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Data-based control design for nonlinear systems with recurrent neural network-based controllers

William D'Amico, Alessio La Bella, Fabio Dercole, Marcello Farina

Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Via Ponzio 34/5, Milano, Italy,
e-mail: `name.surname@polimi.it`.

Abstract: This paper addresses the design of controllers for systems modelled as recurrent neural networks (RNNs). A novel data-based procedure for the design of RNN-based regulators is proposed, guaranteeing closed-loop stability properties and desired performances, conferred by virtual reference feedback tuning. The approach is tested on a realistic nonlinear system.

Keywords: Recurrent neural networks, nonlinear system stability, data-based control.

1. INTRODUCTION

Neural networks (NNs), and especially recurrent neural networks (RNNs) (Bonassi et al., 2022), have gained interest, in the last years, for control applications. In fact, they can be efficiently used, not only to identify unknown systems, but also to design feedback controllers from data.

Data-based methods addressing the control design without previosly identifying the plant model are denoted as *direct*, e.g., virtual reference feedback tuning (VRFT) (Campi and Savaresi, 2006), iterative learning (Gunnarsson et al., 1999), or model-free techniques (Tange et al., 2019).

The use of VRFT to tune controllers in the form of NNs has been marginally investigated, e.g., in (Esparza et al., 2011; Yan et al., 2016; Radac and Precup, 2018). However, these works do not focus on specific properties of NNs, e.g., their stability. In (D'Amico et al., 2022a), VRFT is used for the design of regulators with echo state network (ESN) and long short-term memory (LSTM) structures, devoting specific attention to the stability of the RNN-based controller and to the possibility of enforcing input constraints. Nevertheless, the investigation of the closed-loop stability properties was left to future work.

Despite the increasing popularity and the large potentialities of RNNs in control, few works have been dedicated to their theoretical properties. For example, sufficient conditions ensuring stability-related properties for RNNs are presented in (Miller and Hardt, 2019; Stipanović et al., 2021; Hu and Wang, 2002; Bonassi et al., 2021b,a). The latter works focus on open-loop RNNs, and they do not address the design of stabilizing RNN-based feedback controllers. A fundamental property in this framework, being RNNs nonlinear systems, is incremental input-to-state stability (δISS), (Angeli (2002); Tran et al. (2016)).

A δISS condition for a class of discrete-time RNNs is derived in (D'Amico et al., 2022b). This condition is also suited for guaranteeing stability to control systems where the system and the feedback controller are RNNs. Regarding control systems, in (Yin et al., 2021) the stability is analysed in case of feedforward NN (FFNN) controllers and assuming a linear controlled system. Design conditions for FFNN controllers are also provided in (Vance and Jagannathan, 2008) considering specific classes of second-order nonlinear systems under control. Also model predictive control has been investigated as a method for the design of controllers applicable to RNNs, e.g., (Seel et al., 2021; Bugliari Armenio et al., 2019; Terzi et al., 2021).

In this paper we combine the VRFT approach and the conditions derived in (D'Amico et al., 2022b) to confer performances but, at the same time, stability properties to the control system. We will show that this can be achieved, for specific controller classes (e.g., ESNs) through the solution to a computationally lightweight linear matrix inequality (LMI) problem. Note that the proposed approach combines the advantages of a direct approach (i.e., VRFT) with stability constraints requiring the system model (obtained in a data-based fashion), as typically done in indirect approaches. In our opinion, the advantage of using VRFT despite the knowledge of a system model is that VRFT provides a very general framework for a computationally lightweight performance-oriented controller design, while analytic approaches are not available for general classes of nonlinear systems.

2. NOTATION AND PRELIMINARIES

The ij-th entry of a matrix A is denoted as a_{ij}. The i-th entry of a vector v is indicated as v_i. Given a symmetric matrix P, we use $P \succeq 0$, $P \succ 0$, $P \preceq 0$, and $P \prec 0$ to indicate that it is positive semidefinite, positive definite, negative semidefinite, and negative definite, respectively. Given a sequence of square matrices $A_1, A_2, \ldots, A_n$, $D = \text{diag}(A_1, A_2, \ldots, A_n)$ is a block diagonal matrix having $A_1, A_2, \ldots, A_n$ as main-diagonal blocks. Moreover, $\|v\|$ denotes the 2-norm of a column vector v and $\|v\|_Q = \sqrt{v^T Q v}$ denotes the weighted Euclidean norm of v, being $Q \succeq 0$. Also, $id_n(\cdot)$ denotes a column vector of dimension n with all elements equal to the identity function $id(\cdot)$.

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I and 0 denote identity and zero, respectively, matrices, whose dimensions are clear from the context.
Consider the following discrete-time nonlinear system

$$x(k+1) = f(A\,x(k) + B\,u(k))\,, \tag{1}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the discrete-time index, $x(k) \in \mathbb{R}^n$ is the state of the system, and $u(k) \in \mathbb{R}^m$ is the exogenous variable, $f(\cdot) = [f_1(\cdot)\ \dots\ f_n(\cdot)]^T \in \mathbb{R}^n$ is a vector function applied element-wise, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$.
For the definitions of $\|\boldsymbol{u}\|_\infty$ of a sequence $\boldsymbol{u} = u(0), u(1), ...,$ $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions, and δISS, we defer the reader to (D'Amico et al., 2022b), where we show how the latter property can be enforced to (1) in case $f(\cdot)$ satisfies the following assumption.

Assumption 1. The functions $f_i(\cdot)$ are such that $f_i(0) = 0$, $i = 1, \dots, n$, and they are either globally Lipschitz continuous, with Lipschitz constant L_{pi}, or identity functions.

This class of systems is representative of several RNNs, e.g., ESNs, shallow neural nonlinear auto-regressive exogenous models (NNARX), or the general RNNs considered in (Sontag, 1992), as discussed in (D'Amico et al., 2022b). Under Assumption 1, we can define the set $\mathcal{L} = \{i \in \{1, \dots, n\}\,|\ \ f_i(\cdot) \neq id(\cdot)\}$, the diagonal matrix $W = \text{diag}(L_{p1}, \dots, L_{pn}) \in \mathbb{R}^{n\times n}$, where $L_{pi} = 1$ for all $i \notin \mathcal{L}$, and the matrix $\widetilde{A} = WA$. We are in the position to recall the following result from (D'Amico et al., 2022b).

Theorem 1. (D'Amico et al. (2022b)). Consider system (1) under Assumption 1. Then, the system is δISS if $\exists P = P^T \succ 0$ such that $p_{ij} = p_{ji} = 0\ \forall i \in \mathcal{L}$, $\forall j \in \{1, \dots, n\}$ with $j \neq i$, and it holds that $\widetilde{A}^T P \widetilde{A} - P \prec 0$.

