SUBLINEAR EQUATIONS DRIVEN BY HÖRMANDER OPERATORS

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Abstract. We characterize the existence of a unique positive weak solution for a Dirichlet boundary value problem driven by a linear second order differential operator modeled on Hörmander vector fields, where the right hand side has sublinear growth.

1. Introduction

The aim of this paper is to provide an abstract framework to study the existence of positive weak solutions of the following boundary value problem

$$
\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, $\mathcal{L}$ is the Laplacian built on a family of smooth Hörmander vector fields $X = \{X_1, \ldots, X_m\}$, see Section 2.1 for the relevant definitions, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has a sublinear growth and satisfies a few other technical requirements for which we refer to Section 2.3.

The study of operators driven by Hörmander vector fields has a long history (we refer the reader to the monograph [7] for an historial overview of the subject) and in the last few years there has been a certain interest in the study of nonlinear PDEs settled in this, and sometimes less general, context, see, e.g., [4, 6, 8, 21, 22, 24, 25, 27] and the references therein.

Our compass in this paper is [10] by Brezis and Oswald. Their result, later extended to several different operators (see, e.g., [12, 13, 23, 26, 5]), states that in the particular case in which $X = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$ and $\mathcal{L} = -\Delta$, then (1.1) admits a unique weak solution *if and only if*

$$
\lambda_1(\mathcal{L} - a_0) < 0 < \lambda_1(\mathcal{L} - a_\infty), \tag{1.2}
$$

where $\lambda_1(\mathcal{L} - a_0)$ and $\lambda_1(\mathcal{L} - a_\infty)$ denote, respectively, the smallest eigenvalue of $\mathcal{L} - a_0$ and of $\mathcal{L} - a_\infty$ (both in presence of Dirichlet boundary condition), and $a_0(x)$, $a_\infty(x)$ are possibly unbounded and indefinite weights defined by:

$$
a_0(x) := \lim_{t \to 0^+} \frac{f(x,t)}{t} \quad \text{and} \quad a_\infty(x) := \lim_{t \to +\infty} \frac{f(x,t)}{t}.
$$

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In order to state our main result we first recall that, denoting by $H^1_{0,X}(\Omega)$ the natural Sobolev space associated with the family X (see Section 2.2), a function $u \in H^1_{0,X}(\Omega)$ is called a weak solution of (1.1) if

$$\int_\Omega \langle Xu, Xv\rangle\, dx = \int_\Omega f(x,u)v\, dx, \tag{1.3}$$

for every $v \in H^1_{0,X}(\Omega)$.

Theorem 1.1. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, and let $X = \{X_1, \dots, X_m\}$ be a family of smooth vector fields in $\mathcal{D}$ satisfying the structural assumptions* (H1)-(H2)*. Moreover, let $\Omega \subset \mathcal{D}$ be a bounded set satisfying assumption* (S)*, and let*

$$f : \Omega \times (0,+\infty) \to \mathbb{R}$$

satisfying assumptions (f1)*-to-*(f4)*.*

Then, there exists a solution u to (1.1) if and only if

$$\lambda_1(\mathcal{L} - a_0) < 0 < \lambda_1(\mathcal{L} - a_\infty).$$

Furthermore, this solution u is unique*, and it satisfies the following properties:*

(a) *$u \in L^\infty(\Omega) \cap C(\Omega)$;*
(b) *$u > 0$ pointwise in Ω.*

We point out that (H1) stands for X being a family of Hörmander vector fields. Some comments on Theorem 1.1 are now in order. First, we observe that the proof in [10] is based on several important ingredients, namely:

(a) elliptic regularity theory;
(b) strong maximum principle;
(c) Hopf Lemma;
(d) variational methods.

In particular, (a)-to-(c) are extensively used to prove the uniqueness and the strict positivity of the weak solution. Indeed, the elliptic regularity theory allows to work with *classical solutions* and, in turn, makes it possible to apply the strong maximum principle. Further, thanks to the regularity up to the boundary and the Hopf Lemma, calling u_1 and u_2 two positive weak solutions, one can then consider

$$u_1^2/u_2 \qquad \text{and} \qquad u_2^2/u_1$$

as test functions (i.e., belonging to the Euclidean Sobolev space $H^1_0(\Omega)$) and then conclude that $u_1 = u_2$ thanks to the assumptions made on f. The latter point (d), on purpose vague at this stage, plays a major role while proving that (1.2) is a necessary and sufficient condition to get existence. Unfortunately, in our generality we do not have (a)-to-(c) at our disposal and this makes the strategy described above hard to follow. For this reason, we followed another path that we are going to describe.

In order to prove the positivity of weak solution, we first reduce the problem to prove positivity of subsolutions of a *linear operator* $\mathcal{L} + M$ (where $M > 0$ is a positive constant) and then we apply a strong maximum principle for *viscosity* solutions recently proved in [1]. We stress that the application is not for free, and it requires to pass from weak subsolutions (in the sense specified above), to distributional subsolutions (in the sense of Ishii [19]) and then to viscosity subsolutions. In order to apply [19] we ask X to satisfy assumption (H2), see Section 2.1, which is however satisfied in many relevant cases, and we also need a bit of interior regularity, which can be deduced from [11] once boundedness is established (see Theorem 3.1). Moreover, in the particular case when $\mathcal{L} = -\Delta$, the global boundedness of the weak solutions of (1.1) is proved in [10] by combining a truncation argument with *global* $W^{2,p}$*-estimates*; on the other hand, since it is known that these estimates *cannot hold* in our subelliptic context, see e.g. [20], here we follow a different approach. More precisely, using the sublinear growth of f and mimicking the approach in [5], we obtain the global boundedness of the solutions of (1.1) by adapting the classical method by Stampacchia. In this approach, a key ingredient is the possibility of *continuously* embed the Sobolev space $H^1_{0,X}(\Omega)$ into some $L^q(\Omega)$ with $q > 2$; in its turn, this condition is equivalent to the validity of a *Sobolev-type inequality* of the form

$$\|u\|^2_{L^q(\Omega)} \leq c_q \, \| |Xu| \|^2_{L^2(\Omega)} \qquad \text{for every } u \in C_0^\infty(\Omega). \tag{1.4}$$

While the validity of (1.4) for *general* Hörmander vector fields could be a delicate issue, there are many meaningful situations in which (1.4) holds for an arbitrary bounded open set (see Remark 2.5). However, due to the approach we aim to follow in this paper, we take (1.4) as an axiomatic assumption on the open set Ω; this is assumption (S) in the statement of Theorem 1.1.

Uniqueness is another relevant issue. As briefly explained above, the original argument of Brezis-Oswald cannot be directly repeated because of the lack of regularity at the boundary. In order to avoid such problems, we adapt an approximation argument used in [9] in a different setting. We stress that, compared to [9], the presence of a more general nonlinearity forces to make an extra assumption (f5) on f, see Section 4. Remarkably, Proposition 5.6 shows that this extra assumption is somehow natural when dealing with problem (1.1).

Plan of the paper. In Section 2 we recall all the basic definitions and assumptions needed throughout the paper. In Section 3 we show that any non-negative weak solution of (1.1) is bounded, whereas Section 4 is devoted to the uniqueness of this solution. Finally, in Section 5 we prove Theorem 1.1.

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2. Assumptions, notation and preliminary results

In this section we collect some preliminaries definitions, notation and results which shall be exploited throughout the rest of the paper.

2.1. **The operator $\mathcal{L}$.** Let $\varnothing \neq \mathcal{D} \subseteq \mathbb{R}^n$ be a *fixed* open set. We denote by $\mathcal{X}(\mathcal{D})$ the Lie algebra of the smooth vector fields in $\mathcal{D}$; moreover, if $A \subseteq \mathcal{X}(\mathcal{D})$, we indicate by $\mathrm{Lie}(A)$ the smallest Lie sub-algebra of $\mathcal{X}(\mathcal{D})$ containing A. Finally, if

$$Y = \textstyle\sum_{i=1}^n a_j(x)\partial_{x_j} \in \mathcal{X}(\mathcal{D}) \qquad (\text{for some } a_1, \ldots, a_n \in C^\infty(\mathcal{D})),$$

and if $x \in \mathbb{R}^n$ is arbitrary, we define

$$Y_x := \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix} \in \mathbb{R}^n.$$

Taking into account the above definitions and notation, from now on we fix *once and for all* a family $X = \{X_1, \ldots, X_m\} \subseteq \mathcal{X}(\mathcal{D})$ satisfying the following assumptions:

(H1): $X_1, \ldots, X_m$ satisfy Hörmander's condition on $\mathcal{D}$, that is,

$$\dim\big\{Y_x : Y \in \mathrm{Lie}(X)\big\} = n.$$

(H2): Denoting by $A(x)$ the $n \times n$ symmetric matrix

$$A(x) = S(x) \cdot S(x)^T, \qquad \text{where } S(x) = \big((X_1)_x \cdots (X_m)_x\big), \tag{2.1}$$

we have that $\mathcal{D} \ni x \mapsto A^{1/2}(x)$ is of class (at least) C^1 in $\mathcal{D}$.

