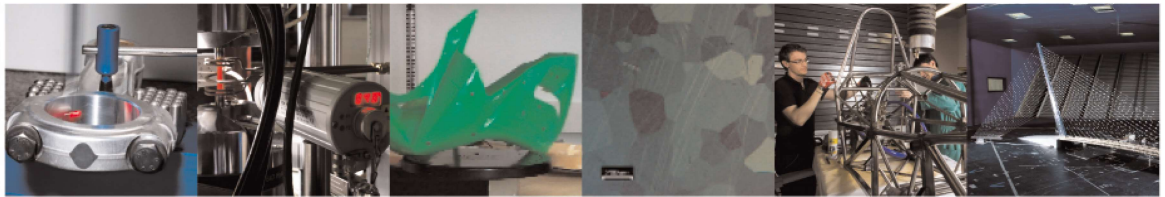




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## **A Meta-Heuristic Optimization Procedure for the Identification of the Nonlinear Model Parameters of Hydraulic Dampers Based on Experimental Dataset of Real Working Conditions**

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This is a post-peer-review, pre-copyedit version of an article published in Journal of Computational and Nonlinear Dynamics. The final authenticated version is available online at: <https://doi.org/10.1115/1.4062541>.

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1       **A meta-heuristic optimization procedure**  
2           **for the identification of the nonlinear**  
3           **model parameters of hydraulic dampers**  
4           **based on experimental dataset of real**  
5           **working conditions**

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23

24       **ABSTRACT**

25

26       *Hydraulic dampers are widely implemented in railway vehicle suspension stages, especially in high-speed*  
27       *passenger trains. They are designed to be mounted in different positions to improve comfort, stability, and*  
28       *safety performances. Numerical simulations are often used to assist the design and optimization of these*  
29       *components. Unfortunately, hydraulic dampers are highly nonlinear due to the complex fluid dynamic*  
30       *phenomena taking place inside the chambers and through the by-pass orifices. This requires accurate*  
31       *damper models to be developed to estimate the influence of the nonlinearities of such components during*  
32       *the dynamic performances of the whole vehicle. This work aims at presenting a new parametric damper*  
33       *model based on a nonlinear lumped element approach. Moreover, a new model tuning procedure will be*

34 *introduced. Differently from the typical sinusoidal characterization cycles, this routine is based on*  
35 *experimental tests of real working conditions. The set of optimal model parameters will be found through*  
36 *a meta-heuristic iterative approach able to minimize the differences between numerical and experimental*  
37 *dampner forces. The performances of the optimal model will be compared with the ones of the most*  
38 *common Maxwell model generally implemented in railway multibody software programs.*

## 39 **1 INTRODUCTION**

40

41           Hydraulic dampers are one of the most diffused types of suspension components  
42 in railway vehicles. The hydraulic dampers are characterized by a dissipative effect  
43 caused by the liquid flowing through the internal orifices of the component. Dampers  
44 are generally implemented in both primary and secondary suspension stages of  
45 passenger vehicles [1] and are oriented in different directions. Different experimental  
46 approaches have been developed to support the design of new suspension components,  
47 such as field tests or Hardware-In-the-Loop (HIL) simulations. The HIL experimental  
48 technique has been adopted in many works to investigate the performances of  
49 prototype suspensions [2] due to its lower cost respect to field tests.

50           Nowadays, in order to further reduce the design cost, the development of new  
51 hydraulic dampers for railway vehicles is largely supported by virtual simulations and  
52 laboratory experimental tests. These procedures aim at evaluating the effects of the  
53 studied component on the dynamic performances of the vehicles. In railway  
54 engineering, the virtual simulations are generally based on the multibody approach,  
55 which allows to represent the dynamic coupling between the vehicle subsystems  
56 (wheelsets, bogies, carbodies, etc.) through the implementation of several suspension

57 elements. The multibody approach proved to be very effective in simulating the  
58 dynamics of railway vehicles, but, according to Evans and Berg [3] and Bruni et al. [4],  
59 there are still several challenges, especially in implementing the correct modelling  
60 approach of the different suspension components.

61         The modelling of hydraulic dampers through accurate virtual models is  
62 fundamental to obtain reliable numerical results. The damper numerical models must  
63 be simple and fast in order to reduce the computational effort of multibody analysis,  
64 especially considering that a railway vehicle contains a large amount of suspension  
65 components to be modelled. Moreover, the use of damper models with low  
66 computational effort is a requirement for the design of vehicle models able to be run in  
67 real time during HIL tests. For these reasons it is important to obtain the maximum  
68 accuracy from simple suspension models before considering switching to more complex  
69 approaches.

70         The dynamics of hydraulic dampers is not trivial due to their nonlinear nature.  
71 For instance, in the most common solution, the damping force, strongly related to the  
72 damper elongation speed, is limited by a blow-off valve which avoids extreme force  
73 development. Furthermore, railway dampers present an asymmetric behavior between  
74 rebound and compression strokes. Asymmetry is generally more relevant in vertical  
75 devices [5]. The damper modelling approach cannot exempt from considering also that  
76 the dynamic behavior of the component is strongly influenced by its overall flexibility.  
77 Indeed, hydraulic dampers are not purely dissipating elements. Oil compressibility,  
78 elasticity of piston and cylinder are the most important contributions to the device

79 flexibility. Moreover, the presence of resilient rubber mounts at the damper-vehicle  
80 interfaces, also known as silent blocks, increases the complexity of the system due to  
81 their nonlinear elastic and dissipating contributions. Several studies investigated the  
82 relevance of a correct damper flexibility modelling. For example, according to [4, 6, 7, 8],  
83 the influence of the yaw damper flexibility is fundamental to correctly estimate the  
84 stability of railway vehicles. Stability assessments performed without considering an in-  
85 series stiffness tends to overestimate the performances of the vehicle [6, 8]. According  
86 to [9, 10], an accurate modelling of the yaw damper is crucial to assess the vehicle  
87 dynamics. The correct estimation of the damper stiffness is also important in the  
88 evaluation of ride comfort performances: different studies [11, 12, 13] considered this  
89 effect when modelling yaw or vertical dampers.

90         In this context, this work aims at introducing a damper model based on lumped  
91 parameters able to represent the nonlinear dynamics of a physical damper prototype.  
92 This model has been developed in Simulink to be co-simulated with a vehicle model in  
93 multibody environment and to be implemented in real time on a HIL test bench.  
94 Together with the parametric model, an optimal identification procedure is presented.  
95 Previous works focused applied Genetic Algorithm or Particle Swarm Optimization [14,  
96 15, 16] to tune lumped element models on real damper prototypes by comparing  
97 numerical and experimental forces obtained by applying sinusoidal characterization  
98 cycles. Moreover, large attention has been given to the application of optimal  
99 identification procedures to numerical models representing magnetorheological  
100 dampers. Rodríguez-Torres et al. [17] tuned nonlinear models for benchmark purposes

101 by considering Genetic Algorithm and Particle Swarm Optimization. Several works  
102 focused their attention on the identification of the best parameters of Bouc-Wen model  
103 for calculating the force provided by magnetorheological dampers [18, 19, 20].  
104 Differently from the previous works, this paper applies the meta-heuristic Firefly  
105 algorithm [21] to identify the optimal parameters of the proposed damper model. This  
106 paper focuses its attention on hydraulic dampers, widely adopted on railway vehicles.  
107 This approach has been designed to optimize an objective function by simulating the  
108 evolution of a set of candidates (in this case, several numerical damper models). This  
109 iterative method can identify the set of parameters able to optimize the accuracy of the  
110 damper model. Differently from the previously works, the procedure proposed in this  
111 paper is based on a set of experimental tests which are similar to the strokes imposed  
112 on railway dampers during the real operating conditions of the vehicles. This approach  
113 differs from the use of simple sinusoidal cycles. Indeed, differently from the typical  
114 sinusoidal tests this approach aims at introducing a tuning of the damper model based  
115 on the experimental reproduction of real maneuvers. The real scenarios will be first  
116 simulated with a multibody vehicle model to obtain the significative strokes able to  
117 characterize the damper in typical conditions (curved or straight track negotiation).  
118 Then, an experimental test rig will impose these reference strokes on the real damper  
119 and will measure the force provided by the prototype. The optimization procedure will  
120 compute the set of model parameters able to minimize the difference between the  
121 measured damper forces and virtual forces provided by the damper model during the  
122 different operating conditions.

123            This paper is organized as follows: section 2 will introduce the damper  
124 parametric model together with the tuning procedure and the experimental setup to  
125 evaluate the benchmark damper forces. Moreover, the parameter identification  
126 procedure will be presented. Section 3 will report the performances of the optimal  
127 damper model and will compare it against a linear Maxwell model, a common damper  
128 modelling approach found in railway dynamics. Section 4 will discuss and conclude the  
129 paper.

