GLOBAL GAUSSIAN ESTIMATES FOR THE HEAT KERNEL OF HOMOGENEOUS SUMS OF SQUARES.

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ABSTRACT. Let $\mathcal{H}=\sum_{j=1}^{m}X_j^2-\partial_t$ be a heat-type operator in $\mathbb{R}^{n+1}$, where $X=\{X_1,\ldots,X_m\}$ is a system of smooth Hörmander's vector fields in $\mathbb{R}^n$, and every X_j is homogeneous of degree 1 with respect to a family of non-isotropic dilations in $\mathbb{R}^n$, while no underlying group structure is assumed. In this paper we prove global (in space and time) upper and lower Gaussian estimates for the heat kernel $\Gamma(t,x;s,y)$ of $\mathcal{H}$, in terms of the Carnot-Carathéodory distance induced by X on $\mathbb{R}^n$, as well as global upper Gaussian estimates for the t- or X-derivatives of any order of Γ. From the Gaussian bounds we derive the unique solvability of the Cauchy problem for a possibly unbounded continuous initial datum satisfying exponential growth at infinity. Also, we study the solvability of the $\mathcal{H}$-Dirichlet problem on an arbitrary bounded domain. Finally, we establish a global scale-invariant Harnack inequality for non-negative solutions of $\mathcal{H}u=0$.

1. Introduction

Let us consider a family $X=\{X_1,\ldots,X_m\}$ of smooth Hörmander's vector fields in $\mathbb{R}^n$ (precise definitions will be given later). The study of the corresponding heat-type operator

$$\mathcal{H}:=\sum_{j=1}^{m}X_j^2-\partial_t \qquad \text{on } \mathbb{R}^{n+1}$$

and its fundamental solution (heat kernel) has a long history and, by now, a vast literature. The study of operators of the kind 'sum of squares of Hörmander's vector fields', $\mathcal{L}=\sum_{j=1}^{m}X_j^2$, as well as their evolutive counterpart, $\mathcal{H}=\mathcal{L}-\partial_t$, is usually characterized by the following dichotomy:

- *local* properties of Hörmander operators of the kind $\mathcal{L}$ or $\mathcal{H}$ have been established for *general* families of Hörmander's vector fields $X_1,\ldots,X_m$ (some cornerstones in this context are [17], [30], [28], [32], [18], [19]), while

- *global* properties of $\mathcal{L}$ or $\mathcal{H}$ have been established almost exclusively when the vector fields $X_1,\ldots,X_m$ are left invariant on some Lie group.

In particular, starting with the famous paper [14] by Folland, a rich theory exists under the assumption that $X_1,\ldots,X_m$ be both left invariant with respect to some group of translations, and homogeneous with respect to some family of dilations (hence, $X_1,\ldots,X_m$ are the generators of a Carnot group $\mathbb{G}$ in $\mathbb{R}^n$). In that context, the heat kernel has the form

$$\Gamma(t,x;s,y)=\gamma(y^{-1}*x,t-s) \tag{1.1}$$

with γ satisfying a two-sided Gaussian bound:

$$\frac{1}{Ct^{Q/2}}\exp\left(-\frac{C\left\Vert x\right\Vert^2}{t}\right)\leq\gamma(x,t)\leq\frac{C}{t^{Q/2}}\cdot\exp\left(-\frac{\left\Vert x\right\Vert^2}{Ct}\right)$$

for every $x\in\mathbb{G}$, $t>0$. Here Q is the homogeneous dimension of the group, and $\left\Vert\cdot\right\Vert$ is a homogeneous norm in $\mathbb{G}$. Analogous upper bounds hold for the derivatives of every order:

$$\left|\partial_t^m X_I\gamma(x,t)\right|\leq\frac{C}{t^{(Q+|I|+2m)/2}}\cdot\exp\left(-\frac{\left\Vert x\right\Vert^2}{Ct}\right)$$

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where $X_I = X_{i_1} X_{i_2} ... X_{i_k}$ with $i_1, ..., i_k \in \{1, 2, ..., m\}$, and $|I| = k$. The above Gaussian bounds on Carnot groups are a special case of the more general results proved for heat kernels corresponding to left invariant, but not necessarily homogeneous, Hörmander's vector fields, by Varopoulos, Saloff-Coste, Coulhon in [37]. They proved that for heat kernels on *nilpotent* Lie groups, a context where one still has (1.1), the function γ satisfies a two sided bound

$$\begin{aligned} &\frac{1}{C\left|B_X(0,\sqrt{t})\right|}\exp\left(-\frac{Cd_X^2(x,0)}{t}\right) \le \gamma(x,t) \\ &\qquad \le \frac{C}{\left|B_X(0,\sqrt{t})\right|}\exp\left(-\frac{d_X^2(x,0)}{Ct}\right), \end{aligned} \tag{1.2}$$

and an upper bound on derivatives of every order:

$$|\partial_t^m X_I \gamma(x,t)| \le \frac{C}{\left|B_X(0,\sqrt{t})\right| t^{(|I|+2m)/2}} \exp\left(-\frac{d_X^2(x,0)}{Ct}\right), \tag{1.3}$$

where d_X is the control distance induced by $X_1, ..., X_m$ and $B_X(0,r)$ the corresponding balls (see [37, Thm. IV.4.2, Thm. IV.4.3]). Also, they proved that on *unimodular Lie groups with polynomial volume growth*, that is satisfying

$$c_1 t^D \le |B_X(0,\sqrt{t})| \le c_2 t^D \qquad \text{for } t \ge 1$$

and some $D > 0$, the above results (1.2) and (1.3) still hold (see [37, Thm. VIII.2.7, Thm. 8.2.9]). For a different approach to Gaussian estimates in the context of Lie groups with polynomial growth, see also the monograph [13] by Dungey, ter Elst, Robinson. For the special case of Gaussian estimates on Carnot groups, that we will explicitly exploit in this paper, we refer to the more recent paper [9] by Bonfiglioli, Lanconelli, Uguzzoni.

For a general system of Hörmander's vector fields, i.e., with *no underlying group structure*, Gaussian bounds for the heat kernel

$$\Gamma(t,x;s,y) = \gamma(t-s,x,y)$$

have been proved by Jerison-Sanchez Calle [19, Thms. 2, 3, 4] in the form:

$$\frac{1}{C\left|B_X(x,\sqrt{t})\right|}\exp\left(-\frac{Cd_X^2(x,y)}{t}\right) \le \gamma(t,x,y) \le \frac{C}{\left|B_X(x,\sqrt{t})\right|}\exp\left(-\frac{d_X^2(x,y)}{Ct}\right) \tag{1.4}$$

$$|\partial_t^m X_I^x X_J^y \gamma(t,x,y)| \le \frac{C}{\left|B_X\left(x,\sqrt{t}\right)\right| t^{(|I|+|J|+2m)/2}} \exp\left(-\frac{d_X\left(x,y\right)^2}{Ct}\right) \tag{1.5}$$

for every multiindices I, J, with x, y ranging in a compact set and $t \in (0,T)$. Using probabilistic techniques, Kusuoka-Stroock have extended the above results to x, y in $\mathbb{R}^N$ and $t \in (0,T)$, in [21], and later to x, y in $\mathbb{R}^N$ and $t > 0$ in [22]. However, Kusuoka-Stroock require that the coefficients of the vector fields belong to $C_b^\infty\left(\mathbb{R}^N\right)$. For instance, vector fields with polynomial coefficients are not covered by their theory (at least as far as global results are concerned). Related results (under the same $C_b^\infty\left(\mathbb{R}^N\right)$ assumptions on the vector fields) have been proved by Léandre in [25], [26]. Davies in [12] has improved the constant in the exponent of the upper bound in (1.4), for a system of Hörmander's vector fields on a compact manifold.

On the other hand, a general setting which allows to develop an interesting *global theory*, without assuming the existence of a group of translations, and allowing unboundedness of the coefficients of $X_1, ..., X_m$ and their derivatives, is that of Hörmander vector fields which are only assumed to be 1-homogeneous with respect to a family of non-isotropic dilations of the form

$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n),$$

where $1 = \sigma_1 \le \dots \le \sigma_n$ are positive integers. In other words,

$$X_j(u \circ \delta_\lambda) = (X_j u) \circ \delta_\lambda$$

for every $j = 1, \dots, m$, every $u \in C^\infty(\mathbb{R}^n)$ and every $\lambda > 0$. Under this assumption (without any underlying group structure), Biagi-Bonfiglioli in [2] have built a global homogeneous fundamental solution for $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ and have studied some of its properties. The idea of this construction is that, according to a procedure originally devised by Folland in [15] and adapted in [2], a system of 1-homogeneous Hörmander's vector fields can always be lifted to a higher dimensional Carnot group where the corresponding sum of squares is known to possess a global, left invariant, homogeneous fundamental solution. Saturating this fundamental solution with respect to the added variables, in [2] a homogeneous fundamental solution for the original operator is produced. More explicit estimates for this kernel have been established in [5], in terms of the distance induced by the vector fields. The general strategy of [2] has been later implemented in [4] for heat operators corresponding to 1-homogeneous vector fields, showing the existence of a global, homogeneous, heat kernel, obtained by saturating the heat kernel of a higher dimensional operator living on a Carnot group.

The aim of this paper is to prove sharp global explicit Gaussian estimates for this heat kernel, in terms of the intrinsic distance induced by the vector fields. More precisely, we will prove Gaussian estimates (1.4)-(1.5) for every $x, y \in \mathbb{R}^n$ and $t > 0$, for heat operators corresponding to 1-homogeneous (but not left invariant) Hörmander's vector fields (see Theorem 2.4).

Our global Gaussian bounds in particular allow to improve known results about the Cauchy problem for this heat operator. In [4, Thm. 4.1] it is proved that for every bounded continuous initial datum f there exists one and only one bounded solution to the Cauchy problem. We will prove that a solution to the Cauchy problem actually exists, at least for small times, as soon as the initial datum f satisfies a growth condition of the kind

$$\int_{\mathbb{R}^n} |f(y)| \exp\big(- \mu\, d_X^2(y,0)\big)\, \mathrm{d}y < +\infty$$

for some constant $\mu > 0$. The solution is unique in the class of functions satisfying a condition

$$\int_0^T \int_{\mathbb{R}^n} \exp\big(- \delta d_X^2(x,0)\big) |u(t,x)|\, \mathrm{d}t\, \mathrm{d}x < +\infty$$

for some $\delta > 0$. Moreover, if f satisfies a stronger bound of the kind

$$\int_{\mathbb{R}^n} |f(y)| \exp\big(- \mu d_X^\alpha(y,0)\big)\, \mathrm{d}y < +\infty \qquad \text{for some } \alpha \in (0,2),$$

then the solution exists for all $t > 0$ (see Theorem 6.2 and Proposition 6.5). In Section 7 we shall present an application of our global Gaussian estimates to the study of the $\mathcal{H}$-Dirichlet problem. In fact, by crucially exploiting these estimates, we shall show that it is possible to apply to our operators $\mathcal{H}$ the axiomatic approach developed in the series of papers [20, 23, 24, 35]; this will lead to some necessary and sufficient conditions for the regularity of boundary points of *any* bounded open set Ω. Finally, in the last part of the paper we will prove a scale-invariant parabolic Harnack inequality for non-negative solutions of $\mathcal{H}u = 0$ (see Theorem 8.1 in Section 8).

We close this introduction with a few remarks about some related fields of research. Gaussian bounds for heat kernels have been studied, besides the Euclidean setting, in the context of Riemannian manifolds. We can quote under this respect the well-known paper [27] by Li-Yau where Gaussian bounds are proved on manifolds with nonnegative curvature (see also the monograph [16] by Grigor'yan and the references therein). Some extensions of these geometric techniques to sub-Riemannian manifolds have been done, see e.g. the paper [6] by Baudoin, Bonnefont, Garofalo. Gaussian bounds have been studied also in the abstract context of Dirichlet forms, see e.g. the papers [33], [34] by Sturm. These researches have made apparent a general relation existing between the validity of Gaussian bounds for the heat kernel, the validiy of global forms of Poincaré's inequality and doubling condition, and the validity of a parabolic Harnack inequality. For a discussion of these general relations see also the monograph [31] by Saloff-Coste. In the context of homogeneous Hörmander vector fields studied in the present paper, global forms of Poincaré's inequality and doubling condition are known, after [5]. Therefore, our results about Gaussian bounds and Harnack inequality are not unexpected. Nevertheless, we have not been able to find in the literature a precise theorem, directly applicable to our context, implying our results. As far as we know, this is the first case of global (in space and time) Gaussian estimates explicitly proved, for both the heat kernel and its derivatives of every order, in the context of Hörmander's vector fields (with possibly unbounded coefficients) in absense of an underlying group structure.

2. Assumptions and statements of Gaussian bounds

We denote by $\mathfrak{X}(\mathbb{R}^n)$ the Lie algebra of the smooth vector fields on $\mathbb{R}^n$ (with $n \geq 2$). Given a set $X \subseteq \mathfrak{X}(\mathbb{R}^n)$, we indicate by $\mathrm{Lie}(X)$ the smallest Lie sub-algebra of $\mathfrak{X}(\mathbb{R}^n)$ containing X. Finally, if $Z \in \mathfrak{X}(\mathbb{R}^n)$ is a smooth vector field of the form

$$Z = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} \qquad \text{for some } a_1, \ldots, a_n \in C^\infty(\mathbb{R}^n)$$

and if $x \in \mathbb{R}^n$, we denote by $Z(x)$ the vector $(a_1(x), \ldots, a_n(x)) \in \mathbb{R}^n$.

Assumptions 2.1. Let $X = \{X_1, \ldots, X_m\}$ (with $m \geq 2$) be a fixed family of *linearly independent* smooth vector fields in Euclidean space $\mathbb{R}^n$ satisfying the following structural assumptions:

(H1): there exists a family $\{\delta_\lambda\}_{\lambda>0}$ of non-isotropic dilations of the form

$$\delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n), \tag{2.1}$$

where $1 = \sigma_1 \leq \ldots \leq \sigma_n$ are positive integers, with respect to which $X_1, \ldots, X_m$ are homogeneous of degree 1. This means that

$$X_j(u \circ \delta_\lambda) = (X_j u) \circ \delta_\lambda \tag{2.2}$$

for every $j = 1, \ldots, m$, every $u \in C^\infty(\mathbb{R}^n)$ and every $\lambda > 0$. We define the δ_λ-*homogeneous dimension* of $\mathbb{R}^n$ as

$$q := \sum_{j=1}^{m} \sigma_j. \tag{2.3}$$

Note that $q \geq n$.

(H2): $X_1, \ldots, X_m$ satisfy Hörmander's rank condition at $x = 0$, that is,

$$\dim\big\{Y(0) : Y \in \mathrm{Lie}(X)\big\} = n. \tag{2.4}$$

Remark 2.2. By combining assumptions (H1) and (H2), it is not difficult to recognize that Hörmander's rank condition is actually satisfied *at every point* $x \in \mathbb{R}^n$, that is,

$$\dim\big\{Y(x) : Y \in \mathrm{Lie}(X)\big\} = n \qquad \text{for all } x \in \mathbb{R}^n$$

(this is proved in [5, Remark 3.2]). Thus, by Hörmander's Hypoellipticity Theorem (see [17]), both the operators $\mathcal{L}$ and $\mathcal{H}$ are C^∞-hypoelliptic in every open subset of $\mathbb{R}^n$.

In order to state our result, we first recall the following standard

Definition 2.3 (Carnot-Carathéodory distance)**.** Let $Y = \{Y_1, \ldots, Y_h\}$ be a family of *smooth* vector fields defined on some space $\mathbb{R}^k$. We assume that the Y_j's satisfy Hörmander's rank condition at every point of $\mathbb{R}^k$. The *Carnot-Carathéodory* (CC, shortly) *distance associated with* Y is defined as

$$d_Y(x, y) = \inf\big\{r > 0 : \text{there exists } \gamma \in C(r) \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\big\},$$

where $C(r)$ is the set of the absolutely continuous curves $\gamma : [0, 1] \to \mathbb{R}^k$ satisfying (a.e. on $[0, 1]$)

$$\gamma'(t) = \sum_{j=1}^{h} a_j(t)\, Y_j(\gamma(t)), \qquad \text{with } |a_j(t)| \leq r \text{ for all } j = 1, \ldots, h.$$

We will denote by $B_Y(x, \rho)$ the metric ball $\big\{y \in \mathbb{R}^k : d_Y(x, y) < \rho\big\}$.

Well-known results assure that under the above assumptions $d_Y(x, y)$ is finite for every couple of points in $\mathbb{R}^k$ and that $(\mathbb{R}^k, d_Y)$ is a metric space; moreover, d_Y is topologically, but not metrically, equivalent to the Euclidean distance.