3. PROBLEM STATEMENT

We consider a nonlinear discrete-time system $\mathcal{S}$ defined by the following model class

$$x_\mathcal{S}(k+1) = f_\mathcal{S}(A_\mathcal{S} x_\mathcal{S}(k) + B_\mathcal{S} u_\mathcal{S}(k))\,, \tag{2a}$$

$$y_\mathcal{S}(k) = C_\mathcal{S} x_\mathcal{S}(k)\,, \tag{2b}$$

$u_\mathcal{S}(k) \in \mathbb{R}^{m_\mathcal{S}}$ is the exogenous variable, $y_\mathcal{S}(k) \in \mathbb{R}^{l_\mathcal{S}}$ is the output vector, $x_\mathcal{S}(k) \in \mathbb{R}^{n_\mathcal{S}}$ is the state vector, $f_\mathcal{S}(\cdot) = [f_{\mathcal{S},1}(\cdot)\ \dots\ f_{\mathcal{S},n_\mathcal{S}}(\cdot)]^T \in \mathbb{R}^{n_\mathcal{S}}$ is a vector function applied element-wise, $A_\mathcal{S} \in \mathbb{R}^{n_\mathcal{S}\times n_\mathcal{S}}$, $B_\mathcal{S} \in \mathbb{R}^{n_\mathcal{S}\times m_\mathcal{S}}$, and $C_\mathcal{S} \in \mathbb{R}^{l_\mathcal{S}\times n_\mathcal{S}}$. The model (2) can be obtained through an identification procedure. We also assume that (2a) is in the class of systems defined by Assumption 1.

We consider the control scheme depicted in Figure 1. It includes a closed-loop regulator $\mathcal{R}$, a discrete-time integrator "$\int$" (to ensure zero steady-state error), a feedforward compensator $\mathcal{C}$ (to enhance the dynamic performances), and a state-observer $\mathcal{O}$, being the states of RNN-based systems not commonly measurable. To make the problem solvable with a lightweight LMI optimization one, the feedback regulator $\mathcal{R}$ and the feedforward compensator $\mathcal{C}$ are selected in the class of ESNs, with equations

$$x_\ell(k+1) = \zeta_\ell(W_{x_\ell} x_\ell(k) + W_{u_\ell} u_\ell(k) + W_{y_\ell} y_\ell(k)), \tag{3a}$$

$$y_\ell(k) = W_{out_{1_\ell}} x_\ell(k) + W_{out_{2_\ell}} u_\ell(k)\,, \tag{3b}$$

where subscript $\ell \in \{\mathcal{C}, \mathcal{R}\}$ is used to refer to the compensator $\mathcal{C}$ and the regulator $\mathcal{R}$, respectively. As discussed in (Bugliari Armenio et al., 2019), the design of $\mathcal{R}$ and $\mathcal{C}$ consists of the tuning of the matrices $W_{out_{1_\ell}}$ and $W_{out_{2_\ell}}$, for $\ell \in \{\mathcal{R}, \mathcal{C}\}$, whereas W_{x_ℓ}, W_{u_ℓ}, and W_{y_ℓ} are fixed.

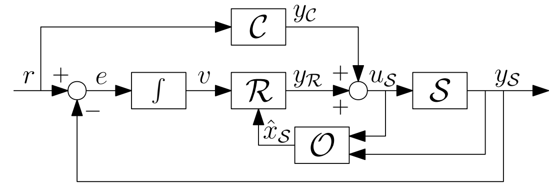

Fig. 1. Control scheme.

In (3), $u_\mathcal{R}(k) = \left[v(k)^T\ \hat{x}_\mathcal{S}(k)^T\right]^T$, $u_\mathcal{C}(k) = r(k)$, the functions $\zeta_{\ell,i}(\cdot)$ of $\zeta_\ell(\cdot)$ are nonlinear globally Lipschitz continuous $\forall i = 1, \dots, n_\ell$, $W_{u_\mathcal{C}} \in \mathbb{R}^{n_\mathcal{C}\times l_\mathcal{S}}$, $W_{out_{2_\mathcal{C}}} \in \mathbb{R}^{m_\mathcal{S}\times l_\mathcal{S}}$, $W_{x_\ell} \in \mathbb{R}^{n_\ell\times n_\ell}$, $W_{y_\ell} \in \mathbb{R}^{n_\ell\times m_\mathcal{S}}$, and $W_{out_{1_\ell}} \in \mathbb{R}^{m_\mathcal{S}\times n_\ell}$. Let us define $W_{u_\mathcal{R}} = [W_{u_{\mathcal{R}v}}\ W_{u_{\mathcal{R}x}}]$, and $W_{out_{2_\mathcal{R}}} = \left[W_{out_{2_{\mathcal{R}v}}}\ W_{out_{2_{\mathcal{R}x}}}\right]$, where $W_{u_{\mathcal{R}v}} \in \mathbb{R}^{n_\mathcal{R}\times l_\mathcal{S}}$, $W_{u_{\mathcal{R}x}} \in \mathbb{R}^{n_\mathcal{R}\times n_\mathcal{S}}$, $W_{out_{2_{\mathcal{R}v}}} \in \mathbb{R}^{m_\mathcal{S}\times l_\mathcal{S}}$, and $W_{out_{2_{\mathcal{R}x}}} \in \mathbb{R}^{m_\mathcal{S}\times n_\mathcal{S}}$.
Also, the discrete-time integrator block "$\int$" has equation

$$\eta(k+1) = \eta(k) + e(k)\,, \tag{4a}$$

$$v(k) = \eta(k) + e(k)\,, \tag{4b}$$

where $e(k) = r(k) - y_\mathcal{S}(k)$.
Finally, the block $\mathcal{O}$ denotes a state observer defined by

$$\hat{x}_\mathcal{S}(k+1) = f_\mathcal{S}(A_\mathcal{S}\hat{x}_\mathcal{S}(k) + B_\mathcal{S} u_\mathcal{S}(k) + L(y_\mathcal{S}(k) - \hat{y}_\mathcal{S}(k)))\,,$$
$$\hat{y}_\mathcal{S}(k) = C_\mathcal{S}\hat{x}_\mathcal{S}(k)\,, \tag{5}$$

where $\hat{y}_\mathcal{S}(k) \in \mathbb{R}^{l_\mathcal{S}}$ is the predicted output vector, $\hat{x}_\mathcal{S}(k) \in \mathbb{R}^{n_\mathcal{S}}$ is the predicted state vector, and the observer gain $L \in \mathbb{R}^{n_\mathcal{S}\times l_\mathcal{S}}$ is a design parameter.