Then, we consider the second-order linear differential operator

$$\mathcal{L} := \sum_{j=1}^{m} X_j^\star \circ X_j, \tag{2.2}$$

where X_j^* is the formal adjoint of X_j in $L^2(\mathcal{D})$, that is,

$$\int_{\mathcal{D}} \varphi\, X_i \psi \,\mathrm{d}x = \int_{\mathcal{D}} \psi\, X_i^* \varphi \,\mathrm{d}x \qquad \forall\ \varphi,\, \psi \in C_0^\infty(\mathcal{D}).$$

The operator $\mathcal{L}$ in (2.2) is usually referred to as a *subelliptic Laplacian*.

Remark 2.1. Taking into account that $X_1, \ldots, X_m$ are *smooth* on $\mathcal{D}$, a direct computation shows that $X_i^* = -X_i - c_i(x)$ (for every $i = 1, \ldots, m$), where

$$c_i(x) = \mathrm{div}\big((X_i)_x\big).$$

As a consequence, the operator $\mathcal{L}$ can be rewritten as follows

$$\mathcal{L} = -\textstyle\sum_{j=1}^m X_j^2 + \sum_{j=1}^n c_j(x) X_j = -\mathrm{div}\big(A(x) \cdot \nabla\big), \tag{2.3}$$

where the matrix $A(x)$ is as in (2.1). In particular, we see that $\mathcal{L}$ is a pure *divergence-form operator* on $\mathcal{D}$, and that assumption (H2) is a *regularity assumption* on the square root of the *principal matrix of* $\mathcal{L}$.

Just about to illustrate the generality of assumptions (H1)-(H2), we provide some concrete examples of families $X \subseteq \mathcal{X}(\mathcal{D})$ satisfying these structural assumptions.

Example 2.2. (1) Let $\mathcal{D} = \mathbb{R}^n$ (with $n \geq 1$), and let

$$X_1 = \partial_{x_1}, \ldots, X_n = \partial_{x_n} \in \mathcal{X}(\mathbb{R}^n).$$

Since, in this case, we have $A(x) = \mathrm{Id}_n$, it is immediate to check that $X_1, \ldots, X_m$ satisfy assumptions (H1)-(H2); moreover, from (2.3) we derive that

$$\mathcal{L} = \textstyle\sum_{j=1}^n X_j^* \circ X_j = -\Delta.$$

(2) Let $\mathbb{G} = (\mathbb{R}^n, *, D_\lambda)$ be a *homogeneous Carnot group* in $\mathbb{R}^n$, and let $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^n)$ be the Lie algebra of $\mathbb{G}$ (see, e.g., [7, Chap. 1] for the relevant definitions). Since $\mathbb{G}$ is a Carnot group, we know that $\mathfrak{g}$ is *nilpotent and stratified*: this means that there exists a suitable vector subspace $\mathfrak{g}_1$ of $\mathfrak{g}$ (the *first layer of* $\mathfrak{g}$) such that

$$\mathfrak{g} = \textstyle\bigoplus_{i=1}^r \mathfrak{g}_i, \tag{2.4}$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$ (for $1 \leq i \leq r-1$), and $[\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}$. Then, if

$$\mathcal{Z} = \{Z_1, \ldots, Z_m\} \subseteq \mathfrak{g}_1$$

is any *linear basis* of $\mathfrak{g}_1$ (so that $m = \dim(\mathfrak{g}_1) \leq n$), it is not difficult to check that the Z_i's satisfy assumptions (H1)-(H2). In fact, since $\mathbb{G}$ is a Lie group, the map

$$E_x : \mathfrak{g} \to \mathbb{R}^n, \qquad E_x(Y) := Y_x \in \mathbb{R}^n,$$

is an isomorphism of vector spaces *for every* $x \in \mathbb{R}^n$ (see, e.g., [7, Prop. 1.2.7]); thus, using (2.4) and the fact that $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ is a basis of $\mathfrak{g}_1$, we get

$$\big\{Y_x : Y \in \mathrm{Lie}(\mathcal{Z})\big\} = \big\{Y_x : Y \in \mathfrak{g}\big\} = E_x\big(\mathfrak{g}\big) = \mathbb{R}^n,$$

from which we derive that Hörmander's condition (H1) is satisfied. As for the validity of assumption (H2), we argue as follows: first of all, since E_x is an isomorphism of vector spaces and $\mathcal{Z}$ is a basis of $\mathfrak{g}_1$, we have

$$\mathrm{rank}\big(S(x)\big) = \mathrm{rank}\big((Z_1)_x \cdots (Z_m)_x\big) = m \qquad \text{for every } x \in \mathbb{R}^n;$$

from this, we derive that $A(x) = S(x) \cdot S(x)^T$ has *constant rank* equal to m. Since the map $x \mapsto S(x)$ is smooth, we then infer from [2, Thm. 1.1] that

$$\mathbb{R}^n \ni x \mapsto A^{1/2}(x) \text{ is smooth as well,}$$

and thus assumption (H2) is satisfied. In this context, the linear operator

$$\Delta_{\mathbb{G}} = \textstyle\sum_{i=1}^m Z_i^* \circ Z_i = \mathrm{div}\big(A(x) \cdot \nabla\big),$$

is usually referred to as a *sub-Laplacian on* $\mathbb{G}$.

(3) Let $\mathcal{D} = \mathbb{R}^2$, and let

$$X_1 = \partial_{x_1}, \qquad X_2 = x_1^2\,\partial_{x_2} \in \mathcal{X}(\mathbb{R}^2).$$

We claim that $X := \{X_1, X_2\}$ satisfies assumptions (H1)-(H2). Indeed, since

$$[X_1, [X_1, X_2]] = 2\partial_{x_2} \in \mathrm{Lie}(X),$$

it is immediate to recognize that Hörmander's condition (H1) is satisfied; on the other hand, since the matrix $A(x) = S(x) \cdot S(x)^T$ is explicitly given by

$$A(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_1^4 \end{pmatrix},$$

we readily see that $\mathbb{R}^2 \ni x \mapsto A^{1/2}(x)$ is smooth, and thus assumption (H2) holds. The linear operator $\mathcal{L}$ associated with $X = \{X_1, X_2\}$, that is,

$$\mathcal{L} = X_1^* \circ X_1 + X_2^* \circ X_2 = \mathrm{div}\big(A(x) \cdot \nabla\big) = \partial_{x_1}^2 + x_1^4\,\partial_{x_2}^2,$$

is usually referred to as the 2-*step Grushin-type operator* (in $\mathbb{R}^2$).

2.2. **The X-Sobolev space.** Now we have properly introduced the operators $\mathcal{L}$ we are interested in, we turn to describe the adequate functional setting for the study of problem (1.1). To begin with, if $\varnothing \neq \Omega \subseteq \mathcal{D}$ is open, we define

$$H^1_X(\Omega) := \big\{u \in L^2(\Omega) : \exists\ X_1 u, \dots, X_m u \in L^2(\Omega)\big\}.$$

Here, $X_1 u, \dots, X_m u$ are assumed to exist in the *weak sense of distributions*; this means, precisely, that $X_j u$ is the unique L^2-function in Ω such that

$$\int_\Omega u\, X_j^* \psi\, dx = \int_\Omega X_j u\, \psi\, dx \qquad \forall\ \psi \in C_0^\infty(\Omega).$$

Throughout what follow, if $u \in H^1_X(\Omega)$ we shall use the notation

$$Xu := (X_1 u, \dots, X_m u).$$

On the *real vector space* $H^1_X(\Omega)$ we consider the Sobolev-type norm

$$\|u\|_{H^1} := \sqrt{\|u\|^2_{L^2(\Omega)} + \||Xu|\|^2_{L^2(\Omega)}}.$$

Moreover, we define

$$H^1_{0,X}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1}}.$$

Clearly, the norm $\|\cdot\|_{H^1}$ is induced by the scalar product

$$\langle u, v\rangle_{H^1} := \int_\Omega uv\,dx + \int_\Omega \langle Xu, Xv\rangle\,dx \qquad (u, v \in H^1_X(\Omega));$$

it is then immediate to recognize that $(H^1_X(\Omega), \|\cdot\|_{H^1})$ is a (real) Hilbert space, and the same is true of $H^1_{0,X}(\Omega)$ (with the induced norm).

