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On p-Brunn–Minkowski inequalities for intrinsic volumes, with $0 \leq p < 1$

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Abstract
We prove the validity of the p-Brunn–Minkowski inequality for the intrinsic volume V_k, $k = 2, \dots, n-1$, of symmetric convex bodies in $\mathbb{R}^n$, in a neighbourhood of the unit ball when one of the bodies is the unit ball, for $0 \leq p < 1$. We also prove that this inequality does not hold true on the entire class of convex bodies of $\mathbb{R}^n$, when p is sufficiently close to 0.

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1 Introduction

The Brunn–Minkowski inequality is one of the cornerstones of convex geometry, the branch of mathematics which studies the theory of convex bodies; in one of its formulations, it states that the volume functional V_n is $(\frac{1}{n})$-concave, that is

$$V_n((1-t)K_0 + tK_1)^{1/n} \geq (1-t)V_n(K_0)^{1/n} + tV_n(K_1)^{1/n}, \quad \forall K_0, K_1 \in \mathcal{K}^n, \ \forall\, t \in [0,1], \tag{1.1}$$

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and equality holds if and only if K_0 and K_1 are homothetic, or contained in parallel hyperplanes. Here V_n is the *volume*, that is, the Lebesgue measure in $\mathbb{R}^n$, K_0, K_1 belong to the set of compact and convex subsets (*convex bodies*) of $\mathbb{R}^n$, denoted by $\mathcal{K}^n$, and the "sum" of sets indicates the *Minkowski linear combination*, that is, the vectorial sum. We refer the reader to the survey paper [15], and to the monograph [30] for a thorough presentation of the Brunn–Minkowski inequality, its numerous connections to other areas of mathematics, and applications.

Inequality (1.1) has a great number of variations and generalizations which consider different kinds of sums and different kinds of shape functionals; here we are interested in extending it to the so called *p-addition* of convex bodies and to the *intrinsic volumes*, rather than the classical volume.

The p-sum of convex bodies was introduced by Firey in [14] for $p \geq 1$, and offers an extension to the Minkowski sum (which represents the case $p = 1$). Its definition is based on the *support function* of a convex body K, which is denoted by $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ (see Sect. 2 for definitions and preliminary results) and finds its motivation starting from the behaviour of the support function with respect to the Minkowski sum of sets. More precisely: for every $K, L \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ the following equality holds:

$$h_{\alpha K+\beta L} = \alpha h_K + \beta h_L.$$

This relation motivates the definition of p-addition, for $p \geq 1$: for $K, L \in \mathcal{K}^n$, both containing the origin (that is, $K, L \in \mathcal{K}^n_0$), and for $\alpha, \beta \geq 0$, the p-combination $\alpha \cdot K +_p \beta \cdot L$, with $p \geq 1$, is defined as the convex body whose support function is given by

$$h_{\alpha\cdot K+_p\beta\cdot L} = (\alpha h_K^p + \beta h_L^p)^{1/p}.$$

This definition is well posed since $(\alpha h_K^p + \beta h_L^p)^{1/p}$ is a (non-negative) support function, by the condition $p \geq 1$. The p-addition is at the core of the branch of convex geometry currently known as L_p-Brunn–Minkowski theory (see [30, Chapter 9]), which received a major impulse by the works of Lutwak (see for instance [22, 23]).

A recent breakthrough in this context is due to the works [5, 6] by Böröczky, Lutwak, Yang and Zhang, where the authors begin the analysis of the range $p < 1$, focusing on the case $p = 0$. In particular, in [5] they establish the following form of the Brunn–Minkowski inequality for the case $p = 0$, called the log-Brunn–Minkowski inequality, which we state in Theorem 1.1. Given K_0 and K_1 in $\mathcal{K}^n_0$, and $t \in [0, 1]$, consider the function $h_t : \mathbb{S}^{n-1} \to \mathbb{R}^n$ defined by

$$h_t := h_{K_0}^{1-t} h_{K_1}^t.$$

Then define the convex body $(1-t)\cdot K_0 +_0 t \cdot K_1$ as the *Aleksandrov body*, or Wulff shape, of h_t; that is:

$$(1-t)\cdot K_0 +_0 t \cdot K_1 := \{x \in \mathbb{R}^n : (x, y) \leq h_t(y)\ \forall\, y \in \mathbb{S}^{n-1}\}, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the standard scalar product of $\mathbb{R}^n$.

Theorem 1.1 (Böröczky, Lutwak, Yang, Zhang) *For every $K_0, K_1 \in \mathcal{K}_0^2$, origin symmetric, and for every $t \in [0, 1]$:*

$$V_2((1-t) \cdot K_0 +_0 t \cdot K_1) \geq V_2(K_0)^{1-t} V_2(K_1)^t.$$

Equality holds if and only if K_0 and K_1 are dilates of each other, or they are parallelotopes.

In [5] the authors conjectured the same result to be valid in arbitrary dimension; this is the well known log-Brunn–Minkowski inequality conjecture.

Conjecture (Log-Brunn–Minkowski inequality–Böröczky, Lutwak, Yang, Zhang) For every $K_0, K_1 \in \mathcal{K}_0^n$, origin symmetric, and for every $t \in [0, 1]$:

$$V_n((1-t) \cdot K_0 +_0 t \cdot K_1) \geq V_n(K_0)^{1-t} V_n(K_1)^t. \tag{1.3}$$

The idea used in (1.2) to define the 0-sum has been extended to define the L_p convex combination of $K_0, K_1 \in \mathcal{K}_0^n$, for $p \in (0, 1)$. Given $t \in [0, 1]$, we set

$$(1-t) \cdot K_0 +_p \ t \cdot K_1 := \left\{ x \in \mathbb{R}^n : (x, y) \leq \left((1-t)h_{K_0}^p(y) + t h_{K_1}^p(y)\right)^{1/p}, \ \forall\, y \in \mathbb{S}^{n-1} \right\}.$$

Note that, by standard properties of p-means, for every $p \geq 0$ it holds:

$$(1-t) \cdot K_0 +_0 t \cdot K_1 \subseteq (1-t) \cdot K_0 +_p \ t \cdot K_1, \quad \forall\, K_0, K_1 \in \mathcal{K}_0^n, \ \forall\, t \in [0, 1] \tag{1.4}$$

(see (2.4) below). Hence, by the previous inclusion and by a standard argument based on the homogeneity of the volume, inequality (1.3) would imply:

$$V_n((1-t) \cdot K_0 +_p t \cdot K_1)^{p/n} \geq (1-t) V_n(K_0)^{p/n} + t V_n(K_1)^{p/n}, \tag{1.5}$$

for every $p \geq 0$, that is a Brunn–Minkowski type inequality for the p-sum, for $p \geq 0$.

The conjectures about the validity of (1.3) and (1.5) originated an intense activity in recent years, and much progress has been made in this area (see [3, 4, 8, 10, 11, 17, 20, 21, 24–27, 29, 31]).

As already mentioned, inequality (1.1) has been generalized in many directions. It became the prototype for many similar inequalities, which bear the name of "Brunn–Minkowski type inequalities", where the volume functional is replaced by other functionals. Among them, we mention those verified by intrinsic volumes as functionals defined on $\mathcal{K}^n$, with respect to the standard Minkowski addition. Indeed, for every $k \in \{0, 1, \ldots, n\}$, the following inequality holds:

$$V_k((1-t)K_0 + tK_1)^{1/k} \geq (1-t)V_k(K_0)^{1/k} + tV_k(K_1)^{1/k}, \quad \forall\, K_0, K_1 \in \mathcal{K}^n, \ \forall\, t \in [0, 1] \tag{1.6}$$

(see [30, Theorem 7.4.5]), where V_k is the k-th intrinsic volume. In particular, when $k = 1$ equality holds in the previous inequality for every K_0, K_1 and t, while for $k = n$ inequality (1.6) is the classical Brunn–Minkowski inequality (1.1). Note that (1.6) implies a corresponding inequality with respect to the p-addition, for every $p \geq 1$, in $\mathcal{K}_0^n$, due to the monotonicity of intrinsic volumes.

The question that we consider in this paper is whether intrinsic volumes verify a Brunn–Minkowski inequality with respect to the p-addition, for $p \in [0, 1)$, in $\mathcal{K}_0^n$.

To begin with, we present the case $k \in \{2, \dots, n-1\}$ (note that, for $k = 0$, V_0 is constant, $k = n$ is the case of the volume, and the case $k = 1$ will be described separately). We prove two types of results, one in the affirmative and one in the negative direction. Our first two results state the validity of the p-Brunn–Minkowski inequality for intrinsic volumes, in a suitable neighborhood of the unit ball B_n of $\mathbb{R}^n$, for every $p \in [0, 1)$. We denote by $\mathcal{K}_{0,s}^n$ the family of origin symmetric convex bodies.

Theorem 1.2 *There exists $\eta > 0$ such that for every $k \in \{2, \dots, n\}$ and $p \in [0, 1)$, if $K \in \mathcal{K}_{0,s}^n$ is of class $C^{2,+}$ and verifies*

$$\|1 - h_K\|_{C^2(\mathbb{S}^{n-1})} \leq \eta,$$

then

$$V_k((1-t) \cdot B_n +_0 t \cdot K) \geq V_k(B_n)^{1-t}\; V_k(K)^t \quad \forall\, t \in (0, 1), \tag{1.7}$$

and

$$V_k((1-t) \cdot B_n +_p t \cdot K)^{p/k} \geq (1-t) V_k(B_n)^{p/k} + t V_k(K)^{p/k} \quad \forall\, t \in (0, 1). \tag{1.8}$$

Moreover, equality holds in (1.7) and in (1.8), if and only if K is a ball centered at the origin.

The proof of (1.7) in this theorem is in the same spirit of analogous (and, in fact, stronger) results concerning the volume, proved in [10, 11, 21]. The argument can be heuristically described as follows: (1.7) is equivalent to the concavity of $\log(V_k)$, with respect to the 0-addition. We compute the second variation of $\log(V_k)$, and we prove that it is negative definite at the unit ball. Then, by a continuity argument, we deduce that such second variation continues to be negative definite in a neighborhood of B_n. As in the case of the volume, determining the sign of the second variation amounts to analysing the spectrum of a second order elliptic operator on $\mathbb{S}^{n-1}$. As it is pointed out in [21], this method dates back to the proof of the standard Brunn–Minkowski inequality for the volume, given by Hilbert (see also [2]).