We can now state our main result:

Theorem 2.4. *Let $X = \{X_1, \dots, X_m\}$ be a family of smooth vector fields in $\mathbb{R}^n$ satisfying Assumptions 2.1, and let $\mathcal{H}$ the heat-type operator*

$$\mathcal{H} := \mathcal{L} - \partial_t = \sum_{j=1}^m X_j^2 - \partial_t \qquad \text{on } \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x. \tag{2.5}$$

Moreover, let $\Gamma(t, x; s, y) := \gamma(t - s, x, y)$ be the global heat kernel of $\mathcal{H}$, that will be precisely defined in (3.10)*. Then, the following facts hold.*

(i) *There exists a constant $\varrho > 1$ such that*

$$\begin{aligned} &\frac{1}{\varrho\,|B_X(x,\sqrt{t})|} \exp\Big(-\frac{\varrho\, d_X^2(x,y)}{t}\Big) \leq \gamma(t,x,y) \\ &\qquad \leq \frac{\varrho}{|B_X(x,\sqrt{t})|} \exp\Big(-\frac{d_X^2(x,y)}{\varrho\, t}\Big), \end{aligned} \tag{2.6}$$

for every $x, y \in \mathbb{R}^n$ and every $t > 0$.

(ii) *For any nonnegative integers k, r there exists $C = C_{k,r} > 0$ such that*

$$\left|\Big(\frac{\partial}{\partial t}\Big)^k Y_1 \cdots Y_r \gamma(t,x,y)\right| \leq C\, \frac{t^{-(k+r/2)}}{|B_X(x,\sqrt{t})|} \exp\Big(-\frac{d_X^2(x,y)}{Ct}\Big), \tag{2.7}$$

for every choice of vector fields $Y_1, \dots, Y_r \in \{X_1^x, \dots, X_m^x, X_1^y, \dots, X_m^y\}$, and every choice of $x, y \in \mathbb{R}^n, t > 0$.

The results about the Cauchy problem for $\mathcal{H}$ will be stated and proved in Section 6, while our scale-invariant Harnack inequality will be stated and proved in Section 8.

3. Preliminaries and known results

3.1. **Carnot groups, lifting and construction of the heat kernel for $\mathcal{H}$.** We begin by recalling the definition of homogeneous Carnot group and some related notions (see, e.g., [8] for an exhaustive treatment of this topic).

We say that $\mathbb{G} = (\mathbb{R}^N, *, D_\lambda)$ is a *homogeneous group* if $(\mathbb{R}^N, *)$ is a Lie group (with group identity $e = 0$) and if there exists a one-parameter family of group automorphisms $\{D_\lambda\}_{\lambda>0}$ acting as in (H1). We shall call the Lie group operation $*$ *'translation'* and the automorphisms D_λ *'dilations'*.

We say that a smooth vector field X is *left invariant* if, for every $f \in C^\infty(\mathbb{R}^N)$, we have

$$X\big(x \mapsto f(y * x)\big) = (Xf)(y * x) \qquad \text{for all } x, y \in \mathbb{G}.$$

For $i = 1, 2, \dots, n$, let X_i be the only left invariant vector field which agrees at the origin with ∂_{x_i}. Assume that for some positive integer $m < N$ we have that $X_1, ..., X_m$ are 1-homogeneous (in the sense of (2.2)) and that $X_1, \dots, X_m$ satisfy Hörmander's condition as in (H2) (at the origin and then, by left invariance, at every point). Then we say that

$$\mathbb{G} \text{ is a } \textit{Carnot group} \text{ and } X_1, ..., X_m \text{ are its } \textit{generators}.$$

A continuous function $\|\cdot\| : \mathbb{G} \to [0, +\infty)$ is called a *homogeneous norm* on $\mathbb{G}$ if there exists $c > 0$ such that, for every $u, v \in \mathbb{G}$, the following hold:

(i) $\|u\| = 0$ if and only if $u = 0$;
(ii) $\|D_\lambda(u)\| = \lambda\|u\|$ for every $\lambda > 0$;
(iii) $\|u * v\| \leq c\big(\|u\| + \|v\|\big)$;
(iv) $\|u^{-1}\| \leq c\|u\|$.

If $X = \{X_1, \dots, X_m\}$ are the generators of a Carnot group $\mathbb{G}$ and d_X the Carnot-Carathéodory distance associated with X, then $\|u\| = d_X(u, 0)$ is a homogeneous norm on $\mathbb{G}$, further satisfying properties (iii)-(iv) with $c = 1$.

A key information for the study of the operator $\mathcal{H}$ (and of its associated heat kernel) is the dimension of the Lie algebra $\mathfrak{a} := \mathrm{Lie}(X)$. Under our assumptions (H1)-(H2), it is easy to see that

$\mathfrak{a}$ has finite dimension: in fact, using [3, Theorem A.11] and [8, Proposition 1.3.10]), one has

$$\mathfrak{a} = \bigoplus_{k=1}^{\sigma_n} \mathfrak{a}_k$$

where $\mathfrak{a}_1 := \mathrm{span}\{X\} = \mathrm{span}\big\{X_1, \ldots, X_m\big\}$ and

$$\mathfrak{a}_k := \mathrm{span}\{[Y, Z] : \, Y \in \mathfrak{a}_1, \, Z \in \mathfrak{a}_{k-1}\} \qquad (\text{for } k \geq 2).$$

In particular, we obtain

$$N = \dim(\mathfrak{a}) \geq \dim\big\{Y(0) : \, Y \mathfrak{a}\big\} = n. \tag{3.1}$$

As a consequence of (3.1), only the following two cases can occur.

(i) $N = n$. In this case, by taking into account the δ_λ-homogeneity of $X_1, \ldots, X_m$, we can apply some results in [7], ensuring the existence of an operation $*$ on $\mathbb{R}^n$ such that

$$\mathbb{F} = (\mathbb{R}^n, *, \delta_\lambda) \text{ is a homogeneous Carnot group with } \mathrm{Lie}(\mathbb{F}) = \mathfrak{a}.$$

Hence, the vector fields $X_1, \ldots, X_m$ are left invariant on $\mathbb{F}$, and the operator $\mathcal{H}$ becomes the *canonical heat operator* on $\mathbb{R} \times \mathbb{F}$. This is a well-studied scenario, in which all the results of this paper are well-known (see, for example, [9]).

(ii) $N > n$. In this case, instead, we derive from [1, Theorem 1.4] that *there cannot exist* any Lie-group structure in $\mathbb{R}^n$ with respect to which $X_1, \ldots, X_m$ are left invariant. In particular, the operator $\mathcal{H}$ *is not* a canonical heat operator on some Carnot group.

In view of the above discussion, throughout the sequel, in the proof of our results, we also make the following 'dimensional' assumption.

(H3): Using the notation $\mathfrak{a} = \mathrm{Lie}(X)$ and $N = \dim(\mathfrak{a})$, we assume that

$$p := N - n \geq 1. \tag{3.2}$$

Remark 3.1. Note that condition (H3) is not a further assumption that we require in order for our results to be true. It is a further condition that is not restrictive to assume within the proofs, because if our Assumptions 2.1 hold and (H3) is *not* true, then our Theorem 2.4 is already known.

Even if assumption (H3) implies that $X_1, \ldots, X_m$ cannot be left invariant with respect to any Lie-group structure in $\mathbb{R}^n$, it is proved in [2] that the X_j's can be lifted (in a suitable sense) to vector fields $Z_1, \ldots, Z_m$ which are left invariant on a *higher-dimensional* Carnot group:

Theorem 3.2 (Lifting, see [2, Theorem 3.1]). *Let us suppose that assumptions* (H1)-*to*-(H3) *are satisfied. Then, it is possible to construct a homogeneous Carnot group* $\mathbb{G} = (\mathbb{R}^N, *, D_\lambda)$ *satisfying the following properties:*

(1) $\mathbb{G}$ *has* m *generators;*

(2) denoting the points of $\mathbb{R}^N$ *as* $u = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p$, *the family of dilations* $\{D_\lambda\}_{\lambda>0}$ *takes the following 'lifted' form:*

$$D_\lambda(u) = D_\lambda(x, \xi) = (\delta_\lambda(x), \delta^*_\lambda(\xi)), \tag{3.3}$$

where $\delta^*_\lambda(\xi) = (\lambda^{\tau_1}\xi_1, \ldots, \lambda^{\tau_p}\xi_p)$ *for some integers* $1 \leq \tau_1 \leq \ldots \leq \tau_p$;

(3) there exists a system of Lie-generators $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ *of* $\mathrm{Lie}(\mathbb{G})$ *s.t.*

$$Z_j(x, \xi) = X_j(x) + R_j(x, \xi), \tag{3.4}$$

where the R_i*'s are smooth vector fields operating only in the variables* $\xi \in \mathbb{R}^p$, *but with coefficient possibly depending on* (x, ξ). *In particular,* $R_1, \ldots, R_m$ *are* D_λ*-homogeneous of degree* 1.

Notation 3.3. Throughout the paper, we will handle points in the 'original' space $\mathbb{R}^n$, and points in the 'lifted' space $\mathbb{R}^N$, according to Theorem 3.2. To this end, we shall use the notation

- $x, y, z, \ldots$ for points in $\mathbb{R}^n$;
- $u = (x, \xi), v = (y, \eta), \ldots$ for points in $\mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^p$,

denoting by Greek letters the added variables in the lifting procedure. The scalar time variables will be denoted by letters t, s, τ. Moreover, we shall indicate by d_X and $d_{\mathcal{Z}}$ the Carnot-Carathéodory distances associated with X and $\mathcal{Z}$, respectively, and with $B_X(x,\rho)$, $B_{\mathcal{Z}}(u,\rho)$ the d_X-ball, $d_{\mathcal{Z}}$-ball, respectively, with centre $x \in \mathbb{R}^n$, $u \in \mathbb{R}^N$, and radius $\rho > 0$.

Since the lifted vector fields $Z_1, \ldots, Z_m$ in Theorem 3.2 are left invariant on $\mathbb{G}$, many properties of $\mathcal{H}_{\mathbb{G}}$ and its associated heat kernel are well-known. In fact, the following theorem holds.

Theorem 3.4 ([9, Theorems 2.1, 2.5]). *There exists a function*

$$\gamma_{\mathbb{G}} : \mathbb{R}^{1+N} \to \mathbb{R},$$

smooth away from the origin, such that

$$\Gamma_{\mathbb{G}}(t,u;s,v) := \gamma_{\mathbb{G}}\left(t-s, v^{-1} * u\right) \tag{3.5}$$

is the global heat kernel of $\mathcal{H}_{\mathbb{G}} = \mathcal{L}_{\mathbb{G}} - \partial_t$; this means, precisely, that

- *for every fixed $(t,z) \in \mathbb{R}^{1+N}$, one has $\Gamma_{\mathbb{G}}(t,z;\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+N})$;*
- *for every $\varphi \in C_0^\infty(\mathbb{R}^{1+N})$ and every $(t,u) \in \mathbb{R}^{1+N}$, one has*

$$\begin{aligned}\mathcal{H}_{\mathbb{G}}&\left(\int_{\mathbb{R}^{1+N}} \Gamma_{\mathbb{G}}(t,u;s,v)\varphi(s,v)\,\mathrm{d}s\,\mathrm{d}v\right)\\ &= \int_{\mathbb{R}^{1+N}} \Gamma_{\mathbb{G}}(t,u;s,v)\mathcal{H}_{\mathbb{G}}\varphi(s,v)\,\mathrm{d}s\,\mathrm{d}v = -\varphi(t,u).\end{aligned}$$

Furthermore, $\gamma_{\mathbb{G}}$ satisfies the following properties:

(i) *$\gamma_{\mathbb{G}} \geq 0$ and $\gamma_{\mathbb{G}}(t,u) = 0$ if and only if $t \leq 0$;*
(ii) *$\gamma_{\mathbb{G}}(t,u) = \gamma_{\mathbb{G}}(t,u^{-1})$ for every $(t,u) \in \mathbb{R}^{1+N}$;*
(iii) *for every $\lambda > 0$ and every (t,u), we have*

$$\gamma_{\mathbb{G}}(\lambda^2 t, D_\lambda(u)) = \lambda^{-Q}\,\gamma_{\mathbb{G}}(t,u),$$

where Q is the homogeneous dimension of the group $\mathbb{G}$, that is,

$$Q := q + q^*, \qquad \text{with } q \text{ as in (2.3) and } q^* := \textstyle\sum_{k=1}^p \tau_k; \tag{3.6}$$

(iv) *$\gamma_{\mathbb{G}}$ vanishes at infinity, that is, $\gamma_{\mathbb{G}}(t,u) \to 0$ as $|(t,u)| \to +\infty$;*
(v) *for every $t > 0$, we have*

$$\int_{\mathbb{R}^N} \gamma_{\mathbb{G}}(t,u)\,\mathrm{d}u = 1.$$

Finally, the following Gaussian estimates for $\gamma_{\mathbb{G}}$ hold:

(a) *there exists a constant $\mathbf{c} \geq 1$, only depending on $\mathbb{G}$ and $\mathcal{Z}$, s.t.*

$$\mathbf{c}^{-1}\, t^{-Q/2} \exp\left(-\frac{\mathbf{c}\,\|u\|^2)}{t}\right) \leq \gamma_{\mathbb{G}}(t,u) \leq \mathbf{c}\, t^{-Q/2} \exp\left(-\frac{\|u\|^2}{\mathbf{c}\, t}\right), \tag{3.7}$$

for every $u \in \mathbb{R}^N$ and every $t > 0$.

(b) *for every nonnegative integers h,k there exists a constant $\widehat{\mathbf{c}} > 0$ s.t.*

$$\left| Z_{i_1} \ldots Z_{i_h} \left(\frac{\partial}{\partial t}\right)^k \gamma_{\mathbb{G}}(t,u)\right| \leq \widehat{\mathbf{c}}\, t^{-(Q+h+2k)/2} \exp\left(-\frac{\|u\|^2}{\widehat{\mathbf{c}}\, t}\right) \tag{3.8}$$

for any $u \in \mathbb{R}^N$, any $t > 0$ and every choice of $i_1, \ldots, i_h \in \{1, \ldots, m\}$.

Now, the 'lifting property' (3.4) contained in Theorem 3.2 easily implies that

$$\mathcal{H}_{\mathbb{G}}\big((t,x) \mapsto u(t,\pi(x))\big) = (\mathcal{H}u)(t,\pi(x)), \qquad \text{for all } u \in C^2(\mathbb{R}^n) \tag{3.9}$$

where π is the projection of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ on $\mathbb{R}^n$. By combining (3.9) with Theorem 3.4, it is proved in [4] the following result.

Theorem 3.5 ([4, Theorem 1.4]). *Let $X = \{X_1, \ldots, X_m\}$ be a set of smooth vector fields on $\mathbb{R}^n$ satisfying axioms* (H1)*-to-*(H3)*, and let $\mathcal{H}$ be the heat-type operator defined in* (2.5)*. Moreover, let $\mathbb{G} = (\mathbb{R}^N, *, D_\lambda)$ and $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ be as in Theorem 3.2.*

Then, if $\gamma_{\mathbb{G}}$ is as in Theorem 3.4, the following facts hold.

(i) *The function* Γ *defined by*

$$\Gamma(t,x;s,y) := \gamma(t-s,x,y) := \int_{\mathbb{R}^p} \gamma_{\mathbb{G}}\big(t-s,(y,0)^{-1}*(x,\eta)\big)\,\mathrm{d}\eta, \tag{3.10}$$

is the global heat kernel of $\mathcal{H}$. *This means, precisely, that*
(i)$_1$ *for any fixed* $(t,x)\in\mathbb{R}^{1+n}$, *we have* $\Gamma(t,x;\cdot)\in L^1_{\mathrm{loc}}(\mathbb{R}^{1+n})$;
(i)$_2$ *for every* $\varphi\in C_0^\infty(\mathbb{R}^{1+n})$ *and every* $(t,x)\in\mathbb{R}^{1+n}$, *we have*

$$\begin{aligned}&\mathcal{H}\Big(\int_{\mathbb{R}^{1+n}}\gamma(t-s,x,y)\varphi(s,y)\,\mathrm{d}s\,\mathrm{d}y\Big)\\ &\qquad=\int_{\mathbb{R}^{1+n}}\gamma(t-s,x,y)\,\mathcal{H}\varphi(s,y)\,\mathrm{d}s\,\mathrm{d}y=-\varphi(t,x).\end{aligned}$$

(ii) *There exists a constant* $\mathbf{c}\geq 1$ *such that*

$$\begin{aligned}&\mathbf{c}^{-1}\,t^{-Q/2}\int_{\mathbb{R}^p}\exp\Big(-\frac{\mathbf{c}\,\|(y,0)^{-1}*(x,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta\leq\gamma(t,x,y)\\ &\qquad\leq\mathbf{c}\,t^{-Q/2}\int_{\mathbb{R}^p}\exp\Big(-\frac{\|(y,0)^{-1}*(x,\eta)\|^2}{\mathbf{c}\,t}\Big)\,\mathrm{d}\eta,\end{aligned}\tag{3.11}$$

for every $x,y\in\mathbb{R}^n$ *and every* $t>0$.