Remark 2.3. In the particular case when $\mathcal{D} = \mathbb{R}^n$ (with $n \geq 1$) and

$$X_{\mathcal{E}} := \{\partial_{x_1}, \ldots, \partial_{x_n}\} \subseteq \mathcal{X}(\mathbb{R}^n),$$

it is very easy to check that $H^1_{X_{\mathcal{E}}}(\Omega)$ and $H^1_{0,X_{\mathcal{E}}}(\Omega)$ do coincide with the classical Sobolev spaces. Hence, we adopt the simplified notation

$$H^1(\Omega) = H^1_{X_{\mathcal{E}}}(\Omega) \qquad \text{and} \qquad H^1_0(\Omega) = H^1_{0,X_{\mathcal{E}}}(\Omega).$$

We also explicitly notice, for a future reference, that

$$X_{\mathcal{E}}u = (\partial_{x_1}u, \ldots, \partial_{x_n}u) = \nabla u \qquad \forall\, u \in H^1(\Omega).$$

Now, in view of the *regularity* of $X_1, \ldots, X_m$, the following *Meyers-Serrin type* result holds true in our context (for a proof see, e.g., [15]):

$$C^\infty(\Omega) \cap H^1_X(\Omega) \text{ is } \textit{dense} \text{ in } H^1_X(\Omega).$$

As a consequence of this *good approximation result*, it is not difficult to extend many properties of classical Sobolev functions to our setting; for example, in the next sections we shall repeatedly exploit the facts listed below:

(1) if $u \in H^1_{0,X}(\Omega)$ and $v \in H^1_X(\Omega)$ is such that $0 \leq v \leq u$, then
$$v \in H^1_{0,X}(\Omega);$$
(2) if $u \in L^\infty(\Omega) \cap H^1_{0,X}(\Omega)$ and $p \geq 1$, then $u^p \in H^1_{0,X}(\Omega)$ and
$$X(u^p) = p\,u^{p-1}\,Xu;$$
(3) if $u \in L^\infty(\Omega) \cap H^1_{0,X}(\Omega)$, $v \in L^\infty(\Omega) \cap H^1_X(\Omega)$ and $v \geq \varepsilon > 0$, then
$$u/v \in H^1_{0,X}(\Omega) \quad \text{and} \quad X(u/v) = \frac{1}{v^2}(vXu - uXv);$$
(4) if $u \in H^1_{0,X}(\Omega)$ and $\kappa \geq 0$, then $u_\kappa := \min\{u, \kappa\} \in H^1_{0,X}(\Omega)$ and
$$X(u_\kappa) = Xu\,\mathbf{1}_{\{u<\kappa\}}.$$

Remark 2.4. Due to its relevance in the sequel, we explicitly highlight the following consequence of (4): *if $u \in H^1_{0,X}(\Omega)$, then u_+, u_-, $|u| \in H^1_{0,X}(\Omega)$, and*

$$Xu_+ = Xu\,\mathbf{1}_{\{u>0\}}, \quad Xu_- = -Xu\,\mathbf{1}_{\{u<0\}} \qquad \text{a.e. in } \Omega.$$

In particular, $Xu = 0$ a.e. on $\{u = 0\}$ (since $u = u_+ - u_-$).

Furthermore, since $X_1, \dots, X_m$ satisfy Hörmander's rank condition (H1) on $\mathcal{D}$, all the results in [18, Sec. 8] do apply to our context; in particular, the following Rellich-Kondrachev type theorem holds true (see also [14, 16]):

$$H^1_{0,X}(\Omega) \text{ is compactly embedded into } L^2(\Omega). \tag{2.5}$$

Together with (2.5), a key ingredient for our approach in the study of problem (1.1) is the validity of some *Sobolev-type embedding theorem* for $H^1_{0,X}(\Omega)$; hence, we fix a definition: *if* $\Omega \subseteq \mathcal{D}$ *is a bounded open set and* $2 \leq q < +\infty$*, we say that* Ω supports a (q, X)-Sobolev type inequality *if there exists* $\mathbf{c} = \mathbf{c}_{q,n} > 0$ *such that*

$$\|u\|^2_{L^q(\Omega)} \leq \mathbf{c}_{q,n}\, \||Xu|\|^2_{L^2(\Omega)} \qquad \forall\, u \in H^1_{0,X}(\Omega).$$

From now on, we then make the following 'structural' assumption:

(S): $\Omega \subseteq \mathcal{D}$ is a *bounded* open set and there exists some $q > 2$ such that Ω supports a (q, X)-Sobolev type inequality (with constant $\mathbf{c} = \mathbf{c}_S > 0$).

Remark 2.5. The validity of (q, X)-Sobolev type inequalities (and, in particular, of X-Poincaré type inequalities, corresponding to the case $q = 2$) is a very interesting problem, which has received a great attention in the literature. While we refer to the survey [18] and to the references therein contained for a thorough investigation on this topic, here we limit ourselves to remind the following facts:

(1) Let $\mathcal{D} = \mathbb{R}^n$ (with $n \geq 3$), and let $X_{\mathcal{E}} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$. On account of Remark 2.3, we see that *every bounded open set* Ω supports a $(q, X_{\mathcal{E}})$-Sobolev type inequality *for every* $1 \leq q \leq 2^*$, where
$$2 \leq q \leq 2^* := \frac{2n}{n-2}.$$

(2) Let $\mathbb{G} = (\mathbb{R}^n, *, D_\lambda)$ be a Carnot group, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Moreover, let $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ be a linear basis of the first layer of $\mathfrak{g}$ (see Example 2.2-(2)). Then, *every bounded open set* $\Omega \subseteq \mathbb{G} \equiv \mathbb{R}^n$ supports a $(q, \mathcal{Z})$-Sobolev type inequality *for every* $1 \leq q \leq 2^*_Q$, where
$$2^*_Q := \frac{2Q}{Q-2},$$
and $Q \geq n$ is the homogeneous dimension of $\mathbb{G}$ (see, e.g., [7, Thm. 5.9.2]).

(3) Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open, and let $X = \{X_1, \dots, X_m\} \subseteq \mathcal{X}(\mathcal{D})$ be *any* family of smooth vector fields satisfying Hörmander's condition (H1) in $\mathcal{D}$. Then, it is proved in [18] that *every open set* $\Omega \Subset \mathcal{D}$ with *sufficiently small diameter* supports a (q, X)-Sobolev type inequality for every $1 \leq q \leq 2^*_Q$, where
$$2^*_Q := \frac{2Q}{Q-2}, \tag{2.6}$$
and $Q \geq n$ is the *local homogeneous dimension* of X on Ω.