## 130 **2 MATERIALS, METHODS AND MODELS**

### 131 **2.1 Damper model**

132            This work introduces a damper model that aims at reproducing the dynamics of  
133 a generic passive hydraulic damper. The lumped elements approach has been chosen to  
134 design the model. This choice reduces the computational effort required, making it  
135 compatible with multibody vehicle simulations and real time procedures, such as HIL  
136 approach.

137            In railway dynamics, numerical simulations require to properly model the  
138 components of the vehicle suspension stages according to a trade-off based on model  
139 accuracy and computational time. The linear dashpot is the simplest modelling approach  
140 for a generic damper. As a matter of fact, this single element can not represent the  
141 complex dynamic of a shock absorber. Indeed, despite dampers are generally defined as  
142 purely dissipative elements, their behavior is also related to an elastic contribution. The  
143 damper virtual model proposed in this paper is reported in figure 1. The model  
144 describes the effects of the silent blocks by means of two sub-models composed by

145 parallel linear terms,  $k_B$  and  $c_B$ . They represent respectively the elastic and dissipative  
 146 actions of the resilient silent blocks. Between the two silent blocks, a main sub-model is  
 147 inserted to simulate the damper structure. In this sub model, three mass elements  $M_C$ ,  
 148  $M_O$  and  $M_P$  are introduced to account for the inertial terms. Generally, such elements  
 149 are neglected in the lumped parameters damper modelling [12]. Nevertheless, they are  
 150 introduced here to allow the general damper model to simulate device layouts  
 151 characterized by very soft damping action where the inertial terms become more  
 152 relevant. Between the mass element, two nonlinear forces are inserted. The  $F_{k,O}$  force is  
 153 introduced to model the elastic contribution, related to both oil compressibility and  
 154 internal structure flexibility. The second force  $F_{c,O}$  aims at describing the nonlinear  
 155 dissipative action due to the oil flow through the piston orifices.

156 The dynamics of the damper model is described by a set of ordinary differential  
 157 equations (ODEs):

$$158 \quad M_C \ddot{x}_4 + k_B x_4 + c_B \dot{x}_4 + F_{k,O} = 0 \quad (1)$$

$$159 \quad M_O \ddot{x}_3 + F_{k,O} + F_{c,O} = 0 \quad (2)$$

$$160 \quad M_P \ddot{x}_2 + F_{c,O} + k_B(x_2 - x_1) + c_B(\dot{x}_2 - \dot{x}_1) = 0 \quad (3)$$

$$161 \quad F = c_B(\dot{x}_2 - \dot{x}_1) + k_B(x_2 - x_1) \quad (4)$$

162 derived by imposing the equilibrium conditions to the elements of the system. The  
 163 presence of  $F_{k,O}$  and  $F_{c,O}$ , which are respectively a function of the relative displacement  
 164  $(x_4 - x_3)$  and of the relative speed  $(\dot{x}_3 - \dot{x}_2)$ , makes the system strongly nonlinear  
 165 [22]. Moreover, these parameters are designed to be asymmetric to represent the



166 differences between compression and rebound strokes. This property is particularly  
167 relevant both in stability [23] and in ride comfort evaluations [5].

## 168 **2.2 Experimental tests based on numerical simulations**

169 In this work, we focused our attention on a yaw damper prototype (twin-tube,  
170 spool-valves Type 06, manufactured by Koni). The choice of a yaw damper has been  
171 made considering that such devices are the most challenging dampers to be modelled in  
172 railway dynamics. The virtual model must accurately reproduce the force of the damper  
173 prototype. For this reason, experimental tests are performed to measure the  
174 benchmark damper force to be used in the tuning of the parametric damper model.

175 Dampers for rail vehicles are typically characterized by imposing sinusoidal cycles  
176 with stroke, speed, and frequency defined according to the EN 13802 standard.  
177 Nevertheless, sinusoidal displacements are far away from the working conditions  
178 experienced in real operating scenarios. Indeed, it is important to consider that tuning a  
179 parametric virtual model on experimental data representing unrealistic conditions might  
180 reduce the capability of the model to accurately reproduce the damper dynamics during  
181 real maneuvers. For this reason, in the tuning procedure proposed in this paper,  
182 experimental force signals have been obtained by imposing on the damper prototype a  
183 set of displacement time histories able to emulate the working conditions of the  
184 damper. The damper forces obtained from the prototype will also be considered as  
185 benchmark to evaluate the accuracy of the virtual damper model.

186 Figure 2 shows the block diagram of the overall tuning procedure, divided into  
187 the upper identification branch and the lower validations stage. The damper strokes

188 during the vehicle maneuvers have been obtained through a single-wagon multibody  
189 model developed with the Simpack software. The model is composed of seven rigid  
190 bodies: a carbody, two bogies, and four wheelsets. The simulated vehicle represents a  
191 generic high-speed train. Besides the four nonlinear yaw dampers, the suspension  
192 components have been modelled with a linear approach. The reference damper strokes  
193 have been obtained from the front-right yaw damper. The damper stroke time histories  
194 are then imposed on the damper prototype to obtain a set of experimental reference  
195 force signals ( $F_{Exp,Id}$  and  $F_{Exp,Val}$ ). For each experiment, one of the reference strokes  
196 obtained by the multibody model is imposed on the damper prototype through a servo-  
197 controlled MTS® actuator (MTS, Type 248.05, rating force: 50 kN), managed by a  
198 SpeedGoat® Real Time Computer. The damper force provided by the prototype is  
199 measured by a load cell (Hottinger Baldwin Messtechnik, Type U10M/50, sensitivity  
200 2.1021 mV/V, adjusted range 50 kN). Figure 3 shows the experimental test bench with  
201 the MTS® Control Unit and the SpeedGoat Real Time Computer.

202         The experimental damper forces of the identification scenarios ( $F_{Exp,Id}$ ) are used  
203 as a benchmark for the identification of the optimal parameters of the damper model,  
204 while the force signals obtained from the tuned damper model simulating the validation  
205 scenarios ( $F_{Num,Val}$ ) are compared to their experimental equivalent ( $F_{Exp,Val}$ ) to estimate  
206 the accuracy of the tuned damper model.

207         In the identification scenario we simulated three different maneuvers to obtain  
208 the reference strokes. The first two scenarios are characterized by track irregularity  
209 profiles defined stochastically considering the superimposition of harmonics with

210 wavelengths between [3 m, 200 m]. The harmonics are characterized by random phases  
211 and deterministic magnitude obtained from the analytical Power Spectral Density (PSD)  
212 functions able to replicate the typical frequency contents of a track irregularity,  
213 according to [1, 24]. The polynomial formulation of the track irregularity PSD is  
214 implemented according to the report B176 of the European Rail Research Institute [25].

215 The three identification scenarios describe:

- 216 • Identification 1: a straight track running in high-speed condition: this test aims  
217 at simulating the vehicle dynamics at speed equal to 250 km/h, with track  
218 irregularity.
- 219 • Identification 2: a low-speed negotiation of a sharp curve: the railway vehicle  
220 simulates the negotiation of a curved track segment composed by a transient  
221 curve entry, a constant curvature segment and a transient curve exit, with  
222 track irregularity. The curve radius is 400 m. The track has been negotiated at  
223 72 km/h.
- 224 • Identification 3: a very low-speed negotiation of a switch: the test is based on  
225 the S-curve maneuver reported in Annex F of EN 14363. The vehicle speed is  
226 set at 43 km/h.

227 The reference damper strokes obtained from the multibody model running the  
228 identification scenarios are reported in figure 4.

229 The selection of the three different scenarios aims at representing the most  
230 important working conditions related to the prototype under investigation (a yaw  
231 damper). The straight running in high-speed conditions is characterized by high

232 frequency oscillations with small amplitudes. In this condition, the yaw dampers are  
233 expected to improve the stability by suppressing the tendency of the vehicle to show  
234 hunting motion [6]. In the negotiation of the curved track, at the constant curvature  
235 gradient of the curve entry, an exit transient can be observed, together with the  
236 constant curvature track segment. Here, yaw dampers provide a negative steering  
237 resistance effect which is responsible for a deterioration of the curving performances of  
238 the vehicle [26]. This negative effect is further amplified during the negotiation of  
239 switches or crossing [13], as shown by the last scenario. The negotiation of the switch is  
240 an interesting condition due to the low-frequency high-amplitude displacement  
241 imposed on the yaw dampers. The damper dynamics in these conditions is significantly  
242 different from the high-frequency low-amplitude oscillations of high-speed running [27].

