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Research article

Lipschitz stable determination of small conductivity inclusions in a semilinear equation from boundary data†

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Abstract: We consider an inverse problem regarding the detection of small conductivity inhomogeneities in a boundary value problem for a semilinear elliptic equation. For such a problem, that is related to cardiac electrophysiology, an asymptotic expansion for the boundary potential due to the presence of small conductivity inhomogeneities was established in [4]. Starting from this we derive Lipschitz continuous dependence estimates for the corresponding inverse problem.

Keywords: inverse problem; semilinear elliptic equation; stability estimate

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded, connected, convex C^1 domain and let

$$\omega_\varepsilon = \bigcup_{i=1}^{m} (z_i + \varepsilon B_i) \tag{1.1}$$

where $B_i = r_i B, i = 1, \dots, m$, B is a given bounded smooth domain containing the origin and the inhomogeneities $(z_i + \varepsilon r_i B), i = 1, \dots, m$ are disjoint. Let K_0 and $K_1, \dots, K_m$ be symmetric, positive

definite tensors and let $f \in L^p(\Omega)$ with $p > d$ be not identically zero.
Consider the solution $u_\varepsilon \in H^1(\Omega)$ of the boundary value problem

$$\begin{cases} -div(K_\varepsilon \nabla u_\varepsilon) + \chi_{\Omega\setminus\omega_\varepsilon} u_\varepsilon^3 = f & \text{in } \Omega \\ K_\varepsilon \nabla u_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\chi_{\Omega\setminus\omega_\varepsilon}$ is the indicator function of the set $\Omega \setminus \omega_\varepsilon$, ν is the outward unit normal vector to $\partial\Omega$ and

$$K_\varepsilon = \begin{cases} K_0 & x \in \Omega \setminus \overline{\omega_\varepsilon}; \\ K_i & x \in (z_i + \varepsilon r_i B),\ i = 1, \dots, m. \end{cases} \tag{1.3}$$

The inverse problem we are interested in is the determination of ω_ε from the knowledge of $u_\varepsilon|_\Gamma$ where $\Gamma \subset \partial\Omega$ is an open arc.

Let $u \in H^1(\Omega)$ be the solution to

$$\begin{cases} -div(K_0 \nabla u) + u^3 = f & \text{in } \Omega \\ K_0 \nabla u \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In [4] the following asymptotic expansion for $(u_\varepsilon - u)|_{\partial\Omega}$ has been derived

$$(u_\varepsilon - u)(y) = \varepsilon^d \sum_{i=1}^m r_i^d \Big(M_i (K_0 - K_i) \nabla u(z_i) \cdot \nabla_x N_u(z_i, y) + u^3(z_i) N_u(z_i, y) \Big) + o(\varepsilon^d), \tag{1.5}$$

for $y \in \partial\Omega$ and where $M_i \in \mathbb{R}^{d\times d}$ are symmetric matrices known as the *polarization tensors* and depending on B, K_0, K_i for $i = 1, \dots, m$; for some specific shapes, the polarization tensors can be explicitly computed (see, e.g., [2] for a detailed derivation). The function N_u appearing in the first order term of the expansion is the Neumann function related to the operator $-div(K_0 \nabla \cdot) + 3u^2$, i.e., the solution, for each $y \in \Omega$, of

$$\begin{cases} -div(K_0 \nabla_x N_u(x, y)) + 3u^2(x) N_u(x, y) = \delta_y & \text{in } \Omega \\ K_0 \nabla_x N_u(x, y) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The inverse problem of determining ω_ε from $u_\varepsilon|_\Gamma$ is related to a simplified version of a model describing the electrical activity of the heart in the presence of small ischemic regions ([4]). In order to detect the set of inhomogeneities in [5] the authors implemented a successful reconstruction algorithm based on the computation of the topological gradient of a suitable boundary misfit functional. In this paper we prove that the centers and the radii of the conductivity inhomogeneities depend in a Lipschitz stable way from the rescaled measurements $\frac{(u_\varepsilon - u)|_\Gamma}{\varepsilon^d}$.