(4) Let $\mathcal{D} = \mathbb{R}^n$, and let $X = \{X_1, \dots, X_m\}$ be a family of smooth vector fields satisfying Hörmander's condition (H1) in $\mathbb{R}^n$. In addition, we assume that the X_i's are *homogeneous of degree* 1 with respect to a family of non-isotropic dilations of the following form
$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n), \qquad \begin{array}{c}\text{where } \sigma_1, \dots, \sigma_n \in \mathbb{N} \text{ and} \\ 1 = \sigma_1 \leq \dots \leq \sigma_n.\end{array}$$

Then, the 'local' result described in (3) turns out to hold in a global form: more precisely, every *bounded open set* $\Omega \subseteq \mathbb{R}^n$ supports a (q, X)-Sobolev type inequality for every $2 \leq q \leq 2^*_Q$, where 2^*_Q is as in (2.6) and

$$Q := \textstyle\sum_{i=1}^n \sigma_i \geq n$$

is the δ_λ-homogeneous dimension of $\mathbb{R}^n$ (see, e.g., [3]). We explicitly point out that, if $X = \{X_1, X_2\}$ is as in Example 2.2-(3), then X_1, X_2 are *homogeneous of degree* 1 with respect to the family of dilations

$$\delta_\lambda(x) = (\lambda x_1, \lambda^3 x_2) \qquad (\lambda > 0);$$

thus, since in this case we have $Q = 4$, every bounded open set $\Omega \subseteq \mathbb{R}^2$ supports a (q, X)-Sobolev type inequality for every $2 \leq q \leq 2^*_Q$, where

$$2^*_Q = 4.$$

2.3. **The $\mathcal{L}$-Dirichlet problem.** Taking into account all the definitions and notation introduced so far, we are finally ready to give precise meaning of *weak solution* for the $\mathcal{L}$-Dirichlet problem (1.1), that is,

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega. \end{cases}$$

To this end, we first fix the relevant assumption on the non-linearity f:

(f1): $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function;

(f2): $f(\cdot, t) \in L^\infty(\Omega)$ for every $t \geq 0$;

(f3): there exists a constant $c > 0$ such that

$$|f(x,t)| \leq c(1+t) \qquad \text{for a.e. } x \in \Omega \text{ and every } t \geq 0; \tag{2.7}$$

(f4): for a.e. $x \in \Omega$, the function $t \mapsto \dfrac{f(x,t)}{t}$ is strictly decreasing in $(0, +\infty)$.

Taking into account assumption (f4), we introduce the functions

$$a_0(x) := \lim_{t \to 0^+} \frac{f(x,t)}{t} \qquad \text{and} \qquad a_\infty(x) := \lim_{t \to +\infty} \frac{f(x,t)}{t}, \tag{2.8}$$

which are allowed to be identically equal to $+\infty$ and $-\infty$, respectively.

Remark 2.6. In this remark we highlight some immediate consequences of assumption (f1)–(f4) which shall be useful in the sequel (for a proof see, e.g., [13]).

(1) By combining (f2) and (f4), we get that

$$\frac{f(x,t)}{t} \geq f(x,1) \geq -\|f(\cdot,1)\|_{L^\infty(\Omega)} =: -c_f > -\infty, \tag{2.9}$$

for a.e. $x \in \Omega$ and every $t \in (0,1]$. In particular, by (f1) and (2.9) we get

$$f(x,0) \geq 0 \text{ for a.e. } x \in \Omega. \tag{2.10}$$

(2) Using again assumption (f4), we have

$$a_0(x) \geq \frac{f(x,t)}{t} \geq a_\infty(x)$$

for a.e. $x \in \Omega$ and every $t > 0$. In particular, by (2.9) we get

$$a_0(x) \geq -c_f \geq a_\infty(x) \quad \text{for a.e. } x \in \Omega. \tag{2.11}$$

We are finally ready to give definition of 'weak solution' of (1.1).

Definition 2.7. Let *all* the above assumptions and notation be in force. We say that a function $u \in H^1_{0,X}(\Omega)$ is a *weak solution* of (1.1) if:

(1) $u \geq 0$ in Ω and $|\{x \in \Omega : u(x) > 0\}| > 0$;
(2) for every function $\varphi \in H^1_{0,X}(\Omega)$ one has

$$\int_\Omega \langle Xu, X\varphi \rangle \, \mathrm{d}x = \int_\Omega f(x,u)\varphi \, dx. \tag{2.12}$$

Remark 2.8. We explicitly point out that the above definition is well-posed: indeed, since we have $H^1_{0,X}(\Omega) \subseteq L^2(\Omega)$, by exploiting (2.7) one has

$$\int_\Omega |f(x,u)\,\varphi| \, dx \leq c_f \bigg(\int_\Omega |\varphi| \, dx + \int_\Omega |u| \, |\varphi| \, \mathrm{d}x \bigg) < +\infty,$$

for every choice of $u, \varphi \in H^1_{0,X}(\Omega)$.

Throughout the rest of the paper, we *tacitly inherit all the assumptions and the notation introduced so far*; in particular, we assume that:

(a) $\mathcal{D} \subseteq \mathbb{R}^n$ is a fixed open set;
(b) $X = \{X_1, \ldots, X_m\} \subseteq \mathcal{X}(\mathcal{D})$ satisfies (H1)-(H2);
(c) $\Omega \subseteq \mathcal{D}$ satisfies (S);
(d) $f : \Omega \times (0, +\infty) \to \mathbb{R}$ satisfies (f1)-to-(f4).

3. Boundedness and Strong Maximum Principle

We start by showing that any non-negative weak solution of (1.1) is bounded. The proof closely follows [5], so we claim no originality. We present it here to make the paper self- contained.

Theorem 3.1. *Let $u_0 \in H^1_{0,X}(\Omega)$ be a weak solution of problem* (1.1) (*according to Definition 2.7*)*, with f satisfying* (f1)*-to-*(f3)*. Then $u_0 \in L^\infty(\Omega)$.*

Proof. Let $\delta \in (0,1)$ be arbitrarily fixed, and let $\tilde{u}_0 = \delta u_0$. For every $k \in \mathbb{N} \cup \{0\}$, we then set $C_k := 1 - 2^{-k}$, and we consider the following functions:

$$v_k := \tilde{u}_0 - C_k, \quad w_k := (v_k)_+ := \max\{v_k, 0\}, \quad U_k := \|w_k\|^2_{L^2(\Omega)}.$$

We explicitly point out that, in view of these definitions, one has

(a) $\|\tilde{u}_0\|^2_{L^2(\Omega)} = \delta^2 \, \|u_0\|^2_{L^2(\Omega)}$);
(b) $w_0 = v_0 = \tilde{u}_0$ (since $C_0 = 0$);
(c) $v_{k+1} \leq v_k$ and $w_{k+1} \leq w_k$ (since $C_k < C_{k+1}$).

Moreover, since $u_0 \in H^1_{0,X}(\Omega)$ (and $w_k \leq u_0$), we infer that

$$w_k \in H^1_{0,X}(\Omega) \text{ and } Xw_k = X\tilde{u}_0 \, \mathbf{1}_{\{\tilde{u}_0 > C_k\}}. \tag{3.1}$$

We are then entitled to use w_k as a *test function* in (2.12), obtaining

$$\int_\Omega \langle X\tilde{u}_0, Xw_k \rangle \, dx = \delta \int_\Omega \langle Xu_0, Xw_k \rangle \, dx = \delta \int_\Omega f(x, u_0) w_k \, dx. \tag{3.2}$$

Now, taking into account (3.1), we get

$$\int_\Omega \langle X\tilde{u}_0, Xw_k \rangle \, dx = \int_{\Omega \cap \{\tilde{u}_0 > C_k\}} \langle Xw_k, Xw_k \rangle \, dx = \int_\Omega |Xw_k|^2 \, dx;$$

this, together with (3.2) and assumption (f3), implies

$$\begin{aligned}\int_\Omega |Xw_k|^2\,dx &\le \delta\int_\Omega |f(x,u_0)|\,w_k\,dx\\ &\le c\int_\Omega (\delta+\delta u_0)\,w_k\,dx \le c\int_\Omega (1+\tilde u_0)w_k\,dx =: (\bigstar).\end{aligned}$$

We then observe that, for every $k\ge 1$, one has:

(i) $\tilde u_0(x) < (2^k-1)w_{k-1}(x)$ for a.e. $x\in\{w_k>0\}$;

(ii) $\{w_k>0\}=\{\tilde u_0>C_k\}\subseteq\{w_{k-1}>2^{-k}\}$

Thus, since $w_k\le w_{k-1}$ a.e. in Ω, for every $k\ge 1$ we obtain

$$\begin{aligned}(\bigstar) &= c\int_{\{w_k>0\}}(1+\tilde u_0)w_k\,dx \le c\int_{\{w_k>0\}}\Big[w_{k-1}+(2^k-1)w_{k-1}^2\Big]\,dx\\ &\le c\int_{\{w_{k-1}>2^{-k}\}}\Big[2^kw_{k-1}^2+(2^k-1)w_{k-1}^2\Big]\,dx\\ &\le c\,2^{2k}\int_{\{w_{k-1}>2^{-k}\}} w_{k-1}^2dx \le c\,2^{2k}\int_\Omega w_{k-1}^2\,dx\\ &= c\,2^{2k}U_{k-1}.\end{aligned}$$

Summing up, we have proved that

$$\int_\Omega |Xw_k|^2\,dx \le c\,2^{2k}U_{k-1} \qquad (\text{for every } k\ge 1).$$

