Calculus of Variations

A Liouville theorem for elliptic equations with a potential on infinite graphs

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Abstract

We investigate the validity of the Liouville property for a class of elliptic equations with a potential, posed on infinite graphs. Under suitable assumptions on the graph and on the potential, we prove that the unique bounded solution is $u \equiv 0$. We also show that on a special class of graphs the condition on the potential is optimal, in the sense that if it fails, then there exist infinitely many bounded solutions.

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1 Introduction

Let (G, ω, μ) be a fixed *infinite weighted* graph, with *edge-weight* ω and *node (or vertex) measure* μ. In this paper we study *bounded* solutions to elliptic equations with a potential of the following form:

$$\Delta u - Vu = 0 \quad \text{in } G, \tag{1.1}$$

where the potential V is a nonnegative function defined in G and Δ denotes the Laplace operator on G.

The uniqueness of solutions of Eq. (1.1) has been investigated by two of us in the recent paper [31]; in this paper, it is proved that $u \equiv 0$ is the only solution to Eq. (1.1), whenever u belongs to a certain $\ell_\varphi^p(G, \mu)$ space, where φ is a weight which tends to 0 at *infinity* and $p \in [1, +\infty)$. Also the case $u \in \ell^\infty(G, \mu)$ can be considered, provided that the graph

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satisfies a suitable property. In any case, an essential hypothesis for the arguments used in [31] is the existence of some $c_1 > 0$ such that

$$V(x) \geq c_1 \quad \text{for all } x \in G. \tag{1.2}$$

Hence, the aim of the present paper is to understand when $u \equiv 0$ is the unique *bounded* solution of problem (1.1), without supposing hypothesis (1.2). To do this, we have to use completely different methods than those exploited in [31].

We say that the *Liouville theorem (or property)* holds for Eq. (1.1), whenever $u \equiv 0$ is the only *bounded* solution of the same equation. Thus, in other terms, we are concerned with the validity of the Liouville theorem for Eq. (1.1).

Before describing our results and methods of proof, let us contextualize our problem within the literature. As it is well-known, many phenomena in various fields of applied sciences can be modeled by means of graphs (see, e.g., [5, 24, 29]). For this reason, partial differential equations posed on graphs have recently attracted the attention of many authors. In particular, qualitative properties of solutions to both elliptic and parabolic equations have been addressed, see e.g. [11–13, 15, 18, 22, 28, 36] and [1, 4, 6, 16, 17, 27, 33, 40, 43], respectively. Moreover, the monographs [10, 23, 34] are important contributions to this topic.

In [19] and in [21], under suitable assumptions on the graph, it is shown that the parabolic problem

$$\begin{cases} \partial_t u = \Delta u & \text{in } G \times (0, \infty) \\ u = 0 & \text{in } G \times \{0\} \end{cases}$$

has at most one solution fulfilling a suitable growth condition. An analogous result can be found in [4], where the time derivative is replaced by discrete differences. Furthermore, in [23, Theorem 12.15, Corollary 12.16, Theorem 12.17] it is shown that, under suitable assumptions on G, if u is a subsolution of Eq. (1.1) with $V \equiv 0$ which satisfies $u \in \ell^p(G, \mu)$ and $u \geq 0$, then u must be constant.

Similar uniqueness results have been established also on manifolds (see, e.g., the seminal L^p-Liouville theorems in [25, 26, 41, 42] and the papers [7–9, 32, 37]), in bounded domains of $\mathbb{R}^n$ (see [2, 3, 35, 38]), and for nonlocal operators (see [30, 39]).

Now, let d denote a *pseudo metric* on G, and let $B_r(x_0)$ be the ball centered at $x_0 \in G$ with radius r (see Sect. 2 below). Concerning the potential V, we suppose that

$$V(x) \geq c_0 \, d^{-\alpha}(x, x_0) \quad \text{for all } x \in G \setminus B_{R_0}(x_0),$$

for some $c_0 > 0$, $R_0 > 0$ and $\alpha \in [0, 1]$. Under this assumption on V we prove that, if there exists some constant $\Lambda \in (0, 1)$ such that

$$\sum_{x \in G \setminus B_1(x_0)} e^{-\Lambda d^\alpha(x, x_0)} \mu(x) < +\infty, \tag{1.3}$$

then $u \equiv 0$ is the only bounded solution of Eq. (1.1). Let us comment that, with respect to the results in [31], we removed hypothesis (1.2), thus allowing the potential V to vanish at infinity with a certain rate; the price to pay for considering such more general potentials is condition (1.3), the geometrical meaning of which will be described Remark 2.5 below.

In order to prove the above-mentioned result, we take inspiration from [7]. However, since in [7] the problem is posed on a Riemannian manifold, many ideas used in that setting cannot be exploited on graphs, hence important differences arise. More precisely, the line of arguments we follow to show our main result is the following: introducing the function

$$v(x, t) := e^t u(x) - 1 \quad \text{for all } x \in G, \ t \in [0, T],$$

we first show that the positive part of v, namely v_+, is a subsolution of Eq. (1.1) in an appropriate sense (see Lemma 4.2). Then, we obtain a key *a priori* estimate for v_+ (see Proposition 4.1), where test functions $\xi = \xi(x, t)$ and $\eta = \eta(x)$ are employed. Then we select a suitable function ξ, which can be regarded as a sort of supersolution of an "adjoint parabolic equation" (see Lemma 5.1). On the other hand, η will be chosen to be a "cut-off" function (see Lemma 5.2). Then the conclusion follows by means of appropriate estimates.

A key point in our strategy of proof of uniqueness is to show that if u is a bounded solution of Eq. (1.1), then there exists a solution v to the same problem such that $0 < v \le 1$ (see Proposition 3.1). To show this, we first need to establish a weak maximum principle, see Lemma 3.3, and a strong maximum principle, see Lemma 3.4. Let us mention that Proposition 3.1, Lemmas 3.3, 3.4, 4.2 can have an independent interest.

We also show that the bound (1.3) with $\alpha \in [0, 1]$ is optimal. More precisely, on a special class of graphs, we prove that if V decays like $d^{-\alpha}(x, x_0)$ for some $\alpha > 1$ as $d(x, x_0) \to +\infty$, then there exist infinitely many bounded solutions to problem (1.1). To be specific, for any $\gamma > 0$ there exists a u solution to (1.1) such that $u(x) \to \gamma$ as $d(x, x_0) \to +\infty$. Its proof is based on the construction of a suitable barrier at infinity, which is related to the class of graphs we consider. To the best of our knowledge, on graphs, such type of results, which consists in prescribing a Dirichlet condition at *infinity*, and the explicit construction of such kind of barrier are totally new.

The paper is organized as follows. In Sect. 2 we describe the graph framework and we state our main result. Section 3 is devoted to the auxiliary results for elliptic equations previously described. In Sect. 4 we obtain the key apriori estimate for $v_+(x, t)$. We introduce and study our test functions in Sect. 5. The main result is proved in Sect. 6. Finally, in Sect. 7 we show the optimality of condition (1.3).

2 Mathematical framework and the main result

2.1 The graph setting

Let G be a countably infinite set and $\mu : G \to (0, +\infty)$ be a given function. Observe that μ can be viewed as a Radon measure on G so that (G, μ) becomes a measure space. Furthermore, let

$$\omega : G \times G \to [0, +\infty)$$

be a symmetric, with zero diagonal and finite sum function, i.e.

$$
\begin{aligned}
&\text{(i) } \omega(x, y) = \omega(y, x) &&\text{for all } (x, y) \in G \times G; \\
&\text{(ii) } \omega(x, x) = 0 &&\text{for all } x \in G; \\
&\text{(iii) } \sum_{y \in G} \omega(x, y) < \infty &&\text{for all } x \in G.
\end{aligned}
\tag{2.1}
$$

Thus, we define *weighted graph* the triplet (G, ω, μ), where ω and μ are the so called *edge weight* and *node measure*, respectively. Observe that assumption (ii) corresponds to ask that G has no loops.

Let x, y be two points in G; we say that

- x is *connected* to y and we write $x \sim y$, whenever $\omega(x, y) > 0$;

- the couple (x, y) is an *edge* of the graph and the vertices x, y are called the *endpoints* of the edge whenever $x \sim y$;
- a collection of vertices $\{x_k\}_{k=0}^n \subset G$ is a *path* if $x_k \sim x_{k+1}$ for all $k = 0, \ldots, n-1$.

We are now ready to list some properties that the weighted graph (G, ω, μ) may satisfy.

Definition 2.1 We say that the weighted graph (G, ω, μ) is

(i) *locally finite* if each vertex $x \in G$ has only finitely many $y \in G$ such that $x \sim y$;
(ii) *connected* if, for any two distinct vertices $x, y \in G$ there exists a path joining x to y;

For any $x \in G$, we define

- the *degree* of x as

$$\deg(x) := \sum_{y \in G} \omega(x, y);$$

- the *weighted degree* of x as

$$\mathrm{Deg}(x) := \frac{\deg(x)}{\mu(x)}.$$

A *pseudo metric* on G is a symmetric, with zero diagonal map, $d : G \times G \to [0, +\infty)$, which also satisfies the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \quad \text{for all } x, y, z \in G.$$

In general, d is not a metric, since we can find points $x, y \in G, x \neq y$ such that $d(x, y) = 0$. Now, let us consider any path $\gamma \equiv \{x_k\}_{k=0}^n \subset G$, and a symmetric function $\sigma : G \times G \to [0, +\infty)$ such that

$$\sigma(x, y) > 0 \quad \text{if and only if} \quad x \sim y.$$

Then we define the *lenght subordinated to σ* as

$$l_\sigma(\gamma) := \sum_{k=0}^{n-1} \sigma(x_k, x_{k+1}).$$

A pseudo metric $d \equiv d_\sigma$ is called a *path pseudo metric* if there exists a symmetric map $\sigma : G \times G \to [0, +\infty)$, with $\sigma(x, y) > 0$ if and only if $x \sim y$, such that

$$d_\sigma(x, y) = \inf \{ l_\sigma(\gamma) : \gamma \text{ is a path between } x \text{ and } y \}.$$

Finally, we define the *jump size $s > 0$* of a pseudo metric d as

$$s := \sup\{d(x, y) : x, y \in G, \omega(x, y) > 0\}. \tag{2.2}$$

For a more detailed understanding of the objects introduced so far, we refer the reader to [14, 20, 21, 31]. We conclude the subsection with the following

Definition 2.2 A pseudo metric d on (G, ω, μ) is said to be *intrinsic* if

$$\frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y) d^2(x, y) \leq 1 \quad \text{for all } x \in G. \tag{2.3}$$

Observe that hypothesis (2.3) can be compared with an analogous condition on Riemannian manifolds. Indeed, given any fixed reference point $x_0 \in G$, let us consider the map

$$G \ni x \mapsto d(x, x_0).$$

Then, we have

$$|\nabla d(x, x_0)|^2 = \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)[d(y, x_0) - d(x, x_0)]^2$$

$$\leq \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)d^2(x, y) \quad \text{for any } x \in G.$$

Therefore, condition (2.3) ensures that

$$|\nabla d(x, x_0)|^2 \leq 1 \quad \text{for any } x \in G.$$

Such a property is clearly fulfilled on Riemannian manifolds.