In this work we propose a novel data-based control design technique for the design of $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{O}$ so as to: *i)* provide global asymptotic stability guarantees for the equilibria of the control system; *ii)* achieve asymptotic tracking of constant reference signals r; *iii)* make the control system as similar as possible to a reference model $\mathcal{M}$.
To do so, the proposed procedure is the following: based on the identified system model (2) and on a dataset of input and output data $u_\mathcal{S}(k), y_\mathcal{S}(k)$, for $k = 0, \dots, N_d$, we design the observer $\mathcal{O}$, the regulator $\mathcal{R}$, and the feedforward compensator $\mathcal{C}$ by enforcing matrix inequality constraints for the δISS of $\mathcal{O}$, of the closed-loop system, and of $\mathcal{C}$, while minimizing a VRFT-based cost function to optimize the performance of the control system.

4. CONTROL DESIGN

4.1 Observer design

We first deal with the design of a δISS observer $\mathcal{O}$ defined in (5). The observer gain L is designed by solving the optimization problem specified in the following lemma.

Lemma 1. Let the system (2) satisfy Assumption 1. Assume that there exist $\sigma > 0$, $H_\mathcal{O}$, and $P_\mathcal{O} = P_\mathcal{O}^T$ such that, by defining $\mathcal{L}_\mathcal{S} = \{i \in \{1, \dots, n_\mathcal{S}\}\,|\ f_{\mathcal{S}i}(\cdot) \neq id(\cdot)\}$, $p_{ij} = p_{ji} = 0\ \forall i \in \mathcal{L}_\mathcal{S}$, $\forall j \in \{1, \dots, n_\mathcal{S}\}$ with $j \neq i$, and such that the following optimization problem is solved.

$$\min_{H_\mathcal{O}, P_\mathcal{O}, \sigma}\ \sigma \tag{6a}$$

subject to

$$\begin{bmatrix} \sigma P_\mathcal{O} - \widetilde{A}_\mathcal{S}^T P_\mathcal{O} \widetilde{A}_\mathcal{S} + C_\mathcal{S}^T H_\mathcal{O}^T \widetilde{A}_\mathcal{S} + \widetilde{A}_\mathcal{S}^T H_\mathcal{O} C_\mathcal{S} & -C_\mathcal{S}^T H_\mathcal{O}^T \\ -H_\mathcal{O} C_\mathcal{S} & P_\mathcal{O} \end{bmatrix} \succ 0\,, \tag{6b}$$

$$\sigma \leq 1\,, \tag{6c}$$

where $\widetilde{A}_\mathcal{S} = W_\mathcal{S} A_\mathcal{S}$, and $W_\mathcal{S} = \text{diag}(L_{p\mathcal{S}1}, \ldots, L_{p\mathcal{S}n_\mathcal{S}})$, $L_{p\mathcal{S}i} > 0\ \forall i$. Then, by setting $L = W_\mathcal{S}^{-1} P_\mathcal{O}^{-1} H_\mathcal{O}$,

$$\|\hat{e}(k+1)\|^2_{P_\mathcal{O}} < \sigma \|\hat{e}(k)\|^2_{P_\mathcal{O}} \to 0 \tag{7}$$

as $k \to \infty$, where $\hat{e}(k) = x_\mathcal{S}(k) - \hat{x}_\mathcal{S}(k)$.

Proof. By following the same steps in (D'Amico et al., 2022b, Proposition 12), from (6b) we can show that $\widetilde{A}^T P_\mathcal{O} \widetilde{A} - P_\mathcal{O} + (1-\sigma) P_\mathcal{O} \prec 0$, where $\widetilde{A} = W_\mathcal{S}(A_\mathcal{S} - LC_\mathcal{S})$. From (6c), it follows that $\widetilde{A}^T P_\mathcal{O} \widetilde{A} - P_\mathcal{O} \prec 0$. Hence, the observer dynamics (5) fulfills the assumptions of Theorem 1 and is δISS. For notational compactness we denote with $\hat{x}_\mathcal{S}(k, \hat{x}_\mathcal{S}(0), \boldsymbol{u}_\mathcal{S}, \boldsymbol{y}_\mathcal{S})$ the evolution of the observer state when the initial condition is $\hat{x}_\mathcal{S}(0)$ and when it is fed by the input and output sequences $\boldsymbol{u}_\mathcal{S}$ and $\boldsymbol{y}_\mathcal{S}$, respectively. Note that $x_\mathcal{S}(k)$ is a possible observer motion, i.e., $\hat{x}_\mathcal{S}(k, x_\mathcal{S}(0), \boldsymbol{u}_\mathcal{S}, \boldsymbol{y}_\mathcal{S}) = x_\mathcal{S}(k)$ for all k. In view of the δISS of the observer, there exists $\beta \in \mathcal{KL}$ such that $\|x_\mathcal{S}(k) - \hat{x}_\mathcal{S}(k, \hat{x}_\mathcal{S}(0), \boldsymbol{u}_\mathcal{S}, \boldsymbol{y}_\mathcal{S})\| = \|\hat{e}(k)\| \leq \beta(\|x_\mathcal{S}(0) - \hat{x}_\mathcal{S}(0)\|, k) \to 0$ as $k \to +\infty$. In particular, using a similar line of reasoning as in the proof of (D'Amico et al., 2022b, Theorem 2), it can be shown that $\|\hat{e}(k)\|^2_{P_\mathcal{O}}$ is the δISS Lyapunov function of (5), and that, from (6b), $\|\hat{e}(k+1)\|^2_{P_\mathcal{O}} \leq \|\hat{e}(k)\|^2_{\widetilde{A}^T P_\mathcal{O} \widetilde{A}} < \sigma \|\hat{e}(k)\|^2_{P_\mathcal{O}}$. □

Note that (6b) is not an LMI since both σ and $P_\mathcal{O}$ are optimization variables. One way to proceed is to remove σ from the set of free variables and to solve the feasibility problem (6b), (6c) for $\sigma = 1$ first. If the solution exists, then we can solve (6b), (6c) by reducing the value of σ, until infeasibility is met, e.g., by bisection.

4.2 Stability guarantees

Here we provide the conditions for the δISS of the control scheme in Figure 1, obtained by conferring, with two separated LMIs, δISS to both $\mathcal{C}$ and the closed-loop system obtained discarding $\mathcal{C}$. This follows from the properties of the cascade of δISS systems, which can be proved by extending to discrete-time systems (Angeli, 2002, Proposition 4.7), provided for continuous-time systems.