To proceed further we observe that, as a consequence of (ii), we have

$$\begin{aligned}U_{k-1} &= \int_\Omega w_{k-1}^2\,dx \ge \int_{\{w_{k-1}>2^{-k}\}} w_{k-1}^2\,dx\\ &\ge 2^{-2k}\,\big|\{w_{k-1}>2^{-k}\}\big| \ge 2^{-2k}\big|\{w_k>0\}\big|.\end{aligned}$$

Thus, using Hölder's inequality and fact that Ω supports a (q,X)-Sobolev type inequality *for some* $q>2$ (see assumption (S)), for every $k\ge 1$ we obtain

$$\begin{aligned}U_k &= \|w_k\|^2_{L^2(\Omega)} \le \left(\int_\Omega w_k^q\,dx\right)^{2/q}\big|\{w_k>0\}\big|^{\frac{q-2}{q}}\\ &\le \mathbf{c}_S\int_\Omega |Xw_k|^2\,dx\cdot\big|\{w_k>0\}\big|^{\frac{q-2}{q}}\\ &\le \mathbf{c}_S\big(c\,2^{2k}\,U_{k-1}\big)\big(2^{2k}U_{k-1}\big)^{\frac{q-2}{q}}\\ &= \mathbf{c}'\left(2^{2+\frac{2(q-2)}{q}}\right)^{k-1}U_{k-1}^{1+\frac{q-2}{q}} \qquad (\text{with } \mathbf{c}' := c\,2^{2+\frac{2(q-2)}{2}}\,\mathbf{c}_S),\end{aligned} \tag{3.3}$$

where $\mathbf{c}_S>0$ is the constant associated with the (q,X)-Sobolev type inequality. Now, estimate (3.3) can be re-written as

$$U_k \le \mathbf{c}'\eta^{k-1}U_{k-1}^{1+\frac{q-2}{q}},$$

where

$$\eta := 2^{2+\frac{2(q-2)}{q}} > 1.$$

Hence, from [17, Lem. 7.1] we get that $U_k \to 0$ as $k \to +\infty$, provided that

$$U_0 = \|\tilde{u}_0\|^2_{L^2(\Omega)} = \delta^2 \|u_0\|^2_{L^2(\Omega)} < (\mathbf{c}')^{-q/(q-2)}\,\eta^{-q^2/(q-2)^2}.$$

As a consequence, if $\delta \in (0,1)$ is small enough, we obtain

$$0 = \lim_{k\to\infty} U_k = \lim_{k\to\infty} \int_\Omega (\tilde{u}_0 - C_k)^2_+ \,\mathrm{d}x = \int_\Omega (\tilde{u}_0 - 1)^2_+ \,dx.$$

Bearing in mind that $\tilde{u}_0 = \delta u_0$ (and $u_0 \geq 0$), we then get

$$0 \leq u_0 \leq \frac{1}{\delta} \qquad \text{a.e. in } \Omega,$$

from which we conclude that $u_0 \in L^\infty(\Omega)$. □

Corollary 3.2. *Let $u_0 \in H^1_{0,X}(\Omega)$ be a weak solution of* (1.1) (*according to Definition 2.7*), *with f satisfying assumptions* (f1)-*to*-(f3). *Then, $u_0 \in C(\Omega)$.*

Proof. On account of Theorem 3.1, we know that $u_0 \in L^\infty(\Omega)$. As a consequence, since $X_1, \ldots, X_m$ satisfy Hörmander's condition (H1) in $\mathcal{D}$, we are entitled to apply the result in [11, Thm. 3.35], ensuring that $u_0 \in C(\Omega)$. □

Remark 3.3. As a matter of fact, the regularity result in [11, Thm. 3.35] ensures that u_0 is *locally Hölder continuous* on Ω with respect to the so-called Carnot-Carathéodory distance d_X associated with $X = \{X_1, \ldots, X_m\}$. However, since the X_i's satisfy Hörmander's condition (H1), it is well-known that d_X is *topologically* (but not metrically) equivalent to the classical Euclidean distance d_E; hence, 'continuous' in Euclidean sense and 'continuous' with respect to d_X are the same.

With Corollary 3.2 at hand, the strong maximum principle for *weak solutions* of (1.1) now follows from [1, Corollary 1.4]. The precise statement is the following.

Theorem 3.4. *Assume that f satisfy* (f1)-*to*-(f4), *and assume that $u_0 \in H^1_{0,X}(\Omega)$ is weak solution of* (1.1) (*according to Definition 2.7*). *Then*

$$u_0 > 0, \quad \textit{in } \Omega.$$

Proof. First of all, by Theorem 3.1 we know that $u_0 \in L^\infty(\Omega)$. As a consequence, by exploiting assumptions (f2) and (f4) on f, we obtain the estimate

$$f(x, u_0) = u_0 \cdot \frac{f(x,u_0)}{u_0} \geq u_0\, \frac{f(x, \|u_0\|_{L^\infty(\Omega)})}{\|u_0\|_{L^\infty(\Omega)}} \geq -M u_0,$$

holding true a.e. on $\{u_0 > 0\}$ and for some $M > 0$. On the other hand, since this estimate also holds when $u_0 = 0$ (see (2.10) in Remark 2.6), and since u_0 is a weak solution of problem (1.1), we then get

$$\int_\Omega \big(\langle X u_0, X\phi\rangle + M u_0\,\phi\big)\,dx \geq 0$$

for every $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$ in Ω. Now, taking into account the definition of $H^1_{0,X}(\Omega)$, the above inequality can be rewritten as

$$\int_\Omega u_0\,(\mathcal{L} + M)\phi\,dx \geq 0 \qquad \forall\, \phi \in C_0^\infty(\Omega),\ \phi \geq 0 \text{ in } \Omega;$$

this, together with the fact that $u_0 \in C(\Omega)$ (as we know from Corollary 3.2), shows that u_0 is a *distributional subsolution* of $\mathcal{L}_M := \mathcal{L} + M \geq 0$ in the sense of Ishii [19]. On account of assumption (H2), we are then entitled to invoke [19, Thm. 2], from which we derive that u_0 is also a (continuous) *viscosity subsolution* of $\mathcal{L}_M \geq 0$. We can now apply to u_0 the strong maximum principle in [1, Cor. 1.4], ensuring that

$$\text{either } u_0 \equiv 0 \text{ in } \Omega \text{ or } u_0 > 0 \text{ in } \Omega.$$

Finally, since $|\{u_0 > 0\}| > 0$, we conclude that $u_0 > 0$ in Ω, as desired. □

Remark 3.5. A closer inspection to the previous proof shows that the only crucial ingredient is the existence of some constant $M > 0$ such that

$$f(x, u_0) \geq -Mu_0 \quad \text{for a.e. } x \in \Omega. \tag{3.4}$$

As a consequence, Theorem 3.4 holds *for every non-negative weak solution* of

$$\mathcal{L}u = g(x, u),$$

provided that $g : \Omega \times (0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfies (3.4). A key example of a non-linearity g satisfying the estimate (3.4) is the following

$$g_\lambda(x, t) := \big(\lambda - a(x)\big)t,$$

where $\lambda \in \mathbb{R}$ and $a \in L^\infty(\Omega)$.

4. Uniqueness

The aim of this section is to establish *uniqueness and boundedness* of weak solutions to problem (1.1). Throughout what follows, we tacitly inherit *all the assumptions and the notation* introduced so far; moreover, to prove *uniqueness* we require the following additional hypothesis of f:

(f5) there exists some $\rho = \rho_f > 0$ such that

$$f(x, t) > 0 \quad \text{for a.e. } x \in \Omega \text{ and every } 0 < t < \rho_f. \tag{4.1}$$

Remark 4.1. We explicitly observe that, in the particular case of power-type linearities $f(x, u) = u^\theta$ (with $0 \leq \theta < 1$), assumption (f5) is trivially satisfied.

We are now ready to state and prove the main result of this section.

Theorem 4.2. *Under the assumptions above, there exists* at most *one weak solution* $u \in H^1_{0,X}(\Omega)$ *of problem* (1.1).

Proof. Let $u_1, u_2 \in H^1_{0,X}(\Omega)$ be two weak solutions of problem (1.1). We then observe that, as a consequence of Theorems 3.1-3.4 and of Corollary 3.2, we have

(a) $u_1, u_2 \in L^\infty(\Omega) \cap C(\Omega)$;

(b) $u_1, u_2 > 0$ pointwise in Ω.