Similar stability results were obtained for the inverse conductivity equation in [8] and [9]. Here, in order to derive the Lipschitz stability, we study the differentiability and injectivity properties of the map $T : \mathbb{R}^{m(d+1)} \to L^\infty(\Gamma)$ defined by

$$T(\mathbf{z}, \mathbf{r}) = \sum_{i=1}^m r_i^d \Big(M_i (K_0 - K_i) \nabla u(z_i) \cdot \nabla_x N_u(z_i, y) + u^3(z_i) N_u(z_i, y) \Big), \quad y \in \Gamma$$

where $\mathbf{z} = (z_1, z_2, \dots, z_m), \mathbf{r} = (r_1, \dots, r_m)$ and apply an abstract theorem derived in [3] and in [6]. The obtained result proves well-posedness of the inverse problem justifying mathematically the successful reconstructions obtained in [5].

2. Statement of the problem and preliminary results

We consider first the solution to the boundary value problem (1.4) where $K_0 \in \mathbb{R}^{d\times d}$ is a symmetric constant matrix satisfying

$$\alpha_0|\xi|^2 \le \xi^T K_0 \xi \le \beta_0|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \tag{2.1}$$

being $\alpha_0 \ge 1$. Problem (1.4) is referred to as the *background* problem, and its solution u as the *unperturbed* potential. Such problem is well posed: as it was proved in [4, Theorem 4.1], for any forcing term $f \in H^{-1}(\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$, continuously depending on f.

Let ω_ε be the set defined in (1.1) consisting of m connected components of small size of prescribed shape B and with centers z_i and radii r_i. We assume that the centers of the inclusions are well-separated from the boundary and between each other; i.e.,

$$\exists d_0 > 0 : \qquad |z_i - z_j| > d_0 \quad i \neq j, \qquad dist(z_i, \partial\Omega) \ge d_0. \tag{2.2}$$

Also the relative sizes $\{r_i\}_{i=1}^m$ of the inclusions satisfy the condition

$$d_0 \le r_k \le d_1 \tag{2.3}$$

and we will assume ε small enough so that the sets $z_k + \varepsilon r_k B$ are disjoint and their distance is larger then $d_0/2$.

Moreover we assume the conductivity in ω_ε is strictly smaller then in $\Omega \setminus \omega_\varepsilon$; more precisely we assume each matrix $K_i \in \mathbb{R}^{d\times d}$ in (1.3) is symmetric and satisfies

$$\alpha_i|\xi|^2 \le \xi^T K_0 \xi \le \beta_i|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \qquad 0 < \alpha_i \le \beta_i < 1. \tag{2.4}$$

We define the *perturbed* potential as the weak solution of (1.2). Also the well-posedness of this problem is proved in [4, Theorem 4.1].

As recalled in the Introduction, in [4] the asymptotic expansion (1.5) has been derived for the difference $u_\varepsilon - u$ for all $y \in \partial\Omega$ under the assumption that $f \in L^p(\Omega)$, $p \ge d$ and $f(x) \ge m > 0$ a.e. in Ω. In [5] the result was extended to the case $f \in L^p(\Omega)$, $\|f\|_{L^p(\Omega)} \neq 0$.

For some specific shapes this formula can be explicitly computed (see, e.g., [2] for a detailed derivation).

Remark 2.1. Using the energy estimate in [4, Theorem 4.3] together with the estimate in [7, Lemma 1], we can conclude that the remainder term appearing in (1.5) satisfies:

$$o(\varepsilon^d) \le C\varepsilon^{d(1+\gamma)}, \quad \gamma \in \left(0, \frac{1}{2}\right].$$

From now on we will assume $f \in C^\alpha(\Omega)$.