Concerning (1.8), we deduce it from (1.7), using (1.4) and an argument based on homogeneity. Moreover, again by homogeneity, (1.8) and (1.7) could be stated replacing B_n with the ball of radius $R > 0$ centered at the origin, for an arbitrary R.

In the case of the volume, $k = n$, and for $n \geq 3$ and $p = 0$, Theorem 1.2 is contained in [10]. For $0 < p < 1$, when p is sufficiently close to 1, the validity of the

p-Brunn–Minkowski inequality for the volume has been proved for the entire class of centrally symmetric convex bodies, with contributions by Kolesnikov and Milman [21] and Chen, Huang, Li, Liu [8] (see also [26]).

We also show by counterexamples that for p sufficiently close to 0, the p-Brunn–Minkowski inequality for V_k does not hold in $\mathcal{K}^n$, for any $k \in \{2, \dots, n-1\}$.

Theorem 1.3 *For every $n \geq 3$, $k \in \{2, \dots, n-1\}$, there exists $\bar{p} \in (0, 1)$ such that for every $p \in (0, \bar{p})$ the p-Brunn–Minkowski inequality for V_k does not hold in $\mathcal{K}^n_{0,s}$. That is, there exist $K_0, K_1 \in \mathcal{K}^n_{0,s}$ such that*

$$V_k \left(\frac{1}{2} \cdot K_0 +_p \frac{1}{2} \cdot K_1 \right)^{\frac{p}{k}} < \frac{1}{2} V_k(K_0)^{\frac{p}{k}} + \frac{1}{2} V_k(K_1)^{\frac{p}{k}}, \tag{1.9}$$

if $0 < p < \bar{p}$.

Indeed we built some counterexamples by considering k-dimensional cubes, with faces parallel to coordinate hyperplanes, embedded in $\mathbb{R}^n$ in such a way that the dimension of their intersection is minimized. The construction shows how the value $\bar{p}$ depends on n and k.

The analysis of the case $k = 1$ yields a reverse Brunn–Minkowski inequality: this is a direct consequence of the linearity of V_1 with respect to the Minkowski addition.

Theorem 1.4 *Let $n \geq 3$, $p \in [0, 1)$, $t \in [0, 1]$. For every $K_0, K_1 \in \mathcal{K}^n_0$ it holds*

$$V_1((1-t) \cdot K_0 +_p t \cdot K_1)^p \leq (1-t) V_1(K_0)^p + t V_1(K_1)^p. \tag{1.10}$$

Moreover, equality holds if and only if either one of the two bodies K_0 and K_1 coincides with $\{0\}$, or they coincide up to a dilation.

As it is well known, Brunn–Minkowski type inequalities (and in particular equality conditions), are often decisive for uniqueness in the corresponding Minkowski problem. The relevant problem for intrinsic volumes is in fact the so-called Christoffel–Minkowski problem, which asks to determine a convex body when one of its area measures is prescribed (see [30, Section 8.4]). As an application of Theorem 1.2, we find the following local uniqueness result for the solution of the L_p version of the Christoffel–Minkowski problem, $0 \leq p < 1$.

Theorem 1.5 *There exists $\eta > 0$ such that for every $k \in \{2, \dots, n-2\}$ and $p \in [0, 1)$, if $K \in \mathcal{K}^n_{0,s}$ is of class $C^{2,+}$ and*

$$\|1 - h_K\|_{C^2(\mathbb{S}^{n-1})} \leq \eta,$$

then the condition

$$h_K^{1-p} \, \mathrm{d}S_{k-1}(K, \cdot) = \mathrm{d}S_{k-1}(B_n, \cdot), \tag{1.11}$$

implies that $K = B_n$ (here S_{k-1} denotes the area measure *of order $(k-1)$ of K).*

A more general result, under the assumption that $h_k \in C^4(\mathbb{S}^{n-1})$, is proved in [7] (see also references contained therein). Moreover, the case $k = n$ of Theorem 1.5 was already established in [10]. Note that (1.11) can be written as a partial differential equation on $\mathbb{S}^{n-1}$:

$$h(x)^{1-p} S_{k-1}(h_{ij}(x) + h(x)\delta_{ij}) = c_{n,k}, \tag{1.12}$$

where h indicates the support function of K, h_{ij}, for $i, j = 1, \dots, n-1$, are the second covariant derivatives of h with respect to a local orthonormal frame on $\mathbb{S}^{n-1}$, δ_{ij} are the usual Kronecker symbols, $S_{k-1}(h_{ij} + h\delta_{ij})$ is the elementary symmetric function of order $(k-1)$ of the eigenvalues of the matrix $(h_{ij} + h\delta_{ij})$ and $c_{n,k} = \binom{n-1}{k-1}$.

In Sect. 5.1 we also present a local uniqueness result for Eq. (1.12), in the more general context of Sobolev spaces. The argument is based on the inverse function theorem for Banach spaces, and it is completely independent of the Brunn–Minkowski inequality.

Organization of the paper After some preliminaries, given in Sect. 2, Sect. 3 concerns properties of intrinsic volumes relevant to the computation of their first and second variations with respect to the p-addition. The proofs of Theorems 1.2 and 1.4 are given in Sect. 4, while Theorem 1.5 is proved in Sect. 5. Eventually, the proof of Theorem 1.3 is contained in Sect. 6.

2 Preliminaries

2.1 Notations

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, endowed with the Euclidean norm $|\cdot|$ and the scalar product $(\cdot, \cdot)$. We denote by $B_n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ the unit ball and the unit sphere, respectively.

2.1.1 Convex bodies and Wulff shapes

The symbol $\mathcal{K}^n$ indicates the set of convex bodies in $\mathbb{R}^n$, that is, convex and compact subsets of $\mathbb{R}^n$. For every $K \in \mathcal{K}^n$, h_K denotes the *support function* of K, which is defined, for every $x \in \mathbb{S}^{n-1}$, as:

$$h_K(x) = \sup\{(x, y) \colon y \in K\}.$$

We say that $K \in \mathcal{K}^n$ is of class $C^{2,+}$ if its boundary ∂K is of class C^2 and the Gauss curvature is positive at every point of ∂K.

We denote by $\mathcal{K}_0^n$ the family of convex bodies containing the origin in their interior and by $\mathcal{K}_{0,s}^n$ the family of those elements of $\mathcal{K}_0^n$ which are origin symmetric. We underline that a convex body contains the origin if and only if its support function is non-negative on $\mathbb{S}^{n-1}$, and the origin is an interior point if and only if the support function is strictly positive on $\mathbb{S}^{n-1}$. Moreover, $K \in \mathcal{K}^n$ is origin symmetric if and only if h_K is even.

For $p \geq 1$, the L_p *Minkowski linear combination* of K and L in $\mathcal{K}_0^n$, with coefficients $\alpha, \beta \geq 0$, denoted by

$$\alpha \cdot K +_p \beta \cdot L,$$

is defined through the relation

$$h_{\alpha \cdot K +_p \beta \cdot L} = (\alpha h_K^p + \beta h_L^p)^{1/p}, \quad \forall\, K, L \in \mathcal{K}_0^n,\ \forall \alpha, \beta \geq 0. \tag{2.1}$$

Given a continuous function $f \in C(\mathbb{S}^{n-1})$ and $f > 0$, we define its *Aleksandrov body*, or *Wulff shape*, as

$$K[f] = \{x \in \mathbb{R}^n : (x, y) \leq f(y),\ \forall\, y \in \mathbb{S}^{n-1}\}.$$

It is not hard to prove that $K[f]$ is a convex body, and

$$h_{K[f]} \leq f.$$

Moreover, equality holds in the previous inequality if f is a support function. Notice that, by definition, if $f \equiv R$, where R is a positive constant, it holds $K[f] = RB_n$.

For $p > 0$, the L_p *Minkowski convex combination* of $K_0, K_1 \in \mathcal{K}_0^n$, with parameter $t \in [0, 1]$, is defined as:

$$(1-t) \cdot K_0 +_p t \cdot K_1 = K[((1-t)h_{K_0}^p + t h_{K_1}^p)^{1/p}], \tag{2.2}$$

and we interpret the case $p = 0$ in the following limiting sense:

$$(1-t) \cdot K_0 +_0 t \cdot K_1 = K[h_{K_0}^{1-t} h_{K_1}^t]. \tag{2.3}$$

Notice that if $p \geq 1$ this coincides with the classical L_p Minkowski linear combination defined by (2.1). Moreover, for all $p \in (0, 1)$, the following chain of inclusions

$$\begin{aligned} (1-t) \cdot K_0 +_0 t \cdot K_1 \subseteq (1-t) \cdot K_0 +_p\ t \cdot K_1 \subseteq (1-t)K_0 +\ tK_1, \\ \forall\, K_0, K_1 \in \mathcal{K}_0^n,\ \forall\, t \in [0, 1] \end{aligned} \tag{2.4}$$

holds. Indeed, by the monotonicity property of the p-means we have

$$h_0^{1-t} h_1^t \leq \left((1-t)h_0^p + t h_1^p\right)^{\frac{1}{p}} \leq (1-t)h_0 + t\, h_1,$$

where h_0 and h_1 denote the support functions of K_0 and K_1, respectively; hence from (2.3) and (2.2) we get (2.4).

For $K \in \mathcal{K}^n$ and $k \in \{0, \ldots, n\}$, $V_k(K)$ denotes the k-th intrinsic volume of K; the intrinsic volumes are described in more details in Sect. 3.