(iii) $\gamma\geq 0$ *and*

$$\gamma(t,x,y)=0 \text{ if and only if } t\leq 0.$$

(iv) γ *is symmetric in the space variables, i.e.*

$$\gamma(t,x,y)=\gamma(t,y,x)\qquad\text{for every } x,y\in\mathbb{R}^n \text{ and every } t>0.$$

(v) Γ *is smooth out of the diagonal of* $\mathbb{R}^{1+n}\times\mathbb{R}^{1+n}$.

(vi) *For every fixed* $(t,x)\in\mathbb{R}^{1+n}$, *with* $t>0$, *we have*

$$\int_{\mathbb{R}^n}\gamma(t,x,y)\,\mathrm{d}y=1.$$

(vii) *If* $\varphi\in C_b^0(\mathbb{R}^n)$, *then the function*

$$u(t,x):=\int_{\mathbb{R}^n}\gamma(t,x,y)\,\varphi(y)\,\mathrm{d}y$$

defined for $(t,x)\in\Omega=(0,+\infty)\times\mathbb{R}^n$ *is the unique bounded classical solution of the homogeneous Cauchy problem for* $\mathcal{H}$, *that is,*

$$\begin{cases}\mathcal{H}u=0 & \text{in } \Omega\\ u(0,x)=\varphi(x) & \text{for } x\in\mathbb{R}^n.\end{cases}$$

(viii) *The function* $\Gamma^*(t,u;s,v)=\Gamma(s,v;t,u)$ *is the global heat kernel of the (formal) adjoint operator* $\mathcal{H}^*:=\mathcal{L}+\partial_t$, *and satisfies dual statements with respect to* (i).

In the above theorem, $\|\cdot\|$ is *any* homogeneous norm on $\mathbb{G}$.

Remark 3.6. Points (ii) and (iv) in the above theorem also imply that, with the same constant $\mathbf{c}\geq 1$ as in (ii), for all $x,y\in\mathbb{R}^n$ and any $t>0$ one has

$$\begin{aligned}&\mathbf{c}^{-1}\,t^{-Q/2}\int_{\mathbb{R}^p}\exp\Big(-\frac{\mathbf{c}\,\|(x,0)^{-1}*(y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta\leq\gamma(t,x,y)\\ &\qquad\leq\mathbf{c}\,t^{-Q/2}\int_{\mathbb{R}^p}\exp\Big(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{\mathbf{c}\,t}\Big)\,\mathrm{d}\eta,\end{aligned}\tag{3.12}$$

(with the switched roles of x,y in the Gaussians). It will be sometimes convenient to use (3.11) in this alternative form.

3.2. **Review of known results on the CC distance.** Throughout the sequel, we will handle two distinct families of Hörmander's vector fields, each one inducing a Carnot-Carathéodory distance:

- the original family of vector fields $X_1, ..., X_m$, defined in $\mathbb{R}^n$, and satisfying (H1)-to-(H3);
- the lifted vector fields $Z_1, ..., Z_m$, defined on the higher dimensional Carnot group $\mathbb{G}$ in $\mathbb{R}^N$.

Both the X_i's and the Z_i's are 1-homogeneous with respect to suitable dilations, which implies some properties of the distances and the corresponding balls. The Z_i's are also left invariant, which implies more properties for the corresponding distance. Finally, the Z_i's are a lifting of the X_i's. The next proposition collects the basic properties which follow from these facts.

Proposition 3.7. *With the previous notation and assumptions about the systems of vector fields X and $\mathcal{Z}$, the following properties hold.*

(i) *Homogeneity:*

$$\begin{aligned} d_X(\delta_\lambda(x), \delta_\lambda(y)) &= \lambda\, d_X(x,y) && \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda > 0 \\ d_{\mathcal{Z}}(D_\lambda(u), D_\lambda(v)) &= \lambda\, d_{\mathcal{Z}}(u,v) && \text{for all } u, v \in \mathbb{R}^N \text{ and } \lambda > 0 \\ \delta_\lambda\big(B_X(x,\rho)\big) &= B_X\big(\delta_\lambda(x), \lambda\rho\big) && \text{for all } x \in \mathbb{R}^n \text{ and } \lambda, \rho > 0 \\ D_\lambda\big(B_{\mathcal{Z}}(u,\rho)\big) &= B_{\mathcal{Z}}\big(D_\lambda(u), \lambda\rho\big) && \text{for all } u \in \mathbb{R}^N \text{ and } \lambda, \rho > 0 \end{aligned}$$

(ii) *Left invariance:*

$$\begin{aligned} d_{\mathcal{Z}}(u,v) &= d_{\mathcal{Z}}(u * w, v * w) && \text{for all } u, v, w \in \mathbb{R}^N \\ u * B_{\mathcal{Z}}(v,\rho) &= B_{\mathcal{Z}}(u * v, \rho) && \text{for all } u, v \in \mathbb{R}^N \text{ and } \rho > 0 \end{aligned}$$

(iii) *Projection:*

$$\begin{aligned} d_X(x,y) &\le d_{\mathcal{Z}}((x,\xi),(y,\eta)) && \text{for all } (x,\xi),(y,\eta) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p \\ \pi\big(B_{\mathcal{Z}}((x,\xi),\rho)\big) &= B_X(x,\rho) && \text{for all } (x,\xi) \in \mathbb{R}^N \text{ and } \rho > 0 \end{aligned}$$

where π is the projection from $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^n$. In particular, since π is surjective, the last equality in (iii) *means that*

$$\forall\, y \in B_X(x,\rho),\ \xi \in \mathbb{R}^p\ \exists\ \eta \in \mathbb{R}^p \text{ s.t. } (y,\eta) \in B_{\mathcal{Z}}(x,\xi),\rho). \tag{3.13}$$

(iv) *Volume of $\mathcal{Z}$-balls: setting $\omega_Q = |B_{\mathcal{Z}}(0,1)|$, we have*

$$|B_{\mathcal{Z}}(u,\rho)| = |B_{\mathcal{Z}}(0,\rho)| = \omega_Q\, \rho^Q \text{ for all } u \in \mathbb{R}^N \text{ and } \rho > 0. \tag{3.14}$$

(v) *Homogeneous norm: if we let*

$$\|u\| = d_{\mathcal{Z}}(u,0) \qquad \text{for every } u \in \mathbb{R}^N,$$

then $\|\cdot\|$ is a homogeneous norm, and we also have

$$d_{\mathcal{Z}}(u,v) = \|v^{-1} * u\| = \|u^{-1} * v\| \qquad \text{for every } u, v \in \mathbb{R}^N.$$

Throughout the following, the symbol $\|\cdot\|$ in $\mathbb{R}^N$ will always denote this special norm.

The proof of Proposition 3.7 can be found in [2], or is immediate.

A much deeper result describes the *volume of X-balls*. The following theorem specializes a celebrated result by Nagel, Stein and Wainger [28] to the case of our 1-homogeneous vector fields X (for a proof see [5, Theorem B]):

Theorem 3.8. *Let $X = \{X_1, \dots, X_m\}$, n and q be as before. Then, there exist constants $\gamma_1, \gamma_2 > 0$ such that, for every $x \in \mathbb{R}^n$ and every $\rho > 0$, one has the estimates*

$$\gamma_1 \sum_{j=n}^{q} f_j(x)\, \rho^j \le |B_X(x,\rho)| \le \gamma_2 \sum_{j=n}^{q} f_j(x)\, \rho^j. \tag{3.15}$$

Here, the functions $f_k, \dots, f_q : \mathbb{R}^n \to \mathbb{R}$ satisfy the following properties:

(1) $f_k, \dots, f_q$ are continuous and non-negative on $\mathbb{R}^n$;

(2) for every $j \in \{n, \dots, q\}$, the function f_j is δ_λ-homogeneous of degree $q - j$.

In particular, $f_q(x)$ is constant in x and strictly positive.

Remark 3.9. From estimate (3.15) it can be easily derived the following notable fact: *for any* $x \in \mathbb{R}^n$ *and any* $0 < r < \rho$, *one has*

$$\gamma_1 \left(\frac{\rho}{r}\right)^n \leq \frac{|B_X(x,\rho)|}{|B_X(x,r)|} \leq \gamma_2 \left(\frac{\rho}{r}\right)^q, \tag{3.16}$$

In particular, the following *global* doubling property holds:

$$|B_X(x,2\rho)| \leq 2^q \gamma_2 \, |B_X(x,\rho)| \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \rho > 0. \tag{3.17}$$

The above facts easily imply that the function

$$\frac{1}{|B_X(x,\sqrt{t})|} \exp\left(- \frac{d_X^2(x,y)}{t}\right)$$

(which plays a key role in our estimates) is not so asymmetric in x, y as could seem. More precisely, we have the following proposition.

Proposition 3.10. *For every* $\theta > 0$ *there exists a constant* $C_1 > 0$ *such that*

$$\frac{1}{|B_X(y,\sqrt{t})|} \exp\left(- \frac{d_X^2(x,y)}{\theta t}\right) \leq \frac{C_1}{|B_X(x,\sqrt{t})|} \exp\left(- \frac{d_X^2(x,y)}{C_1 \theta t}\right), \tag{3.18}$$

for every $x, y \in \mathbb{R}^n$ *and every* $t > 0$.

Proof. To prove this, let us distinguish two cases.

- If $d_X(x,y) \leq \sqrt{t}$, we infer from (3.17) that $|B_X(y,\sqrt{t})|$ and $|B_X(x,\sqrt{t})|$ are equivalent, and thus the above inequality holds.
- If $d_X(x,y) > \sqrt{t}$, then by (3.16) and (3.17) we have

$$\begin{aligned}\frac{1}{|B_X(y,\sqrt{t})|} &\leq \frac{\gamma_2}{|B_X(y,d_X(x,y))|} \cdot \left(\frac{d_X(x,y)}{\sqrt{t}}\right)^q \\ &\leq \frac{2^q\,(\gamma_2)^2}{\big|B_X\big(x,d_X(x,y)\big)\big|} \cdot \left(\frac{d_X(x,y)}{\sqrt{t}}\right)^q \\ &\leq \frac{2^q\,(\gamma_2)^2}{\gamma_1} \cdot \frac{1}{|B_X(x,\sqrt{t})|} \cdot \left(\frac{d_X(x,y)}{\sqrt{t}}\right)^{q-n}.\end{aligned}$$

From this, we readily obtain (3.18) (see, e.g., (4.2)).

This ends the proof. □

Another deep known result that will play a key role in our estimates is the following 'global' version of a well-known result by Sanchéz-Calle [32] (see also [28, Lemma 3.2] and [18]), which compares the volumes of $B_X(x,\rho)$ and $B_{\mathcal{Z}}((x,\xi),\rho)$. For a proof of this result see [5, Theorem C].

Theorem 3.11. *Under the previous assumptions and notation, there exist constants* $\kappa \in (0,1)$ *and* $c_1, c_2 > 0$ *such that, for every* $x \in \mathbb{R}^n$, *every* $\xi \in \mathbb{R}^p$ *and every* $\rho > 0$ *one has the estimates:*

$$\big|\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}((x,\xi),\rho)\}\big| \leq c_1 \frac{|B_{\mathcal{Z}}((x,\xi),\rho)|}{|B_X(x,\rho)|}, \quad \textit{for all } y \in \mathbb{R}^n, \tag{3.19}$$

$$\big|\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}((x,\xi),\rho)\}\big| \geq c_2 \frac{|B_{\mathcal{Z}}((x,\xi),\rho)|}{|B_X(x,\rho)|}, \quad \textit{for all } y \in B_X(x,\kappa\rho). \tag{3.20}$$

We wish to stress that Theorems 3.8 and 3.11 contain *global* results, adapted to our context of homogeneous vector fields. In contrast with this, the original versions of these results, contained in [28], [32], and related to general systems of Hörmander's vector fields, express *local* results.

4. Gaussian estimates for Γ

The aim of this section is to prove upper/lower Gaussian estimates for the global heat kernel $\Gamma(t,x;s,y)$ of $\mathcal{H}$ (or, equivalently, for $\gamma(t,x,y)$) as defined in (3.10)). Broadly put, our approach is the following: on account of (3.12), we already know that Γ satisfies the 'quasi-Gaussian' estimates

$$\gamma(t,x,y) \approx t^{-Q/2} \int_{\mathbb{R}^p} \exp\Big(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta,$$

where Q is as in (3.6); we then derive 'pure' Gaussian estimates for Γ by showing that, for any $x,y\in\mathbb{R}^n$ and any $t>0$, one has

$$t^{-Q/2} \int_{\mathbb{R}^p} \exp\Big(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta \approx \frac{1}{|B_X(x,\sqrt{t})|}\cdot \exp\Big(-\frac{d_X^2(x,y)}{t}\Big). \tag{4.1}$$

To begin with, for a future reference, we state the following lemma.

Lemma 4.1. *The following estimates hold true:*

(i) *for every $\nu>0$ and $\delta\in(0,1)$ there exists $c>0$ such that*

$$\tau^\nu e^{-\tau^2} \le c\, e^{-\delta\tau^2} \quad \text{for every } \tau\ge 0; \tag{4.2}$$

(ii) *for every positive ν,θ there exists $c>0$ such that*

$$\tau^{-\nu} \ge c\, e^{-\theta\tau^2} \quad \text{for every } \tau> 0. \tag{4.3}$$

We then proceed by proving (4.1), and we start with the upper estimate.

Proposition 4.2. *There exists a constant $\kappa>1$ such that*

$$t^{-Q/2} \int_{\mathbb{R}^p} \exp\Big(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta \le \frac{\kappa}{|B_X(x,\sqrt{t})|}\exp\Big(-\frac{d_X^2(x,y)}{2t}\Big), \tag{4.4}$$

for every $x,y\in\mathbb{R}^n$ and every $t>0$.

Proof. Let $x,y\in\mathbb{R}^n$ be arbitrarily fixed, and let $t>0$.

