $\mathcal{P}_{n,s,I}$ in the norm in $\|\cdot\|_{p,\alpha,I}$, that is

$$E_{n,s}(f; \mathbb{C}_I)_{p,\alpha,I} = \inf\{\|f - P\|_{p,\alpha,I}; P \in \mathcal{P}_{n,s,I}\} = \|f - P_{n,s,I}^*\|_{p,\alpha,I}.$$

Note that since $\mathcal{P}_{n,s,I}$ is finite dimensional, for any fixed n, we deduce that this polynomial $P_{n,s,I}^*$ exists.

Since by similar reasonings with those in the complex case in the proof of Theorem 2.6, (ii), we get $V_n(P_{n,s,I}^*) = P_{n,s,I}^*$, for all $q \in \mathbb{C}_I$, it follows

$$\begin{aligned} \|f - V_n(f)\|_{p,\alpha,I} &\leq \|f - P_{n,s,I}^*\|_{p,\alpha,I} + \|V_n(P_{n,s,I}^*) - V_n(f)\|_{p,\alpha,I} \\ &\leq E_{n,s}(f; \mathbb{C}_I)_{p,\alpha,I} + 2^{(p-1)/p} \cdot (2^p + 1)^{1/p} \|P_{n,s,I}^* - f\|_{p,\alpha,I} \\ &= [2^{(p-1)/p} \cdot (2^p + 1)^{1/p} + 1] \cdot E_{n,s}(f; \mathbb{C}_I)_{p,\alpha,I}, \end{aligned}$$

which proves (ii) and the theorem. □

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