2.2 The matrix *Q*

For a function $\varphi \in C^2(\mathbb{S}^{n-1})$ and $i, j \in \{1, \dots, n-1\}$, we denote by φ_i, φ_{ij} the first and second covariant derivatives with respect to a local orthonormal frame on $\mathbb{S}^{n-1}$. Moreover, we set

$$Q[\varphi] = (Q_{ij}[\varphi])_{i,j=1,\dots,n-1} = (\varphi_{ij} + \varphi\delta_{ij})_{i,j=1,\dots,n-1},$$

where δ_{ij}, $i, j \in \{1, \dots, n-1\}$, denotes the Kronecker symbols. Then $Q[\varphi]$ is a symmetric matrix of order $(n-1)$.

The following proposition can be deduced, for instance, from [30, Section 2.5] and it will be used several times in the paper.

Proposition 2.1 *Let $K \in \mathcal{K}^n$. The body K is of class $C^{2,+}$ if and only if its support function h_K is a $C^2(\mathbb{S}^{n-1})$ function and*

$$Q[h_K] > 0 \quad on\ \mathbb{S}^{n-1}.$$

We set

$$C^{2,+}(\mathbb{S}^{n-1}) = \{h \in C^2(\mathbb{S}^{n-1}) \colon Q[h] > 0 \text{ on } \mathbb{S}^{n-1}\};$$

that is, $C^{2,+}(\mathbb{S}^{n-1})$ is the set of support functions of convex bodies of class $C^{2,+}$. We also denote by $C_0^{2,+}(\mathbb{S}^{n-1})$ the set of support functions of convex bodies of class $C^{2,+}$ in $\mathcal{K}_0^n$.

2.3 Elementary symmetric functions of a matrix

Let $N \in \mathbb{N}$ (in most cases we will consider $N = (n-1)$); we denote by $\mathrm{Sym}(N)$ the space of symmetric square matrices of order N. For $A = (a_{ij}) \in \mathrm{Sym}(N)$ and for $r \in \{1, \dots, N\}$, we denote by $S_r(A)$ the *r-th elementary symmetric function* of the eigenvalues $\lambda_1, \dots, \lambda_N$ of A:

$$S_r(A) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le N} \lambda_{i_1} \dots \lambda_{i_r}.$$

For completeness, we set $S_0(A) = 1$. Note that

$$S_N(A) = \det(A), \quad S_1(A) = \mathrm{tr}(A).$$

For $r \in \{1, \dots, N\}$ and $i, j \in \{1, \dots, N\}$, we set

$$S_r^{ij}(A) = \frac{\partial S_r(A)}{\partial a_{ij}}.$$

The symmetric matrix $(S_r^{ij}(A))_{i,j\in\{1,\dots,N\}}$ is called the *r-cofactor* matrix of A. In the special case $r = N$, this is the standard cofactor matrix (in particular $(S_N^{ij}(A))_{i,j\in\{1,\dots,N\}} = A^{-1}\det(A)$, provided $\det(A) \neq 0$). If I_N denotes the identity matrix of order N, then, for every $r \in \{1,\dots,N\}$, we have

$$S_r(\mathrm{I}_N) = \binom{N}{r}. \tag{2.5}$$

Moreover,

$$S_r^{ij}(\mathrm{I}_N) = \binom{N-1}{r-1}\delta_{ij}, \quad \forall\, i,j \in \{1,\dots,N\},$$

i.e.

$$(S_r^{ij}(\mathrm{I}_N))_{i,j\in\{1,\dots,N\}} = \binom{N-1}{r-1}\mathrm{I}_N. \tag{2.6}$$

This follows from (2.5) and from [12, Proposition 2.1]. We also set

$$S_r^{ij,kl}(A) = \frac{\partial^2 S_r(A)}{\partial a_{ij}\partial a_{kl}},$$

for $r \in \{1,\dots,N\}$ and $i,j,k,l \in \{1,\dots,N\}$.

2.4 Integration by parts formulas

The following integration by parts formulas hold.

Proposition 2.2 *For every h, ψ, φ, $\bar\varphi \in C^2(\mathbb{S}^{n-1})$,*

$$\int_{\mathbb{S}^{n-1}} \bar\varphi S_k^{ij}(Q[h])(\varphi_{ij}+\varphi\delta_{ij})\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}} \varphi S_k^{ij}(Q[h])(\bar\varphi_{ij}+\bar\varphi\delta_{ij})\,\mathrm{d}x, \tag{2.7}$$

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} \psi S_k^{ij,kl}(Q[h])(\varphi_{ij}+\varphi\delta_{ij})(\bar\varphi_{ij}+\bar\varphi\delta_{ij})\,\mathrm{d}x\\ &\quad= \int_{\mathbb{S}^{n-1}} \bar\varphi S_k^{ij,kl}(Q[h])(\varphi_{ij}+\varphi\delta_{ij})(\psi_{ij}+\psi\delta_{ij})\,\mathrm{d}x,\end{aligned} \tag{2.8}$$

where we have used the convention of summation over repeated indices.

The proof follows from Lemma 2.3 in [9] (see also [10, (11)]).

2.5 The Poincaré inequality on the sphere

Given a function $\psi \in C^1(\mathbb{S}^{n-1})$, we denote by $\nabla\psi$ the *spherical gradient* of ψ (that is the gradient of ψ as an application from $\mathbb{S}^{n-1}$ to $\mathbb{R}$; see for instance [1]). For

$\psi \in C^2(\mathbb{S}^{n-1})$, we denote by $\|\psi\|_{L^2(\mathbb{S}^{n-1})}$, $\|\nabla\psi\|_{L^2(\mathbb{S}^{n-1})}$, $\|\psi\|_{C^2(\mathbb{S}^{n-1})}$ the standard L^2 and C^2 norms on the sphere, respectively.

Proposition 2.3 (Poincaré inequality on $\mathbb{S}^{n-1}$) *For every $\psi \in C^1(\mathbb{S}^{n-1})$ such that*

$$\int_{\mathbb{S}^{n-1}} \psi(x)\,\mathrm{d}x = 0,$$

it holds

$$\int_{\mathbb{S}^{n-1}} \psi^2(x)\,\mathrm{d}x \le \frac{1}{n-1}\int_{\mathbb{S}^{n-1}} |\nabla\psi(x)|^2\,\mathrm{d}x.$$

The constant in the previous inequality can be improved under a symmetry assumption, as the following result shows (see, e.g., Section 2 in [10] for the proof).

Proposition 2.4 *(*Poincaré inequality on $\mathbb{S}^{n-1}$ with symmetry*) Let $\psi \in C^1(\mathbb{S}^{n-1})$ be even, and such that*

$$\int_{\mathbb{S}^{n-1}} \psi(x)\,\mathrm{d}x = 0.$$

Then

$$\int_{\mathbb{S}^{n-1}} \psi^2(x)\,\mathrm{d}x \le \frac{1}{2n}\int_{\mathbb{S}^{n-1}} |\nabla\psi(x)|^2\,\mathrm{d}x. \tag{2.9}$$

3 Intrinsic volumes

Given a convex body $K \in \mathcal{K}^n$ of class $C^{2,+}$ and $k \in \{0, \dots, n\}$, the k-th intrinsic volume of K can be written in the form:

$$V_k(K) = \frac{1}{k\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} h(x)S_{k-1}(Q[h](x))\,\mathrm{d}x,$$

where $h = h_K \in C^{2,+}(\mathbb{S}^{n-1})$ is the support function of K, and κ_j denotes the j-dimensional volume of the unit ball in $\mathbb{R}^j$ (see formulas (2.43), (4.9) and (5.56) in [30]). Based on the previous formula, we consider the functional

$$F_k\colon C^{2,+}(\mathbb{S}^{n-1}) \to [0,\infty), \quad F_k(h) = \frac{1}{k}\int_{\mathbb{S}^{n-1}} h(x)S_{k-1}(Q[h](x))\,\mathrm{d}x,$$

and we define

$$\mathcal{F}_k\colon C^{2,+}(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{n-1}), \quad \mathcal{F}_k(h) = S_{k-1}(Q[h]). \tag{3.1}$$

The functionals F_k and $\mathcal{F}_k$ have the following properties.

- $F_k(h) = \dfrac{1}{k} \displaystyle\int_{\mathbb{S}^{n-1}} h(x)\mathcal{F}_k(h)(x)\,\mathrm{d}x.$
- $\mathcal{F}_k$ is positively homogeneous of order $(k-1)$, that is:

$$\mathcal{F}_k(th) = t^{k-1}\mathcal{F}_k(h), \quad \forall\, h \in C^{2,+}(\mathbb{S}^{n-1}),\ \forall t > 0.$$

  Consequently, F_k is positively homogeneous of order k.
- For every $h \in C^{2,+}(\mathbb{S}^{n-1})$ there exists a linear functional $L_k(h)\colon C^2(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{n-1})$ such that, for every $\varphi \in C^2(\mathbb{S}^{n-1})$,

$$\lim_{s\to 0} \frac{\mathcal{F}_k(h+s\varphi) - \mathcal{F}_k(h)}{s} = L_k(h)\varphi,$$

  where $L_k(h)\varphi$ denotes $L_k(h)$ applied to φ. L_k admits the following representation:

$$L_k(h)\varphi = S_{k-1}^{ij}(Q[h])(\varphi_{ij} + \varphi\delta_{ij}), \tag{3.2}$$

  where the summation convention over repeated indices is used (see [12, Proposition 4.2]).
- For every $h \in C^{2,+}(\mathbb{S}^{n-1})$, $L_k(h)$ is self-adjoint, that is,

$$\int_{\mathbb{S}^{n-1}} \psi L_k(h)\varphi\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}} \varphi L_k(h)\psi\,\mathrm{d}x,$$

  for every $\varphi, \psi \in C^2(\mathbb{S}^{n-1})$. This follows from Proposition 2.2.