243         Beside the identification scenarios, a second set of maneuvers, known as  
244 validation, is defined with the aim of verifying in different conditions the accuracy of the  
245 damper model tuned by the optimization procedure. The track irregularities of the  
246 validation scenarios are described obtaining new profiles characterized by harmonics  
247 with magnitude obtained from the same PSD analytical description reported in [25] but  
248 with different random phases. This approach generates irregularity signals that are  
249 different between identification and validation scenarios but with frequency contents  
250 that are aligned to the ones observed in real rail tracks. The validation scenarios can be  
251 listed as:

- 252       • Validation 1: a straight track running in high-speed condition. This test aims at  
253           simulating the vehicle dynamics at speed equal to 250 km/h, with a different  
254           track irregularity profile with respect to the identification scenario 1.
- 255       • Validation 2: a high-speed negotiation of a large radius curve with track  
256           irregularity. The curve radius is set equal to 6000 m, while the vehicle speed is  
257           306 km/h.
- 258       • Validation 3: a low-speed negotiation of a sharp curve. The railway vehicle  
259           simulates the negotiation of a curved track segment composed by a transient  
260           curve entry, a constant curvature segment and a transient curve exit. The curve  
261           radius is 500 m. The track is negotiated at 86 km/h and track irregularity is  
262           implemented.

263   The damper strokes of the validation scenarios are reported in figure 5. The  
264   identification and validation scenarios are summarized in table 1.

265       This procedure allows the definition of a set of experimental time histories of the  
266   damper force starting from the damper strokes obtained in the multibody analysis. It is  
267   worth remarking that the approach can be generalized and implemented with any kind  
268   of suspension component as long as it is possible to develop a multibody model of the  
269   vehicle able to simulate the stroke signals of the suspension component of interest in its  
270   most typical operating conditions.

### 271   **2.3 Optimal parameter identification**

#### 272   *Objective function*

273           The identification procedure aims at defining the optimal set of model  
274 parameters (section 2.1) able to guarantee the best accuracy in modelling the damper  
275 force. The experimental procedure illustrated in the previous section gave as results  
276 three force measurements related to typical operating conditions: straight track, small  
277 radius curve, switch negotiation. The three damper strokes can be imposed on the  
278 damper numerical model to simulate the same scenarios in a virtual environment.  
279 Therefore, the accuracy of a generic damper model can be evaluated by comparing the  
280 virtual force with the experimental benchmark signals. The optimal damper model is  
281 then characterized by the set of parameters able to minimize the difference between  
282 experimental force measurements and correspondent virtual forces.

### 283 *Design variables*

284           The set of model parameters represents the design variables of the optimization  
285 problem. The damper model described in section 2 has five constant parameters,  
286 representing the three masses ( $M_C$ ,  $M_O$ ,  $M_P$ ) and the silent blocks elements ( $k_B$ ,  $c_B$ ).  
287 Moreover, the highly nonlinear behavior of the damper oil is represented by two  
288 variable quantities,  $F_{k,O}$ ,  $F_{c,O}$ . They represent the nonlinear damping force related to the  
289 relative speed ( $\dot{x}_3 - \dot{x}_2$ ) and a nonlinear elastic force which is a function of the relative  
290 displacement ( $x_3 - x_4$ ). These functions are respectively described by 14 samples in  
291 the force-speed diagram and 10 samples in the force-displacement one (piecewise  
292 function). The speed and displacement coordinates are set before the optimization  
293 procedure considering the typical working range of the damper under analysis.  
294 Considering the yaw damper prototype, a higher number of points have been

295 introduced in low relative speed or displacement regions, and a larger refinement has  
296 been used in regions with higher relative speed or displacement. The 24 force values,  
297 required to fully describe the two nonlinear relationships, are considered as parameters  
298 to be identified. Therefore, the optimization problem aims at obtaining 29 modelling  
299 parameters.

300 All the design variables have been constrained. The three concentrated masses have  
301 been defined in the range [1 – 20] kg. The bushing properties  $k_B$  and  $c_B$ , have been  
302 respectively constrained between [1e2 – 1e9] N/m and [1e2 – 1e7] Ns/m. The 14  
303 samples illustrating the nonlinear damping effect of the oil have been constrained by  
304 applying a limit to the maximum force values obtained from a preliminary quasi-static  
305 characterization test, performed on the prototype with sinusoidal mono-harmonic  
306 cycles. In this application, the maximum force is set to 3e4 N. The 10 samples related to  
307 the nonlinear elastic effect of the oil have been constrained by applying bounds around  
308 the expected elastic force of the typical linear spring which is implemented in yaw  
309 damper modelling (1.5e7 N/m).

#### 310 *Optimization procedure*

311 The pursuit of the optimal model tuning has been based on a meta-heuristic  
312 iterative approach, the Firefly Algorithm (FA). This procedure, presented in [21], was  
313 inspired by the capability of fireflies to attract other individuals by producing a  
314 bioluminescence from their abdomen. Fireflies with lighter abdomens are more prone  
315 to attract other individuals and are characterized by a large fitness. FA starts with an  
316 initial population of model candidates which maintains a fixed number of individuals. FA

317 simulates the evolution of the population by modifying their parameters. For each  
 318 iteration, the less bright candidates (low fitness individuals) are forced to emulate the  
 319 settings of the shinier candidates (high fitness individuals). The fitness of a model  
 320 candidate is calculated by imposing the three damper strokes to the model and  
 321 comparing the provided virtual forces with the measured experimental forces. The  
 322 experimental damper force  $F_{Exp}$  and the numerical force  $F_{Num}$  are defined as discrete  
 323 time series (length  $N$ ) including the three operating conditions described in section 2.  
 324 Similarly to [17], the fitness  $G$  of a model candidate is calculated according to:

$$325 \quad G = \frac{N}{\sum_{j=1}^N (F_{Exp,j} - F_{Num,j})^2} \quad (5)$$

326 where the  $j$  index specifies the time sample of the two force time series,  $F_{Exp}$   
 327 and  $F_{Num}$ . Within each iteration, the FA modifies the parameters of an individual by  
 328 moving it towards the brighter individuals. Once a  $i$ -esimal model candidate is  
 329 selected, the method updates its 29 parameters of the  $i$ -model according to:

$$330 \quad \check{x}_i^{New} = \check{x}_i + A_{ij} = \check{x}_i + \sum_{j=1}^M \left[ \frac{G_0}{1 + \varepsilon d_{ij}^2} (\check{x}_j - \check{x}_i) + \rho R_j \right] \quad (6)$$

331 The new vector of scaled design variables  $\check{x}_{i,new}$  is obtained starting from the  
 332 original scaled set  $\check{x}_i$ , where  $M$  is the number of individuals showing a fitness  $G_j$   
 333 higher than  $G_i$ . The design variables have been scaled with respect to their  
 334 correspondent lower and upper bounds to avoid discrepancies due to different  
 335 order of magnitude of their scalar values.



336 The second term of equation 6 describes the attraction  $A_{ij}$  of the brighter firefly  
 337  $j$  on the firefly  $i$ . This term forces the parameters of the model  $i$  towards the  
 338 correspondent values of the more accurate model  $j$ . The attractiveness coefficient  
 339  $G_0$  describes the tendency of the fireflies to be attracted by other brighter  
 340 individuals. The attractiveness between fireflies is reduced by the relative distance  
 341 between the two individuals in the design variable domain ( $d_{ij}$ ). This term is based  
 342 on the Cartesian distance between the normalized design variables ( $p$ ) of the two  
 343 models,  $\check{x}_i$  and  $\check{x}_j$ :

$$344 \quad d_{ij} = \sqrt{\sum_{p=1}^{29} (\check{x}_{jp} - \check{x}_{ip})^2} \quad (7)$$

345 The attractiveness is also conditioned by the absorption coefficient  $\varepsilon$ , which  
 346 weights the influence of the distance on the attractiveness across different fireflies.  
 347 A greater value of  $\varepsilon$  brings to a more relevant reduction of attractiveness at higher  
 348 distances. The coefficient  $\varepsilon$  has been set equal to 0.8.