For any $x_0 \in G$ and $r > 0$ we define the ball $B_r(x_0)$ with respect to any pseudo metric d as

$$B_r(x_0) := \{x \in G : d(x, x_0) < r\}.$$

In this paper, we always make the following assumptions:

(i) (G, ω, μ) is a connected, locally finite, weighted graph;

(ii) there exists a *pseudometric* d such that the jump size*s* is finite ;   (2.4)

(iii) the ball $B_r(x)$ with respect to d is a finite set, for any $x \in G, \ r > 0$;

here we have used Definitions 2.1 and 2.2.

2.2 Difference and Laplace operators

Let $\mathfrak{F}$ denote the set of all functions $f : G \to \mathbb{R}$. For any $f \in \mathfrak{F}$ and for all $x, y \in G$, let us give the following

Definition 2.3 Let (G, ω, μ) be a weighted graph. For any $f \in \mathfrak{F}$,

- the *difference operator* is

$$\nabla_{xy} f := f(y) - f(x);$$ (2.5)

- the *(weighted) Laplace operator* on (G, ω, μ) is

$$\Delta f(x) := \frac{1}{\mu(x)} \sum_{y \in G} [f(y) - f(x)]\omega(x, y) \quad \text{for all } x \in G.$$

Clearly,

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \in G} (\nabla_{xy} f)\omega(x, y) \quad \text{for all } x \in G.$$

We also define the *gradient squared* of $f \in \mathfrak{F}$ (see [4])

$$|\nabla f(x)|^2 = \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)(\nabla_{xy} f)^2, \quad x \in G.$$

It is straightforward to show, for any $f, g \in \mathfrak{F}$, the validity of

- the *product rule*

$$\nabla_{xy}(fg) = f(x)(\nabla_{xy}g) + (\nabla_{xy}f)g(y) \quad \text{for all } x, y \in G;$$

- the *integration by parts formula*

$$\sum_{x \in G}[\Delta f(x)]g(x)\mu(x) = -\frac{1}{2}\sum_{x,y \in G}(\nabla_{xy}f)(\nabla_{xy}g)\omega(x, y), \tag{2.6}$$

provided that at least one of the functions $f, g \in \mathfrak{F}$ has *finite* support.

2.3 The main result

We have already stated in (2.4) the main hypotheses on the weighted graph (G, ω, μ). Concerning the *potential V*, we suppose that

$$\begin{aligned}
&V \in \mathfrak{F}, \\
&V(x) \geq 0 \quad \text{for all } x \in G, \\
&V(x) \geq c_0 \, d^{-\alpha}(x, x_0) \quad \text{for all } x \in G \setminus B_{R_0}(x_0),
\end{aligned} \tag{2.7}$$

for some $x_0 \in G$, $R_0 > 0$, $c_0 > 0$ and $\alpha \in [0, 1]$.

We can now state the main result of this paper.

Theorem 2.4 *Let assumptions* (2.3), (2.4) *and* (2.7) *be satisfied. Let u be a bounded solution of problem* (1.1). *Moreover, suppose that there exists $\Lambda \in (0, 1)$ such that* (1.3) *holds. Then*

$$u(x) = 0 \quad \text{for any } x \in G.$$

We end the Introduction with a couple of remarks on Theorem 2.4.

Remark 2.5 It is worth noting that, if condition (1.3) holds, there exists a constant $C > 0$ such that the following estimate holds

$$\text{Vol}(B_r(x_0)) := \sum_{x \in B_r(x_0)} \mu(x) \leq Ce^{\Lambda r^\alpha} \quad \text{for all } r > 1.$$

Hence, condition (1.3) can be regarded as a *volume growth condition*.

Remark 2.6 Observe that condition (1.3) with $\alpha \in [0, 1]$ is sharp. In fact, on a special class of graphs fulfilling this condition, the so-called *model trees*, if V decays like $d^{-\alpha}(x, x_0)$ for some $\alpha > 1$ as $d(x, x_0) \to +\infty$, then there exist infinitely many bounded solutions to problem (1.1). More precisely, for any $\gamma \in (0, +\infty)$ there exists a solution u to (1.1) such that

$$u(x) \to \gamma \quad \text{as } d(x, x_0) \to +\infty.$$

Since $\gamma > 0$ is arbitrary, in particular nonuniqueness for Eq. (1.1) follows. See Sect. 7 for more details. To the best of our knowledge, in the literature such methods used on graphs cannot be found.

3 Auxiliary results

In this section we collect several preliminary results of independent interest which shall be used in the proof of our main result, namely Theorem 2.4.

Proposition 3.1 *Let assumption (2.4)-(i) be fulfilled. Assume that there exists a nontrivial bounded solution of Eq. (1.1). Then there exists a solution v of Eq. (1.1) such that*

$$0 < v \leq 1 \quad in \ G. \tag{3.1}$$

Analogously to [2, 7], the proof of Proposition 3.1 is crucially based on the *unique solvability* of the Dirichlet problem for

$$\mathcal{L} := \Delta - V(x),$$

that is,

$$\begin{cases} \mathcal{L}u = f & in \ \Omega \\ u \equiv g & in \ G \setminus \Omega \end{cases} \tag{3.2}$$

(where $\Omega \subseteq G$ is an arbitrary finite set and $f, g \in \mathfrak{F}$), together with some *maximum principles* for $\mathcal{L}$. Since we were not able to find a precise reference for these results, and in order to make the paper as self-contained as possible, we present here below the full proofs.

To begin with, we give the following definition

Definition 3.2 We say that $u \in \mathfrak{F}$ is a solution of Eq. (3.2) if,

$$\frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y) \left[u(y) - u(x) \right] - V(x)u(x) = f(x) \quad \text{for every } x \in \Omega, \tag{3.3}$$

and $u \equiv g$ in $G \setminus \Omega$. Moreover, we say that u is a supersolution (subsolution) to Eq. (3.2), if the = in (3.3) is replaced by $\leq$ ($\geq$) and $u \geq g$ ($\leq$) in $G \setminus \Omega$.

We now establish the following *Weak Maximum Principle*.

Lemma 3.3 *Let assumption (2.4)-(i) be fulfilled. Let $\Omega \subseteq G$ be a finite set, and let $u \in \mathfrak{F}$ be such that*

$$\begin{cases} \mathcal{L}u \leq 0 & in \ \Omega \\ u \geq 0 & in \ G \setminus \Omega. \end{cases} \tag{3.4}$$

Then

$$u \geq 0 \quad in \ \Omega.$$

Proof We proceed essentially as in the proof of [10, Lemma 1.39]. We set $m := \min_\Omega u$; observe that m is well-defined since the set $\Omega \subseteq G$ is finite. Suppose, by contradiction, that $m < 0$. Then the set

$$F := \{ x \in G : u(x) = m \} \neq \varnothing$$

is such that

$$\text{if } x \in F \text{ and } y \in G, \ y \sim x, \text{ then } y \in F. \tag{3.5}$$

Indeed, let $x \in F$ be fixed, hence $u(x) = m < 0$. Due to (2.7), (3.4) and recalling that $\omega(x, y) > 0$ if $y \sim x$, we have

$$0 \geq \mathcal{L}u(x) = \Delta u(x) - V(x)u(x)$$

$$= \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)(u(y) - u(x)) - V(x)u(x)$$

$$= -\mathrm{Deg}(x)u(x) + \frac{1}{\mu(x)} \sum_{y \sim x} \omega(x, y)u(y) - V(x)u(x)$$

$$\geq -\mathrm{Deg}(x)u(x) + \frac{1}{\mu(x)} \sum_{y \sim x} \omega(x, y)u(y).$$

Therefore, since $u \geq m$ in Ω and $u \geq 0 > m$ in $G \setminus \Omega$, we obtain

$$m\mathrm{Deg}(x) = \mathrm{Deg}(x)u(x) \geq \frac{1}{\mu(x)} \sum_{y \sim x} \omega(x, y)u(y) \geq m\mathrm{Deg}(x),$$

from which we derive that

$$\sum_{y \sim x} \omega(x, y)u(y) = m \deg(x). \tag{3.6}$$

In view of (3.6), since $u \geq m$ in G, we conclude that $u(y) = m$ for every $y \in G$, $y \sim x$, i.e. (3.5).

Now, let us consider some $x \in F$ and $y \in G \setminus \Omega$, hence $u(x) = m < 0$ and $u(y) \geq 0$. Due to (2.4), there exists a path $\{x_k\}_{k=0}^n$ such that

$$x_0 = x, \quad x_n = y.$$

Since $x_0 = x \in F$, we can apply (3.5) and infer that $x_1 \in F$. By repeating this argument, we get that $x_i \in F$ for every $i = 0, ..., n$, hence in particular that $x_n = y \in F$ and thus $u(y) = m < 0$ which yields a contradiction. $\qquad \square$

We now prove the following *Strong Maximum Principle* for $\mathcal{L}$-harmonic functions.

Lemma 3.4 *Let assumption* (2.4)-*(i) be fulfilled. Let* $u \in \mathfrak{F}$ *be such that*

$$\begin{cases} \mathcal{L}u \leq 0 & \text{in } G, \\ u \geq 0 & \text{in } G. \end{cases} \tag{3.7}$$

Then either $u \equiv 0$ *in* G *or* $u > 0$ *in* G.

Proof Let u be a solution of (3.7). Then, due to Lemma 3.3, $u \geq 0$ in G. Let us now assume that there exists $x_0 \in G$ such that $u(x_0) = 0$. Moreover, we consider the set

$$F := \{x \in G : u(x) = 0\} \subseteq G.$$

Observe that $x_0 \in F$. By arguing as in the proof of Lemma 3.3, by using that $\min_G u = 0$, we get, equivalently to (3.5), that

$$\text{if } y \in G \text{ and } y \sim x_0, \text{ then } y \in F.$$

Consequently, since G is connected, we conclude that $F = G$, and hence $u \equiv 0$ in G. $\qquad \square$

Due to Lemmas 3.3 and 3.4, we can now prove the following

Proposition 3.5 *Let assumption* (2.4)-*(i) be fulfilled. Let* $\Omega \subseteq G$ *be a finite set. Let* $f : \Omega \to \mathbb{R}$ *and* $g : G \setminus \Omega \to \mathbb{R}$ *be arbitrary functions. Then there exists a unique solution* $u \in \mathfrak{F}$ *to problem* (3.2) *in the sense of* Definition 3.2.