We will need some properties of the Neumann function N_U defined in (1.6), analogous to the ones that hold in the linear case. First of all notice that we can write

$$N_u(x, y) = \Phi(x, y) + Z(x, y), \tag{2.5}$$

where Φ is a fundamental solution of the operator $-div(K_0\nabla\cdot)$, i.e., (see [10, Chapter 1, Sections 8–10])

$$\Phi(x,y)=\begin{cases}\dfrac{1}{2\pi\sqrt{det\,K_0}}\ln\left[K_0^{-1}(x-y)\cdot(x-y)\right]^{-\frac{1}{2}} & d=2\\ \dfrac{1}{(d-2)\alpha_d\sqrt{det\,K_0}}\left[K_0^{-1}(x-y)\cdot(x-y)\right]^{\frac{2-d}{2}} & d\geq 3.\end{cases} \tag{2.6}$$

α_d being the surface measure of the unit sphere, and $Z(x,y)$ satisfying

$$\begin{cases}-div_x(K_0\nabla_x Z(x,y))+3u^2(x)Z(x,y)=-3u^2(x)\Phi(x,y) & \text{in } \Omega\\ K_0\nabla_x Z(x,y)\cdot\nu=-K_0\nabla_x\Phi(x,y)\cdot\nu & \text{on } \partial\Omega.\end{cases} \tag{2.7}$$

Proposition 2.1. *For the Neumann function N_U defined in* (1.6) *the following hold*

- $N_U(x,y)=N_U(y,x)$ *(symmetry of the Neumann function)*
- $N_U(\cdot,y),N_U(x,\cdot)\in W^{1,p}(\Omega)\ \forall p\in\left[1,\frac{d}{d-1}\right)$
- $\forall\alpha\in\mathbb{R}^n\ \alpha\neq 0,\ \nabla_x N_U(x,\cdot)\cdot\alpha\in L^p(\Omega)\ \forall p\in\left[1,\frac{d}{d-1}\right)$ *and* $\nabla_x N_U(x,\cdot)\cdot\alpha\notin W^{1,1}(\Omega)$,
- $D_x^2 N_U(x,\cdot)\alpha\cdot\beta\notin L^1(\Omega)\ \forall\alpha,\beta\in\mathbb{R}^n,\ \alpha,\beta\neq 0.$

Proof. Let us first observe that u is $C^{2,\alpha}(\Omega)$, $\Phi(\cdot,y)\in W^{1,1}(\Omega)$ and hence $\Phi(\cdot,y)\in L^p(\Omega)$ for $p>1$. Also, by the regularity of $\Phi(x,y)$ for $x\neq y$, it follows that $K_0\nabla_x Z(x,y)\cdot\nu\in W^{1/2,p}(\partial\Omega)$. Hence, by elliptic regularity results for elliptic equations it follows that $Z(\cdot,y)\in W^{2,p}(\Omega)$ and $\nabla Z(\cdot,y)$ is Hölder continuous away from y. The last three properties then clearly follow from the analogous properties of $\Phi(x,y)$. Moreover the differential operator $-div(K_0\nabla\cdot)+3u^2$ is self adjoint and hence applying [10, Corollary II, page 22] it follows that the Neumann function is symmetric, i.e.,

$$N_u(x,y)=N_u(y,x) \tag{2.8}$$

□

3. Main result

In this section, using the asymptotic expansion in (1.5), we derive a Lipschitz stability estimate for the inverse problem under consideration. For seek of simplicity we will prove the result in the case of a single inclusion ω_ε ($m=1$) and, at the end of the section, we will state the general result and sketch the proof.

Besides $f\in C^\alpha(\overline{\Omega})$, we assume $f>0$ in Ω. By elliptic regularity results, we have that $u\in C^{2,\alpha}$ whereas by the maximum principle for elliptic boundary value problems we have that also $u>0$. In addition to this, we also require that u satisfies:

$$\nabla u(z)\neq 0 \qquad \forall z\in\Omega. \tag{3.1}$$

Under these assumptions, it is possible to state our main result:

Theorem 3.1. *Assume $f\in C^\alpha(\overline{\Omega})$, $f>0$ in Ω and let u satisfy* (3.1). *Let Γ be an open non-empty subset of $\partial\Omega$. Then, there exist some positive constants C_1,C_2,ε_0 (depending only on the data) such that, $\forall\varepsilon<\varepsilon_0$*

$$|r-r'|+|z-z'|\leq C_1\varepsilon^{-d}\left\|u_\varepsilon-u'_\varepsilon\right\|_{L^\infty(\Gamma)}+C_2\varepsilon^\gamma,$$

where u_ε and u'_ε are the solutions of (1.2) *in presence of inclusions of the form* $\{z+\varepsilon rB\}$ *and* $\{z'+\varepsilon r'B\}$*, respectively, and* $\gamma \in \left(0, \frac{1}{2}\right)$.