Stability of the closed-loop system. Consider the feedback control system in Figure 1, where $\mathcal{C}$ is discarded, considering $y_\mathcal{C}(k)$ as an independent exogenous variable. Note that $\hat{x}_\mathcal{S}(k) = x_\mathcal{S}(k) - \hat{e}(k)$ and that, in view of Lemma 1, we can account for $\hat{e}(k)$ as an asymptotically vanishing perturbation. By jointly considering (2)-(4) and by recalling that $u_\mathcal{S}(k) = y_\mathcal{R}(k) + y_\mathcal{C}(k)$ and $e(k) = r(k) - y_\mathcal{S}(k)$, we can write the closed-loop system as in (1), where Assumption 1 is satisfied, and $x(k) = \begin{bmatrix} x_\mathcal{R}(k)^T & \eta(k)^T & x_\mathcal{S}(k)^T \end{bmatrix}^T$, $u(k) = \begin{bmatrix} r(k)^T & y_\mathcal{C}(k)^T & \hat{e}(k)^T \end{bmatrix}^T$, $f(\cdot) = \begin{bmatrix} \zeta_\mathcal{R}(\cdot)^T & id_{l_\mathcal{S}}(\cdot)^T & f_\mathcal{S}(\cdot)^T \end{bmatrix}^T$. We compute that $A = F_\mathcal{R} + G_\mathcal{R} K_\mathcal{R}$, being

$$F_\mathcal{R} = \begin{bmatrix} W_{x_\mathcal{R}} & W_{u_{\mathcal{R}v}} & W_{u_{\mathcal{R}x}} - W_{u_{\mathcal{R}v}} C_\mathcal{S} \\ 0 & I & -C_\mathcal{S} \\ 0 & 0 & A_\mathcal{S} \end{bmatrix}, \quad G_\mathcal{R} = \begin{bmatrix} W_{y_\mathcal{R}} \\ 0 \\ B_\mathcal{S} \end{bmatrix},$$

$K_\mathcal{R} = \begin{bmatrix} W_{out_{1_\mathcal{R}}} & W_{out_{2_{\mathcal{R}v}}} & W_{out_{2_{\mathcal{R}x}}} - W_{out_{2_{\mathcal{R}v}}} C_\mathcal{S} \end{bmatrix} = \begin{bmatrix} W_{out_{1_\mathcal{R}}} & W_{out_{2_\mathcal{R}}} \end{bmatrix} E$, where

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -C_\mathcal{S} \\ 0 & 0 & I \end{bmatrix}.$$

Also, let us introduce

$$B = \begin{bmatrix} W_{u_{\mathcal{R}v}} + W_{y_\mathcal{R}} W_{out_{2_{\mathcal{R}v}}} & 0 & -W_{u_{\mathcal{R}x}} - W_{y_\mathcal{R}} W_{out_{2_{\mathcal{R}x}}} \\ I & 0 & 0 \\ B_\mathcal{S} W_{out_{2_{\mathcal{R}v}}} & B_\mathcal{S} & -B_\mathcal{S} W_{out_{2_{\mathcal{R}x}}} \end{bmatrix}.$$

At this point we can apply Theorem 1 to enforce the δISS of the control scheme, as proved in the following lemma.

Lemma 2. If $\exists P_\mathcal{R} = P_\mathcal{R}^T \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ block diagonal such that $p_{ij} = p_{ji} = 0\ \forall i \in \mathcal{L}$, $\forall j \in \{1, \ldots, \tilde{n}\}$ with $j \neq i$ and $\tilde{n} = n_\mathcal{S} + n_\mathcal{R} + l_\mathcal{S}$, and $\exists\, H_\mathcal{R}, Q_\mathcal{R}$ such that

$$\begin{bmatrix} P_\mathcal{R} & \widetilde{F}_\mathcal{R} Q_\mathcal{R} + \widetilde{G}_\mathcal{R} H_\mathcal{R} \\ (\widetilde{F}_\mathcal{R} Q_\mathcal{R} + \widetilde{G}_\mathcal{R} H_\mathcal{R})^T & Q_\mathcal{R} + Q_\mathcal{R}^T - P_\mathcal{R} \end{bmatrix} \succ 0\,, \tag{8}$$

where $\widetilde{F}_\mathcal{R} = W_\mathcal{R} F_\mathcal{R}$, $W_\mathcal{R} = \text{diag}(L_{p\mathcal{R}1}, \ldots, L_{p\mathcal{R}n_\mathcal{R}}, I, W_\mathcal{S})$, and $\widetilde{G}_\mathcal{R} = W_\mathcal{R} G_\mathcal{R}$, then the closed-loop system is δISS by setting $K_\mathcal{R} = H_\mathcal{R} Q_\mathcal{R}^{-1}$. □

Proof. If (8) holds then, as in (De Oliveira et al., 1999, Theorem 1), and since $H_\mathcal{R} = K_\mathcal{R} Q_\mathcal{R}$,

$$P_\mathcal{R} - (\widetilde{F}_\mathcal{R} + \widetilde{G}_\mathcal{R} K_\mathcal{R}) P_\mathcal{R} (\widetilde{F}_\mathcal{R} + \widetilde{G}_\mathcal{R} K_\mathcal{R})^T \succ 0\,.$$

In view of the Schur complement, this is equivalent to

$$P_\mathcal{R}^{-1} - (\widetilde{F}_\mathcal{R} + \widetilde{G}_\mathcal{R} K_\mathcal{R})^T P_\mathcal{R}^{-1} (\widetilde{F}_\mathcal{R} + \widetilde{G}_\mathcal{R} K_\mathcal{R}) \succ 0\,. \tag{9}$$

Note that $P_\mathcal{R}^{-1} = P_\mathcal{R}^{-T} \succ 0$ is block diagonal with the same structure of $P_\mathcal{R}$. From (9) the assumptions of Theorem 1 hold, and the closed-loop control system is δISS. □