Case I: $d_X(x,y)>\sqrt{t}$. In this case, for every $n=0,1,2,\ldots$, we define

$$A_n := \big\{\eta\in\mathbb{R}^p:\, 2^n d_X(x,y) \le \|(x,0)^{-1} * (y,\eta)\| < 2^{n+1} d_X(x,y)\big\}, \tag{4.5}$$

and we observe that, by Proposition 3.7-(iii), it holds $\mathbb{R}^p=\cup_{n\ge 0}A_n$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^p} \exp\Big(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta = \sum_{n=0}^{+\infty}\int_{A_n} \exp\Big(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta\\
&\le \sum_{n=0}^{+\infty} \exp\Big(-\frac{2^{2n}d_X^2(x,y)}{t}\Big)\cdot |A_n|\\
&\le \sum_{n=0}^{+\infty} \exp\Big(-\frac{2^{2n}d_X^2(x,y)}{t}\Big)\cdot \big|\big\{\eta\in\mathbb{R}^p:\,(y,\eta)\in B_{\mathcal{Z}}\big((x,0),2^{n+1}d_X(x,y)\big)\big\}\big|\\
&=: (\bigstar).
\end{aligned}$$

Next, by combining Theorem 3.11 and (3.14), for every $n\ge 0$ we have

$$\begin{aligned}
&\big|\big\{\eta\in\mathbb{R}^p:\,(y,\eta)\in B_{\mathcal{Z}}((x,0),2^{n+1}d_X(x,y))\big\}\big| \le c_1 \frac{\big|B_{\mathcal{Z}}\big((x,0),2^{n+1}d_X(x,y)\big)\big|}{\big|B_X\big(x,2^{n+1}d_X(x,y)\big)\big|}\\
&= c_1\omega_Q\,\frac{2^{Q(n+1)}d_X^Q(x,y)}{\big|B_X\big(x,2^{n+1}d_X(x,y)\big)\big|} \le c_1\omega_Q\,\frac{2^{Q(n+1)}d_X^Q(x,y)}{|B_X(x,d_X(x,y))|}\\
&\le c_1\omega_Q\,\frac{2^{Q(n+1)}d_X^Q(x,y)}{|B_X(x,\sqrt{t})|},
\end{aligned}$$

since $d_X(x,y) > \sqrt{t}$. As a consequence, we obtain

$$
\begin{aligned}
(\bigstar) &\leq c_1\omega_Q \sum_{n=0}^{+\infty} \exp\left(-\frac{2^{2n}d_X^2(x,y)}{t}\right) \cdot \frac{2^{Q(n+1)}d_X^Q(x,y)}{|B_X(x,\sqrt{t})|} \\
&= 2^Q\, c_1\omega_Q\, \frac{t^{Q/2}}{|B_X(x,\sqrt{t})|} \sum_{n=0}^{+\infty} \left(\frac{2^n d_X(x,y)}{\sqrt{t}}\right)^Q \exp\left(-\frac{2^{2n}d_X^2(x,y)}{t}\right) \\
&\qquad \text{(by estimate (4.2), with } \nu = Q \text{ and, e.g., } \delta = 1/2\text{)} \\
&\leq \frac{\alpha_Q\, t^{Q/2}}{|B_X(x,\sqrt{t})|} \sum_{n=0}^{+\infty} \exp\left(-\frac{2^{2n}d_X^2(x,y)}{2t}\right) =: (\bigstar\bigstar),
\end{aligned}
$$

for some constant α_Q depending on Q. On the other hand, since we are assuming that $d_X(x,y) > \sqrt{t}$, for any $n \geq 0$ we have

$$
\begin{aligned}
\exp\left(-\frac{2^{2n}d_X^2(x,y)}{2t}\right) &= \exp\left(-\frac{d_X^2(x,y)}{2t}\right) \cdot \exp\left(-\frac{d_X^2(x,y)}{2t} \cdot \left(2^{2n}-1\right)\right) \\
&\leq \exp\left(-\frac{d_X^2(x,y)}{2t}\right) \cdot \exp\left(-\frac{2^{2n}-1}{2}\right),
\end{aligned}
$$

from which we derive that

$$
\begin{aligned}
(\bigstar\bigstar) &\leq \frac{\alpha_Q\, s^{Q/2}}{|B_X(x,\sqrt{t})|} \exp\left(-\frac{d_X^2(x,y)}{2t}\right) \cdot \sum_{n=0}^{+\infty} \exp\left(-\frac{2^{2n}-1}{2}\right) \\
&= \frac{\alpha'_Q\, t^{Q/2}}{|B_X(x,\sqrt{t})|} \exp\left(-\frac{d_X^2(x,y)}{2t}\right).
\end{aligned}
$$

Finally, using this last estimate, we obtain

$$
t^{-Q/2}\int_{\mathbb{R}^p} \exp\left(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\right) \mathrm{d}\eta \leq \frac{\alpha'_Q}{|B_X(x,\sqrt{t})|} \exp\left(-\frac{d_X^2(x,y)}{2t}\right) \tag{4.6}
$$

which is precisely (4.4) (with $\kappa = \alpha'_Q$).

Case II: $d_X(x,y) \leq \sqrt{t}$. First of all, for every non-negative integer n we consider the set

$$
B_n := \left\{\eta \in \mathbb{R}^p : \, 2^n\sqrt{t} \leq \|(x,0)^{-1} * (y,\eta)\| < 2^{n+1}\sqrt{t}\right\}; \tag{4.7}
$$

moreover, we define

$$
B := \left\{\eta \in \mathbb{R}^p : \, \|(x,0)^{-1} * (y,\eta)\| < \sqrt{t}\right\}. \tag{4.8}
$$

Then we have:

$$
\begin{aligned}
\int_{\mathbb{R}^p} \exp\left(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\right) \mathrm{d}\eta &= \int_B \{\dots\}\,\mathrm{d}\eta + \sum_{n=0}^{+\infty}\int_{B_n} \{\dots\}\,\mathrm{d}\eta \\
&\leq |B| + \sum_{n=0}^{+\infty} \exp\left(-2^{2n}\right) \cdot |B_n| \\
&\leq \left|\left\{\eta \in \mathbb{R}^p : \, (y,\eta) \in B_{\mathbb{Z}}\big((x,0),\sqrt{t}\big)\right\}\right| \\
&\quad + \sum_{n=0}^{+\infty} \exp\left(-2^{2n}\right) \cdot \left|\left\{\eta \in \mathbb{R}^p : \, (y,\eta) \in B_{\mathbb{Z}}\big((x,0),2^{n+1}\sqrt{t}\big)\right\}\right| =: (\bigstar).
\end{aligned}
$$

Now, again by Theorem 3.11 and (3.14), for every $n \geq 0$ we have

$$\big|\{\eta \in \mathbb{R}^p : \, (y,\eta) \in B_{\mathcal{Z}}\big((x,0), 2^n\sqrt{t}\big)\}\big| \leq c_1 \, \frac{\big|B_{\mathcal{Z}}\big((x,0),2^n\sqrt{t}\big)\big|}{\big|B_X\big(x,2^n\sqrt{t}\big)\big|}$$

$$= c_1\omega_Q \, \frac{2^{nQ} t^{Q/2}}{|B_X(x,2^n\sqrt{t})|} \leq c_1\omega_Q \, \frac{2^{nQ}\, t^{Q/2}}{|B_X(x,\sqrt{t})|}.$$

As a consequence, we obtain

$$(\bigstar) \leq c_1\omega_Q \, \frac{t^{Q/2}}{|B_X(x,\sqrt{t})|} + c_1\omega_Q \cdot \sum_{n=0}^{+\infty} \exp\left(-2^{2n}\right) \, \frac{2^{(n+1)Q} t^{Q/2}}{|B_X(x,\sqrt{t})|}$$

$$= c_1\omega_Q \, \frac{t^{Q/2}}{|B_X(x,\sqrt{t})|} \cdot \left(1 + \sum_{n=0}^{+\infty} \exp\left(-2^{2n}\right)\, 2^{Q(n+1)}\right)$$

$$= \frac{\beta_Q \, t^{Q/2}}{|B_X(x,\sqrt{t})|} = (\bigstar\bigstar).$$

On the other hand, since we are assuming that $d_X(x,y) \leq \sqrt{t}$, we have

$$\exp\left(-\frac{d_X^2(x,y)}{2t}\right) \geq e^{-1/2},$$

from which we derive that

$$(\bigstar\bigstar) \leq \beta'_Q \, \frac{t^{Q/2}}{|B_X(x,\sqrt{t})|} \cdot \exp\left(-\frac{d_X^2(x,y)}{2t}\right).$$

Finally, using this last estimate, we obtain

$$t^{-Q/2} \int_{\mathbb{R}^p} \exp\left(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\right) \mathrm{d}\eta \leq \frac{\beta'_Q}{|B_X(x,\sqrt{t})|} \cdot \exp\left(-\frac{d_X^2(x,y)}{2t}\right), \tag{4.9}$$

and this is again (4.4). Gathering (4.6) and (4.9), we conclude that estimate (4.4) holds for every $x, y \in \mathbb{R}^n$ and every $t > 0$ by choosing

$$\kappa := \max\{\alpha'_1, \beta'_Q\} > 1.$$

This ends the proof. □

In order to prove lower estimate of Γ, we need the following property.

Lemma 4.3. *With the above notation and assumption, let $b > a > 0$ be fixed real numbers, and let $x, y \in \mathbb{R}^n$ satisfying*

$$d_X(x,y) < a. \tag{4.10}$$

Then, for every $\xi \in \mathbb{R}^p$ there exists $\overline{\eta} = \overline{\eta}_{x,y,\xi} \in \mathbb{R}^p \setminus \{0\}$ such that

$$\big\{\eta \in \mathbb{R}^p : a \leq d_{\mathcal{Z}}\big((x,\xi),(y,\eta)\big) < b\big\} \supseteq \big\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}\big((y,\overline{\eta}), \tfrac{1}{2}(b-a)\big)\big\}. \tag{4.11}$$

Proof. Since $y \in B_X(x,a)$, if $\xi \in \mathbb{R}^p$ is arbitrarily fixed, by (3.13) there exists

$$\eta_0 \in \big\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}\big((x,\xi),a\big)\big\}. \tag{4.12}$$

In particular, since the set in the right-hand side of (4.12) is open, we can assume that $\eta_0 \neq 0$. We then consider the function $g : [1,+\infty) \to \mathbb{R}$ defined as follows:

$$g(\lambda) := d_{\mathcal{Z}}\big((x,\xi),(y,\delta^*_\lambda(\eta_0))\big),$$

where $\delta^*_\lambda(\eta) = (\lambda^{\tau_1}\eta_1, \ldots, \lambda^{\tau_p}\eta_p)$ is as in (3.3). Clearly, we have that g is continuous on the whole of $[1,+\infty)$; moreover, from (4.12) we infer that

$$g(1) < a. \tag{4.13}$$

We now claim that

$$\lim_{\lambda\to+\infty} g(\lambda) = +\infty. \tag{4.14}$$

To prove (4.14) we first notice that, by triangle's inequality, we have

$$g(\lambda) \geq d_{\mathcal{Z}}\big((0,0),(y,\delta^*_\lambda(\eta_0))\big) - d_{\mathcal{Z}}\big((0,0),(x,\xi)\big) \qquad \text{(for all } \lambda \geq 1\text{);} \tag{4.15}$$

moreover, since the vector fields $Z_1,\ldots,Z_m$ are D_λ-homogeneous of degree 1, by Proposition 3.7-(i) we deduce that

$$\begin{aligned} d_{\mathcal{Z}}\big((0,0),(y,\delta^*_\lambda(\eta_0))\big) &= d_{\mathcal{Z}}\big((0,0),(\delta_\lambda(\delta_{1/\lambda}(y),\delta^*_\lambda(\eta_0))\big) \\ &\quad \text{(setting } y_\lambda = \delta_{1/\lambda}(y)) \\ &= d_{\mathcal{Z}}\big((0,0),D_\lambda(y_\lambda,\eta_0))\big) \\ &= \lambda\, d_{\mathcal{Z}}\big((0,0),(y_\lambda,\eta_0)\big). \end{aligned} \tag{4.16}$$

Since $y_\lambda = \delta_{1/\lambda}(y) \to 0 \in \mathbb{R}^n$ as $\lambda \to +\infty$, and since $\eta_0 \neq 0$, we have

$$\lim_{\lambda\to+\infty} d_{\mathcal{Z}}\big((0,0),(y_\lambda,\eta_0)\big) = d_{\mathcal{Z}}\big((0,0),(0,\eta_0)\big) > 0;$$

as a consequence, taking the limit as $\lambda \to +\infty$ in (4.16) we obtain

$$\lim_{\lambda\to+\infty} d_{\mathcal{Z}}\big((0,0),(y,\delta^*_\lambda(\eta_0))\big) = +\infty. \tag{4.17}$$

Gathering (4.17) and (4.15), we obtain the claimed (4.14).

Next, using the continuity of g, together with (4.13) and (4.14), we infer the existence of a suitable $\overline{\lambda} \in (1,+\infty)$ such that

$$g(\overline{\lambda}) = d_{\mathcal{Z}}\big((x,\xi),(y,\delta^*_{\overline{\lambda}}(\eta_0))\big) = \frac{b+a}{2}. \tag{4.18}$$

Setting $\overline{\eta} := \delta^*_{\overline{\lambda}}(\eta_0)$, we prove (4.11) by showing the stronger inclusion

$$\big\{z \in \mathbb{R}^N : a \leq d_{\mathcal{Z}}((x,\xi),z) < b\big\} \supseteq B_{\mathcal{Z}}\big((x,\overline{\eta}),\tfrac{1}{2}(b-a)\big). \tag{4.19}$$

To this end, let $u \in B_{\mathcal{Z}}\big((y,\overline{\eta}),\frac{1}{2}(b-a)\big)$ be fixed. On the one hand, we have

$$d_{\mathcal{Z}}\big((x,\xi),u\big) \leq d_{\mathcal{Z}}\big((x,\xi),(y,\overline{\eta})\big) + d_{\mathcal{Z}}\big((y,\overline{\eta}),u\big) < \frac{b+a}{2} + \frac{b-a}{2} = b;$$

on the other hand, since we also have

$$d_{\mathcal{Z}}\big((x,\xi),u\big) \geq d_{\mathcal{Z}}\big((x,\xi),(y,\overline{\eta})\big) - d_{\mathcal{Z}}\big(u,(y,\overline{\eta})\big) > \frac{b+a}{2} - \frac{b-a}{2} = a,$$

we conclude that (4.19) holds. This ends the proof. □

We can now prove the estimate from below in (4.1).

Proposition 4.4. *There exists a constant $\vartheta > 1$ such that*

$$t^{-Q/2}\int_{\mathbb{R}^p} \exp\left(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\right) \mathrm{d}\eta \geq \frac{1}{\vartheta\,|B_X(x,\sqrt{t})|}\exp\left(-\frac{\vartheta\, d_X^2(x,y)}{t}\right), \tag{4.20}$$

for every $x,y \in \mathbb{R}^n$ and every $t > 0$.

Proof. Let $x,y \in \mathbb{R}^n$ be arbitrarily fixed, and let $t > 0$.

Case I: $d_X(x,y) > \sqrt{t}$. In this case, we consider the set

$$A := \big\{\eta \in \mathbb{R}^p : 2d_X(x,y) \leq \|(x,0)^{-1}*(y,\eta)\| < 4d_X(x,y)\big\}.$$

By applying Lemma 4.3 (with $a := 2d_X(x,y) > d_X(x,y)$ and $b := 2a$), one has

$$A \supseteq \big\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}\big((y,\overline{\eta}),d_X(x,y)\big)\big\} \tag{4.21}$$

(for a suitable $\overline{\eta} = \overline{\eta}_{x,y} \in \mathbb{R}^p \setminus \{0\}$); as a consequence, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^p} \exp\left(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\right) \mathrm{d}\eta &\geq \int_A \exp\left(-\frac{\|(x,0)^{-1} * (y,\eta)\|^2}{t}\right) \mathrm{d}\eta \\
&\quad (\text{since } \|(x,0)^{-1} * (y,\eta)\|^2 \leq 16\, d_X^2(x,y) \text{ for } \eta \in A) \\
&\geq \exp\left(-\frac{16\, d_X^2(x,y)}{t}\right) \cdot |A| \\
&\geq \exp\left(-\frac{16\, d_X^2(x,y)}{t}\right) \cdot \big|\big\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}\big((y,\overline{\eta}), d_X(x,y)\big)\big\}\big| =: (\bigstar).
\end{aligned}$$

On the other hand, by using Theorem 3.11 (with the choice $(x,\xi) = (y,\overline{\eta})$) and (3.14), we get

$$\begin{aligned}
\big|\big\{\eta \in \mathbb{R}^p : (y,\eta) \in B_{\mathcal{Z}}\big((y,\overline{\eta}), d_X(x,y)\big)\big\}\big| &\geq c_2 \frac{\big|B_{\mathcal{Z}}\big((y,\overline{\eta}), d_X(x,y)\big)\big|}{\big|B_X\big(y, d_X(x,y)\big)\big|} \\
&= c_2\omega_Q \frac{d_X^Q(x,y)}{\big|B_X\big(y, d_X(x,y)\big)\big|} \\
&\quad (\text{since we are assuming that } d_X(x,y) > \sqrt{t}) \\
&> c_2\omega_Q \frac{t^{Q/2}}{\big|B_X\big(y, d_X(x,y)\big)\big|},
\end{aligned}$$

from which we derive the estimate

$$\begin{aligned}
(\bigstar) &\geq c_2\omega_Q \frac{t^{Q/2}}{\big|B_X\big(y, d_X(x,y)\big)\big|} \cdot \exp\left(-\frac{16\, d_X^2(x,y)}{t}\right) \\
&\quad (\text{since } B_X\left(y, d_X(x,y)\right) \subseteq B_X\left(x, 2d_X(x,y)\right)) \\
&\geq c_2\omega_Q \frac{t^{Q/2}}{\big|B_X\big(x, 2d_X(x,y)\big)\big|} \cdot \exp\left(-\frac{16\, d_X^2(x,y)}{t}\right) =: (\bigstar\bigstar).
\end{aligned}$$

We now observe that, since $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree 1, and since we are assuming that $d_X(x,y) > \sqrt{t}$, we can apply (3.16), getting

$$\big|B_X\big(x, 2d_X(x,y)\big)\big| \leq \gamma_2\, |B_X(x,\sqrt{t})| \cdot \left(\frac{2d_X(x,y)}{\sqrt{t}}\right)^q,$$

where q is as in (2.3). As a consequence, we deduce that

$$\begin{aligned}
(\bigstar\bigstar) &\geq \frac{c_2\omega_Q}{2^q\gamma_2} \frac{t^{Q/2}}{|B_X(x,\sqrt{t})|} \cdot \left(\frac{d_X(x,y)}{\sqrt{t}}\right)^{-q} \exp\left(-\frac{16\, d_X^2(x,y)}{t}\right) \\
&\quad (\text{by estimate (4.3), with } \nu = q \text{ and, e.g., } \theta = 4) \\
&\geq \frac{t^{Q/2}}{\alpha_{q,Q}\, |B_X(x,\sqrt{t})|} \exp\left(-\frac{20\, d_X^2(x,y)}{t}\right),
\end{aligned}$$

for some constant $\alpha_{q,Q}$ depending on q, Q. Finally, by exploiting this last estimate, we obtain

$$
\begin{aligned}
&t^{-Q/2}\int_{\mathbb{R}^p}\exp\Big(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta \\
&\qquad\geq t^{-Q/2}\cdot\Big[\frac{t^{Q/2}}{\alpha_{q,Q}\,|B_X(x,\sqrt{t})|}\,\exp\Big(-\frac{20\,d_X^2(x,y)}{t}\Big)\Big] \\
&\qquad(\text{setting } \vartheta_1=\max\{\alpha_{q,Q},20\}) \\
&\qquad\geq \frac{1}{\vartheta_1\,|B_X(x,\sqrt{t})|}\,\exp\Big(-\frac{\vartheta_1\,d_X^2(x,y)}{t}\Big),
\end{aligned}
\tag{4.22}
$$

which exactly the desired (4.20) (with $\vartheta=\vartheta_1>1$).