We conclude this section by recalling the first and second variation of the functional F_k. In order to do this, we fix an element h of $C^{2,+}(\mathbb{S}^{n-1})$ and we consider a differentiable path h_s in $C^{2,+}(\mathbb{S}^{n-1})$, passing through h. In other words, for some $\varepsilon > 0$, we have a map $(-\varepsilon, \varepsilon) \ni s \mapsto h_s \in C^{2,+}(\mathbb{S}^{n-1})$ such that

$$h_0 = h,$$

and the following derivatives exist for every s and for every $x \in \mathbb{S}^{n-1}$:

$$\dot{h}_s(x) := \frac{d}{ds}h_s(x), \quad \ddot{h}_s(x) := \frac{d^2}{ds^2}h_s(x) \quad \text{and} \quad \dddot{h}_s(x) := \frac{d^3}{ds^3}h_s(x).$$

We also set

$$\dot{h} = \dot{h}_s\big|_{s=0}, \quad \ddot{h} = \ddot{h}_s\big|_{s=0} \quad \text{and} \quad \dddot{h} = \dddot{h}_s\big|_{s=0}.$$

We assume that the limits giving the previous derivatives are uniform in x.

In the following proposition we recall the first and second variations of F_k (we refer to [12] for the proofs).

Proposition 3.1 *For every $h \in C^{2,+}(\mathbb{S}^{n-1})$, for every $\varphi \in C^2(\mathbb{S}^{n-1})$ and for every $s \in (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$ sufficiently small, we have*

$$\frac{d}{ds}F_k(h_s) = \int_{\mathbb{S}^{n-1}} \dot{h}_s \mathcal{F}_k(h_s)\,\mathrm{d}x,$$

and

$$\frac{d^2}{ds^2}F(h_s) = \int_{\mathbb{S}^{n-1}} \ddot{h}_s \mathcal{F}(h_s)\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}} \dot{h}_s L(h_s)\dot{h}_s\,\mathrm{d}x.$$

4 Proof of Theorems 1.2 and 1.4

4.1 The case $k \in \{2, \dots, n\}$

In this subsection we prove Theorem 1.2.

4.1.1 Computations and estimates of derivatives

We consider a convex body $K \in \mathcal{K}_0^n$ of class $C^{2,+}$, and we denote by $h \in C_0^{2,+}(\mathbb{S}^{n-1})$ its support function. We fix $\psi \in C^2(\mathbb{S}^{n-1})$ and we define, for sufficiently small $|s|$,

$$h_s = he^{s\psi}. \tag{4.1}$$

The proof of the following result follows from [10, Remark 3.2].

Lemma 4.1 *Let $h \in C_0^{2,+}(\mathbb{S}^{n-1})$ and h_s as in (4.1); there exists $\eta_0 > 0$ (depending on h) with the following property: if $\psi \in C^2(\mathbb{S}^{n-1})$ and*

$$\|\psi\|_{C^2(\mathbb{S}^{n-1})} \le \eta_0,$$

then

$$h_s \in C_0^{2,+}(\mathbb{S}^{n-1}), \quad \forall\, s \in [-2, 2].$$

From here on, we will always assume that h and ψ are such that $h_s \in C_0^{2,+}(\mathbb{S}^{n-1})$ for every $s \in [-2, 2]$. As in Sect. 3, we denote by $\dot{h}_s$, $\ddot{h}_s$, $\dddot{h}_s$ the first, second and third derivatives of h_s with respect to s, respectively. When the index s is omitted, it means that these derivatives are computed at $s = 0$.

Let $k \in \{2, \dots, n\}$; we are interested in the function

$$f_k(s) := \frac{1}{k}\int_{\mathbb{S}^{n-1}} h_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x = \frac{1}{k}\int_{\mathbb{S}^{n-1}} h_s(x)\mathcal{F}_k(h_s)\,\mathrm{d}x. \tag{4.2}$$

Lemma 4.2 *With the notations introduced above, we have, for every s:*

$$f_k'(s) = \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x,$$
$$f_k''(s) = \int_{\mathbb{S}^{n-1}} \ddot{h}_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x)\,\mathrm{d}x,$$

and

$$\begin{aligned} f_k'''(s) &= \int_{\mathbb{S}^{n-1}} \dddot{h}_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x + 2\int_{\mathbb{S}^{n-1}} \ddot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x)\,\mathrm{d}x \\ &+ \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}^{ij,rs}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x) Q_{rs}[\dot{h}_s](x)\,\mathrm{d}x \\ &+ \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\ddot{h}_s](x)\,\mathrm{d}x. \end{aligned}$$

Proof The formulas for the first and the second derivatives follow from Proposition 3.1 and from (3.2). For the third derivative, the proof is similar to the one in [10, Lemma 3.3]:

$$\begin{aligned} f_k'''(s) &= \int_{\mathbb{S}^{n-1}} \dddot{h}_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}} \ddot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x)\,\mathrm{d}x \\ &+ \int_{\mathbb{S}^{n-1}} \ddot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x)\,\mathrm{d}x \\ &+ \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}^{ij,rs}(Q[h_s](x)) Q_{ij}[\dot{h}_s](x) Q_{rs}[\dot{h}_s](x)\,\mathrm{d}x \\ &+ \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\ddot{h}_s](x)\,\mathrm{d}x, \end{aligned}$$

where we have used (3.2) and [10, Remark 3.1]. □

If K is the unit ball, then $h \equiv 1$; consequently

$$h_s = e^{s\psi}, \quad \dot{h}_s = h_s\psi, \quad \ddot{h}_s = h_s\psi^2 \quad \text{and} \quad \dddot{h}_s = h_s\psi^3. \tag{4.3}$$

These formulas, together with Lemma 4.2, lead to the following result.

Corollary 4.3 *Let K be the unit ball. With the notations introduced above we have*

$$f_k(0) = \frac{|\mathbb{S}^{n-1}|}{k}\binom{n-1}{n-k},$$
$$f_k'(0) = \binom{n-1}{k-1}\int_{\mathbb{S}^{n-1}} \psi\,\mathrm{d}x,$$
$$f_k''(0) = \binom{n-2}{n-k}\left[\frac{(n-1)k}{k-1}\int_{\mathbb{S}^{n-1}} \psi^2\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}} \psi\Delta\psi\,\mathrm{d}x\right]$$

(*where* Δ *denotes the spherical Laplacian*).

Proof First note that, from (4.1), $h_0 \equiv 1$. From (4.2) we get that

$$f_k(0) = \frac{1}{k}\int_{\mathbb{S}^{n-1}} S_{k-1}(\mathrm{I}_{n-1})\,\mathrm{d}x = \frac{|\mathbb{S}^{n-1}|}{k}\binom{n-1}{k-1}.$$

Lemma 4.2, (2.6) and (4.1) imply

$$f_k'(0) = \binom{n-1}{k-1}\int_{\mathbb{S}^{n-1}} \psi\,\mathrm{d}x.$$

Moreover, from Lemma 4.2, (2.6) and (4.3) we have

$$\begin{aligned} f_k''(0) &= \binom{n-1}{k-1}\int_{\mathbb{S}^{n-1}} \psi^2\,\mathrm{d}x + \binom{n-2}{k-2}\int_{\mathbb{S}^{n-1}} \psi\delta_{ij}(\psi_{ij}+\psi\delta_{ij})\,\mathrm{d}x \\ &= \frac{(n-2)!}{(k-2)!(n-k)!}\left\{\frac{n-1}{k-1}\int_{\mathbb{S}^{n-1}}\psi^2\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}}\psi\Delta\psi dx + (n-1)\int_{\mathbb{S}^{n-1}}\psi^2\,\mathrm{d}x\right\} \\ &= \binom{n-2}{n-k}\left[\frac{(n-1)k}{k-1}\int_{\mathbb{S}^{n-1}}\psi^2\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}}\psi\Delta\psi\,\mathrm{d}x\right]. \end{aligned}$$

□

Lemma 4.4 *Let $h \in C_0^{2,+}(\mathbb{S}^{n-1})$, and let η_0 be as in Lemma 4.1. There exists a constant $C > 0$, depending on h, n and k, such that if $\psi \in C^2(\mathbb{S}^{n-1})$ and*

$$\|\psi\|_{C^2(\mathbb{S}^{n-1})} \le \eta_0, \tag{4.4}$$

then, the following estimates

$$|f_k(s)| \le C, \quad \text{for all } s \in [-2,2]; \tag{4.5}$$

$$|f_k'(s)| \le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}, \quad \text{for all } s \in [-2,2]; \tag{4.6}$$

$$|f_k''(s)| \le C\left(\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + \|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})}\right), \quad \text{for all } s \in [-2,2]; \tag{4.7}$$

$$|f_k'''(s)| \le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\left(\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + \|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})}\right), \quad \text{for all } s \in [-2,2]; \tag{4.8}$$

hold true.

Proof The proof is similar to the one of Lemma 3.5 in [10]. Throughout the proof, C denotes a positive constant depending on h, n and k.

We firstly observe that, since (4.4) is in force, there exists $C > 0$ such that

$$\|h_s\|_{C^2(\mathbb{S}^{n-1})} \le C, \quad \text{for all } s \in [-2,2].$$

Then,

$$|h_s(x)S_{k-1}(Q[h_s](x))| \le C, \quad \text{for all } s \in [-2,2]; \tag{4.9}$$

and we immediately deduce (4.5).

Now we prove (4.6). From Lemma 4.2 we get

$$|f_k'(s)| = \left| \int_{\mathbb{S}^{n-1}} \dot{h}_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x \right| = \left| \int_{\mathbb{S}^{n-1}} \psi(x) h_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x \right|,$$

and so from (4.9) we obtain the desired estimate (4.6).

Let us now prove (4.7). Again, from Lemma 4.2 and the integration by parts formula (2.7), we have

$$\begin{aligned} |f_k''(s)| \le & \left| \int_{\mathbb{S}^{n-1}} \psi^2(x) h_s(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x \right| \\ & + \left| \int_{\mathbb{S}^{n-1}} \psi(x) h_s(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[\psi h_s](x)\,\mathrm{d}x \right| \\ \le & C\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + \left| \int_{\mathbb{S}^{n-1}} S_{k-1}^{ij}(Q[h_s](x))(\psi h_s)_i(x)(\psi h_s)_j(x)\,\mathrm{d}x \right| \\ \le & C\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + C\|\nabla\psi\|^2_{L^2(\mathbb{S}^{n-1})}, \end{aligned}$$

hence we have the bound in (4.7).