To prove this result we will need the following:

Theorem 3.2. *Let H be a Banach space, $U \subset \mathbb{R}^{d+1}$ ($d = 2, 3$) an open set and $K \subset U$ a convex compact set. Let $T \in C^1(U; H)$ and assume T is injective as well $DT(z, r)$ (for each $(z, r) \in U$) where $DT(z, r) \in \mathcal{L}(\mathbb{R}^{d+1}, H)$ is the Fréchet derivative of T evaluated in $(z, r) \in U$. Then, there exists a constant c such that $\forall (r, z), (r', z') \in K$,*

$$|z - z'| + |r - r'| \leq c\|T(z, r) - T(z', r')\|_H.$$

The above theorem is a particular case of [6, Theorem 2.1] (see also [3]). We will apply it with $H = L^\infty(\Gamma)$, $U = \Omega \times \mathbb{R}^+$ and since Ω is a convex set, we choose $K = \Omega_{d_0} \times [d_0, d_1]$ where $\Omega_{d_0} = \{x \in \Omega : dist(x, \partial\Omega) \geq d_0\}$. Moreover we will let $T : \Omega \times \mathbb{R}^+ \to L^\infty(\Gamma)$ be

$$T(z, r) = F_{z,r}(y) = r^d \left(M(K_0 - K_1)\nabla u(z) \cdot \nabla_x N_u(z, y) + u^3(z) N_u(z, y)\right), \tag{3.2}$$

i.e., the first-order term appearing in the expansion (1.5).

Proof of Theorem 3.1. We first observe that T is a continuous function between $\Omega \times \mathbb{R}^+$ and $L^\infty(\Gamma)$, i.e., the function $F_{z,r}(y)$ is continuous with respect to $(z, r) \in U$ for every $y \in \Gamma$. According to the expression in (3.2), this is a direct consequence of the fact that $u \in C^{2,\alpha}(\Omega)$ and of the smoothness of $N_u(\cdot, y)$ in Ω_{d_0} for $y \in \partial\Omega$ (which can be deduced by local elliptic regularity results since all the coefficients appearing in (1.6) are smooth). For the same reasons, the Fréchet derivative of T is well defined and continuous, being

$$\begin{aligned} DT(z, r)[\Delta z, \Delta r] = DF_{z,r}[\Delta z, \Delta r](y) &= r^d M(K_0 - K_1) D^2 u(z)\Delta z \cdot \nabla_x N_u(z, y) \\ &+ r^d M(K_0 - K_1)\nabla u(z) \cdot D_x^2 N_u(z, y)\Delta z \\ &+ r^d 3u^2(z)\nabla u(z) \cdot \Delta z N_u(z, y) + r^d u^3(z)\nabla_x N_u(z, y) \cdot \Delta z \\ &+ d r^{d-1}\Delta r \left(M(K_0 - K_1)\nabla u(z) \cdot \nabla_x N_u(z, y) + u^3(z) N_u(z, y)\right) \end{aligned} \tag{3.3}$$

where $\nabla_x N_u$ and $D_x^2 N_u$ indicate derivatives with respect to the first variable.