Proof We begin by observing that, given any $u \in \mathfrak{F}$, we can write

$$\mathcal{L}u(x) = \Delta u(x) - V(x)u(x) = \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)(u(y) - u(x)) - V(x)u(x)$$

$$= -\text{Deg}(x)u(x) + \sum_{y \in G} \frac{\omega(x, y)}{\mu(x)}u(y) - V(x)u(x) \qquad (3.8)$$

$$= -(\text{Deg}(x) + V(x))u(x) + \sum_{y \in \Omega} P(x, y)u(y) + \sum_{y \notin \Omega} P(x, y)u(y),$$

where

$$P(x, y) := \frac{\omega(x, y)}{\mu(x)}.$$

Thus, we see that $u \in \mathfrak{F}$ is a solution of (3.2) *if and only if* $u \equiv g$ in $G \setminus \Omega$ and

$$- (\text{Deg}(x) + V(x))u(x) + \sum_{y \in \Omega} P(x, y)u(y) = f(x) - \sum_{y \notin \Omega} P(x, y)g(y) \quad \forall \, x \in \Omega.$$

$$(3.9)$$

We now claim that the validity of (3.9), which only involves the values attained by u on Ω, can be rephrased as a linear equation in a suitable finite-dimensional vector space.

In fact, if we denote by $\mathfrak{F}_\Omega$ the set of all real-valued functions defined on Ω, it is immediate to recognize that $\mathfrak{F}_\Omega$ is a real vector space, and

$$\mathfrak{F}_\Omega = \text{span}\{\chi_{\{x\}} : x \in \Omega\}$$

(here, χ_A stands for the indicator function of the set $A \subseteq G$). Thus, since Ω is finite, we derive that $\mathfrak{F}_\Omega$ has finite dimension $n = \text{card}(\Omega)$. On this space $\mathfrak{F}_\Omega$, we then define the map

$$\mathcal{A} : \mathfrak{F}_\Omega \to \mathfrak{F}_\Omega, \qquad (\mathcal{A}u)(x) := -(\text{Deg}(x) + V(x))u(x) + \sum_{y \in \Omega} P(x, y)u(y).$$

Clearly, $\mathcal{A}$ is linear; moreover, identity (3.9) can be rewritten as

$$\mathcal{A}u = h_{f,g}, \qquad \text{where } h_{f,g} = f(x) - \sum_{y \notin \Omega} P(x, y)g(y) \in \mathfrak{F}_\Omega.$$

Summing up, we have that $u \in \mathfrak{F}$ is a solution of problem (3.2) *if and only if*

$$u \equiv g \text{ in } G \setminus \Omega \quad \text{and} \quad \mathcal{A}(u|_\Omega) = h_{f,g}. \qquad (3.10)$$

Using this 'abstract' formulation of the Dirichlet problem (3.2), we can easily complete the proof of the proposition. First of all we observe that, owing to the Weak Maximum Principle in Lemma 3.3, the linear operator $\mathcal{A}$ is injective: indeed, if $u \in \mathfrak{F}_\Omega$ is such that $\mathcal{A}u = 0$ (that is, $\mathcal{A}u(x) = 0$ for every $x \in \Omega$) and if we define $\hat{u} := u\mathbf{1}_\Omega$, from (3.8) we have

$$\mathcal{L}\hat{u} = \mathcal{A}u = 0 \text{ in } \Omega;$$

thus, since $\hat{u} \equiv 0$ in $G \setminus \Omega$, an immediate application of Lemma 3.3 shows that $\hat{u} \equiv 0$ in G, and hence $u = 0$ in $\mathfrak{F}_\Omega$. From this, since $\mathfrak{F}_\Omega$ has finite dimension, we derive that $\mathcal{A}$ is also surjective, and thus there exists a *unique* function $u \in \mathfrak{F}_\Omega$ such that

$$\mathcal{A}u = h_{f,g}. \tag{3.11}$$

Extending this unique function u by setting $u(x) = g(x)$ for every $x \in G \setminus \Omega$, from (3.11) we conclude that (3.10) is satisfied, and thus u is the unique solution of (3.2). $\qquad \square$

With Proposition 3.5 at hand, we can finally prove Proposition 3.1.

Proof of Proposition 3.1 Let $u \in \mathfrak{F}$ be a non-trivial bounded solution of problem (1.1), and let $\mathbf{s} := \sup_G u \in (0, +\infty)$. Without lost of generality, we may assume $\mathbf{s} = 1$, indeed, it would be sufficient to replace u with $\frac{u}{\mathbf{s}}$. Moreover, if u has *constant sign* on G, then the function

$$v := \mathrm{sgn}(u)u,$$

is a solution of (1.1) satisfying (3.1). Indeed, it is immediate to recognize that v is a non-trivial solution of (1.1), and $0 \leq v \leq 1$ on G. Then, by the Strong Maximum Principle in Lemma 3.4, we conclude that $v > 0$ on G. If, instead, *u changes sign* in G, we fix M such that

$$0 < M < \sup_G u_+ = 1,$$

and, for every $n \in \mathbb{N}$, we let $v_n \in \mathfrak{F}$ be the unique solution of problem

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B_n(p), \\ v = (u - M)_+ & \text{in } G \setminus B_n(p), \end{cases}$$

where $p \in G$ is a point arbitrarily chosen. The existence and uniqueness of v_n for each $n \in \mathbb{N}$ is guaranteed by Proposition 3.5, since the balls $B_r(x)$ are *finite sets*, see (2.4).

We now claim that, for every $n \in \mathbb{N}$, the following properties holds:

(i) $0 \leq v_n \leq 1$ pointwise on G;
(ii) $v_n \leq v_{n+1}$ pointwise on G.

Taking this claim for granted for a moment, we can easily complete the proof of the proposition. In fact, owing to (i)-(ii) we deduce that the sequence $\{v_n\}_n$ is *increasing and bounded* on G; as a consequence, the function

$$v(x) = \lim_{n \to +\infty} v_n(x) \qquad x \in G,$$

is well-defined, and it satisfies $0 \leq v \leq 1$ on G. Moreover, since

$$\mathcal{L}v_n(x) = \Delta v_n(x) - V(x)v_n(x)$$
$$= \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)(v_n(y) - v_n(x)) - V(x)v_n(x) = 0 \quad \forall\, x \in B_n(p),\ n \in \mathbb{N},$$

$$\tag{3.12}$$

and since the sum which defines the Laplacian Δ is actually a *finite sum* (recall that the graph G is locally finite), by letting $n \to +\infty$ in (3.12) we readily obtain

$$\mathcal{L}v(x) = 0 \quad \forall\, x \in \bigcup_{n=1}^{+\infty} B_n(p) = G,$$

and thus v is a solution of problem (1.1). Finally, reminding that $0 \leq v \leq 1$ on G and using the Strong Maximum Principle in Lemma 3.4, we conclude that $v \in \mathfrak{F}$ is a solution of problem (1.1) also fulfilling (3.1). Hence, we are left to prove the claimed (i)-(ii).

Let us show (i). We first observe that, since $\mathcal{L}v_n = 0$ in $B_n(p)$ and since $v_n = (u-M)_+ \geq 0$ in $G \setminus B_n(p)$, from the Weak Maximum Principle in Lemma 3.3 we infer that $v_n \geq 0$ in G. On the other hand, since the constant function $\zeta \equiv 1$ satisfies

$$\begin{cases} \mathcal{L}\zeta = -V(x) \leq 0 & \text{in } B_n(p), \\ \zeta \geq (u-M)_+ & \text{in } G \setminus B_n(p), \end{cases}$$

by applying the Weak Maximum Principle in Lemma 3.3 to the function $\zeta - v_n$ we immediately conclude that $v_n \leq \zeta \equiv 1$ pointwise on G.

Now, let us prove (ii). First of all we observe that, since $\zeta = u - M$ satisfies

$$\begin{cases} \mathcal{L}\zeta = VM \geq 0 & \text{in } B_n(p), \\ \zeta = u - M \leq (u-M)_+ & \text{in } G \setminus B_n(p), \end{cases}$$

by applying the Weak Maximum Principle in Lemma 3.3 to the function $v_{n+1} - \zeta$ we derive that $v_{n+1} \geq \zeta = u - M$ in G; thus, since we already know that $v_{n+1} \geq 0$, we obtain

$$v_{n+1} \geq (u-M)_+ \quad \text{pointwise on } G. \tag{3.13}$$

Owing to (3.13), and applying once again the Weak Maximum Principle in Lemma 3.3 to the function $v_{n+1} - v_n$ with $\Omega = B_n(p)$, we then conclude that $v_n \leq v_{n+1}$ in G.

This ends the proof. $\qquad\blacksquare$

4 A useful apriori estimate

Now we have established Proposition 3.1, we turn to prove an *apriori* estimate for nonnegative and bounded solutions to (1.1) which will play a key role in the proof of Theorem 2.4. Throughout what follows, we set

$$S := G \times [0, +\infty); \tag{4.1}$$

and, for any given $T > 0$,

$$S_T := G \times [0, T].$$

Proposition 4.1 *Let assumption* (2.4) *be in force. Let u be a solution of* Eq. (1.1) *such that* $0 \leq u \leq 1$. *Moreover, let $T > 0$ and define*

$$v(x, t) := e^t u(x) - 1 \quad \text{for all } (x, t) \in S_T.$$

Finally, let $\eta \in \mathfrak{F}$, and $\xi : S_T \to \mathbb{R}$ be such that

(i) $\eta \geq 0$, $\operatorname{supp} \eta$ *is finite* ; $\tag{4.2}$

(ii) $\xi(x, \cdot) \in C^1([0, T))$, $e^{\xi(x, \cdot)} \in C^1([0, T])$ *for any $x \in G$;* $\tag{4.3}$

(iii) $[\eta^2(y) - \eta^2(x)][e^{\xi(y,t)} - e^{\xi(x,t)}] \geq 0$ *for all $x, y \in G$, $x \sim y$, $t \in [0, T]$.* $\tag{4.4}$

Then

$$\sum_{x \in G} V(x)\eta^2(x)v_+^2(x, T)e^{\xi(x,T)}\mu(x) - \sum_{x \in G} V(x)\eta^2(x)v_+^2(x, 0)e^{\xi(x,0)}\mu(x)$$

$$\leq \int_0^T \sum_{x \in G} v_+^2(x, t)\eta^2(x)e^{\xi(x,t)} \left\{ V(x)\xi_t(x, t)\mu(x) + \frac{1}{2}\sum_{y \in G} \omega(x, y)[1 - e^{\xi(y,t)-\xi(x,t)}]^2 \right\} dt$$

$$+ 2\int_0^T \sum_{x,y \in G} v_+^2(x, t)e^{\xi(y,t)}[\eta(y) - \eta(x)]^2 \omega(x, y)\, dt. \tag{4.5}$$

To prove Proposition 4.1 we need an auxiliary result (of independent interest) which shows that, if v is any solution of the parabolic equation

$$V\partial_t u - \Delta u = 0 \quad \text{in } G \times [0, T], \tag{4.6}$$

then $v_+ := \max\{v; 0\}$ is a subsolution of the same equation (in a suitable sense).