Finally, we show that (4.8) holds. From Lemma 4.2 we get

$$\begin{aligned} |f_k'''(s)| \le & \left| \int_{\mathbb{S}^{n-1}} h_s(x)\psi^3(x) S_{k-1}(Q[h_s](x))\,\mathrm{d}x \right| \\ & + 2\left| \int_{\mathbb{S}^{n-1}} h_s(x)\psi^2(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[h_s\psi](x)\,\mathrm{d}x \right| \\ & + \left| \int_{\mathbb{S}^{n-1}} h_s(x)\psi(x) S_{k-1}^{ij,rs}(Q[h_s](x)) Q_{ij}[h_s\psi](x) Q_{rs}[h_s\psi](x)\,\mathrm{d}x \right| \\ & + \left| \int_{\mathbb{S}^{n-1}} h_s(x)\psi(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[h_s\psi^2](x)\,\mathrm{d}x \right|. \end{aligned}$$

Using formula (2.7) from Proposition 2.2, we get

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} h_s(x)\psi(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[h_s\psi^2](x)\,\mathrm{d}x \\ & \quad = \int_{\mathbb{S}^{n-1}} h_s(x)\psi^2(x) S_{k-1}^{ij}(Q[h_s](x)) Q_{ij}[h_s\psi](x)\,\mathrm{d}x, \end{aligned}$$

thus

$$\begin{aligned}|f_k'''(s)| &\le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi^2(x)S_{k-1}(Q[h_s](x))\,\mathrm{d}x\right|\\ &\quad + C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi(x)S_{k-1}^{ij}(Q[h_s](x))Q_{ij}[h_s\psi](x)\,\mathrm{d}x\right|\\ &\quad + \left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi(x)S_{k-1}^{ij,rs}(Q[h_s](x))Q_{ij}[h_s\psi](x)Q_{rs}[h_s\psi](x)\,\mathrm{d}x\right|\\ &\quad + \left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi^2(x)S_{k-1}^{ij}(Q[h_s](x))Q_{ij}[h_s\psi](x)\,\mathrm{d}x\right|.\end{aligned}$$

Now, arguing as we did before, we have that

$$\begin{aligned}|f_k'''(s)| &\le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})}\\ &\quad + \left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi(x)S_{k-1}^{ij,rs}(Q[h_s](x))Q_{ij}[h_s\psi](x)Q_{rs}[h_s\psi](x)\,\mathrm{d}x\right|\\ &\quad + C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})}.\end{aligned}$$

The third term can be estimated, arguing as before, in the following way:

$$\begin{aligned}&\left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi(x)S_{k-1}^{ij,rs}(Q[h_s](x))Q_{ij}[h_s\psi](x)Q_{rs}[h_s\psi](x)\,\mathrm{d}x\right|\\ &\le \left|\int_{\mathbb{S}^{n-1}} h_s^2(x)\psi^2(x)S_{k-1}^{ij,rs}(Q[h_s](x))Q_{rs}[h_s\psi](x)\delta_{ij}\,\mathrm{d}x\right|\\ &\quad + \left|\int_{\mathbb{S}^{n-1}} h_s(x)\psi(x)S_{k-1}^{ij,rs}(Q[h_s](x))Q_{rs}[h_s\psi](x)(\psi h_s)_{ij}(x)\,\mathrm{d}x\right|\\ &\le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\psi\|^2_{L_2(\mathbb{S}^{n-1})}\\ &\quad + C\left|\int_{\mathbb{S}^{n-1}} S_{k-1}^{ij,rs}(Q[h_s](x))Q_{rs}[h_s\psi](x)(\psi h_s)_j(x)(\psi h_s)_i(x)\,\mathrm{d}x\right|\\ &\le C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + C\|\psi\|_{C^2(\mathbb{S}^{n-1})}\|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})},\end{aligned}$$

where we used the definition of $Q_{ij}[h_s\psi_s]$ and the integration by parts formula (2.8). This concludes the proof of (4.8), hence the proof of the lemma. □

4.1.2 Proof of Theorem 1.2

We need one last lemma.

Lemma 4.5 *Let $h\colon \mathbb{S}^{n-1}\to\mathbb{R}$, $h\equiv 1$ and let $\psi\in C^2(\mathbb{S}^{n-1})$ be an even function such that*

$$\|\psi\|_{C^2(\mathbb{S}^{n-1})}\le\eta_0,$$

where η_0 is given by Lemma 4.1. Let $f_k\colon [-2,2] \to \mathbb{R}$ be defined by (4.2), where $k \in \{2,\dots,n\}$. There exists a constant $\eta > 0$, which depends only on n, $\eta \le \eta_0$, such that if

$$\|\psi\|_{C^2(\mathbb{S}^{n-1})} \le \eta, \tag{4.10}$$

then the function

$$s \ \mapsto \ \log f_k(s) \quad \text{is concave in } [-2,2].$$

Moreover, the function is strictly concave, unless ψ is a constant.

Proof We start by computing

$$(\log f_k)' = \frac{f_k'}{f_k}$$

and

$$(\log f_k)'' = \frac{f_k'' f_k - (f_k')^2}{f_k^2}.$$

We show that

$$H(s) := f_k(s) f_k''(s) - (f_k'(s))^2 < 0$$

for all $s \in [-2,2]$, provided $\|\psi\|_{C^2(\mathbb{S}^{n-1})} \le \eta$ and ψ is not constant. We have

$$H(0) = f_k(0) f_k''(0) - (f_k')^2(0).$$

First we assume that

$$\int_{\mathbb{S}^{n-1}} \psi \,\mathrm{d}x = 0. \tag{4.11}$$

According to Corollary 4.3, we have that

$$f_k(0) = \frac{|\mathbb{S}^{n-1}|}{k} \binom{n-1}{n-k}$$

and

$$f_k'(0) = \binom{n-1}{k-1} \int_{\mathbb{S}^{n-1}} \psi \,\mathrm{d}x = 0.$$

Moreover,

$$f_k''(0) = \binom{n-2}{n-k}\left[\frac{(n-1)k}{k-1}\int_{\mathbb{S}^{n-1}}\psi^2\,\mathrm{d}x + \int_{\mathbb{S}^{n-1}}\psi\Delta\psi\,\mathrm{d}x\right]$$
$$= \binom{n-2}{n-k}\left[\frac{(n-1)k}{k-1}\int_{\mathbb{S}^{n-1}}\psi^2\,\mathrm{d}x - \int_{\mathbb{S}^{n-1}}\|\nabla\psi\|^2\,\mathrm{d}x\right].$$

Now, since (4.11) is in force and ψ is even, from Proposition 2.4 we get

$$f_k''(0) \le \binom{n-2}{n-k}\left[\frac{(n-1)k}{2n(k-1)} - 1\right]\int_{\mathbb{S}^{n-1}}\|\nabla\psi\|^2\,\mathrm{d}x.$$

As

$$\frac{(n-1)k}{2n(k-1)} < 1,$$

we may write

$$H(0) \le -\gamma\,\|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})},$$

where $\gamma > 0$ depends only on n. Now, for every $s \in [-2, 2]$ there exists $\bar{s}$ between 0 and s such that

$$H(s) = H(0) + sH'(\bar{s}) = H(0) + s\left[f_k(\bar{s})f_k'''(\bar{s}) - f_k'(\bar{s})f_k''(\bar{s})\right];$$

from Lemma 4.4, we know that (4.5), (4.6), (4.7) and (4.8) hold true, hence

$$|sH'(\bar{s})| \le C\eta\left(\|\psi\|^2_{L_2(\mathbb{S}^{n-1})} + \|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})}\right) \le C\eta\|\nabla\psi\|^2_{L_2(\mathbb{S}^{n-1})},$$

where we used Proposition 2.4 again. We have then proved the concavity of f, and the strict concavity whenever $\|\nabla\psi\|_{L_2(\mathbb{S}^{n-1})} > 0$, i.e. whenever ψ is not a constant.

Now we drop the assumption (4.11). Given $\psi \in C^2(\mathbb{S}^{n-1})$, let

$$m_\psi = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\psi\,\mathrm{d}x \quad\text{and}\quad \bar{\psi} = \psi - m_\psi. \tag{4.12}$$

Clearly $\bar{\psi} \in C^2(\mathbb{S}^{n-1})$ and $\bar{\psi}$ verifies (4.11). Moreover

$$\|\bar{\psi}\|_{C^2(\mathbb{S}^{n-1})} \le \|\psi\|_{C^2(\mathbb{S}^{n-1})} + |m_\psi| \le 2\|\psi\|_{C^2(\mathbb{S}^{n-1})},$$

hence if $\|\psi\|_{C^2(\mathbb{S}^{n-1})} \le \eta_0/2$ then $\bar{\psi}$ satisfies (4.10). Since

$$\bar{h}_s := e^{s\bar{\psi}} = e^{s(\psi - m_\psi)} = e^{-sm_\psi}h_s,$$

we have

$$Q[\bar{h}_s] = e^{-sm_\psi} Q[h_s] \quad \text{and} \quad S_{k-1}(Q[\bar{h}_s](x)) = e^{-(k-1)sm_\psi} S_{k-1}(Q[h_s](x)),$$

thus

$$\bar{f}_k(s) := \frac{1}{k} \int_{\mathbb{S}^{n-1}} \bar{h}_s(x) S_{k-1}(Q[\bar{h}_s](x))\, \mathrm{d}x = e^{-ksm_\psi} f_k(s). \tag{4.13}$$

We conclude that $\log \bar{f}_k$ and $\log f_k$ differ by a linear term, and the concavity (respectively the strict concavity) of $\bar{f}_k$ is equivalent to that of f_k. On the other hand, by the first part of the proof $\log \bar{f}_k$ is concave (and strictly concave unless ψ is constant), as long as $\|\bar{\psi}\|_{C^2(\mathbb{S}^{n-1})}$ is sufficiently small, and this condition is verified if $\|\psi\|_{C^2(\mathbb{S}^{n-1})}$ is sufficiently small.