In order to apply Theorem 3.2, we now need to show the injectivity of T. Assume $T(z, r) = T(z', r')$ as elements of $L^\infty(\Gamma)$. Consider now the expression (3.2) extended for $y \in \Omega$ and the function $G(y) = F_{z,r}(y) - F_{z',r'}(y)$ with $y \in \Omega$. As a first step we show that $G(y)$ is the solution of the following Cauchy problem:

$$\begin{cases} -div(K_0 \nabla G(y)) + 3u^2(y) G(y) = 0 & \text{in } \Omega \setminus \{z, z'\}, \\ G(y) = 0 & \text{on } \Gamma, \\ K_0 \nabla G(y) \cdot \nu = 0 & \text{on } \Gamma. \end{cases} \tag{3.4}$$

Indeed, $G = 0$ on Γ by assumption. To show that the normal derivative vanishes and that the equation is satisfied, being $N_u(z, y)$ and $\nabla N_u(z, y)$ the only terms in the expression of G that depend on y, it is enough to consider the normal derivative and the operator applied to these latest terms. Now, using the boundary conditions in (1.6), the symmetry of N_u, the fact that K_0 is constant and since z, z' are well separated from $\partial\Omega$ we have that, for $y \in \Gamma$

$$K_0 \nabla_y N_u(z, y) \cdot \nu = K_0 \nabla_x N_u(y, z) \cdot \nu = 0$$

and

$$K_0 \nabla_y \nabla_x N_u(z, y) \cdot \nu = \nabla_y K_0 \nabla_x N_u(y, z) \cdot \nu = 0.$$

Moreover, $\forall z \neq y$,

$$-div\left(K_0 \nabla_y N_u(z, y)\right) + 3u^2(y) N_u(z, y) = -div\left(K_0 \nabla_x N_u(y, z)\right) + 3u^2(z) N_u(y, z) = 0$$

and

$$\begin{aligned} -div\left(K_0 \nabla_y \nabla_x N_u(z, y)\right) &+ 3u^2(y) \nabla_x N_u(z, y) = \\ = \nabla_x &\left[-div\left(K_0 \nabla_x N_u(y, z)\right) + 3u^2(z) N_u(y, z)\right] = 0. \end{aligned}$$

Now, according to the unique continuation property for the Cauchy problem for Schrödinger type operators with smooth potential (see [1, Theorem 1.9] applied with $q = 3u^2$), we conclude that $G \equiv 0$ in $\Omega \setminus \{z, z'\}$.

This implies that $z = z'$ and $r = r'$. To see this recall (2.5) and (2.6): the terms $N_u(z, y)$ and $N_u(z', y)$ have a singularity when approaching z and z', of the kind $\ln|y - z|$ if $d = 2$ or $|y - z|^{-1}$ if $d = 3$. Those singularities cannot cancel with the terms $\nabla_x N_u(z, y)$, $\nabla_x N_u(z', y)$, which grow with a different rate (namely, as $|y - z|^{d-1}$). Moreover, the coefficients appearing in front of $N_u(z, y)$ and $N_u(z', y)$ do not vanish because of the positivity of u so that the only way for $T(z, r) \equiv T(z', r')$ is that $z = z'$ and $r = r'$.

We finally need to show that, for each $(z, r) \in U$, $DT(z, r)$ is an injective map from $\mathbb{R}^{d+1}$ to $L^\infty(\Gamma)$. By the linearity of $DT(z, r)[\Delta z, \Delta r]$ with respect to $(\Delta z, \Delta r)$, it is enough to prove that $DT(z, r)[\Delta z, \Delta r] = 0$ implies both $\Delta z = 0$ and $\Delta r = 0$. Define $\widetilde{G}(y) = DF_{z,r}[\Delta z, \Delta r](y)$ $y \in \Omega$. Recalling (3.3), analogously to what was done before, we have that $\widetilde{G}$ is the solution of the following Cauchy problem:

$$\begin{cases} -div(K_0 \nabla \widetilde{G}(y)) + 3u^2(y) \widetilde{G}(y) = 0 & \text{in } \Omega \setminus \{z\}, \\ \widetilde{G}(y) = 0 & \text{on } \Gamma, \\ K_0 \nabla \widetilde{G}(y) \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