Stability of $\mathcal{C}$. Note that (3) is in the class defined by Assumption 1. For reasons related to the fact that the tuning gain matrix $K_\mathcal{C} = \begin{bmatrix} W_{out_{1_\mathcal{C}}} & W_{out_{2_\mathcal{C}}} \end{bmatrix}$ has $n_\mathcal{C} + l_\mathcal{S}$ columns, the state of the compensator is extended with a fictitious variable $\widetilde{x}_\mathcal{C} \in \mathbb{R}^{l_\mathcal{S}}$ evolving as $\widetilde{x}_\mathcal{C}(k+1) = 0$ [1]. In this way we write the compensator equation (3) as in (1) where $x(k) = \begin{bmatrix} x_\mathcal{C}(k)^T & \widetilde{x}_\mathcal{C}(k)^T \end{bmatrix}^T$, $u(k) = r(k)$, and $f(\cdot) = \begin{bmatrix} \zeta_\mathcal{C}(\cdot)^T & id_{l_\mathcal{S}}(\cdot)^T \end{bmatrix}$ satisfies Assumption 1. The dynamic system matrix is $A = F_\mathcal{C} + G_\mathcal{C} K_\mathcal{C}$, being

$$F_\mathcal{C} = \begin{bmatrix} W_{x_\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix}, \; G_\mathcal{C} = \begin{bmatrix} W_{y_\mathcal{C}} \\ 0 \end{bmatrix}, \; B = \begin{bmatrix} W_{u_\mathcal{C}} + W_{y_\mathcal{C}} W_{out_{2_\mathcal{C}}} \\ 0 \end{bmatrix}.$$

The following lemma can be proved.

Lemma 3. If $\exists P_\mathcal{C}$ diagonal and $\exists H_\mathcal{C}, Q_\mathcal{C}$ such that

$$\begin{bmatrix} \widetilde{P}_\mathcal{C} & \widetilde{F}_\mathcal{C} Q_\mathcal{C} + \widetilde{G}_\mathcal{C} H_\mathcal{C} \\ (\widetilde{F}_\mathcal{C} Q_\mathcal{C} + \widetilde{G}_\mathcal{C} H_\mathcal{C})^T & Q_\mathcal{C} + Q_\mathcal{C}^T - \widetilde{P}_\mathcal{C} \end{bmatrix} \succ 0\,, \tag{10}$$

where $\widetilde{F}_\mathcal{C} = W_\mathcal{C} F_\mathcal{C}$, $\widetilde{G}_\mathcal{C} = W_\mathcal{C} G_\mathcal{C}$, $\widetilde{P}_\mathcal{C} = \text{diag}(P_\mathcal{C}, I)$, and $W_\mathcal{C} = \text{diag}(L_{p\mathcal{C}1}, \ldots, L_{p\mathcal{C}n_\mathcal{C}}, I)$, then the compensator $\mathcal{C}$ is δISS by setting $K_\mathcal{C} = H_\mathcal{C} Q_\mathcal{C}^{-1}$. □

Proof. Using the arguments used in the proof of Lemma 2, the LMI (10) implies the δISS of the extended system. This, in turn, implies the δISS of the compensator equation, since $\widetilde{x}_\mathcal{C}(k) = 0$ for all $k > 0$. □

4.3 Tracking of constant signals

In this section we state the main convergence result.

Theorem 2. Assume that there exist σ, $H_\mathcal{O}$, $P_\mathcal{O}$, $H_\mathcal{R}$, $P_\mathcal{R}$, $Q_\mathcal{R}$, $H_\mathcal{C}$, $P_\mathcal{C}$, $Q_\mathcal{C}$ such that (6b)-(6c), (8), and (10) are

[1] This extension does not take any role in the stability condition, but allows us to cast the VRFT problem into an LMI one, which would be otherwise impossible, as shown later in the paper.

feasible. Set $L = W_{\mathcal{S}}^{-1}P_{\mathcal{O}}^{-1}H_{\mathcal{O}}$, $K_{\mathcal{R}} = H_{\mathcal{R}}Q_{\mathcal{R}}^{-1}$, and $K_{\mathcal{C}} = H_{\mathcal{C}}Q_{\mathcal{C}}^{-1}$. Then, for all initial conditions $x_{\mathcal{C}}(0)$, $x_{\mathcal{R}}(0)$, $\eta(0)$, $x_{\mathcal{S}}(0)$, $\hat{x}_{\mathcal{S}}(0)$, for any constant reference signal $\bar{r} \in \mathbb{R}^{l_{\mathcal{S}}}$, if we set $r(k) = \bar{r}$, then $y_{\mathcal{S}}(k) \to \bar{r}$ as $k \to +\infty$.

Proof. The proof requires a number of steps.
- In view of the δISS of the compensator $\mathcal{C}$ (provided by Lemma 3) and of (Angeli, 2002, Proposition 4.2) [2], there exists a unique equilibrium $\bar{x}_{\mathcal{C}}$ obtained with $r(k) = \bar{r}$ such that $x_{\mathcal{C}}(k) \to \bar{x}_{\mathcal{C}}$ as $k \to +\infty$ for all $x_{\mathcal{C}}(0)$. This implies that $y_{\mathcal{C}}(k) \to \bar{y}_{\mathcal{C}}$ as $k \to +\infty$, where $\bar{y}_{\mathcal{C}}$ is the output corresponding to the equilibrium.
- Considering the feedback control system, whose input is $u(k) = \begin{bmatrix} r(k)^T & y_{\mathcal{C}}(k)^T & \hat{e}(k)^T \end{bmatrix}^T$, if $u(k) = \bar{u} = \begin{bmatrix} \bar{r}^T & \bar{y}_{\mathcal{C}}^T & 0 \end{bmatrix}^T$, in view of the δISS (provided by Lemma 2) and of (Angeli, 2002, Proposition 4.2), then there exists a unique equilibrium $\begin{bmatrix} \bar{x}_{\mathcal{R}}^T & \bar{\eta}^T & \bar{x}_{\mathcal{S}}^T \end{bmatrix}^T$. Importantly, in view of the fact that equation (4a) is also in equilibrium, $e(k) = \bar{r} - \bar{y}_{\mathcal{S}} = 0$ for all k, meaning that $\bar{y}_{\mathcal{S}} = \bar{r}$, where $\bar{y}_{\mathcal{S}}$ is the output of $\mathcal{S}$ corresponding to the equilibrium.
- Given the δISS of the control system, by denoting $\delta x(k) = \begin{bmatrix} x_{\mathcal{R}}(k)^T - \bar{x}_{\mathcal{R}}^T, & \eta(k)^T - \bar{\eta}^T, & x_{\mathcal{S}}(k)^T - \bar{x}_{\mathcal{S}}^T \end{bmatrix}^T$ and $\delta u(k) = \begin{bmatrix} r(k)^T - \bar{r}^T, & y_{\mathcal{C}}(k)^T - \bar{y}_{\mathcal{C}}^T, & \hat{e}(k)^T \end{bmatrix}^T$, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\|\delta x(k)\| \leq \beta(\|\delta x(0)\|, k) + \gamma(\|\delta \mathbf{u}\|_\infty)$, where $\delta \mathbf{u}$ denotes the sequence of $\delta u(k)$.
- Finally, note that $r(k) = \bar{r}$ and that $y_{\mathcal{C}}(k) \to \bar{y}_{\mathcal{C}}$ and $\hat{e}(k) \to 0$ as $k \to +\infty$, which imply that $\delta u(k) \to 0$ as $k \to +\infty$; by applying the same arguments used in the proof of (Jiang and Wang, 2001, Lemma 3.8), there exists a function $\gamma_e \in \mathcal{K}_\infty$ such that

$$\overline{\lim}_{k\to+\infty}\|\delta x(k)\| \leq \gamma_e(\overline{\lim}_{k\to+\infty}\|\delta u(k)\|) = 0,$$

where $\overline{\lim}$ denotes the limit superior (Jiang and Wang, 2001). This concludes the proof. □