349 In FA, the modification of the population is also based on a random contribution that  
 350 influences the variation of the design variables of the individuals. This casual effect  
 351 allows the iterative method to better explore the domain of the design variables  
 352 during the search of the optimal solution. Therefore, the random contribution is  
 353 represented by a 29-dimension vector of random variables ( $R_j$ ). This vector is  
 354 calculated for each modification of an  $i$ -esimal individual, while the domains of the  
 355 29 random terms are restricted according to the vectors  $L_{Min}$ ,  $L_{Max}$ , defining the  
 356 lower and upper bounds for each  $p$ -esimal dimension. These limits are defined as:

$$\begin{aligned}
357 \quad L_{Min,i} &= \begin{bmatrix} -\check{x}_{i,1}^{New} \\ \vdots \\ -\check{x}_{i,p}^{New} \\ \vdots \\ -\check{x}_{i,29}^{New} \end{bmatrix}, L_{Max,i} = \begin{bmatrix} 1 - \check{x}_{i,1}^{New} \\ \vdots \\ 1 - \check{x}_{i,p}^{New} \\ \vdots \\ 1 - \check{x}_{i,29}^{New} \end{bmatrix} \quad (8)
\end{aligned}$$

358       The vector  $R_j$  is multiplied by an influence coefficient  $\rho$  which weights the  
359       relevance of the random component on the evolution of the individuals. Random  
360       terms are very important in the first stage of the evolution of the population of the  
361       damper model. On the other hand, high random contributions reduce the  
362       convergence rate of the FA method. For this reason, we introduced a decrement  
363       logarithm law to define a variable influence coefficient  $\rho$  according to the number of  
364       the iteration  $k$ :  $\rho = \rho_0^{k-1}$ , where  $\rho_0$  coefficient has been set equal to 0.9. This  
365       variation aims at preserving the advantages of high random influence in the first  
366       iterations without decreasing the capability of the FA procedure to converge in the  
367       following iterations.

### 368       **3 RESULTS AND DISCUSSION**

369       The results are presented in the following two sections. The first will report the  
370       optimal set of model parameters obtained from the FA, while the second will show  
371       the capability of the optimal damper model to replicate the experimental behavior  
372       among a new set of real maneuvers. The performances of the optimal model will be  
373       also compared with the ones of a Maxwell linear model. This comparative analysis  
374       aims at highlighting the differences between the accuracy of the proposed model  
375       and a damper modelling approach widely implemented in railway dynamics. The

376 linear Maxwell model has been tuned with the same iterative procedure presented  
 377 in section 2.

### 378 **3.1 Optimal damper model**

379 The FA procedure has been implemented with a population of 20 individuals.  
 380 Figure 6 shows the progressive increase of the average population fitness ( $G_{Mean}$ ).  
 381 The approach simulated the evolution of the population during a maximum of 150  
 382 iterations. As we can observe, the algorithm converges towards an optimal solution  
 383 in the last iterations. However, besides the maximum number of iterations, a further  
 384 stopping criterion, based on the gradient of the average fitness of the population, is  
 385 considered starting from the 20<sup>th</sup> iteration. In particular, the algorithm is designed to  
 386 stop when the average fitness of the k-esimal population is minor than a threshold  
 387 defined on the mean of the last 20 average fitness values:

$$388 \quad \text{stop if } G_k < 0.001 \frac{\sum_{t=k-20}^k G_{Mean,t}}{20}$$

389 During evolution, the individual with the best fitness is always stored to obtain  
 390 the best model at the end of the procedure. The storing of the overall best  
 391 candidate avoids excluding eventual optimal solutions found during the initial stages  
 392 of the procedure, where the strong influence of the random effect could lead to a  
 393 loss of this candidate.

394 Figure 7 compares the 5 constant parameters of the initial population (randomly  
 395 defined) with the ones of the final population. As can be observed, the five  
 396 parameters ( $M_C, M_O, M_P, k_B, c_B$ ) are randomly distributed in the initial population but

397 converge to an optimal value in the final population (green markers almost  
398 coincident).

399 Similarly, the nonlinear behavior of the oil is represented in figure 8 by the  
400 elastic  $F_{k,O}$  and the dissipative  $F_{c,O}$  contributions. The two nonlinear relationships are  
401 compared for initial and final populations. We can observe how in the final  
402 population the curves tend to converge to an optimal nonlinear trend.

403 In summary, the optimal parameters of the proposed damper model are reported in  
404 table 1.

### 405 **3.2 Performance of the optimal damper model**

406 As a final step the optimal damper model is simulated in real working conditions.  
407 The validation scenarios have been used to verify the performances of the optimal model  
408 in simulating the dynamics of the physical damper prototype in conditions different with  
409 respect to the dataset used during the identification procedure. To quantify the modelling  
410 accuracy of the proposed model, a linear Maxwell model has also been tuned with the  
411 same procedure. The linear Maxwell model is a common approach when simulating  
412 dampers of rail vehicles dampers [4] and it will be assumed as a reference case.

413 Figure 9 compares the experimental forces obtained from the test rig with the  
414 numerical forces obtained by the proposed numerical model and the optimal tuned  
415 Maxwell linear model in the 3 validation scenarios.

416 In figure 9a, the numerical force obtained by the optimal damper model shows a  
417 very good correlation with respect to the experimental force during the low-speed  
418 negotiation of sharp curves. Moreover, this simulation highlights a typical limitation

419 affecting the linear Maxwell model. Indeed, due to the constant damping ratio of the  
420 Maxwell model, the transient curve segments (seconds 2-8 and seconds 13-19) show poor  
421 performances. This issue is even more critical considering that the lower damper forces  
422 during curve negotiation leads to an overestimation of the curving performance of the  
423 vehicle [26]. On the other side, the use of the proposed model allows to significantly  
424 reduce the error.

425         In figures 9b and 9c, high-speed conditions are analyzed. Both straight track and  
426 large radius curve show a good correlation between the experimental data and the  
427 proposed model. Also in this case, the Maxwell model provides lower damping forces.  
428 This would cause a reduction in the dynamic indexes related to the vehicle stability, such  
429 as bogie lateral accelerations [13]. The difference between the experimental damper  
430 force and the numerical forces obtained with the optimal model are reported in figure  
431 10. The presented results have been quantified in table 2, in which the Mean Squared  
432 Error (MSE) and the Absolute Mean Error (AME) between experimental and numerical  
433 forces, for both the models and all the simulations, are reported. The MSE and AME  
434 formulations reported in table 2 are based on the index  $i$  which defines the  $i$ -esimal time  
435 sample of the two force time histories. The proposed damper model accurately simulates  
436 the forces during all the real operating conditions, with a maximum absolute error lower  
437 than 2400 N (when a peak force of 18 kN is found at 5.5 s of the second validation test).  
438 During negotiation of low radius curves the nonlinear model reduces the error up to 88%  
439 with respect to the Maxwell model and an AME with respect to experimental data lower  
440 than 350 N is observed. In high-speed conditions the AME rises to 500-550 N but a

441 significant improvement of the nonlinear model with respect to the Maxwell one is still  
 442 present (-67%).

443 To better investigate the optimized model, the nonlinear elastic and damping  
 444 terms  $F_{k,O}$  and  $F_{c,O}$  reported in table 2 are analyzed in figure 11 with the aim of quantifying  
 445 their linearity and symmetry. The two terms have been fitted with both a 5<sup>th</sup> order  
 446 polynomial and a linear function. **The equations of the 5<sup>th</sup> order polynomial interpolators**  
 447  **$F_{kO,5th}$  and  $F_{cO,5th}$  are reported:**

$$448 \quad F_{kO,5th} = -9.047e17(x_4 - x_3)^5 + 1.149e14(x_4 - x_3)^4 + 4.689e12(x_4 - x_3)^3 \\ -2.558e8(x_4 - x_3)^2 + 1.335e7(x_4 - x_3) + 12.37 \quad (9)$$

$$449 \quad F_{cO,5th} = 5.932e10(\dot{x}_3 - \dot{x}_2)^5 + 3.884e6(\dot{x}_3 - \dot{x}_2)^4 - 6.911e8(\dot{x}_3 - \dot{x}_2)^3 \\ -1.170e5(\dot{x}_3 - \dot{x}_2)^2 + 1.234e6(\dot{x}_3 - \dot{x}_2) + 57.96 \quad (10)$$

450 By comparing the identified nonlinear trends (green lines) with the linear  
 451 interpolation function (dashed black lines), it can be observed that the damping term  $F_{c,O}$   
 452 is showing nonlinear behavior more relevant than the elastic term  $F_{k,O}$ . Nevertheless, by  
 453 comparing  $F_{k,O}$  with the linear function, it is possible to observe its nonlinear trend,  
 454 especially in the low displacement region (see figure 11c). The 5<sup>th</sup> order polynomial  
 455 function is reported in figure 11 by dividing the even order terms (yellow cross) from the  
 456 odd order terms (red plus). The even order terms can be related to the asymmetry of the  
 457 nonlinear terms between compression and rebound phases. According to figure 11, it can  
 458 be stated that the hydraulic damper under investigation presents a symmetric behavior.  
 459 Indeed, the optimal trends are almost completely defined by the odd order terms of the  
 460 5<sup>th</sup> order polynomial. This last result is aligned to the expectation: the device under

461 investigation is a new yaw damper, designed to provide a symmetric action during both  
462 compression and rebound phases.