Now, in order to show that $u_1 \equiv u_2$ in Ω, we arbitrarily fix $\varepsilon > 0$ and we define

$$\varphi_{1,\varepsilon} := r_{1,\varepsilon} - u_{1,\varepsilon}, \qquad \varphi_{2,\varepsilon} := r_{2,\varepsilon} - u_{2,\varepsilon},$$

where $u_{i,\varepsilon} := \min\{u_i, \varepsilon^{-1}\}$ (for $i = 1, 2$) and

$$r_{1,\varepsilon} := \frac{u_{2,\varepsilon}^2}{u_1 + \varepsilon}, \qquad r_{2,\varepsilon} := \frac{u_{1,\varepsilon}^2}{u_2 + \varepsilon}.$$

Since $u_1, u_2 \in L^\infty(\Omega) \cap H^1_{0,X}(\Omega)$ and $u_1, u_2 > 0$ in Ω, it is easy to see that

$$\varphi_{i,\varepsilon} \in H^1_{0,X}(\Omega) \text{ for every } \varepsilon > 0 \text{ and } i = 1, 2;$$

as a consequence, using $\varphi_{i,\varepsilon}$ as a test function in (1.3) for u_i and adding the resulting identities, we obtain

$$\begin{aligned}&\int_\Omega \langle Xu_1, X\varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega \langle Xu_2, X\varphi_{2,\varepsilon}\rangle\,dx \\ &\quad = \int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx.\end{aligned} \tag{4.2}$$

Now, a direct application of [5, Lemma 4.3] (with $p = 2$) gives

$$\int_\Omega \langle Xu_1, X\varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega \langle Xu_2, X\varphi_{2,\varepsilon}\rangle\,dx \leq 0.$$

As a consequence, identity (4.2) boils down to

$$\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx \leq 0. \tag{4.3}$$

We then aim to pass to the limit as $\varepsilon \to 0^+$ in the above (4.3). First of all, taking into account the very definition of $\varphi_{i,\varepsilon}$, we write

$$\begin{aligned}\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx &= \int_\Omega f(x,u_1)\,r_{1,\varepsilon}\,dx + \int_\Omega f(x,u_2)\,r_{2,\varepsilon}\,dx \\ &\quad - \int_\Omega f(x,u_1)u_{1,\varepsilon}\,dx - \int_\Omega f(x,u_2)u_{2,\varepsilon}\,dx \\ &=: \mathrm{A}_{1,\varepsilon} + \mathrm{A}_{2,\varepsilon} - \mathrm{B}_{1,\varepsilon} - \mathrm{B}_{2,\varepsilon}.\end{aligned}$$

Moreover, since $u_i \in H^1_{0,X}(\Omega)$ and $0 \leq u_{i,\varepsilon} \leq u_i$ (for every $\varepsilon > 0$), a simple dominated-convergence argument based on (2.7) ensures that

$$\mathrm{B}_i := \lim_{\varepsilon\to 0^+} \mathrm{B}_{i,\varepsilon} = \int_\Omega f(x,u_i)u_i\,dx \in \mathbb{R} \qquad (i = 1,2). \tag{4.4}$$

Now, if $\rho = \rho_f > 0$ is as in (4.1), we further split $\mathrm{A}_{i,\varepsilon}$ as

$$\mathrm{A}_{i,\varepsilon} = \int_{\{u_i<\rho_f\}} f(x,u_i)\,r_{i,\varepsilon}\,dx + \int_{\{u_i\geq\rho_f\}} f(x,u_i)\,r_{i,\varepsilon}\,dx =: \mathrm{A}'_{i,\varepsilon} + \mathrm{A}''_{i,\varepsilon}.$$

Furthermore, since for every $\varepsilon > 0$ one also has (again by (2.7))

$$\begin{aligned}&|f(x,u_1)\,r_{1,\varepsilon}| \cdot \mathbf{1}_{\{u_1\geq\rho_f\}} \leq c\big(1+\rho_f^{-1}\big)\,u_2^2 \equiv c_f\,u_2^2 \qquad \text{and} \\ &|f(x,u_2)\,r_{2,\varepsilon}| \cdot \mathbf{1}_{\{u_2\geq\rho_f\}} \leq c_f\,u_1^2,\end{aligned}$$

another application of the Dominated Convergence theorem shows that

$$\begin{aligned}&\mathrm{A}''_1 := \lim_{\varepsilon\to 0^+} \mathrm{A}''_{1,\varepsilon} = \int_{\{u_1\geq\rho_f\}} \frac{f(x,u_1)}{u_1}\,u_2^2\,dx \in \mathbb{R} \qquad \text{and} \\ &\mathrm{A}''_2 := \lim_{\varepsilon\to 0^+} \mathrm{A}''_{2,\varepsilon} = \int_{\{u_2\geq\rho_f\}} \frac{f(x,u_2)}{u_2}\,u_1^2\,dx \in \mathbb{R}.\end{aligned} \tag{4.5}$$

Hence, it remains to study the behavior of $A'_{i,\varepsilon}$ when $\varepsilon \to 0^+$. To this end we observe that, by (4.1) and the fact that $u_{i,\varepsilon}$ is non-negative and monotone increasing with respect to ε, we can apply the Beppo Levi theorem, obtaining

$$
\begin{aligned}
&\mathrm{A}_1' := \lim_{\varepsilon\to 0^+} \mathrm{A}_{1,\varepsilon}' = \int_{\{u_1<\rho_f\}} \frac{f(x,u_1)}{u_1}\, u_2^2\, dx \in [0,+\infty] \qquad \text{and} \\
&\mathrm{A}_2' := \lim_{\varepsilon\to 0^+} \mathrm{A}_{2,\varepsilon}' = \int_{\{u_2<\rho_f\}} \frac{f(x,u_2)}{u_2}\, u_1^2\, dx \in [0,+\infty].
\end{aligned} \tag{4.6}
$$

On the other hand, going back to estimate (4.3) and taking into account the very definitions of the integrals $\mathrm{A}_{i,\varepsilon}', \mathrm{A}_{i,\varepsilon}'', \mathrm{B}_{i,\varepsilon}$, we get

$$
\begin{aligned}
0 \le \mathrm{A}_{1,\varepsilon}', \mathrm{A}_{2,\varepsilon}' &\le \mathrm{A}_{1,\varepsilon}' + \mathrm{A}_{2,\varepsilon}' \\
&\le \mathrm{B}_{1,\varepsilon} + \mathrm{B}_{2,\varepsilon} - \mathrm{A}_{1,\varepsilon}'' - \mathrm{A}_{2,\varepsilon}''.
\end{aligned}
$$

Then, by letting $\varepsilon \to 0^+$ with the aid of (4.4)-(4.5), we obtain

$$
0 \le \mathrm{A}_1', \mathrm{A}_2' \le \mathrm{A}_1' + \mathrm{A}_2' \le \mathrm{B}_1 + \mathrm{B}_2 - \mathrm{A}_1'' - \mathrm{A}_2'',
$$

from which we derive at once that

$$
\mathrm{A}_1', \mathrm{A}_2' < +\infty. \tag{4.7}
$$

Gathering together (4.4)-(4.6), taking into account (4.7) and using in a crucial way that fact that $u_1, u_2 > 0$ pointwise in Ω, we finally have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+} \left(\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\, dx \right) \\
&\quad = \lim_{\varepsilon\to 0^+} \big(\mathrm{A}_{1,\varepsilon}' + \mathrm{A}_{2,\varepsilon}' + \mathrm{A}_{1,\varepsilon}'' + \mathrm{A}_{2,\varepsilon}'' - \mathrm{B}_{1,\varepsilon} - \mathrm{B}_{2,\varepsilon}\big) \\
&\quad = \int_\Omega \Big(\frac{f(x,u_1)}{u_1} u_2^2 + \frac{f(x,u_2)}{u_2} u_1^2 - f(x,u_1)u_1 - f(x,u_2)u_2 \Big)\, \mathrm{d}x \\
&\quad = - \int_\Omega \Big(\frac{f(x,u_1)}{u_1} - \frac{f(x,u_2)}{u_2} \Big) (u_1^2 - u_2^2)\, dx.
\end{aligned} \tag{4.8}
$$