Finally, note that, by (4.1), if $\psi = \psi_0$ is constant then $h_s = e^{\psi_0 s}$. Consequently

$$f_k(s) = ce^{k\psi_0 s}, \quad c > 0,$$

whence $\log f_k(s)$ is linear.

The proof of the lemma is complete. □

We split the proof of Theorem 1.2 in two parts: in the first one we prove the validity of (1.7), while in the second one we deduce (1.8).

Proof of (1.7) in Theorem 1.2 Let $\eta > 0$ be as in Lemma 4.5, and let $K \in \mathcal{K}^n_{0,s}$ be of class $C^{2,+}$ and such that

$$\|1 - h\|_{C^2(\mathbb{S}^{n-1})} \leq \eta, \tag{4.14}$$

where h is the support function of K. This implies that $h > 0$ on $\mathbb{S}^{n-1}$, and therefore we can set $\psi = \log h \in C^2(\mathbb{S}^{n-1})$; thus we may write h in the form $h = e^\psi$. Define, for $t \in [0, 1]$,

$$K_t = (1-t) \cdot B_n +_0 t \cdot K,$$

and let h_t be the support function of K_t; then

$$h_t = 1^{1-t} h^t = e^{t\psi}.$$

Hence $V_k(K_t)$ is concave in $[-2, 2]$, which proves (1.7). Moreover, $V_k(K_t)$ is strictly concave unless ψ is constant, and the latter condition is equivalent to saying that h is constant, i.e. K is a ball centered at the origin. □

The following remark will be useful for the proof of (1.8) in Theorem 1.2.

Remark 4.6 If two convex bodies K_0 and K_1 satisfy the log-Brunn–Minkowski inequality

$$V_k((1-t)\cdot K_0 +_0 t\cdot K_1) \geq V_k(K_0)^{1-t}\ V_k(K_1)^t, \quad \text{for all } t\in[0,1], \tag{4.15}$$

then αK_0 and βK_1 satisfy the same log-Brunn–Minkowski inequality, for $\alpha, \beta > 0$.

Indeed, we consider the convex bodies αK_0 and βK_1; from the definition of 0-sum we have

$$\begin{aligned}(1-t)\cdot\alpha K_0 +_0 t\cdot\beta K_1 &= K[(\alpha h_0)^{1-t}(\beta h_1)^t]\\ &= \alpha^{1-t}\beta^t K[h_0^{1-t}h_1^t]\\ &= \alpha^{1-t}\beta^t[(1-t)\cdot K_0 +_0 t\cdot K_1],\end{aligned} \tag{4.16}$$

where h_0 and h_1 denote the support functions of K_0 and K_1, respectively. Hence, from the fact that V_k is k-homogeneous, (4.16) and (4.15) we obtain

$$\begin{aligned}V_k((1-t)\cdot\alpha K_0 +_0 t\cdot\beta K_1) &= \alpha^{(1-t)k}\beta^{tk}V_k((1-t)\cdot K_0 +_0 t\cdot K_1)\\ &\geq \alpha^{(1-t)k}\beta^{tk}V_k(K_0)^{1-t}\ V_k(K_1)^t\\ &= (\alpha^k V_k(K_0))^{1-t}\ (\beta^k V_k(K_1))^t\\ &= V_k(\alpha K_0)^{1-t}\ V_k(\beta K_1)^t,\end{aligned}$$

i.e. αK_0 and βK_1 satisfy the log-Brunn–Minkowski inequality (4.15) too. By the previous argument, it is clear that we have equality in the inequality for K_0 and K_1 if and only if we have equality in the inequality for αK_0 and βK_1.

Proof of (1.8) in Theorem 1.2 Let B_n and K be as in the statement of Theorem 1.2. In the following we will use an homogeneity argument which is classical in convex geometry; for completeness we write all the details. Let us define

$$\tilde{B_n} := \frac{1}{V_k(B_n)^{1/k}}B_n \quad\text{and}\quad \tilde{K} := \frac{1}{V_k(K)^{1/k}}K;$$

observe that

$$V_k(\tilde{B_n}) = 1 = V_k(\tilde{K}), \tag{4.17}$$

since V_k is k-homogeneous. Because of Theorem 1.2, B_n and K satisfy the log-Brunn–Minkowski inequality

$$V_k((1-t)\cdot B_n +_0 t\cdot K) \geq V_k(B_n)^{1-t}\ V_k(K)^t, \quad \text{for all } t\in[0,1].$$

Thanks to Remark 4.6, $\tilde{B_n}$ and $\tilde{K}$ satisfy the same log-Brunn–Minkowski inequality:

$$V_k((1-t)\cdot \tilde{B_n} +_0 t\cdot \tilde{K}) \geq V_k(\tilde{B_n})^{1-t}\ V_k(\tilde{K})^t, \quad \text{for all } t\in[0,1]. \tag{4.18}$$

Now, for every $t \in [0, 1]$ we define

$$\tilde{t} := \frac{t V_k(K)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}}.$$

Clearly $\tilde{t} \in [0, 1]$ and

$$1 - \tilde{t} = \frac{(1-t)V_k(B_n)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}},$$

hence, applying (4.18) with $t = \tilde{t}$, we find

$$V_k((1-\tilde{t}) \cdot \tilde{B}_n +_0 \tilde{t} \cdot \tilde{K}) \geq V_k(\tilde{B}_n)^{1-\tilde{t}}\, V_k(\tilde{K})^{\tilde{t}} = 1,$$

where we have used (4.17). On the other hand,

$$\begin{aligned}
&V_k((1-\tilde{t}) \cdot \tilde{B}_n +_0 \tilde{t} \cdot \tilde{K}) \\
&\quad = V_k\Bigg(\frac{(1-t)V_k(B_n)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \cdot \frac{1}{V_k(B_n)^{1/k}} B_n \\
&\qquad +_0 \frac{t V_k(K)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \cdot \frac{1}{V_k(K)^{1/k}} K\Bigg) \\
&\quad = V_k\Bigg(\frac{(1-t)V_k(B_n)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \frac{1}{V_k(B_n)^{p/k}} \cdot B_n \\
&\qquad +_0 \frac{t V_k(K)^{p/k}}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \frac{1}{V_k(K)^{p/k}} \cdot K\Bigg) \\
&\quad = V_k\Bigg(\frac{(1-t)}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \cdot B_n +_0 \frac{t}{(1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k}} \cdot K\Bigg) \\
&\quad = \frac{1}{((1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k})^{k/p}} V_k\left((1-t) \cdot B_n +_0 t \cdot K\right),
\end{aligned}$$

where we used the fact that

$$V_k(\lambda \cdot K) = V_k(\lambda^{1/p} K) = \lambda^{k/p} V_k(K), \quad \text{for all } \lambda > 0.$$

Summing up, we have that

$$V_k\left((1-t) \cdot B_n +_0 t \cdot K\right) \geq ((1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k})^{k/p}.$$

The conclusion now follows from the inclusion (2.4), which gives

$$\begin{aligned}
V_k\left((1-t) \cdot B_n +_p t \cdot K\right) &\geq V_k\left((1-t) \cdot B_n +_0 t \cdot K\right) \\
&\geq ((1-t)V_k(B_n)^{p/k} + t V_k(K)^{p/k})^{k/p}. \qquad (4.19)
\end{aligned}$$

Assuming that equality holds in (4.19), we see that we have equality in (4.18) as well. By Remark 4.6 and by the discussion of equality conditions in Theorem 1.2, we obtain that K has to be homothetic to B_n. The vice versa of this statement follows from homogeneity. $\square$

4.2 The case $k = 1$

In this subsection we prove Theorem 1.4.

Proof of Theorem 1.4 We firstly observe that (see Sects. 2 and 3)

$$V_1(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h(x) S_0(Q[h](x))\, \mathrm{d}x = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h(x)\, \mathrm{d}x.$$

With this remark the conclusion of the theorem follows immediately from Minkowski's triangular inequality (if $0 < p < 1$) and from Hölder's inequality (if $p = 0$). Indeed, let $0 < p < 1$ and let $K_0, K_1 \in \mathcal{K}_0^n$, then for any $t \in [0, 1]$

$$\begin{aligned}
&V_1((1-t)\cdot K_0 +_p t\cdot K_1)^p \\
&= \left[\frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h_{(1-t)\cdot K_0 +_p t\cdot K_1}(x)\, \mathrm{d}x\right]^p \\
&\leq \left[\frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} \left((1-t)h_{K_0}^p(x) + t h_{K_1}^p(x)\right)^{1/p} \mathrm{d}x\right]^p \\
&= \left(\frac{1}{\kappa_{n-1}}\right)^p \|(1-t)h_{K_0}^p + t h_{K_1}^p\|_{L^{1/p}(\mathbb{S}^{n-1})} \\
&\leq \left(\frac{1}{\kappa_{n-1}}\right)^p (1-t)\|h_{K_0}^p\|_{L^{1/p}(\mathbb{S}^{n-1})} + \left(\frac{1}{\kappa_{n-1}}\right)^p t\|h_{K_1}^p\|_{L^{1/p}(\mathbb{S}^{n-1})} \\
&= (1-t)V_1(K_0)^p + t V_1(K_1)^p.
\end{aligned}$$

While, if $p = 0$

$$\begin{aligned}
&V_1((1-t)\cdot K_0 +_0 t\cdot K_1) \\
&= \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h_{(1-t)\cdot K_0 +_0 t\cdot K_1}(x)\, \mathrm{d}x \\
&\leq \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h_{K_0}^{1-t}(x) h_{K_1}^t(x)\, \mathrm{d}x \\
&\leq \frac{1}{\kappa_{n-1}} \left[\int_{\mathbb{S}^{n-1}} h_{K_0}\, \mathrm{d}x\right]^{1-t} \frac{1}{\kappa_{n-1}} \left[\int_{\mathbb{S}^{n-1}} h_{K_1}\, \mathrm{d}x\right]^t \\
&= V_1(K_0)^{1-t} V_1(K_1)^t.
\end{aligned}$$

From the characterization of equality conditions in the Minkowski's triangular inequality and in the Hölder's inequality we deduce that $h_{K_0} = \alpha h_{K_1}$, for some $\alpha \geq 0$. This implies that either one of the two bodies K_0 and K_1 coincides with $\{0\}$, or they are homothetic. $\square$

5 Local uniqueness for the L_p Christoffel–Minkowski problem

In this section, we prove Theorem 1.5. We will need the following preliminary result.