This follows again by the fact that $\widetilde{G} = 0$ on Γ by assumption, whereas the differential equation and the Neumann boundary condition are satisfied thanks to the symmetry of $N_u(x, y)$ and because of (1.6). By unique continuation, we conclude that $\widetilde{G}(y) \equiv 0$ for every $y \neq z$. Now, if $\Delta z \neq 0$ in (3.3), being M and $K_0 - K_1$ positive definite and $\nabla u \neq 0$, the last statement in Proposition 2.1 implies that $DT(z, r)[\Delta z, \Delta r] \notin L^1(\Omega)$ and therefore $DT(z, r)[\Delta z, \Delta r] \not\equiv 0$. Moreover if $\Delta z = 0$ and $\Delta r \neq 0$ again Proposition 2.1 together with the positivity of u implies that $DT(z, r)[\Delta z, \Delta r] \notin W^{1,1}(\Omega)$ and therefore cannot be identically zero.

We can now apply Theorem 3.2, obtaining

$$|z - z'| + |r - r'| \leq C_1 \|T(z, r) - T(z', r')\|_{L^\infty(\Gamma)}. \tag{3.5}$$

Recalling that, by (1.5) and Remark 2.1, for $\varepsilon \leq \varepsilon_0$ we have

$$u_\varepsilon(y) - u'_\varepsilon(y) = \left(u_\varepsilon(y) - u(y)\right) - \left(u'_\varepsilon(y) - u(y)\right),$$

we can conclude that

$$\|T(z,r) - T(z',r')\|_{L^\infty(\Gamma)} \leq \varepsilon^{-d}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\Gamma)} + c\varepsilon^{\gamma d},$$

which together with (3.5) gives the thesis of Theorem 3.1. □

Remark 3.1. Observe that the above theorem holds for instance for $u(x,y) = \cos x + \cos y$ in $\Omega = [0,\pi]^2 \subset \mathbb{R}^2$. Let $M_0 \equiv I$ the identity matrix. It can be easily seen that u satisfies the equation in (1.4) in Ω with $f = \cos x + \cos y + (\cos x + \cos y)^3$. Moreover the normal derivative $\nabla u \cdot \nu$ vanishes in $\partial\Omega = \partial[0,\pi]^2$ and $\nabla u \neq 0$.

We want to emphazise that the condition $\nabla u \neq 0$ is necessary to guarantee the reconstruction of the unknown inhomogeneities. In fact, in [8] where similar results as ours have been derived for the conductivity equation with non homogeneous Neumann datum in dimension $n \geq 2$ the authors prove Lipschitz stability estimates assuming the background potential linear.

Guaranteeing the non-vanishing of ∇u in Ω for solutions to boundary value problems for linear elliptic equations is not at all a trivial issue and has been studied in [11] in the two-dimensional setting. The authors proved that a suitable choice of Dirichlet (Neumann) datum guarantees that the solution does not have critical points in the interior. To our knowledge this results cannot be extended semilinear equations and to dimension higher than two.

Finally, the condition $\nabla u \neq 0$ is not always necessary since the theorem holds for instance for $u = c \in \mathbb{R}$.

We are now ready to state the more general stability result when in presence of multiple inhomogeneities.

Theorem 3.3. *Let u_ε and u'_ε be the solutions associated to inclusions ω_ε, ω'_ε satisfying* (2.2), (2.3) *and the disjointness condition thereafter. Then, there exist positive constants C_1, C_2, δ and ε_0 s.t. if $\varepsilon < \varepsilon_0$ and $\epsilon^{-d}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\Gamma)} \leq \delta$,*

(i) $m = m'$ and, after appropriate reordering,

(ii) $|z_i - z'_i| + |r_i - r'_i| < C_1\varepsilon^{-d}\left\|u_\varepsilon - u'_\varepsilon\right\|_{L^\infty(\partial\Omega)} + C_2\varepsilon^{\gamma d}$ for $i = 1,\dots,m$.