Case II: $d_X(x,y)\leq\sqrt{t}$. The proof is similar to that of Case I, letting now

$$A=\big\{\eta\in\mathbb{R}^p:\,2\sqrt{t}\leq\|(x,0)^{-1}*(y,\eta)\|<4\sqrt{t}\big\}.$$

Applying Lemma 4.3 (with $a:=2\sqrt{t}>d_X(x,y)$ and $b:=2a$), we get

$$A\supseteq\big\{\eta\in\mathbb{R}^p:(y,\eta)\in B_{\mathcal{Z}}((y,\overline{\eta}),\sqrt{t})\big\}\neq\varnothing$$

(for a suitable $\overline{\eta}=\overline{\eta}_{x,y}\in\mathbb{R}^p\setminus\{0\}$); as a consequence, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^p}\exp\Big(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta\geq\int_A\exp\Big(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\Big)\,\mathrm{d}\eta \\
&\geq e^{-16}\cdot|A|\geq e^{-16}\cdot\big|\big\{\eta\in\mathbb{R}^p:(y,\eta)\in B_{\mathcal{Z}}((y,\overline{\eta}),\sqrt{t})\big\}\big|=:(\bigstar).
\end{aligned}
$$

On the other hand, by using Theorem 3.11 (with the choice $(x,\xi)=(y,\overline{\eta})$) and (3.14), we get

$$
\begin{aligned}
\big|\big\{\eta\in\mathbb{R}^p:(y,\eta)\in B_{\mathcal{Z}}((y,\overline{\eta}),\sqrt{t})\big\}\big|&\geq c_2\,\frac{\big|B_{\mathcal{Z}}((y,\overline{\eta}),\sqrt{t})\big|}{|B_X(y,\sqrt{t})|} \\
&=c_2\omega_Q\,\frac{t^{Q/2}}{|B_X(y,\sqrt{t})|},
\end{aligned}
$$

from which we derive the estimate (remind that we are assuming $d_X(x,y)\leq\sqrt{t}$)

$$
\begin{aligned}
(\bigstar)&\geq\frac{c_2\omega_Q}{e^{16}}\,\frac{t^{Q/2}}{|B_X(y,\sqrt{t})|} \\
&\big(\text{since } B_X(y,\sqrt{t})\subseteq B_X(x,d_X(x,y)+\sqrt{t})\subseteq B_X(x,2\sqrt{t})\big) \\
&\geq\frac{c_2\omega_Q}{e^{16}}\,\frac{t^{Q/2}}{|B_X(x,2\sqrt{t})|}=:(\bigstar\bigstar).
\end{aligned}
$$

By (3.17), we have

$$|B_X(x,2\sqrt{t})|\leq\gamma_2 2^q\,|B_X(x,\sqrt{t})|\qquad(\text{where } q \text{ is as in (2.3)});$$

as a consequence, we deduce that

$$(\bigstar\bigstar)\geq\frac{c_2\omega_Q}{2^q e^{16}}\,\frac{t^{Q/2}}{|B_X(x,\sqrt{t})|}\geq\frac{t^{Q/2}}{\beta_{q,Q}\,|B_X(x,\sqrt{t})|}\cdot\exp\Big(-\frac{d_X^2(x,y)}{t}\Big),$$

for some constant $\beta_{q,Q}$ depending on q, Q. Using this last estimate, we get

$$
\begin{aligned}
& t^{-Q/2}\int_{\mathbb{R}^p}\exp\left(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\right)\mathrm{d}\eta \\
& \qquad \geq t^{-Q/2}\cdot\left[\frac{t^{Q/2}}{\beta_{q,Q}\,|B_X(x,\sqrt{t})|}\cdot\exp\left(-\frac{d_X^2(x,y)}{t}\right)\right] \\
& \qquad \text{(setting } \vartheta_2 := \max\{\beta_{q,Q},1\}\text{)} \\
& \qquad \geq \frac{1}{\vartheta_2\,|B_X(x,\sqrt{t})|}\exp\left(-\frac{\vartheta_2\, d_X^2(x,y)}{t}\right),
\end{aligned}
\tag{4.23}
$$

and this is again the desired (4.20) (this time with $\vartheta = \vartheta_2 \geq 1$). Gathering (4.22) and (4.23), we conclude that estimate (4.20) holds for every $x, y \in \mathbb{R}^n$ and every $t > 0$ by choosing

$$\vartheta := \max\{\vartheta_1, \vartheta_2\} > 1.$$

This ends the proof. □

Thanks to Propositions 4.2 and 4.4, we can now (2.6) in Theorem 2.4.

Proof of Theorem 2.4-(i). For every $x, y \in \mathbb{R}^n$ and every $t > 0$, we set

$$H(x,y,t) := t^{-Q/2}\int_{\mathbb{R}^p}\exp\left(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{t}\right)\mathrm{d}\eta.$$

On account of (3.12), we know that there exists a constant $\mathbf{c} \geq 1$, only depending on $\mathbb{G}$ and on $\mathcal{Z}$ (which, in their turn, only depend on the set X), such that

$$\mathbf{c}^{-1-Q/2}\,H\left(x,y,\mathbf{c}^{-1}t\right) \leq \gamma(t,x,y) \leq \mathbf{c}^{-1-Q/2}\,H\left(x,y,\mathbf{c}t\right) \tag{4.24}$$

for every $x, y \in \mathbb{R}^n$ and every $t > 0$. These bounds, together with the preceding Propositions 4.2 and 4.4, immediately give (2.6). □

5. Estimates for the derivatives of Γ

The aim of this section is to establish (upper) Gaussian estimates for the space derivatives along $X_1, \ldots, X_m$ and for the 'time derivatives' of arbitrary order of γ, that is Theorem 2.4-(ii). To begin with, we state the following theorem proved in [4], which provides integral representations (analogous to formula (3.10)) for any space/time derivative of γ.

Theorem 5.1 (See [4, Theorem 3])**.** *Under the previous assumption, and keeping the notation of Theorem 3.5, for any nonnegative integers α, h, k and any choice of indexes $i_1, \ldots, i_h, j_1, \ldots, j_k$ in $\{1, \ldots, m\}$, we have the following representation formulas*

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}\right)^{\alpha} X_{i_1}^x\cdots X_{i_h}^x\gamma(t,x,y) \\
& \qquad = \int_{\mathbb{R}^p}\left(\left(\frac{\partial}{\partial t}\right)^{\alpha} Z_{i_1}\cdots Z_{i_h}\gamma_{\mathbb{G}}\right)\left(t,(y,0)^{-1}*(x,\eta)\right)\mathrm{d}\eta;
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}\right)^{\alpha} X_{j_1}^y\cdots X_{j_k}^y\gamma(t,x,y) \\
& \qquad = \int_{\mathbb{R}^p}\left(\left(\frac{\partial}{\partial t}\right)^{\alpha} Z_{j_1}\cdots Z_{j_k}\gamma_{\mathbb{G}}\right)\left(t,(x,0)^{-1}*(y,\eta)\right)\mathrm{d}\eta;
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}\right)^{\alpha} X_{j_1}^y\cdots X_{j_k}^y X_{i_1}^x\cdots X_{i_h}^x\gamma(t,x,y) \\
& \qquad = \int_{\mathbb{R}^p}\left(\left(\frac{\partial}{\partial t}\right)^{\alpha} Z_{j_1}\cdots Z_{j_k}\left((Z_{i_1}\cdots Z_{i_h}\gamma_{\mathbb{G}})\circ\widetilde{\iota}\,\right)\right)\left(t,(x,0)^{-1}*(y,\eta)\right)\mathrm{d}\eta\,,
\end{aligned}
\tag{5.3}
$$

holding true for every $(t,x)\neq(0,y)$ *in* $\mathbb{R}^{1+n}$*. Here* $\widetilde{\iota}:\mathbb{R}^{1+N}\to\mathbb{R}^{1+N}$ *is the map defined by*

$$\widetilde{\iota}(t,u)=(t,u^{-1})$$

and u^{-1} *is the inverse of* u *in* $\mathbb{G}=(\mathbb{R}^N,*)$.

While the proof of our Gaussian estimates for the derivatives appearing in (5.2) and (5.1) is, by now, quite straightforward, for the mixed case in (5.3) it will require some extra work. We start establishing the following proposition, which will be useful for the case of mixed derivatives.

Proposition 5.2. *With the above notation, for any nonnegative integers* α,h,k *and any choice of indexes* $i_1,\dots,i_h,j_1,\dots,j_k\in\{1,\dots,m\}$*, there exists* $c_1,c_2>0$ *such that*

$$\left|\left(\frac{\partial}{\partial t}\right)^{\alpha} Z_{j_1}\cdots Z_{j_k}\big((Z_{i_1}\cdots Z_{i_h}\gamma_{\mathbb{G}})\circ\widetilde{\iota}\,\big)(t,u)\right|\le c_1\,t^{-(Q+2\alpha+h+k)/2}\exp\left(-\frac{\|u\|^2}{c_2\,t}\right),$$

for every $u\in\mathbb{G}$ *and every* $t>0$.

In turn, Proposition 5.2 follows from two facts which are stated separately in the next two lemmas, since they may be of independent interest.

Lemma 5.3. *Let* Y *be a* 1*-homogeneous (but not necessarily left invariant) smooth vector field on* $\mathbb{G}$*. Then, it is possible to find another* 1*-homogeneous smooth vector field* $\widetilde{Y}$ *such that*

$$Y(f\circ\iota)=(\widetilde{Y}f)\circ\iota\qquad\text{for every } f\in C^\infty(\mathbb{R}^N),$$

where $\iota(u)=u^{-1}$ *is the inversion map on* $\mathbb{G}$.

Proof. First of all, let us write the dilations on $\mathbb{G}$ as:

$$D_\lambda(u_1,...,u_N)=(\lambda^{\alpha_1}u_1,...,\lambda^{\alpha_N}u_N)\qquad(\text{for any }\lambda>0\text{ and }u\in\mathbb{G}).$$

We can write

$$Y=\sum_{j=1}^N b_j(u)\,\frac{\partial}{\partial u_j}$$

where $b_j(u)$ is a (α_j-1)-homogeneous polynomial function. Moreover, using the structure of the inversion map on homogeneous groups (see, e.g., [8, Corollary 1.3.16]), we know that the k-th component of $\iota(u)$ is a α_k-homogeneous polynomial function. Therefore

$$\begin{aligned}Y(f\circ\iota)(u)&=\sum_{j=1}^N b_j(u)\sum_{k=1}^N\frac{\partial f}{\partial u_k}\big(\iota(u)\big)\frac{\partial\iota_k}{\partial u_j}(u)\\&=\sum_{k=1}^N\left(\sum_{j=1}^N b_j(u)\frac{\partial\iota_k}{\partial u_j}(u)\right)\frac{\partial f}{\partial u_k}\big(\iota(u)\big)\equiv\sum_{k=1}^N c_k(u)\frac{\partial f}{\partial u_k}\big(\iota(u)\big),\end{aligned}$$

and c_k is a homogeneous polynomial function of degree

$$(\alpha_j-1)+(\alpha_k-\alpha_j)=\alpha_k-1.$$

Next, we define $\widetilde{c}_k=c_k\circ\iota$. Since the dilations D_λ are group automorphisms, we have

(i) $D_\lambda\big(\iota(u)\big)=\iota\big(D_\lambda(u)\big)$;

(ii) $\widetilde{c}_k\big(D_\lambda(u)\big)=c_k\big(D_\lambda\big(\iota(u)\big)\big)=\lambda^{\alpha_k-1}c_k\big(\iota(u)\big)=\lambda^{\alpha_k-1}\widetilde{c}_k(u)$.

Hence, $\widetilde{c}_k$ is (α_k-1)-homogeneous as well, and

$$Y(f\circ\iota)(u)=\sum_{k=1}^N\widetilde{c}_k\big(\iota(u)\big)\frac{\partial f}{\partial u_k}\big(\iota(u)\big)\equiv(\widetilde{Y}f)\big(\iota(u)\big),$$

where $\widetilde{Y}:=\sum_{k=1}^N\widetilde{c}_k(u)\,\partial_{u_k}$ is a 1-homogeneous vector field (in view of (ii)). □

Next, let us prove the following:

Proposition 5.4. *Let α, r be nonnegative integers, and let $Y_1, ..., Y_r$ be 1-homogeneous (but not necessarily left invariant) smooth vector fields on $\mathbb{G}$. Then, there exist constants c_1, $c_2 > 0$ such that, for every $u \in \mathbb{G}$ and every $t > 0$, the following Gaussian bound holds*

$$\left|\left(\frac{\partial}{\partial t}\right)^{\alpha} Y_1 \cdots Y_r \gamma_{\mathbb{G}}(t,u)\right| \le c_1\, t^{-(Q/2+\alpha+r/2)} \exp\left(-\frac{\|u\|^2}{c_2\, t}\right).$$

Proof. If $Y_1, ..., Y_r$ are 1-homogeneous *and left invariant* vector fields on $\mathbb{G}$, this result is proved by [9, Theorem. 2.5] (see also (3.8) in Theorem 3.4). We are going to show that the result for left invariant 1-homogeneous vector fields easily implies our more general statement.

In fact, let $X_1, ..., X_N$ be the canonical basis of $\mathbb{G}$, i.e., X_i is the unique left invariant vector field on $\mathbb{G}$ such that $X_i(0) = \partial_{u_i}$. Up to possibly reordering the X_i's, we can assume that X_i is α_i-homogeneous, with $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \le \alpha_{m+2} .... \le \alpha_N = s$, and s is the step of $\mathbb{G}$ (that is, $\mathrm{Lie}(\mathbb{G})$ is nilpotent of step s). Then, for homogeneity reasons, we have

$$X_i = \partial_{u_i} + \sum_{\substack{k=1\\ \alpha_k > \alpha_i}}^{N} b_{ik}(u)\partial_{u_k} \qquad (\text{for } i = 1, 2, ..., N),$$

where $b_{ik}(u)$ is a $(\alpha_k - \alpha_i)$-homogeneous polynomial function. In particular, since $X_N = \partial_{u_N}$, we can solve the above system in $\partial_{u_1}, ..., \partial_{u_N}$ using backward substitution, thus writing

$$\partial_{u_i} = X_i + \sum_{\substack{k=1\\ \alpha_k > \alpha_i}}^{N} c_{ik}(u) X_k \qquad (\text{for } i = 1, 2, ..., N), \tag{5.4}$$

where $c_{ik}(u)$ is a $(\alpha_k - \alpha_i)$-homogeneous polynomial function.