Lemma 5.1 *There exists $\eta > 0$ such that for every $p \in (0, 1)$ and if $K \in \mathcal{K}^n_{0,s}$ is of class $C^{2,+}$ and*

$$\|1 - h_K\|_{C^2(\mathbb{S}^{n-1})} \leq \eta,$$

then for every $t \in [0, 1]$ the function $h_t : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by

$$h_t = [(1 - t) + t h_K^p]^{1/p}$$

is the support function of a convex body $K_t \in \mathcal{K}^n_{0,s}$, of class $C^{2,+}$.

Proof Note that $h_t \in C^2(\mathbb{S}^{n-1})$ for every $t \in [0, 1]$. By Proposition 2.1, we need to prove that

$$Q[h_t] > 0 \quad \text{on } \mathbb{S}^{n-1}$$

for every $t \in [0, 1]$. By contradiction, assume that there exist a sequence $\eta_j > 0$, $j \in \mathbb{N}$, converging to 0, a sequence of convex bodies $K_j \in \mathcal{K}^n_{0,s}$, of class $C^{2,+}$, a sequence $t_j \in [0, 1]$ and a sequence $x_j \in \mathbb{S}^{n-1}$, such that, denoting by h_j the support function of K_j,

$$\|1 - h_j\|_{C^2(\mathbb{S}^{n-1})} \leq \eta_j,$$

and

$$Q[h_j(x_j)] \leq 0.$$

Clearly h_j converges to the constant function $h_0 \equiv 1$ in $C^2(\mathbb{S}^{n-1})$, and hence h_j converges to $h_0 \equiv 1$ in $C^2(\mathbb{S}^{n-1})$. Up to subsequences, we may also assume that t_j and x_j converge to $\bar{t} \in [0, 1]$ and $\bar{x} \in \mathbb{S}^{n-1}$, respectively. As a consequence of these facts, by the continuity of Q we get

$$Q[h_0(\bar{x})] \leq 0,$$

which is a contradiction, as $Q[h_0]$ is the identity matrix. ◻

Proof of Theorem 1.5 We first consider the case $p > 0$. Let $\bar{\eta} > 0$ be smaller than the two positive quantities, both called η, appearing in Theorem 1.2 and Lemma 5.1. Let $K \in \mathcal{K}^n_{0,s}$ be of class $C^{2,+}$ and such that

$$\|1 - h_K\|_{C^2(\mathbb{S}^{n-1})} \leq \eta.$$

Up to replacing η with a smaller constant, we may assume that $h_K > 0$ on $\mathbb{S}^{n-1}$. For simplicity, in the rest of the proof we will write h instead of h_K. We also set

$$K_t = (1-t)\cdot B_n +_p t\cdot K, \quad \forall t \in [0,1].$$

By the definition of p-addition and Lemma 5.1, the support function h_t of K_t is given by:

$$h_t = \left[(1-t) + th^p\right]^{1/p}.$$

We now consider the functions $f, g\colon [0,1] \to \mathbb{R}$ defined by:

$$f(t) = [V_k(K_t)]^{p/k} = \left[\frac{1}{k\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} h_t S_{k-1}(Q[h_t])\,\mathrm{d}x\right]^{p/k},$$
$$g(t) = (1-t)[V_k(B_n)]^{p/k} + t[V_k(K)]^{p/k}.$$

By (1.8) in Theorem 1.2,

$$f(t) \geq g(t), \quad \forall\, t \in [0,1].$$

Moreover $f(0) = g(0)$, $f(1) = g(1)$, so that

$$f'(0) \geq g'(0), \quad f'(1) \leq g'(1). \tag{5.1}$$

Note that

$$g'(0) = g'(1) = [V_k(K)]^{p/k} - [V_k(B_n)]^{p/k}. \tag{5.2}$$

By Lemma 4.2, we have:

$$\begin{aligned} f'(0) &= \frac{p}{k} V_k(B_n)^{\frac{p}{k}-1}\left[\frac{1}{p\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} (h^p-1)S_{k-1}(Q[h_0])\,\mathrm{d}x\right] \\ &= [V_k(B_n)]^{\frac{p}{k}-1}\frac{1}{k\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} h^p S_{k-1}(Q[h_0])\,\mathrm{d}x \\ &\quad -[V_k(B_n)]^{\frac{p}{k}-1}\frac{1}{k\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} S_{k-1}(Q[h_0])\,\mathrm{d}x \\ &= [V_k(B_n)]^{\frac{p}{k}-1}\frac{1}{k\kappa_{n-k}}\int_{\mathbb{S}^{n-1}} hS_{k-1}(Q[h])\,\mathrm{d}x - [V_k(B_n)]^{\frac{p}{k}} \\ &= [V_k(B_n)]^{\frac{p}{k}-1}[V_k(K)] - [V_k(B_n)]^{\frac{p}{k}}, \end{aligned} \tag{5.3}$$

where we have used (1.11). From (5.1), (5.2) and (5.3), we get

$$V_k(K) \geq V_k(B_n).$$

In a similar way, from the comparison $g'(1) \geq f'(1)$ we obtain the reverse inequality. Hence $V_k(K) = V_k(B_n)$, which implies that $f'(0) = f'(1) = 0$. As f is concave, we have that f is constant in $[0, 1]$, which means that the inequality (1.8) becomes an equality for K. By Theorem 1.2, K is a dilation of B_n. On the other hand, (1.11) implies $K = B_n$.

The proof in the case $p = 0$ is similar; (1.7) in Theorem 1.2 and Lemma 4.1 will have to be used instead of (1.8) in Theorem 1.2 and Lemma 5.1, respectively. □

5.1 Local uniqueness via a different approach

In this subsection we obtain a local uniqueness result for Eq. (1.11), in the context of Sobolev spaces, via a different argument.

Let $k \in \{1, \dots, n-1\}$ and let q be any real number such that

$$q > \max\left\{2k, \frac{n-1}{2}\right\}. \tag{5.4}$$

We consider the Sobolev space $W^{2,q}(\mathbb{S}^{n-1})$. As $q > \frac{n-1}{2}$, by the Sobolev embedding theorem, $C(\mathbb{S}^{n-1})$ continuously embeds in this space. Let $X := \{h \in W^{2,q}(\mathbb{S}^{n-1}) \colon h > 0\}$. For $p \in [0, 1)$, consider the functional

$$\mathcal{G} \colon X \to L^2(\mathbb{S}^{n-1})$$

defined by

$$\mathcal{G}(h) = h^{1-p} S_k(Q[h])$$

(with the obvious extension of the matrix Q to functions having second derivatives defined a.e.). $\mathcal{G}$ is differentiable in X, and its differential at $h \in X$ is given by

$$D\mathcal{G}(h)\phi = (1-p)h^{-p} S_k(Q[h])\phi + h^{1-p} S_k^{ij}(Q[h])(\phi_{ij} + \phi\delta_{ij}),$$

and depends continuously on $h \in X$. In particular, at $h = h_{B^n} \equiv 1$ we have, by the results of Sect. 2.3,

$$\begin{aligned} D\mathcal{G}(h_{B^n})\phi &= (1-p)\binom{n}{k}\phi + \binom{n-1}{k-1}(\Delta\phi + (n-1)\phi) \\ &= \binom{n-1}{k-1}\left[\Delta\phi + \phi\left(\frac{n(1-p)}{k} + n - 1\right)\right]. \end{aligned}$$

The value

$$\frac{n(1-p)}{k} + n - 1$$

does not belong to the spectrum of the spherical Laplacian, for any value of $k \in \{1, \dots, n\}$. Therefore $D\mathcal{G}(h_{B^n})$ is invertible. By the inverse function theorem in Banach spaces (see e.g. [28]), $\mathcal{G}$ is injective in a neighborhood of h_{B^n} in $W^{2,k}(\mathbb{S}^{n-1})$.

Let $K \in \mathcal{K}_0^n$, and let $h = h_K$; assume that $h \in W^{2,q}(\mathbb{S}^{n-1})$. By a standard approximation argument, and by weak continuity of area measures of convex bodies (see e.g. [30, Chapter 4]), it follows that

$$dS_k(K, x) = S_k(h_{ij} + h\delta_{ij})dx.$$

Assume that $h > 0$ on $\mathbb{S}^{n-1}$ (i.e. the origin is an interior point of K) and that it verifies (1.11). Then (1.12) holds as well, a.e. on $\mathbb{S}^{n-1}$. Therefore $\mathcal{G}(h) = \mathcal{G}(h_{B^n})$. If, in addition, $\|h - 1\|_{W^{2,q}(\mathbb{S}^{n-1})} \leq \eta$, then $K = B^n$.

We have then proved the following result.

Theorem 5.2 *Let $k \in \{1, \dots, n-1\}$ and let $q \in \mathbb{R}$ verify (5.4). There exists a constant $\eta > 0$ such that if K is a convex body containing the origin in its interior, verifying (1.11), with support function $h \in W^{2,q}(\mathbb{S}^{n-1})$ and*

$$\|h - 1\|_{W^{2,q}(\mathbb{S}^{n-1})} \leq \eta,$$

then $K = B^n$.

6 Proof of Theorem 1.3: counterexamples

In this section we show that for every $k \in \{2, \dots, n-1\}$ there exists $\bar{p}$ such that the p-Brunn–Minkowski inequality for the intrinsic volumes does not hold, for every $p < \bar{p}$, that is, (1.9) is satisfied for suitable $K_0, K_1 \in \mathcal{K}_{0,s}^n$.