Proof. (i) Let us show that $m = m'$ following the ideas contained in [9]. In order to simplify the notation sometimes in what follows we will write $\mathbf{z} = (z_1,\dots,z_m)$, $\mathbf{r} = (r_1,\dots,r_m)$, $\mathbf{z}' = (z'_1,\dots,z'_{m'})$ and $\mathbf{r}' = (r'_1,\dots,r'_{m'})$. Recall that

$$u_\varepsilon(y) - u'_\varepsilon(y) = \varepsilon^d\left(T(\mathbf{z},\mathbf{r})(y) - T(\mathbf{z}',\mathbf{r}')(y)\right) + o(\epsilon^d) \tag{3.6}$$

Assume first $u_\varepsilon(y) - u'_\varepsilon(y) = 0$ for $y \in \Gamma$; then from (3.6) we must have

$$T(\mathbf{z},\mathbf{r})(y) - T(\mathbf{z}',\mathbf{r}')(y)\Big|_\Gamma = 0.$$

With an argument analogous to the proof of theorem 3.1, letting $G(y) = T(\mathbf{z},\mathbf{r})(y) - T(\mathbf{z}',\mathbf{r}')(y)$ for $y \in \Omega$, we have that $G(y)$ satisfies

$$\begin{cases} -div(K_0\nabla G(y)) + 3u^2(y)G(y) = 0 & \text{in } \Omega \setminus \{z_1,\dots,z_m,z'_1\dots,z'_{m'}\}, \\ G(y) = 0 & \text{on } \Gamma, \\ K_0\nabla G(y)\cdot\nu = 0 & \text{on } \Gamma. \end{cases} \tag{3.7}$$

and so that, by unique continuation, necessarily $G \equiv 0$ in $\Omega \setminus \{z_1, \ldots, z_m, z'_1 \ldots, z'_{m'}\}$. This clearly implies $m = m'$ since if it were not so the singularities of $N_u(z_i, y)$, $\nabla N_u(z_i, y)$ $i = 1, \ldots, m$ and $N_u(z'_i, y)$, $\nabla N_u(z'_i, y)$ $i = 1, \ldots, m'$, as y approches z_i and z'_i respectively, could not cancel out.

Let now $u_\varepsilon(y) - u'_\varepsilon(y) \neq 0$. The condition $\epsilon^{-d}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\Gamma)} \leq \delta$ together with (3.6) implies that for $\varepsilon \leq \varepsilon_0(\delta)$ we have

$$\|T(\mathbf{z}, \mathbf{r}) - T(\mathbf{z}', \mathbf{r}')\|_{L^\infty(\Gamma)} \leq \frac{3}{2}\delta. \tag{3.8}$$

If the thesis *(i)* did not hold, this would imply that for every $\delta_n = 2/3n$, $n \in \mathbb{N}$, there exist $\left\{z_i^{(n)}\right\}_{i=1}^m = \mathbf{z}^{(n)}$, $\left\{r_i^{(n)}\right\}_{i=1}^m = \mathbf{r}^{(n)}$, $\left\{z_i'^{(n)}\right\}_{i=1}^{m'} = \mathbf{z}'^{(n)}$ and $\left\{r_i'^{(n)}\right\}_{i=1}^{m'} = \mathbf{r}'^{(n)}$ such that

$$\|T(\mathbf{z}^{(n)}, \mathbf{r}^{(n)}) - T(\mathbf{z}'^{(n)}, \mathbf{r}'^{(n)})\|_{L^\infty(\Gamma)} \leq \frac{1}{n} \tag{3.9}$$

with $m \neq m'$. Observe that m and m' can be chosen independent of n since (2.3) together with the disjointness assumption implies that the total number of inhomogeneity is bounded. By compactness, up to subsequences we have

$$\mathbf{z}^{(n)} \to \mathbf{z} \quad \mathbf{r}^{(n)} \to \mathbf{r} \quad \mathbf{z}'^{(n)} \to \mathbf{z}' \quad \mathbf{r}'^{(n)} \to \mathbf{r}'$$

and (3.9) implies that

$$T(\mathbf{z}, \mathbf{r})(y) - T(\mathbf{z}', \mathbf{r}')(y)\Big|_\Gamma = 0.$$

From now on the proof proceeds as in the case $u_\varepsilon(y) - u'_\varepsilon(y) = 0$ for $y \in \Gamma$.