With (4.8) at hand, we can easily conclude the proof of the theorem. Indeed, using these cited identities we can let $\varepsilon \to 0^+$ in (4.3), obtaining

$$
- \int_\Omega \Big(\frac{f(x,u_1)}{u_1} - \frac{f(x,u_2)}{u_2} \Big) (u_1^2 - u_2^2)\, dx \le 0.
$$

From this, by crucially exploiting assumption (f4) and again the fact that $u_1, u_2 > 0$ in Ω, we conclude that $u_1 \equiv u_2$ in Ω. This ends the proof. □

5. Necessary and sufficient conditions

In this last section we exploit all the results established so far in order to give the proof of Theorem 1.1. Throughout what follows, we tacitly adopt all the notation introduced in Sections 2.1-4: in particular,

- $\Omega \subseteq \mathbb{R}^n$ is a bounded open set which satisfies (S);
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (f1)-to-(f4);
- a_0 and a_∞ are the functions defined in (2.8);

Furthermore, similarly to Brezis-Oswald [10], we define the *smallest* (*Dirichlet*) *eigenvalue* of $\mathcal{L}-a_0$ and of $\mathcal{L}-a_\infty$ as follows:

$$\lambda_1(\mathcal{L}-a_0) := \inf_{\substack{u\in H^1_{0,X}(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \left\{ \int_\Omega |Xu|^2\,dx - \int_{\{u\neq 0\}} a_0|u|^2\,dx \right\},$$

$$\lambda_1(\mathcal{L}-a_\infty) := \inf_{\substack{u\in H^1_{0,X}(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \left\{ \int_\Omega |Xu|^2\,dx - \int_{\{u\neq 0\}} a_\infty|u|^2\,dx \right\}.$$

It should be noticed that, since a_0 and a_∞ can be *unbounded* in Ω, one *cannot define* $\lambda_1(\mathcal{L}-a_0)$ and $\lambda_1(\mathcal{L}-a_\infty)$ by using the standard variational approach.

Although it is not highly involved, the proof of Theorem 1.1 is quite long, and it articulates into different steps; as a consequence, for the Reader's convenience we split such a proof into several independent results, of interest on their own.

To begin with, we let

$$F(x,u) = \int_0^u f(x,t)\,dt,$$

and we consider the functional $E : H^1_{0,X}(\Omega) \to \mathbb{R}$ defined as follows:

$$E(u) := \frac{1}{2}\int_\Omega |Xu|^2\,dx - \int_\Omega F(x,u)\,dx, \tag{5.1}$$

The functional E is well-defined, differentiable and its critical points are weak solutions of problem (1.1). Moreover, we have the following key result.

Proposition 5.1. *Let E be the functional defined in* (5.1)*, and assume that*

$$\lambda_1(\mathcal{L}-a_0) < 0 < \lambda_1(\mathcal{L}-a_\infty).$$

Then, the following facts hold:

(a) *E is coercive on $H^1_{0,X}(\Omega)$.*
(b) *E is weakly l.s.c. in $H^1_{0,X}(\Omega)$, so it has a minimum point in $H^1_{0,X}(\Omega)$.*
(c) *There exists $\phi \in H^1_{0,X}(\Omega)$ such that $E(\phi) < 0$, so that*

$$\min_{u\in H^1_{0,X}(\Omega)} E(u) < 0,$$

and the minimum point is a solution to (1.1)*.*

Proof. (a) The coercivity of E can be proved by arguing exactly as in [10], and so we omit the details. We limit ourselves to mention that condition

$$\lambda_1(\mathcal{L}-a_\infty) > 0$$

is used at this stage.

(b) Let $u \in H^1_{0,X}(\Omega)$ be fixed, and let $\{u_n\}_n$ be a sequence in $H^1_{0,X}(\Omega)$ which weakly converges to u as $n \to +\infty$. Owing to assumption (2.7), we have

$$|F(x,u)| \leq c(|u| + |u|^2);$$

moreover, since $H^1_{0,X}(\Omega)$ is compactly embedded into $L^2(\Omega)$, see (2.5), by possibly passing to a sub-sequence we can assume that

$$u_n \to u \text{ strongly in } L^2(\Omega) \text{ and pointwise a.e. in } \Omega.$$

Gathering these facts, we easily conclude that

$$\lim_{n\to+\infty}\int_\Omega F(x,u_n)\,dx = \int_\Omega F(x,u)\,dx,$$

and this immediately implies the claim.

(c) To prove this assertion, we follow the argument originally presented in [10]. Since $\lambda_1(\mathcal{L}-a_0)<0$, there exists $\phi\in H^1_{0,X}(\Omega)$ such that $\|\phi\|_{L^2(\Omega)}=1$ and

$$\int_\Omega |X\phi|^2\,dx < \int_{\{\phi\neq 0\}} a_0\,|\phi|^2 dx. \tag{5.2}$$

We then claim that it is not restrictive to assume that $\phi\geq 0$ and $\phi\in L^\infty(\Omega)$. In fact, reminding that $|\phi|\in H^1_{0,X}(\Omega)$ and $|X|\phi||=|X\phi|$ a.e. in Ω, we find

$$\int_\Omega |X|\phi||^2\,dx = \int_\Omega |X\phi|^2\,dx < \int_{\{\phi\neq 0\}} a_0\,|\phi|^2\,dx,$$

and thus we can suppose $\phi\geq 0$. As for the assumption $\phi\in L^\infty(\Omega)$, we define

$$\phi_M = \min\{\phi, M\} \qquad (\text{for } M>0).$$

Since $\phi_M\in H^1_{0,X}(\Omega)$ and $X\phi_M = X\phi\,\mathbf{1}_{\{u<M\}}$, we get

$$\int_\Omega |X\phi_M|^2\,dx \leq \int_\Omega |X\phi|^2\,dx < \int_{\{\phi\neq 0\}} a_0\,|\phi|^2\,dx.$$

On the other hand, since a_0 is bounded from below (see (2.11)), we have

$$\int_\Omega a_0\phi^2 \leq \liminf_{M\to+\infty}\int_\Omega a_0\phi_M^2\,dx;$$

as a consequence, there exists $M>0$ such that

$$\int_\Omega |X\phi_M|^2\,dx < \int_{\{\phi\neq 0\}} a_0\,\phi_M^2\,dx.$$

Summing up, by possibly replacing ϕ with $\tilde\phi=\min\{|\phi|,M\}$ (for $M>0$ sufficiently large), we can choose $\phi\geq 0$ and bounded.

Now, we have that

$$\liminf_{u\to 0}\frac{F(x,u)}{u^2}\geq \frac{a_0(x)}{2};$$

thus, proceeding exactly as in [10, Proof of (15)], we get

$$\liminf_{\varepsilon\to 0}\int_\Omega \frac{F(x,\varepsilon\phi)}{\varepsilon^2} \geq \frac{1}{2}\int_{\{\phi\neq 0\}} a_0\phi^2\,dx.$$

From this, using (5.2) we conclude that

$$2E(\varepsilon\phi) = \int_\Omega |X\phi|^2\,dx - 2\int_\Omega \frac{F(x,\varepsilon\phi)}{\varepsilon^2}\,dx < 0$$

for every $\varepsilon>0$ small enough, and the proof is complete. □

Since Proposition 5.1 contains the 'sufficiency' part of Theorem 1.1, we now turn our attention to the 'necessity' part. To begin with, we prove the next result.

Proposition 5.2. *Let $u\in H^1_{0,X}(\Omega)$ be a solution of* (1.1). *Then*

$$\lambda_1(\mathcal{L}-a_0)<0. \tag{5.3}$$

Proof. On the one hand, by the very definition of $\lambda_1(\mathcal{L}-a_0)$ we have

$$\lambda_1(\mathcal{L}-a_0) \leq \frac{1}{\|u\|_{L^2(\Omega)}}\Big(\int_\Omega |Xu|^2\,dx - \int_{\{u\neq 0\}} a_0\,|u|^2\Big).$$

On the other hand, since $u \in H^1_{0,X}(\Omega)$ solves (1.1), we have that

$$\int_\Omega |Xu|^2\,dx = \int_\Omega f(x,u)u\,dx.$$

Now, using Corollary 3.2 and the strong maximum principle in Theorem 3.4, we infer that $u \in C(\Omega)$ and $u > 0$ in Ω. As a consequence, using assumption (f4) and bearing in mind the definition of a_0, we get

$$\frac{f(x,u)}{u} < a_0(x) \text{ a.e. in } \Omega,$$

Thus, we obtain

$$\int_\Omega f(x,u)u\,dx < \int_\Omega a_0 u^2\,dx,$$

and the proof is complete. □

We now prove the 'dual' statement of Proposition 5.2. Before doing this, we briefly study the eigenvalue problem associated with $\mathcal{L}$.