Given $k \in \{2, \dots, n-1\}$, we consider

$$\begin{aligned} K_0 := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, & x_j = 0, \ \forall\, j = 1, \dots, n-k \\ & \text{and } |x_i| \leq 1, \ \forall\, i = n-k+1, \dots, n\}, \end{aligned}$$

and

$$K_1 := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, |x_i| \leq 1, \ \forall\, i = 1, \dots, k \text{ and } x_j = 0, \ \forall\, j = k+1, \dots, n\}.$$

K_0 and K_1 are k-dimensional cubes of side length 2; therefore $V_k(K_0) = V_k(K_1) = 2^k$. We set, for $p \in (0, 1)$,

$$K_p := \frac{1}{2} \cdot K_0 +_p \frac{1}{2} \cdot K_1,$$

so that its support function h_{K_p} satisfies

$$h_{K_p}(x) := h_{K[(\frac{1}{2}h^p_{K_0}+\frac{1}{2}h^p_{K_1})^{1/p}]}(x) \leq \left(\frac{1}{2}h^p_{K_0}(x) + \frac{1}{2}h^p_{K_1}(x)\right)^{1/p} \quad \text{for all } x \in \mathbb{R}^n. \tag{6.1}$$

We denote by $\{e_1, \ldots, e_n\}$ the standard orthonormal basis of $\mathbb{R}^n$, and we treat the cases $k > n/2$ and $k \leq n/2$ separately.

- *Case* $k > n/2$. In this case we have

$$h_{K_0}(\pm\, e_i) = \begin{cases} 0 & \text{if } i \in \{1, \ldots, n-k\} \\ 1 & \text{if } i \in \{n-k+1, \ldots, k\} \\ 1 & \text{if } i \in \{k+1, \ldots, n\}, \end{cases}$$
$$\text{and} \quad h_{K_1}(\pm\, e_i) = \begin{cases} 1 & \text{if } i \in \{1, \ldots, n-k\} \\ 1 & \text{if } i \in \{n-k+1, \ldots, k\} \\ 0 & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

Therefore, from (6.1),

$$h_{K_p}(\pm\, e_i) \leq \begin{cases} 2^{-1/p} & \text{if } i \in \{1, \ldots, n-k\} \\ 1 & \text{if } i \in \{n-k+1, \ldots, k\} \\ 2^{-1/p} & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

We deduce that

$$K_p \subseteq K := \left[-2^{-1/p}, 2^{-1/p}\right]^{n-k} \times [-1, 1]^{2k-n} \times \left[-2^{-1/p}, 2^{-1/p}\right]^{n-k},$$

where $[x_0, y_0]^m$ indicates the m-dimensional cube given by the product of m copies of $[x_0, y_0]$. This implies

$$\begin{aligned} V_k(K_p) \leq V_k(K) &= V_k\left(\left[-2^{-1/p}, 2^{-1/p}\right]^{2n-2k} \times [-1, 1]^{2k-n}\right) \\ &= V_k\left(\prod_{i=1}^{n}[-a_i, a_i]\right) = 2^k \sum_{1\leq i_1<\ldots<i_k\leq n} a_{i_1}\ldots a_{i_k}, \end{aligned}$$

where

$$a_i = \begin{cases} 2^{-1/p} & \text{if } i \in \{1, \ldots, 2n-2k\} \\ 1 & \text{if } i \in \{2n-2k+1, \ldots, n\}. \end{cases}$$

Notice that, since $k < n$, we have $n - (2n - 2k) = 2k - n < k$, hence when choosing k intervals among $\{[-a_i, a_i]\}_{i=1,\ldots,n}$, at least one of the them is of the form $\left[-2^{-1/p}, 2^{-1/p}\right]$. We are going to discuss separately the cases $k \leq \frac{2}{3}n$ and $k > \frac{2}{3}n$.

If $k \leq \frac{2}{3}n$, then $2n - 2k \geq k$, and

$$\begin{aligned} V_k(K_p) &\leq 2^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1} \ldots a_{i_k} = 2^k \sum_{i=1}^{k} \binom{2n-2k}{i} 2^{-i/p} \\ &\leq 2^{k-1/p} \sum_{i=1}^{k} \binom{2n-2k}{i} =: C_{n,k}\; 2^{k-1/p}, \end{aligned}$$

whereas if $k > \frac{2}{3}n$, then $2n - 2k < k$ and

$$\begin{aligned} V_k(K_p) &\leq 2^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1} \ldots a_{i_k} = 2^k \sum_{i=1}^{2n-2k} \binom{2n-2k}{i} 2^{-i/p} \\ &\leq 2^{k-1/p} \sum_{i=1}^{2n-2k} \binom{2n-2k}{i} = 2^{k-1/p}(2^{2n-2k} - 1). \end{aligned}$$

Since $V_k(K_0) = V_k(K_1) = 2^k$, we have

$$\left(\frac{1}{2} V_k(K_0)^{\frac{p}{k}} + \frac{1}{2} V_k(K_1)^{\frac{p}{k}} \right)^{\frac{k}{p}} = 2^k,$$

while

$$V_k(K_p) \leq \begin{cases} C_{n,k}\; 2^{k-1/p} & \text{if } \frac{n}{2} < k \leq \frac{2}{3}n \\ 2^{k-1/p}(2^{2n-2k} - 1) & \text{if } \frac{2}{3}n < k \leq n-1 \end{cases}$$

If $\frac{n}{2} < k \leq \frac{2}{3}n$, consider

$$\bar{p} = \frac{1}{\log_2(C_{n,k})},$$

and let $p < \bar{p}$ (note that $C_{n,k} > 1$, so that $\bar{p} > 0$). Hence

$$2^k > C_{n,k}\; 2^{k-1/p},$$

that is, the p-Brunn–Minkowski inequality fails.

If $\frac{2}{3}n < k \leq n - 1$, we choose

$$\bar{p} = \frac{1}{\log_2\left(2^{2n-2k} - 1\right)}$$

and we take $p < \bar{p}$. Hence

$$2^k > 2^{k-\frac{1}{p}}(2^{2n-2k} - 1),$$

that is, the p-Brunn–Minkowski inequality fails.
• *Case* $k \leq n/2$. In this case we have

$$h_{K_0}(\pm e_i) = \begin{cases} 0 & \text{if } i \in \{1, \dots, k\} \\ 0 & \text{if } i \in \{k+1, \dots, n-k\} \\ 1 & \text{if } i \in \{n-k+1, \dots, n\}, \end{cases}$$
$$\text{and} \quad h_{K_1}(\pm e_i) = \begin{cases} 1 & \text{if } i \in \{1, \dots, k\} \\ 0 & \text{if } i \in \{k+1, \dots, n-k\} \\ 0 & \text{if } i \in \{n-k+1, \dots, n\}. \end{cases}$$

Consequently, from (6.1),

$$h_{K_p}(\pm e_i) \leq \begin{cases} 2^{-1/p} & \text{if } i \in \{1, \dots, k\} \\ 0 & \text{if } i \in \{k+1, \dots, n-k\} \\ 2^{-1/p} & \text{if } i \in \{n-k+1, \dots, n\} \end{cases}$$

and we deduce that

$$K_p \subseteq K := \left[-2^{-1/p}, 2^{-1/p}\right]^k \times \{0\}^{n-2k} \times \left[-2^{-1/p}, 2^{-1/p}\right]^k.$$

Therefore,

$$V_k(K_p) \leq V_k\left(\left[-2^{-1/p}, 2^{-1/p}\right]^{2k} \times \{0\}^{n-2k}\right) = \binom{2k}{k} 2^{k-\frac{k}{p}}.$$

Let $\bar{p} = \frac{k}{\log_2 \binom{2k}{k}}$ and consider $p < \bar{p}$. We have

$$\binom{2k}{k} < 2^{\frac{k}{p}},$$

which is equivalent to $2^k > \binom{2k}{k} 2^{k-\frac{k}{p}}$, which entails that the p-Brunn–Minkowski inequality fails.

Summing things up, we have the following result: for every fixed $k \in \{2, \dots, n-1\}$, let

$$\bar{p}_k = \begin{cases} \dfrac{k}{\log_2 \binom{2k}{k}} & \text{for } 1 < k \leq \frac{n}{2}, \\ \dfrac{1}{\log_2 \sum_{i=1}^{k} \binom{2(n-k)}{i}} & \text{for } \frac{n}{2} < k \leq \frac{2}{3}n, \\ \dfrac{1}{\log_2[2^{2(n-k)} - 1]} & \text{for } \frac{2}{3}n < k \leq n-1; \end{cases}$$

then the p-Brunn Minkowski for intrinsic volumes does not hold if $p < \bar{p}_k$. In particular, this proves Theorem 1.3.

Remark 6.1 The value of $\bar{p}_k$ is bounded away from 1, as n and k range in $\mathbb{N}$ and $\{2, \dots, n-1\}$, respectively. We analyse its value and its asymptotic behaviour in high dimension, in the three cases $1 < k \leq \frac{n}{2}$, $\frac{n}{2} < k \leq \frac{2}{3}n$ and $\frac{2}{3}n < k \leq n-1$.
Case $1 < k \leq \frac{n}{2}$. The sequence

$$b_k = \binom{2k}{k} 2^{-k},$$

is strictly increasing, hence $b_k > b_1 = 1$ for $k \geq 2$, which implies that

$$\bar{p}_k = \frac{k}{\log_2 \binom{2k}{k}} < 1;$$

moreover $\lim_{k\to\infty} \bar{p}_k = \frac{1}{2}$.
Case $\frac{n}{2} < k \leq \frac{2}{3}n$. We notice that, since $k > 1$,

$$\sum_{i=1}^{k} \binom{2(n-k)}{i} > \binom{2(n-k)}{1} = 2n - 2k \geq 2n - \frac{4}{3}n = \frac{2}{3}n, \tag{6.2}$$

hence

$$\bar{p}_k < \frac{1}{\log_2((2n)/3)}$$

for every $k \leq \frac{2}{3}n$ and for every $n \geq 3$. Hence the asymptotic behavior of $\bar{p}_k$ as n tends to infinity is infinitesimal for every $\frac{n}{2} < k \leq \frac{2}{3}n$.
Case $\frac{2}{3}n < k \leq n-1$. Since $n-k \geq 1$, we have $\bar{p}_k \leq 1/\log_2 3 < 1$.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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