This is the content of the following

Lemma 4.2 *Let $v : S_T \to \mathbb{R}$ be a solution of Eq. (4.6) such that the map $t \mapsto v(x, t)$ is $C^1([0, T])$ for any $x \in G$. Then, for any fixed $x \in G$,*

$$V\partial_t v_+(x, t) - \Delta v_+(x, t) \leq 0 \quad \text{for a.e. } t \in [0, T].$$

Proof We separately consider three cases.

Case 1. Let $(x, t) \in G \times (0, T)$ be such that

$$v(x, t) > 0. \tag{4.7}$$

Then, by the continuity of the map $t \mapsto v(x, t)$, there exists $\delta > 0$ such that $v(x, \tau) > 0$ for any $\tau \in (t - \delta, t + \delta)$. Therefore, $v(x, \tau) = v_+(x, \tau)$ for any $\tau \in (t - \delta, t + \delta)$. Consequently, since the map $t \mapsto v(x, t)$ is $C^1([0, T])$, we also have that

$$\partial_t v(x, t) = \partial_t v_+(x, t). \tag{4.8}$$

Now, due to the fact that $v_+(y, t) \geq v(y, t)$ for any $y \in G$, by (4.7), (4.8) and (4.6), we have

$$V\partial_t v_+(x, t) - \Delta v_+(x, t) = V\partial_t v(x, t) - \frac{1}{\mu(x)}\sum_{y \in G}[v_+(y, t) - v_+(x, t)]\,\omega(x, y)$$

$$\leq V\partial_t v(x, t) - \frac{1}{\mu(x)}\sum_{y \in G}[v(y, t) - v(x, t)]\,\omega(x, y) = 0.$$

Case 2. Let $(x, t) \in G \times (0, T)$ be such that

$$v(x, t) < 0. \tag{4.9}$$

Then, by the continuity of the map $t \mapsto v(x, t)$, there exists $\delta > 0$ such that $v(x, \tau) < 0$ for any $\tau \in (t - \delta, t + \delta)$. Therefore, $v_+(x, \tau) = 0$ for any $\tau \in (t - \delta, t + \delta)$. Consequently, since the map $t \mapsto v(x, t)$ is $C^1([0, T])$, we also have that

$$\partial_t v_+(x, t) = 0. \tag{4.10}$$

Now, due to the fact that $v_+(y, t) \geq 0$ for any $y \in G$, by (4.9) and (4.10), we have

$$V \partial_t v_+(x, t) - \Delta v_+(x, t) = -\frac{1}{\mu(x)} \sum_{y \in G} [v_+(y, t) - v_+(x, t)] \omega(x, y)$$

$$= -\frac{1}{\mu(x)} \sum_{y \in G} [v_+(y, t)] \omega(x, y) \leq 0.$$

Case 3. Let $x \in G$ fixed. We consider the set $\mathcal{U}_x = \{t \in (0, T) : v(x, t) = 0\}$. We have that

$$\partial_t v(x, t) = 0 \quad \text{for a.e. } t \in \mathcal{U}_x. \tag{4.11}$$

Then we can find $\mathcal{V}_x \subset \mathcal{U}_x$ such that

$$\lambda(\mathcal{V}_x) = \lambda(\mathcal{U}_x) \text{ and } \partial_t v(x, t) = 0 \text{ for every } t \in \mathcal{V}_x \tag{4.12}$$

(here, λ denotes the one-dimensional Lebesgue measure). Thus, since $v_+(y, t) \geq 0$ pointwise on $G \times (0, T)$, by (4.11) and (4.12), for any $t \in \mathcal{V}_x$, we have

$$V \partial_t v_+(x, t) - \Delta v_+(x, t) = -\frac{1}{\mu(x)} \sum_{y \in G} [v_+(y, t) - v_+(x, t)] \omega(x, y)$$

$$= -\frac{1}{\mu(x)} \sum_{y \in G} [v_+(y, t)] \omega(x, y) \leq 0.$$

Hence the thesis follows from the combination of the three cases.   □

Proof of Proposition 4.1 We first observe that, due to Lemma 4.2, for any $x \in G$,

$$V(x) \partial_t v_+(x, t) - \Delta v_+(x, t) \leq 0 \quad \text{for a.e. } t \in [0, T]. \tag{4.13}$$

We multiply (4.13) by the function $(x, t) \mapsto v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x)$ and we integrate on the time interval $[0, T]$. Thus we get

$$\int_0^T V(x) \partial_t v_+(x, t) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt$$

$$\leq \int_0^T \Delta(v_+(x, t)) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt.$$

We sum over $x \in G$, so

$$\sum_{x \in G} \int_0^T V(x) \partial_t v_+(x, t) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt$$

$$\leq \sum_{x \in G} \int_0^T \Delta(v_+(x, t)) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt.$$

Notice that, since $\eta(x)$ is finitely supported, the series in the latter inequality are actually finite sums. Therefore, we obtain

$$\int_0^T \sum_{x \in G} V(x) \partial_t v_+(x, t) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt$$

$$\leq \int_0^T \sum_{x \in G} \Delta(v_+(x, t)) v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt. \tag{4.14}$$

Set

$$I := \sum_{x \in G} \Delta(v_+(x, t)) \, v_+(x, t) \, \eta^2(x) e^{\xi(x,t)} \mu(x).$$

Then, by (2.5) and (2.6),

$$
\begin{aligned}
I &= -\frac{1}{2} \sum_{x, y \in G} \left(\nabla_{xy} v_+\right) \nabla_{xy} \left[v_+ \eta^2 e^{\xi}\right] \omega(x, y) \\
&= -\frac{1}{2} \sum_{x, y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \, \omega(x, y) \\
&\quad - \frac{1}{2} \sum_{x, y \in G} v_+(x, t) \eta^2(x) \left(\nabla_{xy} v_+\right) \left(\nabla_{xy} e^{\xi}\right) \omega(x, y) \\
&\quad - \frac{1}{2} \sum_{x, y \in G} v_+(x, t) e^{\xi(y,t)} \left(\nabla_{xy} v_+\right) \left(\nabla_{xy} \eta^2\right) \omega(x, y) =: J_1 + J_2 + J_3.
\end{aligned}
\tag{4.15}
$$

In view of (2.5), we obviously have

$$\nabla_{xy} \eta^2 = [\eta(y) + \eta(x)][\eta(y) - \eta(x)] \quad \text{for all } x, y \in G. \tag{4.16}$$

By Young's inequality with exponent 2, for any $\delta_1 > 0$, we have, for every $t \in [0, T]$,

$$
\begin{aligned}
J_2 &= -\frac{1}{2} \sum_{x, y \in G} v_+(x, t) \eta^2(x)(\nabla_{xy} v_+)(\nabla_{xy} e^{\xi}) \, \omega(x, y) \\
&= -\frac{1}{2} \sum_{x, y \in G} v_+(y, t) \eta^2(y)(\nabla_{xy} v_+) \left[e^{\xi(y,t)} - e^{\xi(x,t)}\right] \omega(x, y) \\
&= -\frac{1}{2} \sum_{x, y \in G} v_+(y, t) \eta^2(y)(\nabla_{xy} v_+) e^{\xi(y,t)} \left[1 - e^{\xi(x,t) - \xi(y,t)}\right] \omega(x, y) \\
&\leq \frac{\delta_1}{4} \sum_{x, y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \, \omega(x, y) \\
&\quad + \frac{1}{4\delta_1} \sum_{x, y \in G} v_+^2(y, t) \left[1 - e^{\xi(x,t) - \xi(y,t)}\right]^2 \eta^2(y) e^{\xi(y,t)} \, \omega(x, y) \\
&= \frac{\delta_1}{4} \sum_{x, y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \, \omega(x, y) \\
&\quad + \frac{1}{4\delta_1} \sum_{x, y \in G} v_+^2(x, t) \left[1 - e^{\xi(y,t) - \xi(x,t)}\right]^2 \eta^2(x) e^{\xi(x,t)} \, \omega(x, y).
\end{aligned}
\tag{4.17}
$$

Similarly, due to (4.16), by Young's inequality with exponent 2, we have, for every $\delta_2 > 0$ and for every $t \in [0, T]$,

$$
\begin{aligned}
J_3 &= -\frac{1}{2} \sum_{x, y \in G} v_+(x, t) e^{\xi(y,t)} (\nabla_{xy} v_+) \left[\eta(y) + \eta(x)\right] \left[\eta(y) - \eta(x)\right] \omega(x, y) \\
&\leq \frac{\delta_2}{4} \sum_{x, y \in G} e^{\xi(y,t)} \left[\nabla_{xy} v_+\right]^2 \left[\eta(y) + \eta(x)\right]^2 \omega(x, y)
\end{aligned}
$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} e^{\xi(y,t)} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 \omega(x,y)$$

$$\leq \frac{\delta_2}{2} \sum_{x,y \in G} e^{\xi(y,t)} \left[\nabla_{xy} v_+\right]^2 \left[\eta^2(y) + \eta^2(x)\right] \omega(x,y)$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} e^{\xi(y,t)} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 \omega(x,y)$$

$$= \frac{\delta_2}{4} \sum_{x,y \in G} \left[e^{\xi(y,t)} + e^{\xi(x,t)}\right] \left[\nabla_{xy} v_+\right]^2 \left[\eta^2(y) + \eta^2(x)\right] \omega(x,y)$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} e^{\xi(y,t)} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 \omega(x,y). \tag{4.18}$$