4.4 VRFT cost minimization

To confer suitable dynamic performances to the control system the VRFT approach (Campi and Savaresi, 2006) is adopted. With reference to Figure 1, the problem addressed in this section consists of the design of the regulator/compensator gains $W_{out_{1_\ell}}$ and $W_{out_{2_\ell}}$, $\ell \in \{\mathcal{R}, \mathcal{C}\}$ such that the system whose input is $r(k)$ and whose output is $y_{\mathcal{S}}(k)$ be as close as possible to the reference one $\mathcal{M}$. To keep the problem linear, we split the design problem in two parts: (i) the design of $\mathcal{R}$ and (ii) the design of $\mathcal{C}$, which are discussed in two separated sections.

VRFT-based design of $\mathcal{R}$. In this section, we discard $\mathcal{C}$ and set $y_{\mathcal{C}}(k) = 0$, $\forall k$. Consider the dataset $u_{\mathcal{S}}(k), y_{\mathcal{S}}(k)$, for $k = 0, \ldots, N_d$. To apply VRFT we need to
- compute the *virtual reference* $r^*(k) = \mathcal{M}^{-1}y_{\mathcal{S}}(k)$;
- compute the *virtual error* $e^*(k) = r^*(k) - y_{\mathcal{S}}(k)$;
- compute the *integrated virtual error* according to the recursive equation $v^*(k) = v^*(k-1) + e^*(k)$, with initial condition $v^*(0) = 0$;
- compute the state estimate $\hat{x}_{\mathcal{S}}(k)$ using (5) with initial condition $\hat{x}_{\mathcal{S}}(0) = 0$.

At this point, we need to identify the regulator model $\mathcal{R}$ whose input consists of $u_{\mathcal{R}}^*(k) = \begin{bmatrix} v^*(k)^T & \hat{x}_{\mathcal{S}}(k)^T \end{bmatrix}^T$ and whose output is known, i.e., $y_{\mathcal{R}}(k) = u_{\mathcal{S}}(k)$. This, thanks to the fact that $\mathcal{R}$ is an ESN, boils down to a simple LS problem, where the unknowns are $W_{out_{1_\mathcal{R}}}$ and $W_{out_{2_\mathcal{R}}}$. To do so, we compute $x_{\mathcal{R}}^*(k)$ according to (3a), with inputs $u_{\mathcal{R}}^*(k)$, $y_{\mathcal{R}}(k) = u_{\mathcal{S}}(k)$, and initial condition $x_{\mathcal{R}}^*(0) = 0$.
The VRFT problem consists in the minimization of

$$J_{VR}(W_{out_{1_\mathcal{R}}}, W_{out_{2_\mathcal{R}}}) = = \frac{1}{N_d}\sum_{k=k_0}^{N_d-1}\left\| u_{\mathcal{S}}(k) - \begin{bmatrix} W_{out_{1_\mathcal{R}}} & W_{out_{2_\mathcal{R}}} \end{bmatrix}\begin{bmatrix} x_{\mathcal{R}}^*(k) \\ u_{\mathcal{R}}^*(k) \end{bmatrix}\right\|^2 \qquad (11)$$

where the initial instant k_0 is considered to discard the initial transient. However we cannot, at the same time, directly minimize J_{VR} and enforce the stability constraint (8) since the free variables of (8) are $Q_{\mathcal{R}}$, $H_{\mathcal{R}}$, and $P_{\mathcal{R}}$, and $\begin{bmatrix} W_{out_{1_\mathcal{R}}} & W_{out_{2_\mathcal{R}}} \end{bmatrix} = K_{\mathcal{R}}E^{-1} = H_{\mathcal{R}}Q_{\mathcal{R}}^{-1}E^{-1}$.
To find a unifying LMI problem, we need to define
$U_{N_d} = [u_{\mathcal{S}}(k_0) \ \ldots \ u_{\mathcal{S}}(N_d-1)]^T$,

$$X_{N_d} = \begin{bmatrix} x_{\mathcal{R}}^*(k_0)^T & u_{\mathcal{R}}^*(k_0)^T \\ \vdots & \vdots \\ x_{\mathcal{R}}^*(N_d-1)^T & u_{\mathcal{R}}^*(N_d-1)^T \end{bmatrix}$$

and $\widetilde{X}_{N_d} = E^T(X_{N_d}^TX_{N_d})^{-1}X_{N_d}^T$, and we introduce the following "identifiability" assumption.

Assumption 2. Matrix $X_{N_d}^TX_{N_d}$ is invertible.

Along the lines of (D'Amico and Farina, 2022, Theorem 1), it is possible to show that, under Assumption 2, the following optimization problem

$$\min_{H_{\mathcal{R}},\Phi_{\mathcal{R}}} \ \mathrm{tr}(\Phi_{\mathcal{R}}) \qquad (12a)$$

$$\text{subject to} \quad \begin{bmatrix} \Phi_{\mathcal{R}} - \Delta_{\mathcal{R}} & H_{\mathcal{R}} \\ H_{\mathcal{R}}^T & Q_{\mathcal{R}} \end{bmatrix} \succcurlyeq 0 \qquad (12b)$$

where
$\Delta_{\mathcal{R}} = U_{N_d}^T\widetilde{X}_{N_d}^TQ_{\mathcal{R}}\widetilde{X}_{N_d}U_{N_d} - H_{\mathcal{R}}\widetilde{X}_{N_d}U_{N_d} - U_{N_d}^T\widetilde{X}_{N_d}^TH_{\mathcal{R}}^T$
is equivalent to minimizing (11) if, for any scalar $\gamma > 0$,

$$Q_{\mathcal{R}} = \gamma E^{-1}X_{N_d}^TX_{N_d}E^{-T}. \qquad (12c)$$

VRFT-based design of $\mathcal{C}$. The design of $\mathcal{C}$ is performed as a second step, after the regulator $\mathcal{R}$ is obtained as described above. Considering that the input-output data sequences and $\mathcal{R}$ are available, to apply VRFT we need to
- compute $y_{\mathcal{R}}^*(k)$ according to (3), with input $u_{\mathcal{R}}^*(k)$, and initial condition $x_{\mathcal{R}}^*(0) = 0$;
- compute $y_{\mathcal{C}}^*(k) = u_{\mathcal{S}}(k) - y_{\mathcal{R}}^*(k)$.