(ii) The proof follows the one in the case of a single inhomogeneity being just formally more complicated. In fact, since $m = m'$ we have that

$$\begin{aligned} G(y) = T(\mathbf{z}, \mathbf{r})(y) - T(\mathbf{z}', \mathbf{r}')(y) \;&=\; \sum_{i=1}^m r_i^d \left(M_i(K_0 - K_i)\nabla u(z_i) \cdot \nabla_x N_u(z_i, y) + u^3(z_i)N_u(z_i, y)\right) \\ &-\; \sum_{i=1}^m (r'_i)^d \left(M_i(K_0 - K_i)\nabla u(z'_i) \cdot \nabla_x N_u(z'_i, y) + u^3(z'_i)N_u(z'_i, y)\right) \end{aligned}$$

Using now (3.7) with $m = m'$ by unique continuation we get $G \equiv 0$ in $\Omega \setminus \{z_1, \ldots, z_m, z'_1 \ldots, z'_m\}$. Then using assumption 2.2 we proceed exactly as in Theorem 3.1 showing that $z_i = z_i, r_i = r'_i$ for $i = 1, \cdots, m$. To prove the differentiability properties of T and injectivity of $DT(\mathbf{z}, \mathbf{r})$ note that

$$\begin{aligned} DT(\mathbf{z}, \mathbf{r})[\Delta\mathbf{z}, \Delta\mathbf{r}] = &\sum_{i=1}^m r_i^d M_i(K_0 - K_i)D^2u(z_i)\Delta z_i \cdot \nabla_x N_u(z_i, y) \\ &+ r_i^d M_i(K_0 - K_i)\nabla u(z_i) \cdot D_x^2 N_u(z_i, y)\Delta z_i \\ &+ r_i^d 3u^2(z_i)\nabla u(z_i) \cdot \Delta z N_u(z_i, y) + r_i^d u^3(z_i)\nabla_x N_u(z_i, y) \cdot \Delta z_i \\ &+ d r_i^{d-1}\Delta r_i \left(M(K_0 - K_i)\nabla u(z_i) \cdot \nabla_x N_u(z_i, y) + u^3(z_i)N_u(z_i, y)\right) \end{aligned} \tag{3.10}$$

Reasoning as in Theorem 3.1 the the Fréchet derivative of T $DT(\mathbf{z}, \mathbf{r})$ is well defined and continuous. Defining $T(\mathbf{z}, \mathbf{r})(y) = F_{\mathbf{z},\mathbf{r}}(y)$ let $\widetilde{G}(y) = DF_{\mathbf{z},\mathbf{r}}[\Delta\mathbf{z}, \Delta\mathbf{r}](y)$ $y \in \Omega$. Then from the expression of the derivative and analogously to what was done in Theorem 3.1, we have that $\widetilde{G}$ is the solution of the following Cauchy problem:

$$\begin{cases} -div(K_0\nabla\widetilde{G}(y)) + 3u^2(y)\widetilde{G}(y) = 0 & \text{in } \Omega \setminus \{z_1, z_2, \ldots, z_m\}, \\ \widetilde{G}(y) = 0 & \text{on } \Gamma, \\ K_0\nabla\widetilde{G}(y)\cdot\nu = 0 & \text{on } \Gamma. \end{cases}$$

This follows again by the fact that $\widetilde{G} = 0$ on Γ by assumption, whereas the differential equation and the Neumann boundary condition are satisfied thanks to the symmetry of $N_u(x, y)$ and because of (1.6). By unique continuation, we conclude that $\widetilde{G}(y) \equiv 0$ for every $y \in \Omega \setminus \{z_1, z_2, \ldots, z_m\}$ and then using the expression of $DT(\mathbf{z}, \mathbf{r})$ and proceeding as in the last part of Theorem 3.1 injectivity of $DT(\mathbf{z}, \mathbf{r})$ and estimate *(ii)* follow concluding the proof.

□

Conflict of interest

The authors declare no conflict of interest.

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