#### 463 **4 CONCLUSIONS**

464 In this work, a new parametric nonlinear model for hydraulic dampers is proposed. The  
465 model, based on a lumped parameter approach, is designed to simulate a generic  
466 hydraulic damper in different operating conditions and guarantee good accuracy with  
467 reduced computational efforts. The model features two nonlinear terms which aim at  
468 representing the intrinsic nonlinear behavior of hydraulic dampers.

469         The proposed methodology tunes the damper model on a physical prototype of a  
470 yaw damper. The training dataset for the model tuning has been obtained with a specific  
471 test rig designed to impose on the prototype different strokes and measure the damper  
472 force. These target strokes have been obtained from multibody analysis simulating real  
473 operating conditions of a high-speed rail vehicle.

474         The tuning of the damper model has been performed by introducing an iterative  
475 optimization procedure based on the meta-heuristic Firefly Algorithm. This routine,  
476 focused on the minimization of the differences between virtual and experimental forces,  
477 gave as result an optimal set of model parameters.

478         The performances of the optimal damper model have been compared with a best  
479 tuned Maxwell model, a typical damper modelling approach implemented in railway  
480 dynamics. The proposed nonlinear model has proved to be able at simulating the damper  
481 dynamics in very different conditions, simulating the maneuvers performed by rail  
482 vehicles. This model reduced by one order of magnitude the mean squared error of the

483 linear Maxwell models, returning an absolute mean error between numerical and  
484 experimental forces below 600 N. Moreover, the optimal nonlinear terms of the damper  
485 model have been investigated through the use of polynomial interpolations. A nonlinear  
486 behavior has been identified on the dissipative nonlinear term while the elastic term is  
487 characterized by a smaller nonlinearity, localized in the low displacement region. This  
488 analysis also highlighted the symmetric behavior of the tested damper.

489         As a conclusion, the nonlinear model proved to be a good solution for the  
490 simulation of the dynamics of a real damper prototype in different conditions. The  
491 optimal procedure demonstrated to be an interesting approach for optimizing the  
492 modelling capabilities of generic dynamic models. The definition of a training dataset  
493 based on real operating conditions maximized the capability of the damper model of  
494 simulating real working conditions. In suspension modelling the use of the proposed  
495 procedure represents a good solution to increase the model accuracy, both in straight  
496 track (stability) and curve negotiation (wheel-rail wear) analysis, preserving the  
497 computational effort.

498

#### 499 **FUNDING**

500 The authors received no financial support for the research and/or publication of this  
501 article.

502



503 **NOMENCLATURE**

504

$A_{ij}$	Attraction of the brighter firefly $j$ on the firefly $i$
$c_B$	Linear damping related to the silent block
$d$	Distance between two fireflies
$F_{c,O}$	Nonlinear damping force acting between elements O, P
$F_{cO,5th}$	Expression of the 5 <sup>th</sup> order polynomial interpolator of $F_{c,O}$
$F_{Exp,Id}$	Time histories of the experimental damper forces measured from the damper prototype during the replication of the identification scenarios on the test bench
$F_{Exp,Val}$	Time histories of the experimental damper forces measured from the damper prototype during the replication of the validation scenarios on the test bench
$F_{k,O}$	Nonlinear elastic force acting between elements C, O
$F_{kO,5th}$	Expression of the 5 <sup>th</sup> order polynomial interpolator of $F_{k,O}$
$F_{Num,Id}$	Time histories of the numerical damper forces obtained from the damper model in the identification scenarios
$F_{Num,Val}$	Time histories of the numerical damper forces measured from the damper prototype during the replication of the identification scenarios on the test bench
$G$	Fitness of a single damper model candidate

$G_0$	Attractiveness coefficient
$k_B$	Linear stiffness related to the silent block
$L_{Max}$	Design variable upper bound
$L_{Min}$	Design variable lower bound
$M_C$	Concentrated mass, element C
$M_O$	Concentrated mass, element O
$M_P$	Concentrated mass, element P
$R$	Random contribution
$x_{1,Id}$	Generic time history of the damper stroke during identification scenarios
$x_{1,Val}$	Generic time history of the damper stroke during validation scenarios
$\check{x}_i$	Vector of the design variables of the i-esimal damper model candidate scaled on the proposed lower and upper bounds.
$\varepsilon$	Absorption coefficient
$\rho$	Influence coefficient of the random term

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510 **Bibliography**

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#### Figure Captions List

- Fig. 1 Damper dynamic model: the overall device composed by a main structure and two silent blocks.
- Fig. 2 Block diagram of the overall tuning procedure of the damper model.
- Fig. 3 Experimental test bench: (a) MTS<sup>®</sup> actuator to impose the reference strokes obtained from the multibody model to the yaw damper prototype. The actual damper force is measured by the load cell on the right side of the bench; (b) The SpeedGoat<sup>®</sup> Real Time Computer to acquire the output

signal (damper force) and to send the input signal (reference stroke) to the actuator through the MTS® Control Unit.

- Fig.4 Damper stroke during the identification scenarios simulated with the multibody model ( $x_{1,id}$ ). (a) Straight track running at 250 km/h; (b) Low-speed negotiation of a sharp curve; (c) Switch negotiation.
- Fig. 5 Damper stroke during the validation scenarios simulated with the multibody model ( $x_{1,val}$ ). (a) Straight track running at 250 km/h; (b) High-speed negotiation of large radius curve; (c) Low speed curve negotiation.
- Fig. 6 Evolution of the mean fitness of the population.
- Fig. 7 Comparison between the concentrated parameters of the initial and final population of damper models.
- Fig. 8 Comparison of the nonlinear elastic and damping contributes between initial and final populations. The small amplitude region has been zoomed in both graphs.
- Fig. 9 Comparison between experimental and optimal numerical force of the tested yaw damper. (a) Negotiation of sharp curve with radius 500 m. (b) Negotiation of large radius curve (6000 m) at 306 km/h. (c) Straight track high-speed running (250 km/h).
- Fig. 10 Differences between experimental and optimal numerical force of the tested yaw damper. (a) Negotiation of sharp curve with radius 500 m. (b)

Negotiation of large radius curve (6000 m) at 306 km/h. (c) Straight track high-speed running (250 km/h).

Fig.11 Analysis of the nonlinear terms  $F_{k,0}$  and  $F_{c,0}$ : the optimal terms (solid green line) are compared with a linear interpolation function (black dashed line) and with the even (yellow cross) and odd terms (red plus) of a 5th order polynomial function fitted on the data reported in table 2.

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#### Table Caption List

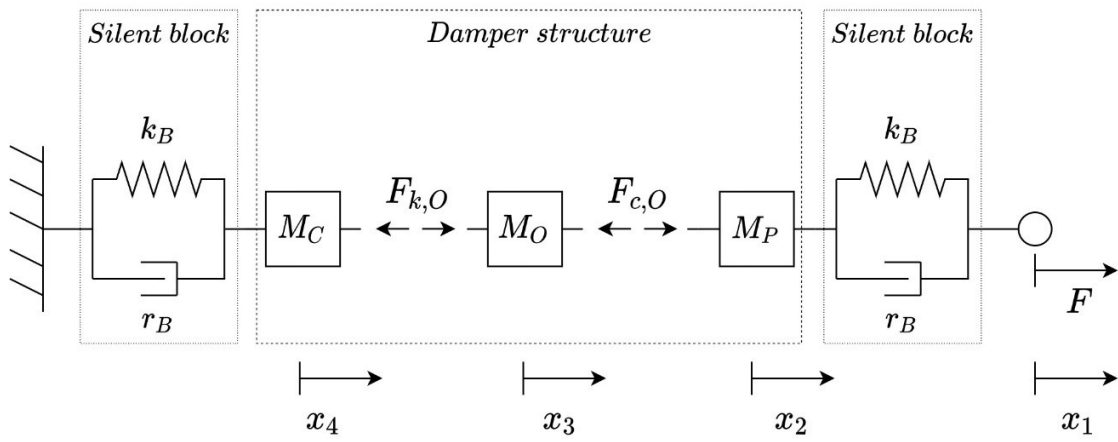
Table 1	Summary of the main characteristics of identification and validation scenarios. The curvature and the rail cant are linearly varied along the clothoid transient segments.
Table 2	Optimal set of model parameters obtained after the implementation of the Firefly Algorithm.
Table 3	Numerical resume of the modelling performances of the proposed optimal damper model and the Maxwell linear model.