At this point, we need to identify the compensator model $\mathcal{C}$ whose input consists of $u_{\mathcal{C}}^*(k) = r^*(k)$ and whose output is $y_{\mathcal{C}}^*(k)$. This, thanks to the fact that $\mathcal{C}$ is an ESN, boils down to a simple LS problem, where the unknowns are $W_{out_{1_\mathcal{C}}}$ and $W_{out_{2_\mathcal{C}}}$. To do so, we compute $x_{\mathcal{C}}^*(k)$ according to (3a), with inputs $u_{\mathcal{C}}^*(k) = r^*(k)$, $y_{\mathcal{C}}^*(k)$, and initial condition $x_{\mathcal{C}}^*(0) = 0$. The VRFT problem consists of minimizing

$$J_{VR}^{\mathcal{C}}(W_{out_{1_\mathcal{C}}}, W_{out_{2_\mathcal{C}}}) = = \frac{1}{N_d}\sum_{k=k_0}^{N_d-1}\left\| y_{\mathcal{C}}^*(k) - \begin{bmatrix} W_{out_{1_\mathcal{C}}} & W_{out_{2_\mathcal{C}}} \end{bmatrix}\begin{bmatrix} x_{\mathcal{C}}^*(k) \\ r^*(k) \end{bmatrix}\right\|^2 . \qquad (13)$$

As done in case of the regulator $\mathcal{R}$ we cannot, at the same time, directly minimize $J_{VR}^{\mathcal{C}}$ and enforce the stability

[2] Proposition 4.2 is stated for continuous-time systems but it can be readily extended to the discrete-time case due to δISS.

constraint (10) since the free variables of (10) are $Q_{\mathcal{C}}$, $H_{\mathcal{C}}$, and $P_{\mathcal{C}}$, and $\left[W_{out_{1\mathcal{C}}} \; W_{out_{2\mathcal{C}}}\right] = K_{\mathcal{C}} = H_{\mathcal{C}} Q_{\mathcal{C}}^{-1}$.To find a unifying LMI problem, we apply again the arguments used in (D'Amico and Farina, 2022). First we define $U_{N_d}^{\mathcal{C}} = [y_{\mathcal{C}}^*(k_0) \; \dots \; y_{\mathcal{C}}^*(N_d-1)]^T$,

$$X_{N_d}^{\mathcal{C}} = \begin{bmatrix} x_{\mathcal{C}}^*(k_0)^T & r^*(k_0)^T \\ \vdots & \vdots \\ x_{\mathcal{C}}^*(N_d-1)^T & r^*(N_d-1)^T \end{bmatrix},$$

and $\widetilde{X}_{N_d}^{\mathcal{C}} = ((X_{N_d}^{\mathcal{C}})^T X_{N_d}^{\mathcal{C}})^{-1} (X_{N_d}^{\mathcal{C}})^T$, where the following is assumed to hold.

Assumption 3. Matrix $(X_{N_d}^{\mathcal{C}})^T X_{N_d}^{\mathcal{C}}$ is invertible.

As done for the design of $\mathcal{R}$ it is possible to show that, under Assumption 3, the optimization problem

$$\min_{H_{\mathcal{C}}, \Phi_{\mathcal{C}}} \; \mathrm{tr}(\Phi_{\mathcal{C}}) \tag{14a}$$

$$\text{subject to} \quad \begin{bmatrix} \Phi_{\mathcal{C}} - \Delta_{\mathcal{C}} & H_{\mathcal{C}} \\ H_{\mathcal{C}}^T & Q_{\mathcal{C}} \end{bmatrix} \succcurlyeq 0 \tag{14b}$$

where $\Delta_{\mathcal{C}} = (U_{N_d}^{\mathcal{C}})^T (\widetilde{X}_{N_d}^{\mathcal{C}})^T Q_{\mathcal{C}} \widetilde{X}_{N_d}^{\mathcal{C}} U_{N_d}^{\mathcal{C}} - H_{\mathcal{C}} \widetilde{X}_{N_d}^{\mathcal{C}} U_{N_d}^{\mathcal{C}} - (U_{N_d}^{\mathcal{C}})^T (\widetilde{X}_{N_d}^{\mathcal{C}})^T H_{\mathcal{C}}^T$ is equivalent to minimizing (13) if, for any scalar $\gamma > 0$,

$$Q_{\mathcal{C}} = \gamma (X_{N_d}^{\mathcal{C}})^T X_{N_d}^{\mathcal{C}}, \tag{14c}$$

4.5 Design procedure

Here we briefly sketch the overall design procedure.

1. Collect an input-output dataset from the plant.
2. Identify a model in the class (2).
3. Design the observer gain L by solving (6).
4. Design $\mathcal{R}$ by solving (12) subject also to (8). Note that, setting $Q_{\mathcal{R}}$ as in (12c) may result in an infeasible problem. In this case, (12c) can be relaxed by defining the matrix $Q_{\mathcal{R}}$ as an optimization variable and replacing (12c) with

$$Q_{\mathcal{R}} - \gamma E^{-1} X_{N_d}^T X_{N_d} E^{-T} + \lambda_{\mathcal{R}} I \succcurlyeq 0 \tag{15a}$$

$$-Q_{\mathcal{R}} + \gamma E^{-1} X_{N_d}^T X_{N_d} E^{-T} + \lambda_{\mathcal{R}} I \succcurlyeq 0 \tag{15b}$$

where the scalar $\lambda_{\mathcal{R}} \geq 0$ has to be minimized together with $\mathrm{tr}(\Phi_{\mathcal{R}})$, i.e., making the cost $\mathrm{tr}(\Phi_{\mathcal{R}}) + c_{\mathcal{R}} \lambda_{\mathcal{R}}$, where $c_{\mathcal{R}} > 0$ is a user-defined constant.