Lemma 5.3. *Let $a \in L^\infty(\Omega)$ be fixed. Then, the problem*

$$\begin{cases} \mathcal{L}u + a(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

admits a smallest eigenvalue $\lambda_1(\mathcal{L}+a) \in \mathbb{R}$; moreover, there exists an associated eigenfunction which is strictly positive *almost everywhere in Ω.*

Proof. Let $\gamma : H^1_{0,X}(\Omega) \to \mathbb{R}$ be the C^1-functional defined as

$$\gamma(u) = \int_\Omega |Xu|^2 \mathrm{d}x + \int_\Omega a(x)u^2 \mathrm{d}x \qquad (u \in H^1_{0,X}(\Omega)).$$

Moreover, we consider the *Banach manifold*

$$M := \big\{u \in H^1_{0,X}(\Omega) : \|u\|^2_{L^2(\Omega)} = 1\big\} \subseteq H^1_{0,X}(\Omega).$$

Since $a \in L^\infty(\Omega)$, it is immediate to recognize that γ is *bounded from below on M*, that is, there exists $m \in \mathbb{R}$ such that $\gamma \geq m$ on M; hence, we define

$$\lambda_1(\mathcal{L}+a) := \inf\Big\{\gamma(u) : u \in M\Big\} \in \mathbb{R}. \tag{5.5}$$

Let now $\{u_n\}_{n\geq 1} \subseteq M$ be a minimizing sequence for (5.5). Since $\|u_n\|_{L^2(\Omega)} = 1$ for every $n \geq 1$ and $\gamma(u_n) \to \lambda_1(\mathcal{L}+a) \in \mathbb{R}$ as $n \to +\infty$, we easily see that

$$\{u_n\}_{n\geq 1} \subseteq H^1_{0,X}(\Omega) \text{ is } \textit{bounded};$$

as a consequence, by possibly passing to a sub-sequence we can assume that there exists a function $e_1 \in M$ such that

$$u_n \rightharpoonup e_1 \text{ in } H^1_{0,X}(\Omega) \qquad \text{as } n \to +\infty. \tag{5.6}$$

Moreover, since $H^1_{0,X}(\Omega)$ is compactly embedded into $L^2(\Omega)$, by possibly choosing a further sub-sequence we infer that

$$u_n \to e_1 \text{ in } L^2(\Omega). \tag{5.7}$$

By (5.6)-(5.7), and since γ is weakly lower continuous, we have

$$\gamma(e_1) = \int_\Omega |Xe_1|^2 \mathrm{d}x + \int_\Omega a(x) e_1^2 \,\mathrm{d}x \leq \liminf_{n\to+\infty} \gamma(u_n) = \lambda_1(\mathcal{L}+a).$$

From this, taking into account (5.5) and the fact that $e_1 \in M$, we get

$$\gamma(e_1) = \lambda_1(\mathcal{L}+a),$$

and thus e_1 is a *minimum point for* γ *constrained to* M. By the well-known Lagrange multiplier rule, we then infer that $\lambda_1(\mathcal{L}+a)$ is the smallest eigenvalue for problem (5.4), with associated eigenfunction $e_1 \in H^1_{0,X}(\Omega)$. Finally, we observe that

$$\gamma(|u|) = \gamma(u) \quad \text{for every } u \in H^1_{0,X}(\Omega),$$

and thus we may assume that $e_1 \geq 0$ a.e. in Ω. As a consequence, since $e_1 \in M$, the strong maximum principle stated in Remark 3.5 ensures that

$$e_1(x) > 0 \quad \text{for a.e. } x \in \Omega,$$

and the proof is complete. □

Remark 5.4. By proceeding essentially as in the proof of [5, Prop. 5.1], it is possible to prove the following additional results for the $\mathcal{L}$-eigenvalue problem (5.4):

(a) the eigenspace associated with $\lambda_1(\mathcal{L}+a)$ is *one-dimensional*;
(b) every eigenfunction associated with an eigenvalue $\lambda > \lambda_1(\mathcal{L}+a)$ is *nodal*, that is, it changes sign in Ω.

However, we do not need (a)-(b) for our arguments.

Thanks to Lemma 5.3, we can prove the following result.

Proposition 5.5. *Let* $u \in H^1_{0,X}(\Omega)$ *be solution of* (1.1). *Then*

$$\lambda_1(\mathcal{L} - a_\infty) > 0. \tag{5.8}$$

Proof. We first observe that, by Theorem 3.1 and Corollary 3.2, we have

$$u \in L^\infty(\Omega) \cap C(\Omega).$$

Arguing as in [10], we then define the weight

$$\overline{a}(x) := \frac{f(x, \|u\|_{L^\infty(\Omega)} + 1)}{\|u\|_{L^\infty(\Omega)} + 1}.$$

On account of assumption (f2), it is readily seen that $\overline{a} \in L^\infty(\Omega)$; as a consequence, from Lemma 5.3 we infer the existence of $\mu \in \mathbb{R}$ and $\psi \in H^1_{0,X}(\Omega)$ such that

$$\begin{cases} \mathcal{L}\psi - \overline{a}(x)\psi = \mu\psi & \text{in } \Omega, \\ \psi > 0 & \text{a.e. in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, using ψ as test function for (1.1), we get

$$\int_\Omega u\psi\,(\overline{a} + \mu)\,\mathrm{d}x = \int_\Omega f(x,u)\psi\,\mathrm{d}x.$$

On the other hand, since $u > 0$ pointwise in Ω (as we know from the strong maximum principle in Theorem 3.4), by assumption (f4) we have

$$\int_\Omega f(x,u)\psi\,\mathrm{d}x > \int_\Omega \overline{a}(x) u\psi\,\mathrm{d}x.$$

As a consequence, we obtain

$$\mu \int_\Omega u\psi \,\mathrm{d}x > 0,$$

and the desired (5.8) follows by arguing exactly as in [10]. □

Finally, we prove that (5.3) implies the validity of assumption (f5). This will allow us to obtain the *uniqueness* of the solutions for problem (1.1) *without any additional assumption on the function* f.

Proposition 5.6. *If* $\lambda_1(\mathcal{L} - a_0) < 0$*, then* (f5) *holds.*

Proof. We first observe that, in view of assumption (f4), $f(x,\cdot)$ can change sign *at most once* in $(0,+\infty)$. Arguing by contradiction, we then assume that $f(x,\cdot)$ is *negative* in a right neighborhood of $t = 0$. Hence, in particular,

$$a_0(x) \leq 0 \text{ for a.e. } x \in \Omega. \tag{5.9}$$

As a consequence of (5.9), we obtain

$$\lambda_1(\mathcal{L} - a_0) = \inf_{\substack{u \in H^1_{0,X}(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \left\{ \int_\Omega |Xu|^2 \,dx - \int_{\{u\neq 0\}} a_0\,|u|^2\,dx \right\} \geq 0,$$

but this is clearly a contradiction. □

By combining the results in this section, we can easily prove Theorem 1.1.

Proof (of Theorem 1.1). We first assume that there exists a solution $u \in H^1_{0,X}(\Omega)$ for problem (1.1). Then, by Propositions 5.2 and 5.5 we get

$$\lambda_1(\mathcal{L} - a_0) < 0 < \lambda_1(\mathcal{L} - a_\infty). \tag{5.10}$$

Moreover, from Theorems 3.1-3.4 and Corollary 3.2 we derive that

(a) $u \in L^\infty(\Omega) \cap C(\Omega)$;
(b) $u > 0$ pointwise in Ω.

Finally, by combining (5.10), Proposition 5.6 and Theorem 4.2 we conclude that u is *unique*, and this completes the proof of the 'necessity' part of Theorem 1.1.

As for the 'sufficiency' part, let us assume that (5.10) is satisfied. Then, by Proposition 5.1 we know that there exists (at least) one solution $u \in H^1_{0,X}(\Omega)$ for problem (1.1). From this, by proceeding exactly as in the first part of the proof, we deduce that u is *unique* and it satisfies (a)-(b). This ends the demonstration. □

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