Let now Y be a 1-homogeneous vector field. Owing to (5.4), we have

$$Y = \sum_{i=1}^{N} \beta_i(u)\, \partial_{u_i} = \sum_{i=1}^{N} \beta_i(u) \left(X_i + \sum_{\substack{k=1\\ \alpha_k > \alpha_i}}^{N} c_{ik}(u) X_k \right) \equiv \sum_{i=1}^{N} \gamma_i(u) X_i,$$

where $\gamma_i(u)$ is a $(\alpha_i - 1)$-homogeneous polynomial function. Notice that, since $X_1, ..., X_m$ are generators of $\mathrm{Lie}(\mathbb{G})$, every X_i with $i > m$ can be written as a linear combination (with constant coefficients) of commutators of $X_1, ..., X_m$, of length α_i. Thus, since the Gaussian bound holds for left invariant vector fields (see (3.8) in Theorem 3.4), we obtain

$$\begin{aligned}
|Y\gamma_{\mathbb{G}}(t,u)| &\le \sum_{i=1}^{N} |\gamma_i(u)| \cdot |X_i \gamma_{\mathbb{G}}(t,u)| \le \widehat{\mathbf{c}} \sum_{i=1}^{N} |\gamma_i(u)| \cdot t^{-(Q+\alpha_i)/2} \exp\left(-\frac{\|u\|^2}{\widehat{\mathbf{c}}\, t}\right) \\
&\le \kappa \sum_{i=1}^{N} \|u\|^{\alpha_i - 1} \cdot t^{-(Q+\alpha_i)/2} \exp\left(-\frac{\|u\|^2}{\widehat{\mathbf{c}}\, t}\right) \\
&= \kappa\, t^{-(Q+1)/2} \sum_{i=1}^{N} \left(\frac{\|u\|}{\sqrt{t}}\right)^{\alpha_i - 1} \cdot \exp\left(-\frac{\|u\|^2}{\widehat{\mathbf{c}}\, t}\right) \\
&\le c_1\, t^{-(Q+1)/2} \exp\left(-\frac{\|u\|^2}{c_2\, t}\right)
\end{aligned}$$

where the last inequality follows from (4.2). The general case then follows by iteration. □

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. By repeatedly applying Lemma 5.3, we can rewrite

$$\left(\frac{\partial}{\partial t}\right)^{\alpha} Z_{j_1} \cdots Z_{j_k}\big((Z_{i_1} \cdots Z_{i_h} \gamma_{\mathbb{G}}) \circ \widetilde{\iota}\,\big) = \left\{ \left(\frac{\partial}{\partial t}\right)^{\alpha} \widetilde{Z}_{j_1} \cdots \widetilde{Z}_{j_k}\big(Z_{i_1} \cdots Z_{i_h} \gamma_{\mathbb{G}}\big) \right\} \circ \widetilde{\iota}$$

with $\widetilde{\iota}(t,u) = (t, u^{-1})$. Here the Z_i's are 1-homogeneous and left invariant, whereas the $\widetilde{Z}_i$'s are just 1-homogeneous. Anyhow, we can apply Proposition 5.4 and get the desired result. □

With Proposition 5.2 in hand, we can prove the Gaussian estimates on the derivatives.

Proof of Theorem 2.4-(ii). We distinguish three different cases.

CASE 1. $Y_1,\ldots,Y_r = X^x_{i_1}\cdots X^x_{i_r}$. Then, by (5.1), (3.8) and Proposition 4.2 we have

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial t}\right)^{\alpha} X^x_{i_1}\cdots X^x_{i_r}\gamma(t,x,y)\right|\\ &\qquad\leq \widehat{\mathbf{c}}\,t^{-(Q/2+\alpha+r/2)}\int_{\mathbb{R}^p}\exp\left(-\frac{\|(y,0)^{-1}*(x,\eta)\|}{\widehat{\mathbf{c}}\,t}\right)\mathrm{d}\eta\\ &\qquad\leq c\,t^{-(\alpha+r/2)}\,\frac{1}{|B_X(y,\sqrt{t})|}\,\exp\left(-\frac{d^2_X(x,y)}{C\,t}\right).\end{aligned}$$

The assertion then follows by Remark 3.10.

CASE 2. $Y_1,\ldots,Y_r = X^y_{j_1}\cdots X^y_{j_r}$. Then, by (5.2), (3.8) and Proposition 4.2, we have

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial t}\right)^{\alpha} X^y_{j_1}\cdots X^y_{j_r}\gamma(t,x,y)\right|\\ &\qquad\leq \widehat{\mathbf{c}}\,t^{-(Q/2+\alpha+r/2)}\int_{\mathbb{R}^p}\exp\left(-\frac{\|(x,0)^{-1}*(y,\eta)\|}{\widehat{\mathbf{c}}\,t}\right)\mathrm{d}\eta\\ &\qquad\leq c\,t^{-(\alpha+r/2)}\,\frac{1}{|B_X(x,\sqrt{t})|}\,\exp\left(-\frac{d^2_X(x,y)}{C\,t}\right).\end{aligned}$$

CASE 3. $Y_1,\ldots,Y_r = X^y_{j_1}\cdots X^y_{j_k}X^x_{i_1}\cdots X^x_{i_h}$ (with $k+h=r$). In this last case, by exploiting (5.3), Proposition 5.2 and again Proposition 4.2, we obtain

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial t}\right)^{\alpha} X^y_{j_1}\cdots X^y_{j_k}X^x_{i_1}\cdots X^x_{i_h}\gamma(t,x,y)\right|\\ &\qquad\leq c_1\,t^{-(Q/2+\alpha+r/2)}\int_{\mathbb{R}^p}\exp\left(-\frac{\|(x,0)^{-1}*(y,\eta)\|^2}{c_2\,t}\right)\mathrm{d}\eta\\ &\qquad\leq c\,t^{-(\alpha+r/2)}\,\frac{1}{|B_X(x,\sqrt{t})|}\,\exp\left(-\frac{d^2_X(x,y)}{C\,t}\right).\end{aligned}$$

This ends the proof. □

6. AN APPLICATION TO THE CAUCHY PROBLEM FOR $\mathcal{H}$

As anticipated in the Introduction, in this section we exploit the global Gaussian bounds of Γ to study the unique solvability of the Cauchy problem for $\mathcal{H}$. More precisely, we extend the result proved in [4, Thm. 4.1], where the Cauchy problem is studied for bounded continuous initial data, to possibly unbounded continuous initial data, fastly growing at infinity.

As for the proof of Gaussian estimates, we will make Assumptions 2.1 and will also assume (H3) (see Section 3). As noted before, condition (H3) amounts to assuming that we are *not* in a Carnot group (see also Remark 6.4 after the proof of our result for some explanation on this point).

We start with the following

Definition 6.1. Let $S_\tau := (0,\tau)\times\mathbb{R}^n$, for some fixed $\tau\in(0,+\infty]$. Given any function $f\in C(\mathbb{R}^n)$, we say that $u: S_\tau\to\mathbb{R}$ is a *classical solution* of the Cauchy problem

$$\begin{cases}\mathcal{H}u = 0 & \text{in } S_\tau,\\ u(0,x) = f(x) & \text{for } x\in\mathbb{R}^n\end{cases}\tag{6.1}$$

if it satisfies the following properties:

(1) $u\in C^2(S_\tau)$ and $\mathcal{H}u = 0$ pointwise on S_τ;
(2) $\lim_{t\to 0^+} u(t,x) = f(x)$ for every fixed $x\in\mathbb{R}^n$.

Using the upper Gaussian estimates of Γ, we are able to prove that (6.1)) admits (at least) one classical solution when the initial datum f grows at most exponentially. In what follows, we set

$$\rho_X(x) := d_X(0,x) \qquad (x \in \mathbb{R}^n).$$

Theorem 6.2. *There exists $T > 0$ such that, if $f \in C(\mathbb{R}^n)$ satisfies the growth condition*

$$\int_{\mathbb{R}^n} |f(y)| \exp\big(-\mu\rho_X^2(y)\big)\,\mathrm{d}y < +\infty \tag{6.2}$$

for some constant $\mu > 0$, then the function

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\, f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \gamma(t,x,y)\, f(y)\,\mathrm{d}y \tag{6.3}$$

is a classical solution of (6.1) *on the strip $S_{T/\mu}$.*

Furthermore, it is possible to find constants $\tau, \delta > 0$ (depending on μ) such that

$$\int_{S_\tau} \exp\big(-\delta\rho_X^2(x)\big)\, |u(t,x)|\,\mathrm{d}t\,\mathrm{d}x < +\infty. \tag{6.4}$$

Finally, if u_1, u_2 are two classical solutions of (6.1) *with the same continuous initial datum f, and if u_1, u_2 satisfy condition* (6.4) *in two strips S_{τ_1}, S_{τ_2}, respectively, then*

$$u_1 \equiv u_2 \text{ in } S_\tau, \qquad \text{for } \tau = \min\{\tau_1, \tau_2\}.$$

Before proving Theorem 6.2 we establish, for a future reference, the following easy lemma.

Lemma 6.3. *For every fixed $\theta > 0$, we have*

$$\phi(y) := \exp\big(-\theta\rho_X^2(y)\big) \in L^1(\mathbb{R}^n). \tag{6.5}$$

Proof. Since ϕ is bounded, it suffices to show that ϕ is integrable at infinity. To this end, if $\sigma_1, \ldots, \sigma_n$ are as in (2.1), we consider the *homogeneous norm*

$$\mathcal{N}(y) := \sum_{j=1}^n |y_j|^{1/\sigma_j} \qquad (y \in \mathbb{R}^n),$$

and we prove that ϕ is integrable on the set $\mathcal{O} := \{\mathcal{N} \geq 1\}$. Now, using Lemma 4.1, and taking into account that both $\mathcal{N}$ and ρ_X are δ_λ-homogeneous of degree 1, we have

$$\int_{\mathcal{O}} \phi(y)\,\mathrm{d}y \leq c\int_{\mathcal{O}} \frac{1}{\rho_X^{2q}(y)}\,\mathrm{d}y = c\sum_{k=0}^{+\infty}\int_{\{2^k\leq\mathcal{N}<2^{k+1}\}} \frac{1}{\rho_X^{2q}(y)}\,\mathrm{d}y$$

(performing the change of variable $y = \delta_{2^k}(u)$)

$$= c\left(\int_{\{1\leq\mathcal{N}<2\}} \frac{1}{\rho_X^{2q}(u)}\,\mathrm{d}u\right)\cdot\sum_{k=0}^{+\infty}\frac{1}{2^{qk}} < +\infty.$$

This ends the proof. □

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Assume that $f \in C(\mathbb{R}^n)$ satisfies (6.2), and let u be as in (6.3).

Step I. Let us show that u is well defined and solves (6.1) in some S_T. To this end, let $R > 0$ be arbitrarily fixed, and let $\phi_R \in C_0^0(\mathbb{R}^n)$ satisfy the following properties

- $\phi_R \equiv 1$ on $\{\rho_X < R\}$;
- $\phi_R \equiv 0$ on $\{\rho_X > 2R\}$;
- $0 \leq \phi_R \leq 1$ on $\mathbb{R}^n$.

Then, we can write

$$u(t,x) = \int_{\mathbb{R}^n} \gamma(t,x,y)\, f(y)\phi_R(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \gamma(t,x,y)\, f(y)\big(1-\phi_R(y)\big)\,\mathrm{d}y \equiv u_1(t,x) + u_2(t,x).$$

Since $f\phi_R$ is bounded continuous, by [4, Theorem 4.1] we know that u_1 is well defined for every $t>0$, and it solves (6.1) with initial datum $f\phi_R$ on the whole of on $(0,+\infty)\times\mathbb{R}^n$. In particular, since $\phi_R\equiv 1$ on the set $\{\rho_X<R\}$, for every $x\in\mathbb{R}^n$ with $\rho_X(x)<R$ we have

$$\lim_{t\to 0^+} u_1(t,x) = (f\phi_R)(x) = f(x).$$

We now prove that there exists a suitable $T>0$, *independent of the chosen* R, such that the following facts hold on the bounded stripe $S_{T/\mu,R} := (0,T/\mu)\times\{\rho_X<R\}$:

(i) u_2 is well defined; (ii) u_2 it solves the equation $\mathcal{H}u=0$; (iii) $u_2(t,x)\to 0$ as $t\to 0^+$.

As for (i)-(ii) we observe that, by the Gaussian estimate (2.6), we have

$$\begin{aligned}
|u_2(t,x)| &\leq \frac{\varrho}{|B(x,\sqrt{t})|}\int_{\{\rho_X(y)>2R\}} \exp\left(-\frac{d_X^2(x,y)}{\varrho t}\right)|f(y)|\,\mathrm{d}y \\
&= \frac{\varrho}{|B(x,\sqrt{t})|}\int_{\{\rho_X(y)>2R\}} \exp\left(-\frac{d_X^2(x,y)}{\varrho t}+\mu\rho_X^2(y)\right)|f(y)|\,\exp\big(-\mu\rho_X^2(y)\big)\,\mathrm{d}y.
\end{aligned} \tag{6.6}$$

On the other hand, for every $x,y\in\mathbb{R}^n$ satisfying $\rho_X(x)<R$ and $\rho_X(y)>2R$, one has

$$d_X(x,y)\geq \rho_X(y)-\rho_X(x)\geq \frac{\rho_X(y)}{2};$$

as a consequence, we get

$$\exp\left(-\frac{d_X^2(x,y)}{\varrho t}+\mu\rho_X^2(y)\right)\leq \exp\left(-\rho_X^2(y)\left(\frac{1}{4\varrho t}-\mu\right)\right)\leq 1, \tag{6.7}$$

as soon as $x\in\{\rho_X<R\}$ and $\frac{1}{4\varrho t}-\mu>0$, that is (setting $T_1:=1/(4\varrho)$)

$$t<\frac{T_1}{\mu}.$$

Gathering together all these facts, for fixed $(t,x)\in S_{T_1/\mu,R}$ we obtain

$$|u_2(t,x)|\leq c_{t,x}\int_{\mathbb{R}^n}|f(y)|\,\exp\big(-\mu\rho_X^2(y)\big)\,\mathrm{d}y<+\infty.$$

Now, using the Gaussian estimates (2.7) for the derivatives of γ, and arguing exactly as above, one can easily prove that $u_2\in C^2(S_{T/\mu,R})$ and $\mathcal{H}u_2=0$ on $S_{T/\mu,R}$, where

$$T:=\min\left\{T_1,\frac{1}{4C}\right\}\quad\text{and } C \text{ is as in (2.7).} \tag{6.8}$$

Next, we show that for $t\to 0^+$ we have $u_2(t,x)\to 0$ if $x\in\mathbb{R}^n$ satisfies $\rho_X(x)<R$. To this end we first observe that, by (3.15), for every $t>0$ and $x\in\mathbb{R}^n$ we have

$$|B_X(x,\sqrt{t})|\geq \gamma_1\sum_{h=n}^{q} f_h(y)\,t^{h/2}\geq \gamma_1\,f_q\,t^{q/2}=\kappa_q\,t^{q/2}, \tag{6.9}$$

with $\kappa_q:=\gamma_1 f_q$ (remind that $f_n,\ldots,f_q\geq 0$ and f_q is a positive constant). As a consequence, by combining (6.9), (6.6) and (6.7), for every $(t,x)\in S_{T/\mu,R}$ we obtain the estimate

$$|u_2(t,x)|\leq c\int_{\{\rho_X(y)>2R\}}\frac{e^{-\rho_X^2(y)\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}\cdot|f(y)|\,\exp\big(-\mu\rho_X^2(y)\big)\,\mathrm{d}y.$$

We are going to show that, by Lebesgue's theorem, the last integral goes to zero as $t\to 0^+$. On the one hand, for every fixed $y\in\mathbb{R}^n$ with $\rho_X(y)>2R$, we have

$$\lim_{t\to 0^+}\left(\frac{e^{-\rho_X^2(y)\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}\cdot|f(y)|\,e^{-\mu\rho_X^2(y)}\right)\leq c_y\cdot\lim_{t\to 0^+}\frac{e^{-4R^2\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}=0.$$

On the other hand, for every $t>0$ and every $y\in\mathbb{R}^n$ satisfying $\rho_X(y)>2R$, one has

$$\frac{e^{-\rho_X^2(y)\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}\cdot|f(y)|\,e^{-\mu\rho_X^2(y)}\leq\frac{e^{-4R^2\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}\cdot|f(y)|\,e^{-\mu\rho_X^2(y)}$$
$$=\sup_{t>0}\left(\frac{e^{-4R^2\left(\frac{1}{4\varrho t}-\mu\right)}}{t^{q/2}}\right)\cdot|f(y)|\,e^{-\mu\rho_X^2(y)}\equiv c\,|f(y)|\,e^{-\mu\rho_X^2(y)}\in L^1(\mathbb{R}^n),$$

and thus, by Lebesgue's theorem, we conclude that

$$\lim_{t\to0^+}u_2(t,x)=0\qquad\text{for every } x\in\mathbb{R}^n \text{ with } \rho_X(x)<R.$$

Summing up, we have proved that u_2 satisfies (i)-to-(iii) on $S_{T/\mu,R}$, as desired.

Finally, due to the arbitrariness of $R>0$, we then conclude that u is a classical solution of problem (6.1) on the stripe $S_{T/\mu}$ (with T as in (6.8)).