From (4.4) we can easily infer that

$$\left[\eta^2(y) + \eta^2(x)\right] \left[e^{\xi(y,t)} + e^{\xi(x,t)}\right] \leq 2 \left\{\eta^2(y) e^{\xi(y,t)} + \eta^2(x) e^{\xi(x,t)}\right\}. \tag{4.19}$$

By using (4.18) and (4.19),

$$J_3 \leq \frac{\delta_2}{2} \sum_{x,y \in G} \left[\nabla_{xy} v_+\right]^2 \left\{\eta^2(y) e^{\xi(y,t)} + \eta^2(x) e^{\xi(x,t)}\right\} \omega(x,y)$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} e^{\xi(y,t)} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 \omega(x,y)$$

$$= \delta_2 \sum_{x,y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \omega(x,y)$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} e^{\xi(y,t)} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 \omega(x,y). \tag{4.20}$$

By combining (4.15), (4.17) and (4.20) we get, for every $t \in [0, T]$,

$$\sum_{x \in G} \Delta v_+(x,t) v_+(x,t) \eta^2(x) e^{\xi(x,t)} \mu(x)$$

$$\leq -\frac{1}{2} \sum_{x,y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \omega(x,y)$$

$$+ \frac{\delta_1}{4} \sum_{x,y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \omega(x,y)$$

$$+ \frac{1}{4\delta_1} \sum_{x,y \in G} v_+^2(x,t) \left[1 - e^{\xi(y,t) - \xi(x,t)}\right]^2 \eta^2(x) e^{\xi(x,t)} \omega(x,y)$$

$$+ \delta_2 \sum_{x,y \in G} \left[\nabla_{xy} v_+\right]^2 \eta^2(y) e^{\xi(y,t)} \omega(x,y)$$

$$+ \frac{1}{4\delta_2} \sum_{x,y \in G} v_+^2(x,t) \left[\eta(y) - \eta(x)\right]^2 e^{\xi(y,t)} \omega(x,y). \tag{4.21}$$

We now choose $\delta_1 = 1$ and $\delta_2 = \frac{1}{4}$ in such a way that $-\frac{1}{2} + \frac{\delta_1}{4} + \delta_2 = 0$. Consequently, (4.21) yields, for every $t \in [0, T]$,

$$
\begin{aligned}
I &= \sum_{x \in G} \Delta v_+(x, t) v_+(x, t) \, \eta^2(x) e^{\xi(x,t)} \mu(x) \\
&\leq \frac{1}{4} \sum_{x,y \in G} v_+^2(x, t) \left[1 - e^{\xi(y,t) - \xi(x,t)} \right]^2 \eta^2(x) e^{\xi(x,t)} \, \omega(x, y) \\
&\quad + \sum_{x,y \in G} v_+^2(x, t) \, [\eta(y) - \eta(x)]^2 \, e^{\xi(y,t)} \, \omega(x, y).
\end{aligned}
\tag{4.22}
$$

By substituting (4.22) into (4.14), due to the linearity of the integral operation, we get

$$
\begin{aligned}
\int_0^T & \sum_{x \in G} V(x) \, \partial_t v_+(x, t) \, v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt \\
&\leq \int_0^T \frac{1}{4} \sum_{x,y \in G} v_+^2(x, t) \left[1 - e^{\xi(y,t) - \xi(x,t)} \right]^2 \eta^2(x) e^{\xi(x,t)} \, \omega(x, y) \, dt \\
&\quad + \int_0^T \sum_{x,y \in G} v_+^2(x, t) [\eta(y) - \eta(x)]^2 e^{\xi(y,t)} \omega(x, y) \, dt.
\end{aligned}
\tag{4.23}
$$

We now consider the left-hand side of (4.23). Thus we have

$$
\begin{aligned}
\int_0^T & \sum_{x \in G} V(x) \, \partial_t v_+(x, t) \, v_+(x, t) \eta^2(x) e^{\xi(x,t)} \mu(x) \, dt \\
&= \sum_{x \in G} V(x) \eta^2(x) \mu(x) \int_0^T (v_+)_t(x, t) \, v_+(x, t) e^{\xi(x,t)} \, dt \\
&= \frac{1}{2} \sum_{x \in G} V(x) \eta^2(x) \mu(x) \int_0^T (v_+^2)_t(x, t) \, e^{\xi(x,t)} \, dt \\
&= \frac{1}{2} \sum_{x \in G} V(x) \eta^2(x) v_+^2(x, T) \, e^{\xi(x,T)} \mu(x) \\
&\quad - \frac{1}{2} \sum_{x \in G} V(x) \eta^2(x) v_+^2(x, 0) \, e^{\xi(x,0)} \mu(x) \\
&\quad - \frac{1}{2} \int_0^T \sum_{x \in G} V(x) \eta^2(x) v_+^2(x, t) \xi_t(x, t) \, e^{\xi(x,t)} \mu(x) \, dt.
\end{aligned}
\tag{4.24}
$$

By combining together (4.23) and (4.24), we then obtain

$$
\begin{aligned}
\sum_{x \in G} & V(x) \eta^2(x) v_+^2(x, T) \, e^{\xi(x,T)} \mu(x) - \sum_{x \in G} V(x) \eta^2(x) v_+^2(x, 0) \, e^{\xi(x,0)} \mu(x) \\
&\leq \int_0^T \sum_{x \in G} \eta^2(x) v_+^2(x, t) \, e^{\xi(x,t)} \left\{ V(x) \xi_t(x, t) \mu(x) + \frac{1}{2} \sum_{y \in G} \left[1 - e^{\xi(y,t) - \xi(x,t)} \right]^2 \omega(x, y) \right\} dt \\
&\quad + 2 \int_0^T \sum_{x,y \in G} v_+^2(x, t) [\eta(y) - \eta(x)]^2 e^{\xi(y,t)} \omega(x, y) \, dt.
\end{aligned}
$$

This is precisely (4.5), and the proof is complete. $\qquad\square$

5 Some distinguished test functions

Let us now prove the existence of suitable test functions ξ and η which are *admissible* in (4.5) and which satisfy some *ad-hoc* properties.

Throughout the sequel, we let $x_0 \in G$ and $R_0 > 0$ be as in assumption (2.7); furthermore, we choose parameters $M > 0$, $T > 0$, $\beta \in (0, 1]$, $\lambda > 1$ and

$$r \geq 2s + R_0, \tag{5.1}$$

where s is defined in (2.2). Setting

$$\mathbf{d}(x) := d(x, x_0) \quad \text{for every } x \in G,$$

we then define

$$\xi(x, t) := -M \frac{\rho(x)}{\lambda T - t} \quad \text{for any } x \in G, \ t \in [0, T),$$

$$\text{where } \rho(x) := \max\{\mathbf{d}^\beta(x), r^\beta\} = \begin{cases} r^\beta & \text{if } \mathbf{d}(x) \leq r, \\ \mathbf{d}^\beta(x) & \text{if } \mathbf{d}(x) > r. \end{cases} \tag{5.2}$$

Concerning the function ξ, we have the following key lemma.

Lemma 5.1 *Let hypotheses* (2.3) *and* (2.4) *be fulfilled; suppose that* (2.7) *is satisfied with* $\alpha \in [0, 1]$. *Let ξ be the function defined in* (5.2). *Then*

$$V(x)\,\xi_t(x, t)\mu(x) + \frac{1}{2} \sum_{y \in G} \omega(x, y) \left[1 - e^{\xi(y,t) - \xi(x,t)}\right]^2 \leq 0 \quad \text{for all } x \in G, \ t \in (0, T),$$

$$\tag{5.3}$$

provided that $M = T > 0$ *is sufficiently small and* $\beta = \alpha$.

Proof To ease the readability, we split the proof into two steps.

STEP I: In this first step we prove the following estimate

$$|\rho(x) - \rho(y)| \leq \beta R_0^{\beta-1} d(x, y) \quad \text{for every } x, y \in G \text{ with } x \sim y. \tag{5.4}$$

To this end, it is useful to distinguish two cases.

(i) $x \in B_{r-s}(x_0)$. In this case, by triangle's inequality and (2.2) we have

$$\mathbf{d}(y) < r \quad \text{for every } y \in G, \ y \sim x;$$

as a consequence, from the very definition of ρ we derive

$$|\rho(x) - \rho(y)| = r^\beta - r^\beta = 0,$$

and this trivially implies (5.4).

(ii) $x \in G \setminus B_{r-s}(x_0)$. In this case we first notice that, since the function

$$\mathbb{R} \ni t \mapsto \max\{t, r^\beta\}$$

is Lipschitz-continuous with Lipschitz constant $L = 1$, by the Mean Value Theorem and again the triangle inequality we can write

$$|\rho(x) - \rho(y)| \leq |\mathbf{d}^\beta(x) - \mathbf{d}^\beta(y)| \leq \beta \sigma^{\beta-1}|\mathbf{d}(x) - \mathbf{d}(y)|$$

$$\leq \beta \sigma^{\beta-1} d(x, y) \quad \forall \ y \in G, \tag{5.5}$$

where $\sigma \geq 0$ is a suitable point between $\mathbf{d}(x)$ and $\mathbf{d}(y)$. On the other hand, since we assuming that $x \notin B_{r-s}(x_0)$ (hence, $\mathbf{d}(x) \geq r - s$), by (2.2) we have

$$\mathbf{d}(y) \geq r - 2s \geq R_0 \quad \forall\, y \in G, \; y \sim x. \tag{5.6}$$

Recalling that $\beta \leq 1$, from (5.5) to (5.6) we immediately obtain

$$|\rho(x) - \rho(y)| \leq \beta R_0^{\beta-1} d(x, y) \quad \forall\, y \in G, \; y \sim x,$$

which is exactly the desired (5.4).