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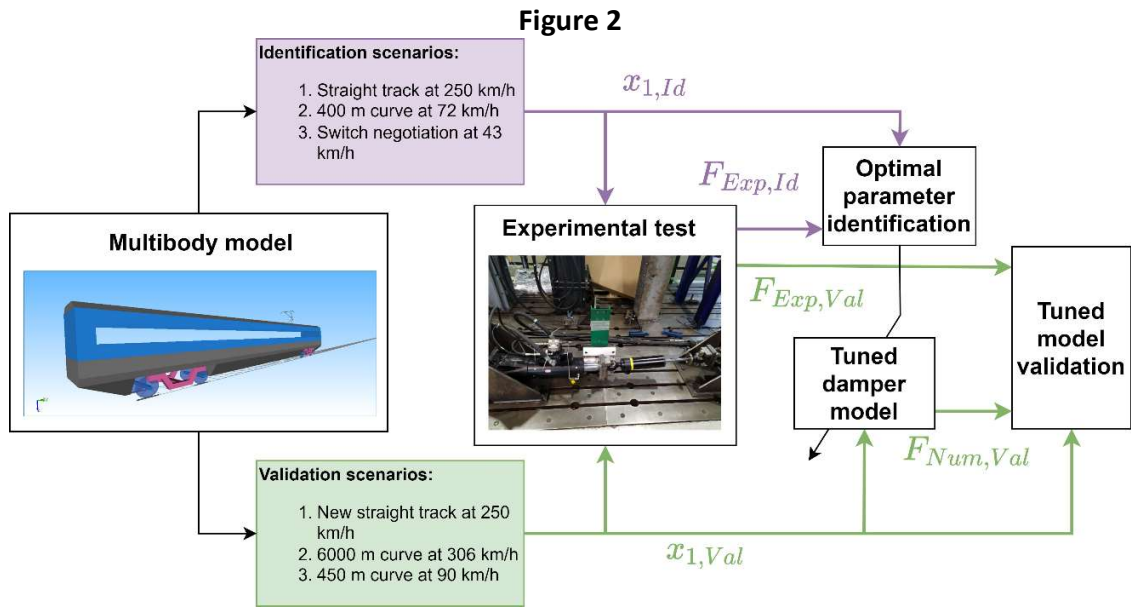
**Figure 1**



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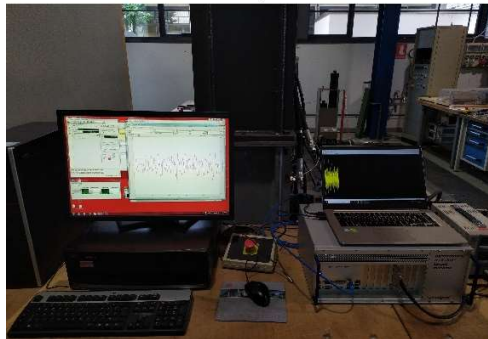
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**Figure 3**

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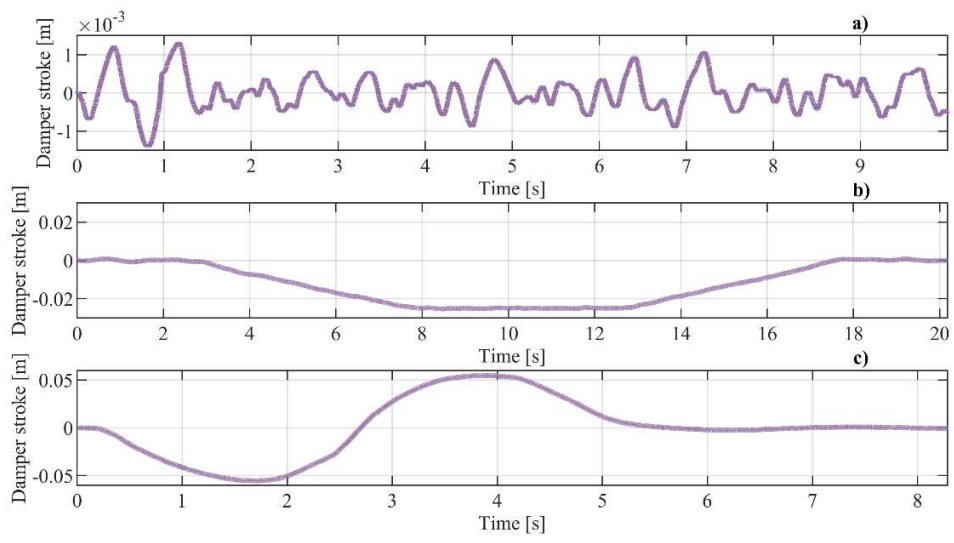
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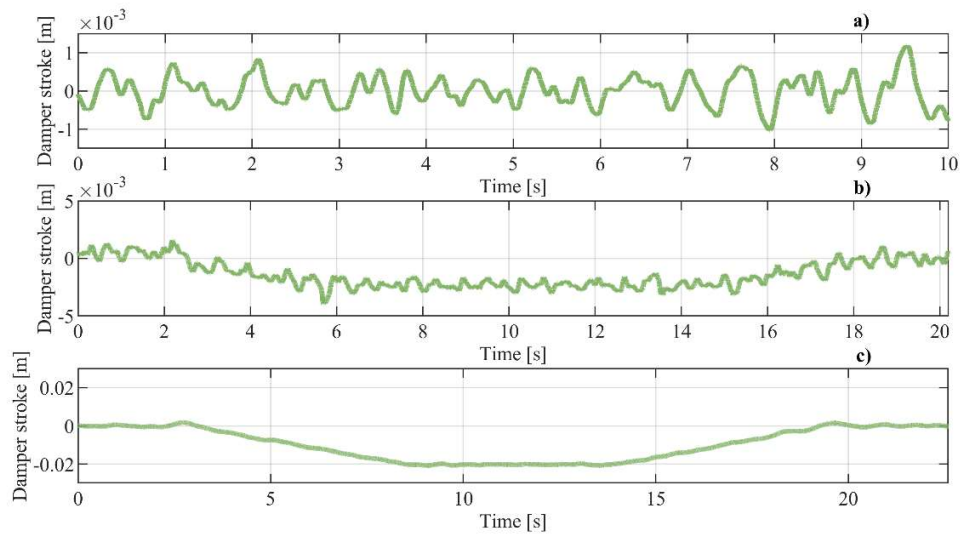
**Figure 4**



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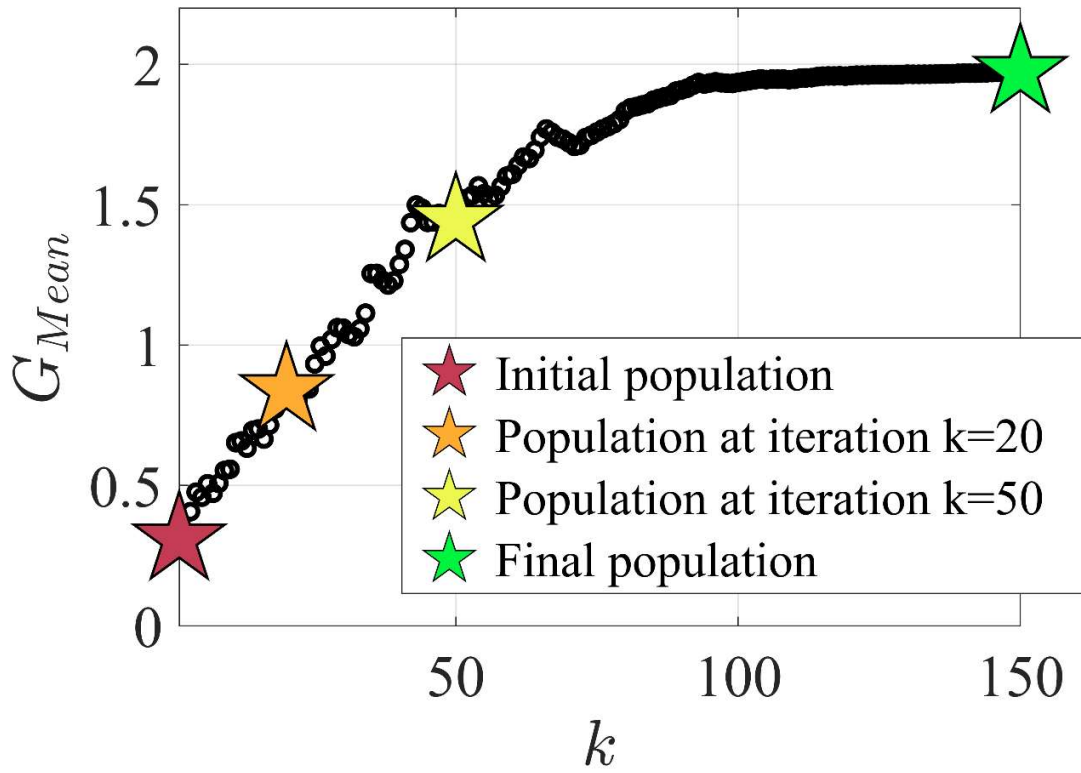
Figure 5