STEP II. Let us show that u satisfies a bound (6.4) for some $\delta,\tau>0$. Since u is a continuous function on the stripe $S_{T/\mu}$, the integral

$$\int_0^\tau\int_{\{\rho_X(x)\leq R\}}|u(t,x)|\,\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}t\,\mathrm{d}x$$

is finite for every choice of $\delta,R>0$ and every $0<\tau<T/\mu$. So, it is enough to show that there exist suitable $\delta\in(0,+\infty)$ and $0<\tau<T/\mu$ such that

$$\int_0^\tau\int_{\{\rho_X(x)>1\}}|u(t,x)|\,\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}t\,\mathrm{d}x<+\infty.$$

By the very definition of u in (6.3), we have

$$\int_0^\tau\int_{\{\rho_X(x)>1\}}|u(t,x)|\,\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq\int_0^\tau\int_{\{\rho_X(x)>1\}}\left(\int_{\mathbb{R}^n}\gamma(t,x,y)\,|f(y)|\,\mathrm{d}y\right)\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}t\,\mathrm{d}x.$$

We then split the space integral as follows

$$\int_{\{\rho_X(x)>1\}}\left(\int_{\mathbb{R}^n}\gamma(t,x,y)\,|f(y)|\,\mathrm{d}y\right)\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}x$$
$$=\int_{\{\rho_X(x)>1\}}\left(\int_{\{\rho_X(y)\geq2\rho_X(x)\}}\gamma(t,x,y)\,|f(y)|\,\mathrm{d}y\right)\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}x$$
$$+\int_{\{\rho_X(x)>1\}}\left(\int_{\{\rho_X(y)<2\rho_X(x)\}}\gamma(t,x,y)\,|f(y)|\,\mathrm{d}y\right)\exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}x$$
$$\equiv A(t)+B(t).$$

As for $A(t)$, by combining the Gaussian estimate (2.6) with (6.9), we get

$$A(t)\leq\frac{c\varrho}{t^{q/2}}\int_{\{\rho_X(x)>1\}}\left(\int_{\{\rho_X(y)\geq2\rho_X(x)\}}e^{-\frac{d_X^2(x,y)}{\varrho t}+\mu\rho_X^2(y)}\cdot e^{-\mu\rho_X^2(y)}\,|f(y)|\,\mathrm{d}y\right)\cdot e^{-\delta\rho_X^2(x)}\,\mathrm{d}x;$$

moreover, using the fact that $d_X(x,y)\geq\rho_X(y)-\rho_X(x)$ for every $x,y\in\mathbb{R}^n$, one has

$$\exp\left(-\frac{d_X^2(x,y)}{\varrho t}+\mu\rho_X^2(y)\right)\leq\exp\left(\frac{\rho_X^2(x)}{\varrho t}\right)\cdot\exp\left(-\rho_X^2(y)\Big(\frac{1}{2\varrho t}-\mu\Big)\right)=(\bigstar).$$

As a consequence, since in $A(t)$ we have $\rho_X(y)\geq2\rho_X(x)$ and $\rho_X(x)>1$, we obtain

$$(\bigstar)\leq\exp\left(\frac{\rho_X^2(x)}{\varrho t}-4\rho_X^2(x)\Big(\frac{1}{2\varrho t}-\mu\Big)\right)=\exp\left(-\rho_X^2(x)\Big(\frac{1}{\varrho t}-4\mu\Big)\right)$$
$$\leq\exp\left(-\Big(\frac{1}{\varrho t}-4\mu\Big)\right),$$

provided that $t \in (0, T/\mu)$, see (6.8). Using this last estimate, we get

$$\begin{aligned} A(t) &\leq \frac{c\varrho}{t^{q/2}}\, e^{-\left(\frac{1}{\varrho t} - 4\mu\right)} \left(\int_{\mathbb{R}^n} |f(y)| \exp\big(-\mu\rho_X^2(y)\big)\,\mathrm{d}y\right)\left(\int_{\mathbb{R}^n} \exp\big(-\delta\rho_X^2(x)\big)\,\mathrm{d}x\right) \\ &= \frac{c'}{t^{q/2}} e^{-\left(\frac{1}{\varrho t} - 4\mu\right)} \end{aligned}$$

where we have exploited Lemma 6.3. From this, we finally obtain

$$\int_0^\tau A(t)\,\mathrm{d}t \leq \int_0^\tau \frac{c_1}{t^{q/2}} e^{-\left(\frac{1}{\varrho t} - 4\mu\right)}\,\mathrm{d}t < +\infty, \quad \text{for any } \tau \in (0, T/\mu) \text{ and any } \delta > 0.$$

As for $B(t)$, since $\rho_X(y) < 2\rho_X(x)$, we have

$$\begin{aligned} B(t) &= \int_{\{\rho_X(x)>1\}} \left(\int_{\{\rho_X(y)<2\rho_X(x)\}} \gamma(t,x,y)\, e^{\mu\rho_X^2(y)} \cdot |f(y)|\, e^{-\mu\rho_X^2(y)}\,\mathrm{d}y\right) e^{-\delta\rho_X^2(x)}\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} |f(y)| \exp\big(-\mu\rho_X^2(y)\big)\left(\int_{\mathbb{R}^n} \gamma(t,x,y)\, e^{(4\mu-\delta)\rho_X^2(x)}\,\mathrm{d}x\right)\mathrm{d}y. \end{aligned}$$

Thus, if we choose $\delta \geq 4\mu$, from Theorem 3.4-(iv) and (vi) we obtain

$$\int_{\mathbb{R}^n} \gamma(t,x,y)\, e^{-(\delta-4\mu)\rho_X^2(x)}\,\mathrm{d}x \leq \int_{\mathbb{R}^n} \gamma(t,x,y)\,\mathrm{d}x = 1.$$

As a consequence, we get

$$B(t) \leq \int_{\mathbb{R}^n} |f(y)|\, e^{-\mu\rho_X^2(y)}\,\mathrm{d}y =: c < +\infty,$$

from which we derive that

$$\int_0^\tau B(t)\,\mathrm{d}t < +\infty \qquad \text{for any } \tau \in (0, T/\mu) \text{ and any } \delta \geq 4\mu.$$

Summing up, we conclude that u satisfies (6.4) for every $\delta \geq 4\mu$ and every $\tau \in (0, T/\mu)$.

Step III. Let us prove the uniqueness result. By linearity, it is enough to show that if for some $\tau > 0$ the function $u \in C^2(S_\tau)$ is a classical solution of

$$\text{(6.10)} \qquad \begin{cases} \mathcal{H}u = 0 & \text{in } S_\tau, \\ u(0,x) = 0 & \text{for } x \in \mathbb{R}^n \end{cases}$$

and satisfies (6.4), then $u \equiv 0$ on S_τ. Denoting again by π_n the projection of $\mathbb{R}^N$ onto $\mathbb{R}^n$, we set

$$\widehat{u} : \widehat{S}_\tau := (0,\tau) \times \mathbb{R}^N \to \mathbb{R}, \qquad \widehat{u}(t,z) := u\big(t, \pi_n(z)\big).$$

Obviously, $\widehat{u} \in C^2(\widehat{S}_\tau)$; moreover, since u solves (6.10) and $\mathcal{H}_{\mathbb{G}} = \sum_{j=1}^m Z_j^2 - \partial_t$ is a lifting of $\mathcal{H}$ (see (3.9)), it is easy to check that $\widehat{u}$ is a classical solution of

$$\text{(6.11)} \qquad \begin{cases} \mathcal{H}_{\mathbb{G}}\widehat{u} = 0 & \text{in } \widehat{S}_\tau, \\ \widehat{u}(0,z) = 0 & \text{for } z \in \mathbb{R}^N. \end{cases}$$

We claim that there exists $\widehat{\delta} > 0$ such that

$$\text{(6.12)} \qquad \int_{\widehat{S}_\tau} \exp\big(-\widehat{\delta}\,\|z\|^2\big)\,|\widehat{u}(t,z)|\,\mathrm{d}t\,\mathrm{d}z < +\infty.$$

Once this is proved, by [9, Theorem 6.5] we derive that $\widehat{u} \equiv 0$ on $\widehat{S}_\tau$, and thus $u \equiv 0$ on S_τ.

To prove (6.12), let $\widehat{\nu} > 0$ to be fixed in a moment. By using Proposition 4.2 (with $x = 0$ and $t = \widehat{\delta}^{-1} > 0$), we obtain the following computation

$$\begin{aligned}\int_{\widehat{S}_\tau} \exp\big(-\widehat{\delta}\,\|z\|^2\big)\,|\widehat{u}(t,z)|\,\mathrm{d}t\,\mathrm{d}z &= \int_{\widehat{S}_\tau} \exp\big(-\widehat{\delta}\,\|(x,\xi)\|^2\big)\,|\widehat{u}(t,(x,\xi))|\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_{S_\tau}\left(\int_{\mathbb{R}^p} \exp\big(-\widehat{\delta}\,\|(x,\xi)\|^2\big)\,\mathrm{d}\xi\right)|u(t,x)|\,\mathrm{d}t\,\mathrm{d}x \qquad (6.13)\\ &\le \frac{\kappa}{\widehat{\delta}^{Q/2}\,|B_X(0,\widehat{\delta}^{-1/2})|}\int_{S_\tau}\exp\left(-\frac{\widehat{\delta}\,\rho_X^2(x)}{\kappa}\right)|u(t,x)|\,\mathrm{d}t\,\mathrm{d}x,\end{aligned}$$

for a suitable constant $\kappa > 1$. As a consequence, if we choose $\widehat{\delta} := \delta\cdot\kappa$, with δ as in (6.4), from (6.13) we immediately deduce the claimed (6.12). This ends the proof. □

Remark 6.4. In the special case of Carnot groups, the *uniqueness* part of our result was already known, after [9, Thm. 6.5], and in our proof (Step III) we have explicitly exploited that result, relying on the assumption (H3) and the lifting technique. On the other hand, in the proof of our existence result (Steps I-II) we have never exploited assumption (H3) and the lifing technique. Actually, our proof in Steps I-II works also in Carnot groups, and our existence result extends the one proved in [9, Corollary 6.2], where a stronger pointwise (instead of integral) bound was assumed on f.

If the initial datum satisfies a slightly stronger assumption than (6.2), we can refine the previous results getting existence and uniqueness of the solution for every $t > 0$:

Proposition 6.5. *Let $f \in C(\mathbb{R}^n)$ satisfy the growth assumption* (6.2) *in the following stronger form:* there exist $\alpha \in (0,2)$ and $\mu > 0$ such that

$$\int_{\mathbb{R}^n} |f(y)|\,\exp\big(-\mu\rho_X^\alpha(y)\big)\,\mathrm{d}y < +\infty. \qquad (6.14)$$

Then, the function u defined by (6.3) *is a classical solution of* (6.1) *on $S_\infty := (0,+\infty)\times\mathbb{R}^n$.*

Proof. Using assumption (6.14), it is easy to see that for *every* fixed $\theta > 0$ one has

$$\int_{\mathbb{R}^n} |f(y)|\,\exp\big(-\theta\rho_X^2(y)\big)\,\mathrm{d}y < +\infty. \qquad (6.15)$$

As a consequence, from Theorem 6.2 we derive that the function u in (6.3) is a classical solution of (6.1) on $S_{T/\theta}$ for every $\theta > 0$, hence on the whole of S_∞. □

7. An application to the Dirichlet problem for $\mathcal{H}$

The aim of this section is to show how our global Gaussian estimates for Γ can be used to study the solvability of the $\mathcal{H}$-Dirichlet problem on an *arbitrary bounded domain* $\Omega \subseteq \mathbb{R}^{1+n}$. All the results we are going to present basically follow by combining the results of the previous sections with the investigations carried out (in an abstract framework) in [20, 23, 24, 35].

To begin with, we need to establish the following proposition.

Proposition 7.1. *The CC distance d_X associated with our system $X = \{X_1,\ldots,X_m\}$ of homogeneous Hörmander's vector fields satisfies the so-called* segment property: *for every fixed $x, y \in \mathbb{R}^n$ there exists a continuous path $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$ and*

$$d_X(x,y) = d_X\big(x,\gamma(t)\big) + d_X\big(\gamma(t),y\big) \qquad \textit{for all } 0 \le t \le 1.$$

Proof. This fact has been proved in [8, Corollary 5.15.6] in the context of Carnot groups. Actually, the same proof can be repeated in our setting; the only nontrivial point that must be checked is that the d_X-balls $B_X(x,\rho)$ are bounded in the Euclidean sense (for all $x \in \mathbb{R}^n$ and $\rho > 0$).

To prove this fact, we argue as follows. First of all, since the distance d_X is topologically equivalent to the Euclidean distance, there exists some $r > 0$ such that the Euclidean ball $B_E(0,1)$ contains the d_X-ball $B_X(0,r)$. On the other hand, for every $R > 0$ we have

$$\delta_{r/R}\big(B_X(0,R)\big) = B_X\,(0,r) \subseteq B_E(0,1);$$

hence, $\delta_{r/R}(B_X(0,R))$ is bounded in the Euclidean sense and, by the explicit form of $\delta_{r/R}$, the same is true for $B_X(0,R)$. From this, since for any $x \in \mathbb{R}^n$ and $\rho > 0$ we have $B_X(x,\rho) \subseteq B_X(0,R)$, with $R = \rho + d_X(x,0)$, we conclude that every d_X-ball is bounded in the Euclidean sense. $\square$

Using the segment property of d_X, jointly with the properties of Γ listed in Theorem 3.5 and the global Gaussian estimates (2.6) in Theorem 2.4, we can apply the axiomatic approach developed in [23]: denoting by H the sheaf of functions defined as

$$\Omega \mapsto H(\Omega) := \big\{u \in C^\infty(\Omega) : \mathcal{H}u = 0 \text{ in } \Omega\big\},$$

we have that $(\mathbb{R}^n, H)$ is a β-harmonic space satisfying the Doob convergence property. In this context, given a fixed open set $\Omega \subseteq \mathbb{R}^{1+n}$, we say that

- a function $u : \Omega \to \mathbb{R}$ is *$\mathcal{H}$-harmonic* in Ω if $u \in H(\Omega)$;
- a function $u : \Omega \to (-\infty, +\infty]$ is *$\mathcal{H}$-superharmonic in* Ω if
  - (a) u is lower semi-continuous (l.s.c., for short) in Ω;
  - (b) the set $\{x \in \Omega : u(x) < +\infty\}$ is dense in Ω;
  - (c) for every $v \in C(\overline{\Omega})$ such that $v\big|_\Omega \in H(\Omega)$ and $v \le u$ on $\partial\Omega$ one has $v \le u$ on Ω.
- a function $u : \Omega \to [-\infty, +\infty)$ is *$\mathcal{H}$-subharmonic in* Ω if $-u$ is $\mathcal{H}$-superharmonic in Ω.

We denote by $\overline{H}(\Omega)$ (resp. $\underline{H}(\Omega)$) the (convex) cone of the $\mathcal{H}$-superharmonic (resp. $\mathcal{H}$-subharmonic) functions in Ω. Obviously, we have $\underline{H}(\Omega) = -\overline{H}(\Omega)$ and $\overline{H}(\Omega) \cap \underline{H}(\Omega) = H(\Omega)$.

Let now $\Omega \subseteq \mathbb{R}^{1+n}$ be a fixed open set, and let $\varphi \in C(\partial\Omega)$. We say that a function $u : \Omega \to \mathbb{R}$ is a *classical solution* of the $\mathcal{H}$-Dirichlet problem

$$\begin{cases} \mathcal{H}u = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = \varphi \end{cases} \tag{7.1}$$

if it satisfies the following properties:

- $u \in C(\overline{\Omega})$ and $u\big|_\Omega \in C^2(\Omega)$;
- $\mathcal{H}u = 0$ in Ω and $u\big|_{\partial\Omega} = \varphi$.

Since $\mathcal{H} = \mathcal{L} - \partial_t$ satisfies the Weak Maximum Principle on every open subset of $\mathbb{R}^{1+n}$ (see, e.g., [3, Example 8.20]), there exists at most one classical solution of the Dirichlet problem (7.1); however, the *existence* of such a solution for a general $\varphi \in C(\partial\Omega)$ is not guaranteed. For this reason, we introduce the so-called *PerronâĂŞWienerâĂŞBrelotâĂŞBauer* (PWBB, in short) solution of (7.1).