STEP II: In this second step we establish (5.3). To begin with, we point out that

$$(e^a - 1)^2 \leq a^2 e^{2|a|} \quad \forall\, a \in \mathbb{R};$$

this inequality, together with (5.4), allows us to write

$$\begin{aligned}
\left(1 - e^{\xi(y,t)-\xi(x,t)}\right)^2 &\leq |\xi(y, t) - \xi(x, t)|^2 e^{2|\xi(y,t)-\xi(x,t)|} \\
&= \frac{M^2}{(\lambda T - t)^2} |\rho(y) - \rho(x)|^2 e^{\frac{2M}{\lambda T - t}|\rho(y)-\rho(x)|} \\
&\leq \frac{\beta^2 R_0^{2\beta-2} M^2}{(\lambda T - t)^2} e^{\frac{2\beta R_0^{\beta-1} M d(x,y)}{T(\lambda-1)}} d(x, y)^2,
\end{aligned}$$

and this estimate holds for every $x, y \in G$ and every $t \in (0, T)$. Then, using (2.2) and recalling that d is *intrinsic* (hence, (2.3) holds), for every $x \in G$ and $t \in (0, T)$ we obtain

$$\begin{aligned}
V(x)\, \xi_t(x, t)\mu(x) &+ \frac{1}{2} \sum_{y \in G} \omega(x, y) \left[1 - e^{\xi(y,t)-\xi(x,t)}\right]^2 \\
&\leq -\frac{M\rho(x)}{(\lambda T - t)^2} V(x)\mu(x) + \frac{\beta^2 R_0^{2\beta-2} M^2}{2(\lambda T - t)^2} \sum_{y \in G} e^{\frac{2\beta R_0^{\beta-1} M d(x,y)}{T(\lambda-1)}} \omega(x, y)d(x, y)^2 \\
&\leq -\frac{M\rho(x)}{(\lambda T - t)^2} V(x)\mu(x) + \frac{\beta^2 R_0^{2\beta-2} M^2}{2(\lambda T - t)^2} e^{\frac{2\beta R_0^{\beta-1} Ms}{T(\lambda-1)}} \sum_{y \in G} \omega(x, y)d(x, y)^2 \\
&\leq \frac{M\mu(x)}{(\lambda T - t)^2} \left\{ -V(x)\rho(x) + \frac{M}{2} \beta^2 R_0^{2\beta-2} e^{\frac{2\beta R_0^{\beta-1} Ms}{T(\lambda-1)}} \right\}.
\end{aligned} \tag{5.7}$$

To proceed further, we now *fix $\beta = \alpha$* and we exploit assumption (2.7): taking into account the piecewise definition of ρ, see (5.2), it is easy to recognize that

$$V(x)\rho(x) \geq c_0 \mathbf{d}(x)^{-\alpha} \rho(x) = c_0 \mathbf{d}(x)^{-\alpha} \cdot \max\{\mathbf{d}^\alpha(x), r^\alpha\} \geq c_0 \quad \forall\, x \in G; \tag{5.8}$$

as a consequence, by combining (5.7), (5.8) we conclude that

$$\begin{aligned}
V(x)\, \xi_t(x, t)\mu(x) &+ \frac{1}{2} \sum_{y \in G} \omega(x, y) \left[1 - e^{\xi(y,t)-\xi(x,t)}\right]^2 \\
&\leq \frac{M\mu(x)}{(\lambda T - t)^2} \left\{ -c_0 + \frac{M}{2} \beta^2 R_0^{2\beta-2} e^{\frac{2\beta R_0^{\beta-1} Ms}{T(\lambda-1)}} \right\} \leq 0,
\end{aligned}$$

provided that

$$M = T \quad \text{and} \quad 0 < M \leq \frac{2c_0 R_0^{2-2\beta} e^{-\frac{2}{\lambda-1}\beta R_0^{\beta-1} s}}{\beta^2}.$$

This ends the proof.   □

Now that we have proved Lemma 5.1, we turn to prove the existence of a suitable 'cut-off' function η. To this end, taking for fixed all the notation introduced so far, we choose

$$r_1 > 2r + 8s$$

and we define the function

$$\eta(x) := \min\left\{\frac{2\,[r_1 - s - \mathbf{d}(x)]_+}{r_1},\; 1\right\} \quad \text{for any } x \in G. \tag{5.9}$$

Owing to [30, Lemma 5.2] (with the choice $\delta = 1/2$), we obtain the following result.

Lemma 5.2 *Let assumptions* (2.3), (2.4) *be satisfied. Then, the function η defined in* (5.9) *satisfies the following properties:*

(i) $|\nabla_{xy}\eta| \leq \dfrac{2}{r_1}d(x, y)\chi_{\left\{\frac{r_1}{2} - 2s \leq \mathbf{d}(x) \leq r_1\right\}} \quad \text{for any } x \in G;$

(ii) $\displaystyle\sum_{y \in G}\left(\nabla_{xy}\eta\right)^2 \omega(x, y) \leq \dfrac{4}{(r_1)^2}\mu(x)\chi_{\left\{\frac{r_1}{2} - 2s \leq \mathbf{d}(x) \leq r_1\right\}} \quad \text{for any } x \in G. \tag{5.10}$

6 Proof of Theorem 2.4

Due to the results established in Sects. 3, 4 and 5, we are ready to provide the proof of Theorem 2.4. In what follows, we take for fixed all the notation introduced so far.

Proof of Theorem 2.4 By contradiction, suppose that there exists a non-trivial bounded solution of Eq. (1.1). Then, due to Proposition 3.1, we know that there exists a solution u to the same Eq. (1.1) such that

$$0 < u \leq 1 \quad \text{pointwise in } G. \tag{6.1}$$

Let us now define $v(x, t) := e^t u(x) - 1$, for any $(x, t) \in S$, for S as in (4.1). We want to show that

$$v(x, t) \leq 0 \quad \text{for every } x \in \text{supp}(V) \text{ and } t > 0. \tag{6.2}$$

To do so, let us fix $T > 0$ (to be chosen conveniently small in a moment), and we arbitrarily choose $r > 2s + R_0$, for $R_0 > 0$ and $0 < s < +\infty$ as in (2.7) and (2.2), respectively. Let ξ be as in (5.2) with $\beta = \alpha$, $\lambda > 1$ and with $M = T$ chosen as in Lemma 5.1. Moreover, let us fix $r_1 > 0$ in such a way that

$$r_1 \geq 2r + 8s, \tag{6.3}$$

and let η be as in Lemma 5.2. Now, we observe that η and ξ obviously satisfy conditions (4.2) and (4.3); furthermore, also (4.4) is fulfilled, since both η and $\xi(\cdot, t)$ are *non-increasing functions* of $\mathbf{d}$. Therefore, from (4.5) we obtain

$$\sum_{x \in G} V(x)\eta^2(x)v_+^2(x, T)e^{\xi(x,T)}\mu(x) - \sum_{x \in G} V(x)\eta^2(x)v_+^2(x, 0)e^{\xi(x,0)}\mu(x)$$

$$\leq \int_0^T \sum_{x \in G} v_+^2(x, t)\eta^2(x)e^{\xi(x,t)}\left\{V(x)\xi_t(x, t)\mu(x) - \frac{1}{2}\sum_{y \in G}\omega(x, y)[1 - e^{\xi(y,t) - \xi(x,t)}]^2\right\}dt$$

$$+ 2 \int_0^T \sum_{x,y \in G} v_+^2(x,t) e^{\xi(y,t)} [\eta(y) - \eta(x)]^2 \omega(x,y) \, dt. \tag{6.4}$$

On the other hand, by Lemma 5.1 there exists $T_0 = T_0(\lambda) > 0$, such that

$$V(x) \xi_t(x,t) \mu(x) - \frac{1}{2} \sum_{y \in G} \omega(x,y)[1 - e^{\xi(y,t) - \xi(x,t)}]^2 \leq 0 \qquad \forall \, x \in G, \, t \in (0,T),$$
$$\tag{6.5}$$

provided that $T \leq T_0$. As a consequence, by combining (6.4) and (6.5), we obtain

$$\sum_{x \in G} V(x) \eta^2(x) v_+^2(x,T) e^{\xi(x,T)} \mu(x) - \sum_{x \in G} V(x) \eta^2(x) v_+^2(x,0) e^{\xi(x,0)} \mu(x)$$
$$\leq 2 \int_0^T \sum_{x,y \in G} v_+^2(x,t) e^{\xi(y,t)} [\eta(y) - \eta(x)]^2 \omega(x,y) \, dt. \tag{6.6}$$

We then proceed by estimating both sides of (6.6).

- *Estimate of the left-hand side.* First of all we observe that, owing to the definition of η in (5.9), we have $\eta \geq 0$ pointwise on G and $\eta \equiv 1$ on $B_r(x_0) = \{x : \mathbf{d}(x) < r\}$. Now we observe that, due to (6.1), $v_+(x,0) = 0$ in G. Moreover, since $\rho(x) = \mathbf{d}(x)^\alpha$ in $B_r(x_0)$, we obtain

$$\sum_{x \in G} V(x) \eta^2(x) v_+^2(x,T) e^{\xi(x,T)} \mu(x) - \sum_{x \in G} V(x) \eta^2(x) v_+^2(x,0) e^{\xi(x,0)} \mu(x)$$
$$= \sum_{x \in G} V(x) \eta(x)^2 v_+^2(x,T) e^{\xi(x,T)} \mu(x)$$
$$\geq \sum_{x \in B_r(x_0)} V(x) v_+^2(x,T) e^{\xi(x,T)} \mu(x)$$
$$= \gamma_{r,\lambda} \sum_{x \in B_r(x_0)} V(x) \, v_+^2(x,T) \mu(x), \tag{6.7}$$

where we have used the shorthand notation $\gamma_{r,\lambda} = e^{-\frac{r^\alpha}{\lambda-1}} > 0$.

- *Estimate of the right-hand side.* We first observe that, using the definition of ξ given in (5.2), since $0 \leq v_+(x,t) \leq e^t$ on S, for S as in (4.1), and by exploiting (5.10), we have

$$\int_0^T \sum_{x,y \in G} v_+^2(x,t) e^{\xi(y,t)} [\eta(y) - \eta(x)]^2 \omega(x,y) \, dt$$
$$= \int_0^T \sum_{x,y \in G} v_+^2(x,t) e^{\xi(y,t)} (\nabla_{xy} \eta)^2 \omega(x,y) \, dt$$
$$= \int_0^T \sum_{x,y \in G} v_+^2(y,t) e^{\xi(x,t)} (\nabla_{xy} \eta)^2 \omega(x,y) \, dt$$
$$\leq e^{2T} \int_0^T \sum_{x,y \in G} \exp\left(-\frac{T \rho(x)}{\lambda T - t}\right) (\nabla_{xy} \eta)^2 \omega(x,y) \, dt$$
$$\leq T e^{2T} \sum_{x,y \in G} \exp\left(-\frac{\rho(x)}{\lambda}\right) (\nabla_{xy} \eta)^2 \omega(x,y)$$

$$\leq \frac{4Te^{2T}}{(r_1)^2} \sum_{x \in G} \exp\left(-\frac{\rho(x)}{\lambda}\right) \chi_{\left\{\frac{r_1}{2}-2s \leq d(x,x_0) \leq r_1\right\}}(x)\mu(x)$$

$$\leq \frac{4Te^{2T}}{(r_1)^2} \sum_{x \in G} \exp\left(-\frac{\rho(x)}{\lambda}\right) \chi_{\{r < d(x,x_0) \leq r_1\}}(x)\mu(x).$$