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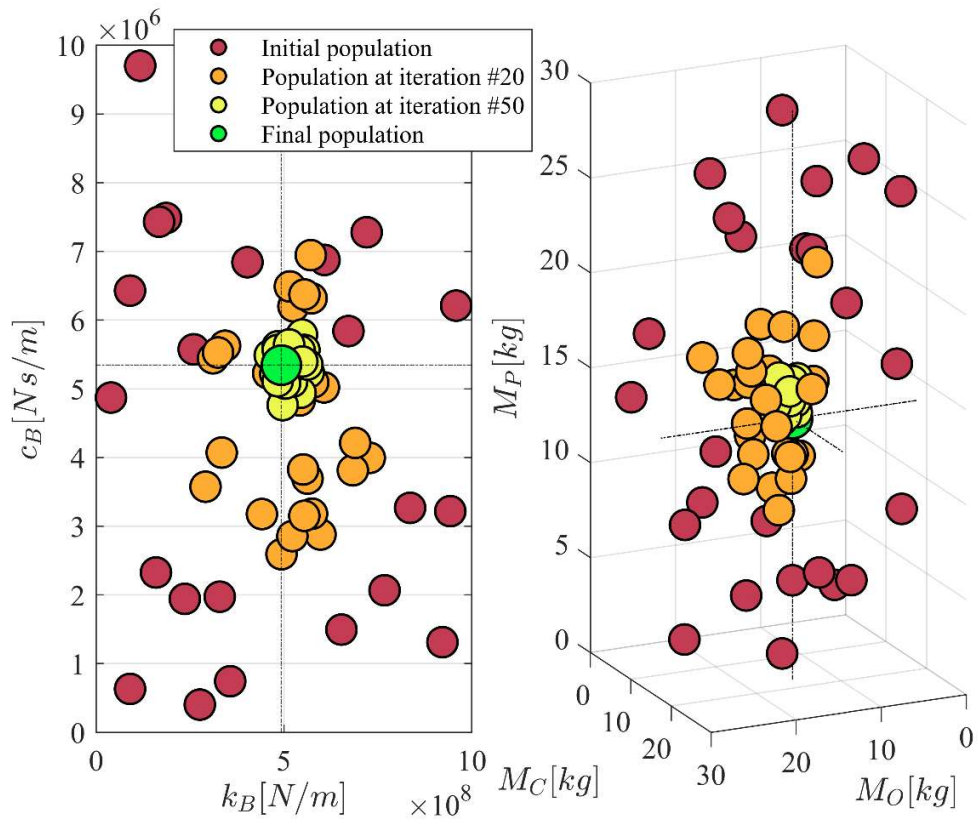
Figure 6



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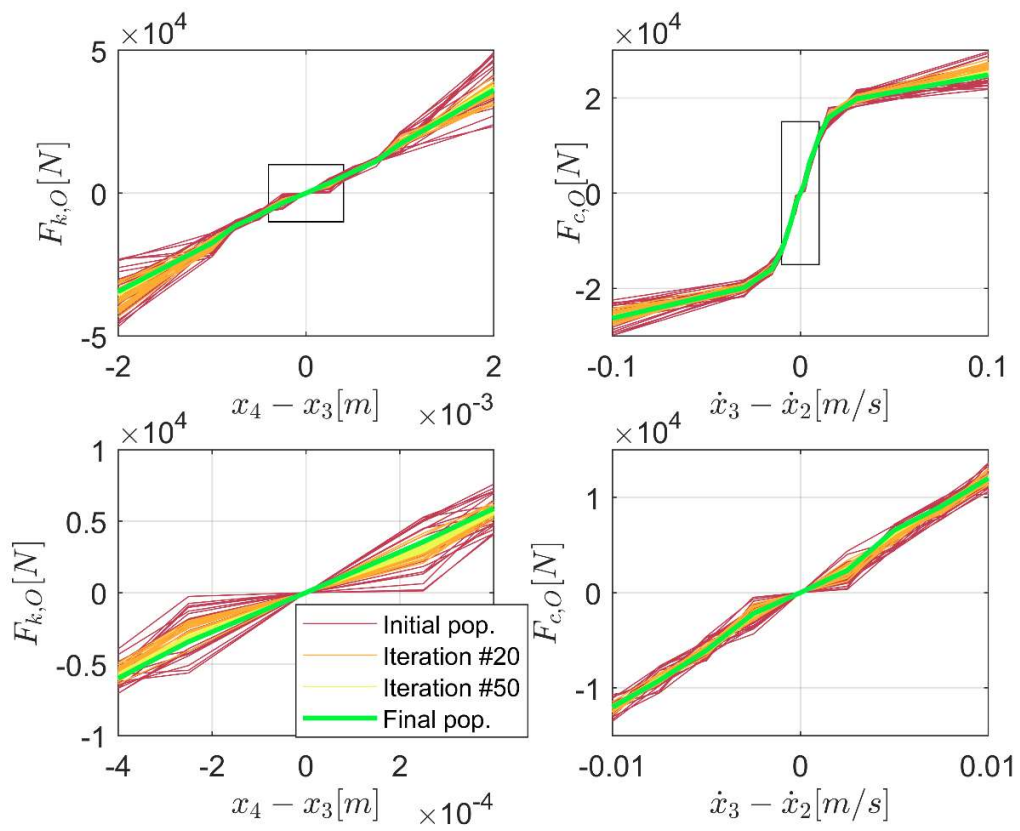
Figure 7



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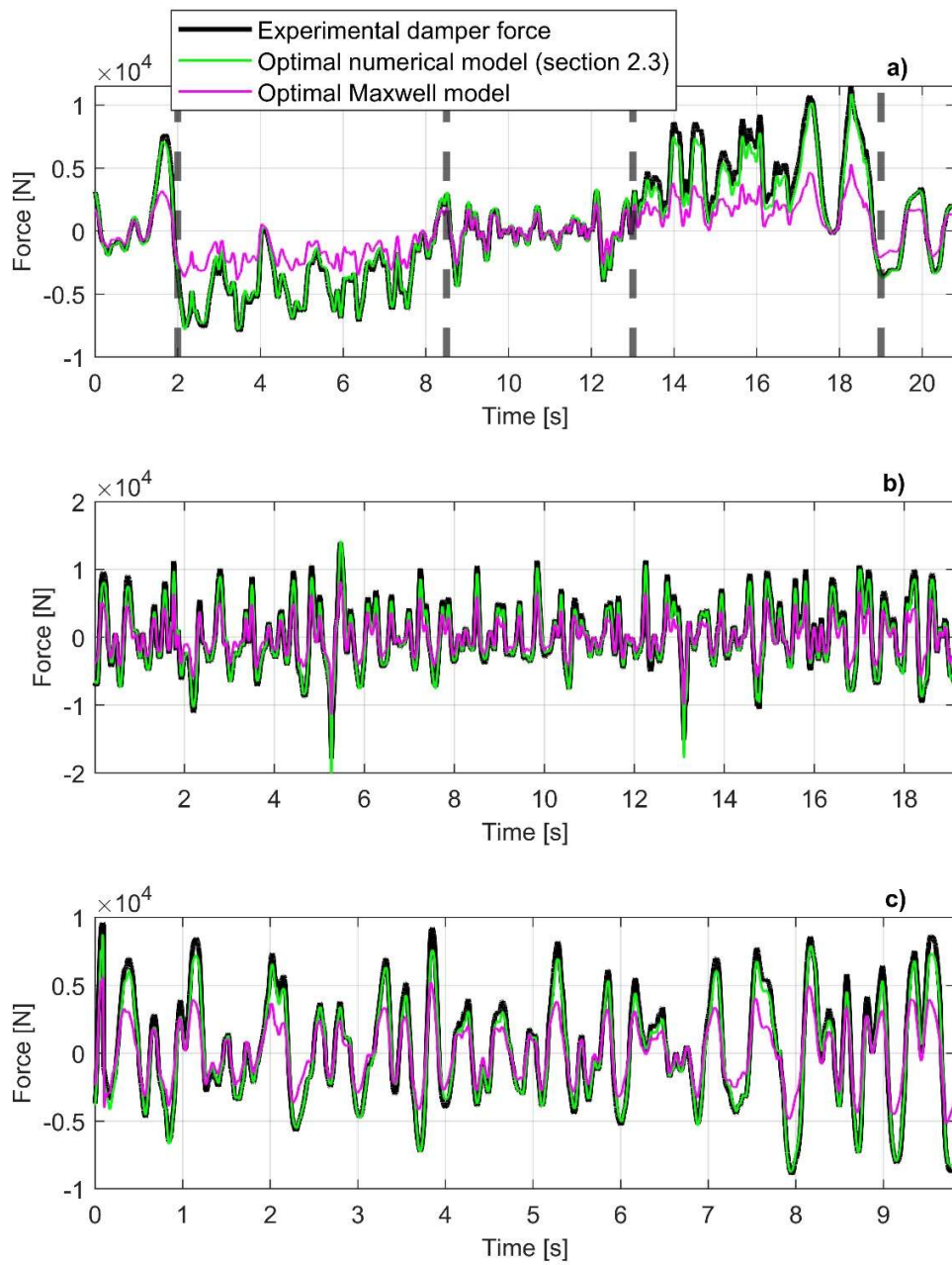
Figure 8