Following [23], we first consider the functions

$$\overline{H}^\Omega_\varphi(x) := \inf\big\{u(x) : u \in \overline{H}(\Omega) \text{ and } \liminf_{\omega\to\omega_0} u(\omega) \ge \varphi(\omega_0) \text{ for all } \omega_0 \in \partial\Omega\big\} \qquad \text{and}$$

$$\underline{H}^\Omega_\varphi(x) := \sup\big\{u(x) : u \in \underline{H}(\Omega) \text{ and } \limsup_{\omega\to\omega_0} u(\omega) \le \varphi(\omega_0) \text{ for all } \omega_0 \in \partial\Omega\big\}.$$

Then, since $(\mathbb{R}^{1+n}, H)$ satisfies Doob'sconvergence property, it can be proved that

$$\overline{H}^\Omega_\varphi \equiv \underline{H}^\Omega_\varphi =: H^\Omega_\varphi \in H(\Omega).$$

We shall call this function the PWBB solution of (7.1). Obviously, if u is the classical solution of (7.1), one has $u \equiv H^\Omega_\varphi$ on Ω; on the other hand, even if H^Ω_φ can be constructed for an arbitrary $\varphi \in C(\partial\Omega)$ and it is always $\mathcal{H}$-harmonic in Ω, one cannot expect (in general) that

$$\lim_{\omega\to\omega_0} H^\Omega_\varphi(\omega) = \varphi(\omega_0) \qquad \text{for } \omega_0 \in \partial\Omega.$$

The following definition is thus plainly justified.

Definition 7.2. A point $\omega_0 \in \partial\Omega$ is called *$\mathcal{H}$-regular* if

$$\lim_{\omega\to\omega_0} H^\Omega_\varphi(\omega) = \varphi(\omega_0) \qquad \text{for all } \varphi \in C(\partial\Omega). \tag{7.2}$$

Due to the validity of the segment property for d_X, the 'good' behavior of Γ in Theorem 3.5, and the validity of *global* Gaussian estimates for Γ, we are entitled to apply to our context all the abstract results established in [20, 23, 24, 35]. As a consequence, we obtain several necessary/sufficient conditions for a point $\omega_0 \in \partial\Omega$ to be $\mathcal{H}$-regular (in the sense of Definition 7.2).

Throughout the sequel, given any compact set $K\subseteq\mathbb{R}^{1+n}$, we define

$$\begin{aligned} &V_K(\omega)=\liminf_{z\to\omega}\big(W_K(z)\big), \qquad \text{where} \\ &W_K(z):=\inf\big\{v(z):\,v\in\overline{H}(\mathbb{R}^n), v\geq 0 \text{ on } \mathbb{R}^{1+n} \text{ and } v\geq 1 \text{ on } K\big\}. \end{aligned} \tag{7.3}$$

The function V_K is usually referred to as the $\mathcal{H}$-balayage of $u_0\equiv 1$ on K.

Theorem 7.3. [23, Thm.s 4.6 and 4.11] *Let* $\Omega\subseteq\mathbb{R}^{1+n}$ *be* a bounded open set, *and let* $\omega_0=(t_0,x_0)$ *be a fixed point of* $\partial\Omega$. *For any* $r>0$, *we define*

$$\Omega_r'(\omega_0):=\big\{\omega=(t,x)\in\mathbb{R}^{1+n}\setminus\Omega:\,t\leq t_0,\ \big(d_X(x,x_0)^4+|t-t_0|^2\big)^{1/4}\leq r\big\},$$

and we denote by V_r *the so-called* $\mathcal{H}$*-balayage of* $u_0\equiv 1$ *on* $\Omega_r'(\omega_0)$, *that is,*

$$V_r:=V_{\Omega_r'(\omega_0)}. \tag{7.4}$$

Then, following assertions are equivalent:

- ω_0 is not $\mathcal{H}$*-regular;*
- *there exists* $r>0$ *such that* $V_r(\omega_0)<1$*;*
- $V_r(\omega)\to 0$ *as* $r\to 0^+$.

On the other hand, if there exist real constants $M,\rho,\theta>0$ *such that*

$$\big|\big\{x\in\overline{B_X(x_0,M\rho)}:\,(t_0-\rho^2,x)\notin\Omega\big\}\big|\geq\theta\,|B_X(x_0,M\rho)|,$$

then ω_0 *is* $\mathcal{H}$*-regular.*

Another *sufficient* condition for $\mathcal{H}$-regularity is the following.

Theorem 7.4. [20, Theorem 5.1] *Let* $\Omega\subseteq\mathbb{R}^{1+n}$ *be an open set, and let* $\omega_0=(t_0,x_0)\in\partial\Omega$ *be fixed. Moreover, let* $\{B_\lambda\}_{0<\lambda<1}$ *be a basis of closed neighborhoods of* x_0 *in* $\mathbb{R}^n$ *such that*

$$B_\lambda\subseteq B_\mu \text{ if } 0<\lambda<\mu\leq 1.$$

For every $\lambda\in(0,1)$, *we define*

$$\Omega_\lambda^c(\omega_0):=\big([t_0-\lambda,t_0]\times B_\lambda\big)\setminus\Omega \quad \text{and} \quad T_\lambda(\omega_0):=\big\{x\in\mathbb{R}^n:\,(t_0-\lambda,x)\in\Omega_\lambda^c(\omega_0)\big\}.$$

Then the point ω_0 *is* $\mathcal{H}$*-regular if*

$$\limsup_{\lambda\searrow 0^+}\int_{T_\lambda(\omega_0)}\gamma(\lambda,x_0,\xi)\,\mathrm{d}\xi>0.$$

By making use of the so-called $\mathcal{H}$-Wiener function (associated with the open set Ω and the point $\omega_0\in\partial\Omega$), it is possible to derive a *necessary and sufficient* condition for ω_0 to be regular.

Theorem 7.5. [23, Theorem 5.4] *Let* $\Omega\subseteq\mathbb{R}^{1+n}$ *be a bounded open set, and let* $\omega_0=(t_0,x_0)\in\partial\Omega$ *be fixed. Moreover, given a number* $p>0$ *and a sequence* $\{r_k\}_{k\in\mathbb{N}}$ *converging to* 0 *as* $k\to+\infty$, *we define the* $\mathcal{H}$-Wiener function (*associated with* Ω *and* ω_0) *as*

$$\mathcal{W}(\omega):=\sum_{k=1}^{+\infty}\frac{1-V_k(\omega)}{p^k}, \tag{7.5}$$

where $V_k=V_{r_k}$ *and, for every* $r>0$, *the function* V_r *is as in* (7.4). *Then*

$$\omega_0 \text{ is } \mathcal{H}\text{-regular if and only if } \mathcal{W}(\omega)\to 0 \text{ as } \omega\to\omega_0.$$

Finally, by making explicit use of our global Gaussian estimates for Γ, we can obtain criteria for $\mathcal{H}$-regularity which are resemblant to the classical results proved by Wiener and Landis for the heat operator $\Delta-\partial_t$. In order to clearly state these criteria, we first fix some notation.

Given a compact set $K\subseteq\mathbb{R}^{1+n}$, let V_K be the $\mathcal{H}$-balayage of $u_0\equiv 1$ on K defined in (7.3). By classical results of Potential Theory, it is known that V_K is $\mathcal{H}$-superharmonic on $\mathbb{R}^{1+n}$; as a consequence, there exists a unique positive Radon measure $\mu=\mu_K$ on $\mathbb{R}^{1+n}$ such that

$$\mathcal{H}V_k=-\mu_K \text{ in } \mathcal{D}'(\mathbb{R}^{1+n}) \qquad \text{and} \qquad \operatorname{supp}(\mu_K)=K.$$

(see, e.g., [29]). We then define the $\mathcal{H}$-*capacity* of K as follows

$$\mathcal{C}_{\mathcal{H}}(K):=\mu_K(K).$$

Moreover, if $\mathcal{M}^+(K)$ denotes the set of non-negative Radon measures on $\mathbb{R}^{1+n}$ with support contained in K, we also define the *a-Gaussian capacity* of K as follows

$$\mathcal{C}_a(K) := \sup\left\{\nu(K) :\ \nu \in \mathcal{M}^+(K) \text{ and } \int_K G_a(t,x;s,y)\,d\mu(s,y) \leq 1 \text{ for all } (t,x) \in R^{1+n}\right\},$$

where for every $a > 0$ we have used the notation

$$G_a(t,x;s,y) := \begin{cases} 0, & \text{if } t \leq s, \\ \dfrac{1}{|B_X(x,\sqrt{t-s})|}\exp\left(-a\,\dfrac{d_X^2(x,y)}{t-s}\right), & \text{if } t > s. \end{cases} \tag{7.6}$$

Notice that, using (7.6), our Gaussian estimates (2.6) reads as

$$\frac{1}{\varrho}\,G_\varrho(t,x;s,y) \leq \Gamma(t,x;s,y) \leq \varrho\, G_{1/\varrho}(t,x;s,y) \qquad (\text{for all } (t,x),(s,y) \in \mathbb{R}^{1+n}).$$

Here is a 'Wiener-type' test for $\mathcal{H}$-regularity.

Theorem 7.6. [24, Theorem 1.1] *Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0,x_0) \in \partial\Omega$. For every fixed $\lambda \in (0,1)$ and every $h,k \in \mathbb{N}$, we define*

$$\Omega_k^h(\omega_0,\lambda) := \Big\{\omega = (t,x) \in \mathbb{R}^{1+n}\setminus\Omega :\ \lambda^{k+1} \leq t_0 - t \leq \lambda^k,$$
$$\frac{1}{\lambda^{h-1}} \leq \exp\left(\frac{d_X^2(x_0,x)}{t_0-t}\right) \leq \frac{1}{\lambda^h},\ \left(d_X(x,x_0)^4 + |t-t_0|^2\right)^{1/4} \leq \sqrt{\lambda}\Big\}.$$

Then, if $\varrho > 0$ is as in (2.6), the following facts hold.

- *if there exist $0 < a \leq 1/\varrho$ and $b > \varrho$ such that*

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_a\big(\Omega_k^h(\omega_0,\lambda)\big)}{\big|B_X(x_0,\lambda^{k/2})\big|}\,\lambda^{bh} = +\infty,$$

  then the point ω_0 is $\mathcal{H}$-regular.
- *If the point ω_0 is $\mathcal{H}$-regular, then*

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_b\big(\Omega_k^h(\omega_0,\lambda)\big)}{\big|B_X(x_0,\lambda^{k/2})\big|}\,\lambda^{ah} = +\infty,$$

  for every $0 < a \leq 1/\varrho$ and $b \geq \varrho$.

Finally, a 'Landis-type' condition for $\mathcal{H}$-regularity is given by the following theorem.

Theorem 7.7. [35, Theorem 1.3] *Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0,x_0) \in \partial\Omega$. For every fixed $\lambda \in (0,1)$ and every $k \in \mathbb{N}$, we consider the set*

$$\Omega_k^c(\omega_0) := \left\{\omega = (t,x) \in \mathbb{R}^{1+n}\setminus\Omega :\ \frac{1}{\lambda^{k\log(k)}} \leq \Gamma(t_0,x_0;t,x) \leq \frac{1}{\lambda^{(k+1)\log(k+1)}}\right\} \cup \{(t_0,x_0)\},$$

where Γ is the global heat kernel of $\mathcal{H}$. Then ω_0 is $\mathcal{H}$-regular if and only if

$$\sum_{k=1}^{+\infty} V_{\Omega_k^c(\omega_0)}(\omega_0) = +\infty,$$

where $V_{\Omega_k^c(\omega_0)}$ is the $\mathcal{H}$-balayage of $u_0 \equiv 1$ on $\Omega_k^c(\omega_0)$, see (7.3).

8. Scale-invariant Harnack inequality for $\mathcal{H}$

In this last section we prove a scale-invariant Harnack inequality for non-negative solutions of $\mathcal{H}u = 0$. This fact easily follows, via the lifting procedure, from the analogous result proved on Carnot groups in [9, Corollary 4.5]. It is however a result which is worthwhile to be pointed out.

Given any point $\omega_0 = (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n$, any number $r > 0$, we define

$$C(\omega_0, r) := \big\{(t, x) \in \mathbb{R}^{1+n} : \, d_X(x, x_0) < r, \; |t - t_0| < r^2\big\}.$$

Furthermore, for every $\lambda \in (0, 1/2)$, we set

$$S_\lambda(\omega_0, r) := \big\{(t, x) \in \mathbb{R}^{1+n} : \, d_X(x, x_0) < (1 - \lambda)r, \; \lambda r^2 < t_0 - t < (1 - \lambda)r^2\big\}.$$

We are ready to state our result.

Theorem 8.1. *For every h, $k = 0, 1, 2, \dots$ and every fixed $\lambda \in (0, 1/2)$, it is possible to find a positive constant $\nu = \nu_{h,k,\lambda} > 0$ such that, for every $\omega_0 = (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n$, every $r > 0$, and every nonnegative function $u \in C^2(C(\omega_0, r))$ satisfying $\mathcal{H}u = 0$ on $C(\omega_0, r)$,*

$$\sup_{S_\lambda(\omega_0,r)} \big|X_{i_1} \cdots X_{i_h} (\partial_t)^k u\big| \le \nu \, r^{-(h+2k)} \, u(\omega_0), \tag{8.1}$$

for every $i_1, \dots, i_h \in \{1, \dots, m\}$.

Proof. Letting $v_0 := (x_0, 0) \in \mathbb{R}^N$ and $\widehat{\omega}_0 := (t_0, v_0) \in \mathbb{R}^{1+N}$, we define

$$\widehat{C}(\widehat{\omega}_0, r) := \big\{(t, v) \in \mathbb{R}^{1+N} : \, d_{\mathcal{Z}}(v, v_0) < r, \; |t - t_0| < r^2\big\} \qquad \text{and}$$

$$\widehat{S}_\lambda(\widehat{\omega}_0, r) := \big\{(t, v) \in \mathbb{R}^{1+N} : \, d_{\mathcal{Z}}(v, v_0) < (1 - \lambda)r, \; \lambda r^2 < t_0 - t < (1 - \lambda)r^2\big\}.$$

Let then $u \in C^2(C(\omega_0, r))$ be any non-negative function satisfying of $\mathcal{H}u = 0$ on $C(\omega_0, r)$. Denoting by $\pi_n : \mathbb{R}^N \to \mathbb{R}^n$ the canonical projection of $\mathbb{R}^N$ onto $\mathbb{R}^n$, we set

$$\widehat{u}(t, v) := u\big(t, \pi_n(v)\big) \qquad (v \in \mathbb{R}^N).$$

Since $B_{\mathcal{Z}}(v_0, r) \subseteq \pi_n^{-1}\big(B_X(x_0, r)\big)$ (see Proposition 3.7-(iii)), we have

$$\widehat{u} \in C^2\big(\widehat{C}(\widehat{\omega}_0, r)\big).$$

Moreover, since $u \ge 0$ and $\mathcal{H}u = 0$ on $C(\omega_0, r)$, from the lifting property (3.9) we derive that

$$\widehat{u} \ge 0 \quad \text{and} \quad \mathcal{H}_{\mathbb{G}}\widehat{u} = 0 \qquad \text{on } \widehat{C}(\widehat{\omega}_0, r).$$

Putting together these facts, we are entitled to apply [9, Corollary 4.5], obtaining

$$\sup_{\widehat{S}_\lambda(\widehat{\omega}_0,r)} \big|Z_{i_1} \cdots Z_{i_h} (\partial_t)^k \widehat{u}\big| \le \nu \, r^{-(h+2k)} \, \widehat{u}(t_0, v_0), \tag{8.2}$$

where $\nu > 0$ is an absolute constant only depending on h, k and λ. We now claim that the above (8.2) is precisely the desired (8.1). In fact, by the very definition of $\widehat{u}$, we have

$$\widehat{u}(t_0, v_0) = u(t_0, x_0) = u(\omega_0); \tag{8.3}$$

moreover, by repeatedly exploiting (3.4), we get

$$\begin{aligned} Z_{i_1} \cdots Z_{i_h} (\partial_t)^k \widehat{u}(t, v) &= (\partial_t)^k \Big(Z_{i_1} \cdots Z_{i_h}\big(v \mapsto u(t, \pi_n(v))\big)\Big) \\ &= (\partial_t)^k \Big(Z_{i_1} \cdots Z_{i_{h-1}}\big(v \mapsto (X_{i_h}u)(t, \pi_n(v))\big)\Big) \\ &= \ldots = \big((\partial_t)^k X_{i_1} \cdots X_{i_h} u\big)(t, \pi_n(v)) \qquad \text{for all } (t, v) \in \widehat{C}(\widehat{\omega}_0, r). \end{aligned}$$

From this, taking into account that $\pi_n\left(B_{\mathcal{Z}}(v_0, (1 - \lambda)r)\right) = B_X(x_0, (1 - \lambda)r)$, we readily obtain

$$\sup_{\widehat{S}_\lambda(\widehat{\omega}_0,r)} \big|Z_{i_1} \cdots Z_{i_h} (\partial_t)^k \widehat{u}\big| = \sup_{S_\lambda(\omega_0,r)} \big|X_{i_1} \cdots X_{i_h} (\partial_t)^k u\big|. \tag{8.4}$$

By combining (8.2), (8.3) and (8.4), we finally derive (8.1), with an absolute constant $\nu > 0$ which depends on the chosen h, k and λ (but not on ω_0, r nor u). This ends the proof. ∎

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