Now, since $\rho(x) = \mathbf{d}^\alpha(x)$ when $\mathbf{d}(x) > r$, by using (6.3), we obtain that

$$\int_0^T \sum_{x,y \in G} v_+^2(x,t)e^{\xi(y,t)}[\eta(y) - \eta(x)]^2 \omega(x,y)\,dt$$

$$\leq \frac{c(T)}{(r_1)^2} \sum_{x \in G} \exp\left(-\frac{\mathbf{d}^\alpha(x)}{\lambda}\right) \chi_{\{r < d(x,x_0) \leq r_1\}}(x)\mu(x). \tag{6.8}$$

Then, by combining (6.7) and (6.8) with (6.6), we obtain

$$\sum_{x \in B_r(x_0)} V(x)\,v_+^2(x,T)\mu(x) \leq \frac{c(T,\lambda,r)}{(r_1)^2} \sum_{x \in G} \exp\left(-\frac{\mathbf{d}^\alpha(x)}{\lambda}\right) \chi_{\{r < \mathbf{d}(x) \leq r_1\}}(x)\mu(x), \tag{6.9}$$

where $c(T,\lambda,r) > 0$ is a constant only depending on T, λ and r. Now, if $0 < \Lambda < 1$ is as in (1.3), and if we set

$$\lambda = \frac{1}{\Lambda} > 1,$$

then estimate (6.9) boils down to

$$\sum_{x \in B_r(x_0)} V(x)\,v_+^2(x,T)\mu(x) \leq \frac{c(T,\lambda,r)}{(r_1)^2} \sum_{x \in G} e^{-\Lambda \mathbf{d}^\alpha(x)} \chi_{\{r < \mathbf{d}(x) \leq r_1\}}(x)\mu(x)$$

$$= \frac{c(T,\lambda,r)}{(r_1)^2} \sum_{x \in G} e^{-\Lambda d^\alpha(x,x_0)} \chi_{\{r < \mathbf{d}(x) \leq r_1\}}(x)\mu(x). \tag{6.10}$$

Furthermore, we recall that r_1 was arbitrarily fixed, hence by taking the limit as $r_1 \to +\infty$ in (6.10) and by assumption (1.3), we get

$$\sum_{x \in B_r(x_0)} V(x)\,v_+^2(x,T)\mu(x) = 0.$$

Thus, we readily derive that

$$v(x,T) \leq 0 \quad \forall\, x \in B_r(x_0) \cap \mathrm{supp}(V), \ \ 0 \leq T \leq T_0(\lambda),$$

where we recall that the number T_0 depends on λ, which is by now fixed. From this, recalling also that $r \geq 2s + R_0$ was arbitrarily fixed, we then obtain

$$v(x,T) \leq 0 \quad \forall\, x \in \mathrm{supp}(V), \ \ 0 \leq T \leq T_0(\lambda). \tag{6.11}$$

Now, let us introduce the *'shifted' function*

$$v_1(x,t) = v(x, t + T_0(\lambda)).$$

Clearly, it is still a solution of the parabolic problem (4.6). In addition, by (6.11), $v_1(x,0) \leq 0$ for every $x \in \mathrm{supp}(V)$. By applying the very same argument exploited so far, we can infer that

$$v_1(x,T) \leq 0 \quad \forall\, x \in \mathrm{supp}(V),\ 0 \leq T \leq T_0(\lambda),$$

hence

$$v(x, T) \leq 0 \quad \forall\, x \in \mathrm{supp}(V), \;\; 0 \leq T \leq 2T_0(\lambda).$$

By iterating this argument, and by using in a crucial way the fact that $T_0(\lambda) > 0$ is a *universal number remaining unchanged at any iteration* (as $\lambda > 0$ is fixed), we conclude that

$$v(x, T) \leq 0 \quad \forall\, x \in \mathrm{supp}(V), \; T \geq 0.$$

This yelds (6.2).

We can now easily conclude the proof of the theorem, in fact, due to (6.2) and exploiting the definition of v, we have

$$0 < u \leq e^{-t} \quad \text{for every } x \in \mathrm{supp}(V) \text{ and } t > 0.$$

Then, by letting $t \to +\infty$, we deduce that $u \leq 0$ on $\mathrm{supp}(V) \neq \varnothing$, but this is clearly in contradiction with (6.1). This completes the proof.

7 Optimality on model trees

We start by showing a general non-uniqueness criterium which holds for any graph (G, ω, μ) such that (2.4) is fulfilled. We write $x \to \infty$ whenever $d(x, x_0) \to +\infty$, for some reference point $x_0 \in G$.

7.1 A general non-uniqueness criterium

Let us consider Eq. (1.1). We can prove the following result.

Proposition 7.1 *Let assumption (2.4) be in force. Moreover let $V \in \mathfrak{F}$, $V > 0$ and $\hat{R} > 0$. If there exists a supersolution to problem*

$$\begin{aligned}
\frac{1}{V}\Delta h &= -1 \quad \text{in } G \setminus B_{\hat{R}}, \\
h &> 0 \quad \text{in } G \setminus B_{\hat{R}}, \\
\lim_{x \to \infty} h(x) &= 0,
\end{aligned} \tag{7.1}$$

then there exist infinitely many bounded solutions u of problem (1.1). In particular, for any $\gamma \in \mathbb{R}$, $\gamma > 0$, there exists a solution u to problem (1.1) such that

$$\lim_{x \to \infty} u(x) = \gamma.$$

Proof Let $\gamma \in \mathbb{R}$, $\gamma > 0$. For any $j \in \mathbb{N}$, let us consider the following problem

$$\begin{cases}
\Delta u - V(x)u &= 0 \quad \text{in } B_j \\
u &= \gamma \quad \text{in } G \setminus B_j.
\end{cases} \tag{7.2}$$

Due to assumption (2.4), existence and uniqueness of a solution u_j to problem (7.2), in the sense of Definition 3.2, for any $j \in \mathbb{N}$ is granted by Proposition 3.5. We now claim that

$$0 \leq u_j(x) \leq \gamma \quad \text{for any } x \in G \text{ and for any } j \in \mathbb{N}. \tag{7.3}$$

In fact, since

$$\Delta u_j - V(x)u_j = 0 \quad \text{in } B_j,$$

and since $u_j = \gamma \geq 0$ in $G \setminus B_j$, from Lemma 3.3, we can infer that $u_j \geq 0$ in G. On the other hand, let $\overline{v}(x) := \gamma$ for any $x \in G$. Then, since $V(x) > 0$ for any $x \in G$, and since $\gamma > 0$

$$\Delta\overline{v}(x) - V(x)\overline{v}(x) = \frac{1}{\mu(x)} \sum_{y \in G} \omega(x, y)[\overline{v}(y) - \overline{v}(x)] - V(x)\overline{v}(x) \tag{7.4}$$

$$= -\gamma\, V(x) \leq 0 \quad \text{for any } x \in G.$$

For any $j \in \mathbb{N}$, let $w := \overline{v} - u_j$. Due to (7.2) and (7.4),

$$\Delta w - V(x)w \leq 0 \quad \text{in } G.$$

Moreover, $w \geq 0$ in $G \setminus B_j$. Hence, by Lemma 3.3, $w \geq 0$ in G and, in particular,

$$u_j \leq \gamma \quad \text{in } G.$$

Therefore, (7.3) follows. Furthermore, for any $j \in \mathbb{N}$, let u_{j+1} be the solution to problem (7.2) in B_{j+1}. Thus, in particular, observe that

$$\Delta u_{j+1} - V(x)u_{j+1} = 0 \quad \text{in } B_j,$$

and, by (7.3), $u_{j+1}(x) \leq \gamma$ for any $x \in G \setminus B_j$. Let $v := u_j - u_{j+1}$, then

$$\begin{cases} \Delta v - V(x)v = 0 & \text{in } B_j, \\ v = \gamma - u_{j+1} \geq 0 & \text{in } G \setminus B_j. \end{cases}$$

Therefore, by Lemma 3.3, $v \geq 0$ in G and, in particular, for any $j \in \mathbb{N}$,

$$u_{j+1} \leq u_j \quad \text{in } G. \tag{7.5}$$

Hence, from (7.3) and (7.5), we deduce that the sequence $\{u_j\}_{j \in \mathbb{N}}$ is *decreasing and bounded* on G. Therefore, there exists $u \in \mathfrak{F}$ such that

$$u(x) = \lim_{j \to +\infty} u_j(x), \quad x \in G.$$

Moreover, u_j solves problem (7.2) for any $j \in \mathbb{N}$ and since the sum which defines the Laplacian Δ is finite, by letting $j \to +\infty$ we obtain that u is a solution to problem (1.1). Let h be a supersolution to problem (7.1). Then we define, for any $x \in G \setminus B_{\hat{R}}$, with $\hat{R} > 0$ as in the assumptions,

$$\underline{w}(x) := -C\, h(x) + \gamma. \tag{7.6}$$

For any $j \in \mathbb{N}$ such that $B_j \supset B_{\hat{R}}$, we show that $\underline{w}$ is a subsolution to problem (7.2) in the sense of Definition 3.2. Due to (7.1) and since $V > 0$ in G, we have that

$$\Delta\underline{w}(x) - V(x)\underline{w}(x) = \frac{C}{\mu(x)} \sum_{y \in G} \omega(x, y)[(-h(y) + \gamma) - (-h(x) + \gamma)]$$

$$- V(x)(-Ch(x) + \gamma)$$

$$= -C\Delta h(x) + V(x)C\, h(x) - V(x)\gamma$$

$$\geq V(x)\,(C - \gamma)$$

$$\geq 0 \quad \text{for any } x \in B_j \setminus B_{\hat{R}},$$

provided that $C > 0$ is big enough. Moreover, $u_j \geq \underline{w}$ in $G \setminus B_j$ since $u_j = \gamma$ in $G \setminus B_j$. Therefore, by Lemma 3.3, we get

$$u_j(x) \geq \underline{w}(x) \quad \text{for any } x \in G \setminus B_{\hat{R}}. \tag{7.7}$$

By combining together (7.3) and (7.7) we get

$$-C h(x) + \gamma = \underline{w}(x) \leq u_j(x) \leq \gamma \quad \text{for any } x \in G \setminus B_{\hat{R}}.$$

By letting $j \to \infty$ we get

$$-C h(x) + \gamma \leq u(x) \leq \gamma \quad \text{for any } x \in G \setminus B_{\hat{R}},$$

thus, in particular, due to (7.1),

$$\lim_{x \to \infty} u(x) = \gamma.$$

$\square$

7.2 Model trees and a special supersolution

In this subsection we consider a special kind of graphs, the so called *model trees*, and we show that the uniqueness result in Theorem 2.4 is sharp for this choice of graph. More precisely, we show that the choice $\alpha \in [0, 1]$ in assumption (2.7) is optimal, indeed infinitely many bounded solutions exist whenever $\alpha > 1$. Let us first define a model tree.