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Figure 9

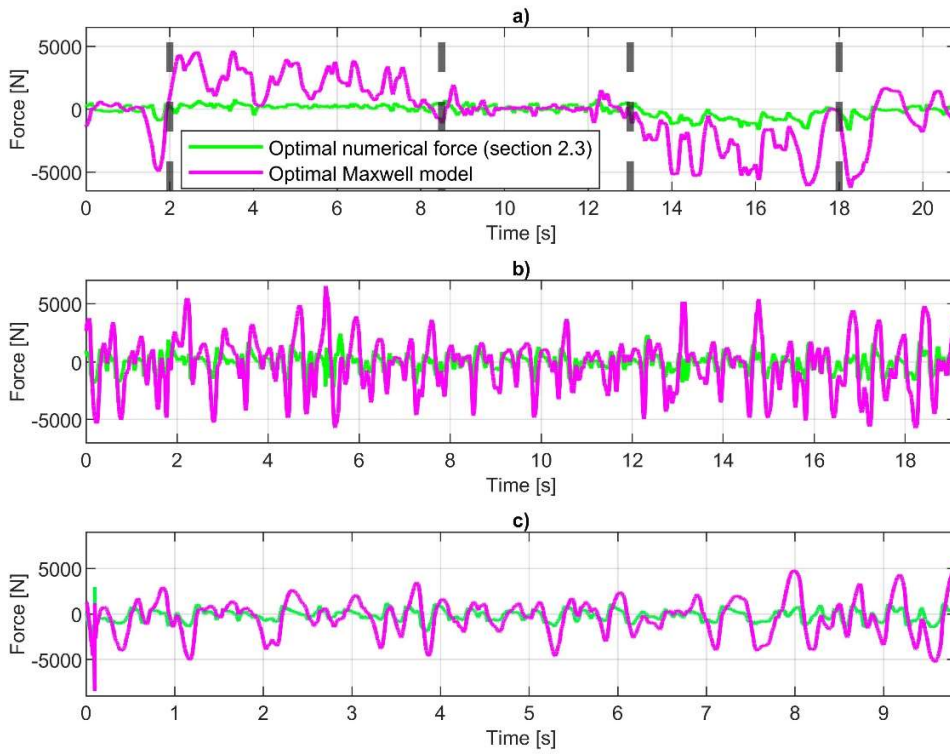


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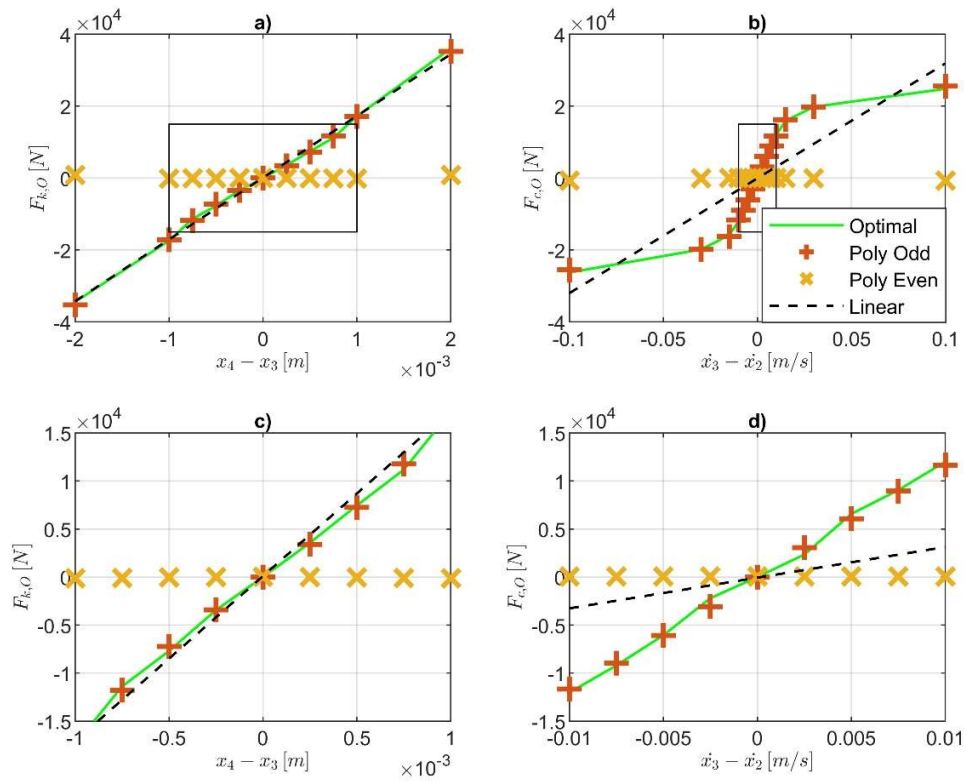
Figure 10



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Figure 11



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**Table1**

Scenario	Vehicle speed [km/h]	Track irregularity	Track segments			
			#	Type	Curve radius [m]	Rail cant [m]
Identification 1	250 km/h	Present	1	Straight	$\infty$	0
Identification 2	72 km/h	Present	1	Straight	$\infty$	0
			2	Clothoid transient curve	From $\infty$ to 400	From 0 to 0.06
			3	Constant curve	400	0.06
			4	Clothoid transient curve	From 400 to $\infty$	From 0.06 to 0
			5	Straight	$\infty$	0
Identification 3	43 km/h	Not present	1	Straight	$\infty$	0
			2	Constant curve	190	0
			3	Straight	$\infty$	0
			4	Constant curve	190	0
			5	Straight	$\infty$	0
Validation 1	250 km/h	Present	1	Straight	$\infty$	0
Validation 2	306 km/h	Present	1	Straight	$\infty$	0
			2	Clothoid transient curve	From $\infty$ to 6000	From 0 to 0.09
			3	Constant curve	6000	0.09
			4	Clothoid transient curve	From 6000 to $\infty$	From 0.09 to 0
			5	Straight	$\infty$	0
Validation 3	86 km/h	Present	1	Straight	$\infty$	0
			2	Clothoid transient curve	From $\infty$ to 500	From 0 to 0.084
			3	Constant curve	500	0.084
			4	Clothoid transient curve	From 500 to $\infty$	From 0.084 to 0
			5	Straight	$\infty$	0

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**Table 2**

Parameter	Value	E.I.
$M_O$	16.5	[kg]
$M_C$	15.8	[kg]
$M_P$	17.3	[kg]
$k_B$	4.08e8	[N/m]
$c_B$	4.17e6	[Ns/m]
$[F_{k,0}]$	[-34 -17 -11 -7.7 -3.4 0 3.6 7.5 11 17 36]	[kN]
$[x_4 - x_3]$	[-2 -1 -0.75 -0.5 -0.25 0 0.25 0.5 0.75 1 2]	[mm]
$[F_{c,0}]$	[-26 -20 -16 -12 -9.2 -6.1 -2.2 0 2.3 6.6 9.1 12 16 20 25]	[kN]
$[\dot{x}_3 - \dot{x}_2]$	[-100 -30 -15 -10 -7.5 -5 -2.5 -0 -2.5 5 7.5 10 15 30 100]	[mm/s]

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**Table 3**

	<b>Maxwell model</b>	<b>Proposed model</b>	<b>Error Percentage Variation</b>	<b>Maxwell model</b>	<b>Proposed model</b>	<b>Error Percentage Variation</b>
<b>Index</b>	$MSE = \frac{\sum_{i=1}^N (F_{Exp,i} - F_{Num,i})^2}{N}$		$\frac{MSE_{Prop} - MSE_{Max}}{MSE_{Max}} 100$	$AME = \frac{\sum_{i=1}^N  F_{Exp,i} - F_{Num,i} }{N}$		$\frac{AME_{Prop} - AME_{Max}}{AME_{Max}} 100$
<b>Test 1</b>	5.88e6 N <sup>2</sup>	2.18e5 N <sup>2</sup>	-96.3 %	1.80e3 N	346 N	-80.8 %
<b>Test 2</b>	5.05e6 N <sup>2</sup>	5.32e5 N <sup>2</sup>	-89.5 %	1.73e3 N	569 N	-67.1 %
<b>Test 3</b>	4.07e6 N <sup>2</sup>	4.19e5 N <sup>2</sup>	-89.7 %	1.55e3 N	526 N	-66.1 %

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