DE GRUYTER Adv. Nonlinear Stud. 2021; 21(3): 523–555

Research Article

Filomena Feo, Juan Luis Vázquez* and Bruno Volzone

Anisotropic p-Laplacian Evolution of Fast Diffusion Type

https://doi.org/10.1515/ans-2021-2136
Received June 2, 2021; revised June 19, 2021; accepted June 20, 2021

Abstract: We study an anisotropic, possibly non-homogeneous version of the evolution p-Laplacian equation when fast diffusion holds in all directions. We develop the basic theory and prove symmetrization results from which we derive sharp L^1-L^∞ estimates. We prove the existence of a self-similar fundamental solution of this equation in the appropriate exponent range, and uniqueness in a smaller range. We also obtain the asymptotic behaviour of finite mass solutions in terms of the self-similar solution. Positivity, decay rates as well as other properties of the solutions are derived. The combination of self-similarity and anisotropy is not common in the related literature. It is however essential in our analysis and creates mathematical difficulties that are solved for fast diffusions.

Keywords: Nonlinear Parabolic Equations, p-Laplace Diffusion, Anisotropic Equation, Symmetrization, Fundamental Solutions, Asymptotic Behaviour

MSC 2010: 35K55, 35K65, 35A08, 35B40

Communicated by: Julián López Gómez

1 Introduction

This paper focuses on the study of the existence of self-similar fundamental solutions to the following "anisotropic p-Laplacian equation" (APLE for short):

$$u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} \quad \text{posed in } Q := \mathbb{R}^N \times (0, +\infty), \tag{1.1}$$

and their role to describe the long-time behaviour of general classes of finite-mass of the initial-value problem. Fundamental solutions are solutions of the equation for all times $t > 0$ that take a point mass (i.e., a Dirac delta) as initial data. In the process, we construct a theory of existence and uniqueness for initial data in L^q spaces, $1 \le q < +\infty$, and we prove important results on symmetrization, boundedness, barriers and positivity.

We are specially interested in the presence of different growth exponents p_i. We take $N \ge 2$ and $p_i > 1$ for $i = 1, \dots, N$. Therefore, this equation is an anisotropic relative of the standard isotropic p-Laplacian equation

$$u_t = \Delta_p u := \sum_{i=1}^{N} (|\nabla u|^{p-2} u_{x_i})_{x_i}, \tag{1.2}$$

***Corresponding author: Juan Luis Vázquez,** Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain, e-mail: juanluis.vazquez@uam.es. https://orcid.org/0000-0002-9871-257X
Filomena Feo, Dipartimento di Ingegneria, Università degli Studi di Napoli "Parthenope", Centro Direzionale Isola C4, 80143 Napoli, Italy, e-mail: filomena.feo@uniparthenope.it. https://orcid.org/0000-0002-1748-6292
Bruno Volzone, Dipartimento di Scienze e Tecnologie, Università degli Studi di Napoli "Parthenope", Centro Direzionale Isola C4, 80143 Napoli, Italy, e-mail: bruno.volzone@uniparthenope.it

that has been extensively studied in the literature as the standard model for gradient dependent nonlinear diffusion equation, with possibly degenerate or singular character. Though most the attention has been given to the elliptic counterpart, $-\Delta_p u = f$, the parabolic case is also treated; see e.g. the well-known [30, 40, 41] among the many references.

Even in the case where all the exponents p_i in (1.1) are the same, we obtain an alternative version $u_t = L_{p,h}(u)$ with a homogeneous but non-isotropic spatial operator

$$L_{p,h}(u) := \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i}, \tag{1.3}$$

which appears quite early in the literature; cf. [41, 65, 66]; see also [14]. This operator has been sometimes named “pseudo-p-Laplacian operator” [10], and more recently, “orthotropic p-Laplacian operator” [16, 17], due to the invariance of $L_{p,h}$ with respect to the dihedral group for $N = 2$. This will be our preferred denomination. The parabolic version appears in [36, 51, 52]. In the general studies of nonlinear diffusion, the case where the exponents p_i are different falls into the category of “structure conditions with non-standard growth”. The anisotropic equation was also studied in a number of references like [39, 53]. Actually, a more general doubly nonlinear model was introduced in those references; see also [2]. Very general structure conditions are considered by various authors like [57], specially in elliptic problems. Our interest here differs from those works.

The Setting. We consider solutions to the Cauchy problem for equation (1.1) with nonnegative initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.4}$$

We assume that $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \ge 0$, and put $M := \int_{\mathbb{R}^N} u_0(x)\,dx$, the so-called total mass. The reader is here reminded that the strong qualitative and quantitative separation between the two exponent ranges, $p > 2$ and $p < 2$, is a key feature of the isotropic p-Laplacian equation (1.2). We recall that, in the isotropic equation, the range $p > 2$ is called the slow gradient-diffusion case (with finite speed of propagation and free boundaries), while the range $1 < p < 2$ is called the fast gradient-diffusion case (with infinite speed of propagation); cf. [30] and [61, Section 11].