Let (T, ω_0, μ_c) be a weighted graph as in Sect. 2, with $\mu_c(x) \equiv c$ (for $c > 0$). We assume that the *edge weight* ω_0 satisfies, together with (2.1), the following additional property:

$$\omega_0(x, y) \in \{0, 1\} \text{ for every} x, y \in G.$$

and we denote by d the distance on T *subordinated to* ω_0, that is,

$$d(x, y) = \inf \left\{ l_{\omega_0}(\gamma) : \gamma = \{x_k\}_{k=0}^n \text{ is a path between } x \text{ and } y \right\},$$

where $l_{\omega_0}(\gamma) = \sum_{k=0}^{n-1} \omega(x_k, x_{k+1})$. Accordingly, we define

$$S_r(x_0) = \{x \in T : d(x, x_0) = r\} \quad (x_0 \in G, \ r > 0).$$

We then say that the graph T is a *model tree* if the following properties holds:

a) T contains a vertex x_0, known as the root of the model;
b) denoting by $m(x)$ the number of edges which have x as endpoint (for every $x \in T$), the number $m(x)$ is constant on $S_r(x_0) = S_r$, that is

$$\text{if } x \in S_r(x_0) = \{x \in T : d(x, x_0) = r\}, \text{ then } m(x) = m(r).$$

If T is a model tree, we define the *branching* at the distance r from the root, and we denote it by $b(r)$, as the number of edges connecting each vertex in S_r to a vertex in S_{r+1}; we also agree to set $b(0) = m(x_0)$. Thus, for any $r > 0$, we have that

$$m(r) = b(r) + 1.$$

We say that the model tree is *homogeneous* if the branching is constant, that is,

$$b(r) = b \text{ for every } r \geq 0, \text{ for some } b \in \mathbb{N}.$$

Throughout what follows, we will deal with a *model homogeneous tree with branching b*, which will be denoted by (T_b, ω_0, μ_c); accordingly, we denote the ball $B_r(x_0)$ of radius r centered at the root x_0 of T_b simply by B_r. We explicitly notice that, by definition, we have

$$\ast)\ \omega_0(x, y) = 1 \text{ if and only if } d(x, y) = 1; \tag{7.8}$$

$$\ast)\ \text{if } \omega_0(x, y) = 1, \text{ then } d(y, x_0) = d(x, x_0) \pm 1; \tag{7.9}$$

Moreover, for every $x \in T_b \setminus \{x_0\}$ we also have

$$\ast)\ \text{card}\{y \in T_b : d(x, y) = 1, d(y, x_0) = d(x, x_0) + 1\} = b,$$
$$\ast)\ \text{card}\{y \in T_b : d(x, y) = 1, d(y, x_0) = d(x, x_0) - 1\} = 1. \tag{7.10}$$

We can now prove the following lemma.

Lemma 7.2 *Let (T_b, ω_0, μ_c) be a graph as above with branching $b \geq 2$, and let $V \in \mathfrak{F}$, $V > 0$ on T_b. Assume that, for some $\alpha > 1$ and $R_0 \geq 2$, we have*

$$V(x) \leq C_0\, d^{-\alpha}(x, x_0) \quad \text{for any } x \in G \setminus B_{R_0}. \tag{7.11}$$

Then there exist $\hat{C} > 0$, $\hat{R} > 0$ and $\beta > 0$ such that

$$h(x) := \hat{C} d^{-\beta}(x, x_0), \quad x \in G \setminus B_{\hat{R}}$$

is a supersolution to problem (7.1).

Proof Let $x \in G \setminus B_{R_0}$ be fixed. Setting $\mathbf{d}(\cdot) := d(\cdot, x_0)$, and observing that

$$\mathbf{d}(y) \geq \mathbf{d}(x) - 1 \geq 1 \quad \forall\, y \in T_b,\ y \sim x$$

(recall that $R_0 \geq 2$ and see (7.9)), by the Mean Value Theorem we have

$$\begin{aligned}
\Delta h(x) &= \frac{\hat{C}}{\mu_c} \sum_{y \in G} \omega_0(x, y)[\mathbf{d}^{-\beta}(y) - \mathbf{d}^{-\beta}(x)] \\
&= -\beta \frac{\hat{C}}{\mu_c} \sum_{y \in G} \omega_0(x, y)\xi^{-\beta-1}[\mathbf{d}(y) - \mathbf{d}(x)],
\end{aligned} \tag{7.12}$$

for some ξ in between $\mathbf{d}(y)$ and $\mathbf{d}(x)$. Clearly,

$$\begin{aligned}
\sum_{y \in G} \omega_0(x, y)\xi^{-\beta-1}[\mathbf{d}(y) - \mathbf{d}(x)] &= \sum_{\mathbf{d}(y) > \mathbf{d}(x)} \omega_0(x, y)\xi^{-\beta-1}[\mathbf{d}(y) - \mathbf{d}(x)] \\
&\quad + \sum_{\mathbf{d}(y) < \mathbf{d}(x)} \omega_0(x, y)\xi^{-\beta-1}[\mathbf{d}(y) - \mathbf{d}(x)] \\
&=: S^+ + S^-. \tag{7.13}
\end{aligned}$$

Moreover, by exploiting (7.9), (7.10) (and since $\beta > 0$), we have

(i) $S^+ \geq \displaystyle\sum_{\mathbf{d}(y) > \mathbf{d}(x)} \omega_0(x, y)\mathbf{d}^{-\beta-1}(y)[\mathbf{d}(y) - \mathbf{d}(x)] = b\,\mathbf{d}^{-\beta-1}(y) = b[\mathbf{d}(x) + 1]^{-\beta-1};$

(ii) $S^- \geq \displaystyle\sum_{\mathbf{d}(y) < \mathbf{d}(x)} \omega_0(x, y)\mathbf{d}^{-\beta-1}(x)[\mathbf{d}(y) - \mathbf{d}(x)] = -\mathbf{d}^{-\beta-1}(x).$

$$\tag{7.14}$$

By combining (7.12), (7.13) and (7.14), we then obtain

$$\Delta h(x) \le \frac{\hat{C}\beta}{\mu_c} \left\{ -b[\mathbf{d}(x) + 1]^{-\beta-1} + \mathbf{d}^{-\beta-1}(x) \right\}. \tag{7.15}$$

We finally observe that, given any $\varepsilon > 0$, there exists $\hat{R} > R_0$ such that

$$1 < \frac{[\mathbf{d}(x) + 1]^{\beta+1}}{\mathbf{d}^{\beta+1}(x)} < 1 + \varepsilon \quad \text{whenever } \mathbf{d}(x) > \hat{R}; \tag{7.16}$$

thus, if $x \in G \setminus B_{\hat{R}} \subseteq G \setminus B_{R_0}$, by (7.15), (7.16) and (7.11) we obtain

$$\frac{1}{V(x)} \Delta h(x) \le \frac{\hat{C}\beta}{\mu_c C_0} \mathbf{d}^\alpha(x) \mathbf{d}^{-\beta-1}(x) \left(-\frac{b}{1+\varepsilon} + 1 \right) \le -\frac{\hat{C}\hat{R}^{\alpha-\beta-1}\beta}{2C_0\mu_c} = -1,$$

provided that

$$\varepsilon = \frac{2b - 3}{3} > 0, \ 0 < \beta \le \alpha - 1, \ \hat{C} = \frac{2C_0\mu_c}{\hat{R}^{\alpha-\beta-1}\beta}.$$

Note that $\varepsilon > 0$ since $b \ge 2$, while $\beta > 0$ since $\alpha > 1$. This completes the proof. $\square$

From Proposition 7.1 and Lemma 7.2 we obtain the next

Corollary 7.3 *Let* (T_b, ω_0, μ_c) *be a graph as above with branching* $b \ge 2$, *and let* $V \in \mathfrak{F}$, $V > 0$ *on* T_b. *Assume that condition* (7.11) *is fulfilled.*

Then, there exist infinitely many bounded solutions u *of problem* (1.1). *In particular, for any* $\gamma \in \mathbb{R}$, $\gamma > 0$, *there exists a solution* u *to problem* (1.1) *such that*

$$\lim_{x \to \infty} u(x) = \gamma.$$

7.3 A counterexample

On account of Corollary 7.3, we can easily show that the requirement $\alpha \in [0, 1]$ in assumption (2.7) *cannot be dropped*, that is, this assumption is *optimal* if one restricts to a particular class of graphs.

To illustrate this fact, let (T_b, ω_0, μ_1) be a homogeneous model tree *with branching* $b \ge 2$ (and weighted counting measure $\mu_c \equiv 1$), and let x_0 be the root of the model. Given any number $\alpha > 1$, we then consider the function $V \in \mathfrak{F}$ defined as follows:

$$V(x) := (1 + d(x, x_0))^{-\alpha}$$

(where d is the usual distance on trees). Clearly, assumptions (2.3) and (2.4) are satisfied in this context; moreover, $V : T_b \to \mathbb{R}$ is a *strictly positive potential* on T_b, but the last condition in assumption (2.7) is obviously violated (since $\alpha > 1$).

We now observe that, since $\alpha > 1$, the series (1.3) is convergent *for every choice of* $\Lambda \in (0, 1)$: in fact, recalling that $\mu_1 \equiv 1$, we have the following computation

$$\sum_{x \in G \setminus B_1(x_0)} e^{-\Lambda d^\alpha(x, x_0)} \mu(x) = \sum_{n=1}^\infty \left(\sum_{\{x : d(x, x_0) = n\}} e^{-\Lambda d^\alpha(x, x_0)} \mu(x) \right)$$

$$= \sum_{n=1}^\infty e^{-\Lambda n^\alpha} \mu(\{x : d(x, x_0) = n\}) = \sum_{n=1}^\infty b^n e^{-\Lambda n^\alpha},$$

and the series $\sum_n b^n e^{-\Lambda n^\alpha}$ converges by the Ratio Test.

On the other hand, since $b \geq 2$ and since V is a strictly positive potential on T_b satisfying condition (7.11) (with $R_0 = 1$ and $C_0 = 1$), we are entitled to apply Corollary 7.3, ensuring that there exist *infinitely many* non-trivial bounded solutions to Eq. (1.1). Hence, the condition $\alpha \in [0, 1]$ is optimal in this context.

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