5. Design $\mathcal{C}$ by solving (14) subject also to (10). As in case of $\mathcal{R}$, (14c) can be relaxed by defining the matrix $Q_{\mathcal{C}}$ as a free optimization variable and replacing (14c) with

$$Q_{\mathcal{C}} - \gamma (X_{N_d}^{\mathcal{C}})^T X_{N_d}^{\mathcal{C}} + \lambda_{\mathcal{C}} I \succcurlyeq 0 \tag{16a}$$

$$-Q_{\mathcal{C}} + \gamma (X_{N_d}^{\mathcal{C}})^T X_{N_d}^{\mathcal{C}} + \lambda_{\mathcal{C}} I \succcurlyeq 0 \tag{16b}$$

where the scalar $\lambda_{\mathcal{C}} \geq 0$ has to be minimized together with $\mathrm{tr}(\Phi_{\mathcal{C}})$, i.e., making the cost $\mathrm{tr}(\Phi_{\mathcal{C}}) + c_{\mathcal{C}} \lambda_{\mathcal{C}}$, where $c_{\mathcal{C}} > 0$ is a user-defined constant.

5. SIMULATION RESULTS

The system to be controlled is a simulated pH neutralization process (see Bugliari Armenio et al. (2019) for details). A noiseless dataset containing $N_d = 100000$ normalized input-output data is collected with a sampling time $T_s = 25\ s$ from the simulated process. The input data consist of a multilevel pseudo-random signal (MPRS), whose amplitude lies in $[12, 16]\ mL/s$. The identified model with $n_{\mathcal{S}} = 10$ states is

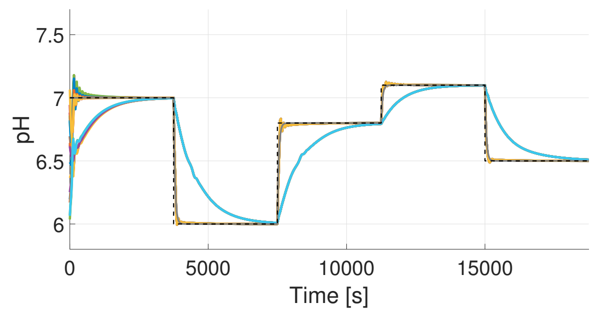

Fig. 2. Output trajectories. Black dashed line: reference trajectory; grey line: reference model output trajectory; light blue line: only $\mathcal{R}$ is used; yellow line: both $\mathcal{R}$ and $\mathcal{C}$ are used.

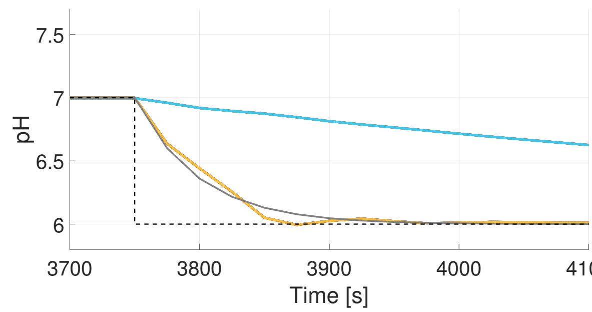

Fig. 3. Zoom of the output trajectories. Black dashed line: reference trajectory; grey line: reference model output trajectory; light blue line: only $\mathcal{R}$ is used; yellow line: both $\mathcal{R}$ and $\mathcal{C}$ are used.

$$\begin{aligned} x_{\mathcal{S}}(k+1) &= \zeta_{\mathcal{S}}(W_{x_{\mathcal{S}}} x_{\mathcal{S}}(k) + W_{u_{\mathcal{S}}} u_{\mathcal{S}}(k) + W_{y_{\mathcal{S}}} y_{\mathcal{S}}(k)), \\ y_{\mathcal{S}}(k) &= W_{out_{1\mathcal{S}}} x_{\mathcal{S}}(k), \end{aligned} \tag{17}$$

where $\zeta_{\mathcal{S}_i}(\cdot) = id(\cdot)$ for $i = 1, \dots, 5$, whereas $\zeta_{\mathcal{S}_i}(\cdot) = \tanh(\cdot)$ for $i = 6, \dots, 10$. The structure of the state equations, both nonlinear and linear, is such that the output can take unbounded values, allowing us to the possible inclusion of an explicit integral action, e.g., see (D'Amico et al., 2022b, Proposition 15). The desired reference model $\mathcal{M}$ is the first-order asymptotically stable and unitary-gain LTI system $y(k) = -ay(k-1) + br(k-1)$, with settling time $10T_s = 250$ s (being $a = -0.6$ and $b = 0.4$), whereas the open-loop system settles in about 500 s.

The observer gain L is computed by solving (6b), where the minimum value of σ for which we have feasibility is 0.28. The regulator $\mathcal{R}$ is designed with $n_{\mathcal{R}} = 5$ states using YALMIP and MOSEK (Lofberg, 2004; ApS, 2019), with $c_{\mathcal{R}} = 0.01$. The feedforward compensator $\mathcal{C}$ is designed with $n_{\mathcal{C}} = 5$ states, where $c_{\mathcal{C}} = 100$. For both the controllers $\mathcal{R}$ and $\mathcal{C}$ we set $k_0 = 500$, $\gamma = \frac{1}{N_d - k_0}$, and $\zeta_{\mathcal{C}_i}(\cdot) = \zeta_{\mathcal{R}_i}(\cdot) = \tanh(\cdot)$ for all i.

The control scheme in Figure 1 is tested on the physics-based simulator, where a denormalization of the control variable $u_{\mathcal{S}}$ and a normalization of the output $y_{\mathcal{S}}$ are carried out upstream and downstream of the process, respectively, with the same normalization parameters applied in the model identification phase. A normalization of the reference signal is also performed. In Figure 2 the reference tracking results of the control system using the simulated pH process and starting from 20 different random initial conditions are represented in case only $\mathcal{R}$ is used or both $\mathcal{R}$

and $\mathcal{C}$ are employed, respectively. The output trajectories converge to the reference value. However, in the former case the response is quite slow, whereas in the latter the desired performance is achieved due to the feedforward compensator, designed using the proposed VRFT procedure. Figure 3 reports a zoom of the trajectories depicted in Figure 2.

6. CONCLUSION

In this paper we proposed a novel data-based method for the design of controllers for RNNs. Our method guarantees stability properties of the closed-loop and desired static and dynamic performances. This is done by applying the VRFT approach for the data-driven design of the ESN-based regulator and feedforward compensator. In future work we will address the problem of guaranteeing stability robustly, with possible model mismatches due to the noise on the available data.

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