In this paper, we will focus on the case where *fast diffusion holds in all directions*, i.e.,

$$1 < p_i < 2 \quad \text{for all } i = 1, \dots, N. \tag{H1}$$

We recall that, in the orthotropic fast diffusion equation (i.e., equation (1.1) with $p_1 = p_2 = \cdots = p_N = p < 2$, hence p-homogeneous), there is a critical exponent

$$p_c(N) := \frac{2N}{N+1}$$

such that $p > p_c$ is a necessary and sufficient condition for the existence of fundamental solutions; cf. [61]. Note that $1 < p_c(N) < 2$ for $N \ge 2$.

Moreover, we will always assume the condition

$$\sum_{i=1}^{N} \frac{1}{p_i} < \frac{N+1}{2}, \tag{H2}$$

that is crucial in what follows. We we may also write it in terms of p_c as $\bar{p} > p_c$, where $\bar{p}$ is the inverse average

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}. \tag{1.5}$$

We point out that (H2) excludes the presence of (many) small exponents $1 < p_i < p_c$ close to 1. On the contrary, condition (H2) would obviously be in force under the assumptions of slow diffusion in all directions: $2 \le p_i < +\infty$ for all $i = 1, \dots, N$ (a situation we will not consider here). However, in the fast diffusion range, we have to impose it; otherwise, the results we expect to obtain would be false.

Finally, it is well known in the literature on operators with non-standard growth that some control on the difference of diffusivity exponents is needed; see for instance [9, 15, 42]. Here, we will only need the condition

$$p_i \leq \frac{N+1}{N}\bar{p} \tag{H3}$$

(see Section 2). It is remarkable that this condition is automatically satisfied if (H1) and (H2) are in force.

Under these conditions on the exponents, we develop a theory of existence, regularity, symmetrization, and upper and lower estimates for the Cauchy problem. We prove the existence of a self-similar solution starting from a Dirac mass, so-called fundamental solution or Barenblatt solution. Moreover, in the particular orthotropic case where $p_i = p$ for all i, thanks to extra regularity results that we derive, it is possible to prove uniqueness of the fundamental solution, and the theory goes on to show the asymptotic behaviour of all nonnegative finite mass solutions in the sense that they are attracted by the corresponding Barenblatt solution with same mass as $t \to \infty$. This set of results shows that the ideas proposed by Barenblatt in his classical work [8] are valid for our equation too.

Outline of the Paper by Sections. Here is a detailed summary of the contents. In Section 2, we examine the form of the possible self-similar solutions, the a priori conditions on the exponents, and we also introduce the renormalized equation and its elliptic counterpart. The role of assumptions (H1), (H2) and (H3) is examined.

In Section 3, we review the basic existence and uniqueness theory for the Cauchy Problem using the theories of monotone and accretive operators in L^q spaces. This general theory is valid in the whole range $p_i > 1$, with no further restriction on the exponents. The L^1 theory is examined in detail in Section 4.

In Section 5, we develop the technique of Schwarz symmetrization for our anisotropic equation, and we prove sharp comparison results by using the concept of mass concentration as explained in [60]. Symmetrization is an important topic in itself with a huge literature, specially when anisotropy is mild; see [4, 5, 56]. The passage from anisotropic to isotropic is based on a sharp elliptic result by Cianchi [22] that we develop in this setting using mass comparison, a strong tool used in some of our previous papers. The topic has independent interest, and the theory and results are proved for all $p_i > 1$ under assumption (H2).

The theory developed up to this point (including symmetrization) is used in Section 6 to obtain a uniform L^∞ bound for solutions with L^1 data, the so-called L^1-L^∞ effect. Theorem 6.1 is a key estimate in what follows.

We begin at this moment the construction of the self-similar fundamental solution under conditions (H1) and (H2). In a preparatory section, Section 7, we construct the sharp anisotropic upper barrier for the solutions of our problem; this is another key tool that we need. The theory is now ready to tackle the construction of the special solution. The existence result, Theorem 8.1, is maybe the main result of the paper. In Section 9, we construct the lower barrier and prove global positivity, an important additional information on the obtained solution.

The very delicate question of uniqueness of the fundamental solutions is solved only for the orthotropic case, $p_i = p$, in Section 10.2, and as a consequence, we establish the asymptotic behaviour of general solutions of the Cauchy problem in that case; see Section 10.3. Both questions remain open for the anisotropic non-orthotropic equations.

As supplementary information, we discuss in Section 12 the necessary control on the anisotropy for the theory to work. We devote Section 13 to introduce the study of self-similarity for anisotropic doubly nonlinear equations. Finally, we add a section on comments and open problems.

Some Related Works. This work follows the study of self-similarity for the anisotropic porous medium equation (APME) in the fast diffusion range done by the authors in [34], where previous references to the literature are mentioned. Though it is well known that the PME and the PLE are closely related as models of nonlinear diffusion of degenerate type (see for instance [63]), the theories and the results differ in many important details, hence the interest on this investigation.

In a recent paper, Ciani and Vespri [23] study the existence of Barenblatt solutions for the same anisotropic p-Laplace equation (1.1) posed also in the whole space, but they consider the slow diffusion case in all directions, i.e., $p_i > 2$ for all i. They exploit the property of finite propagation that holds in that exponent range. Uniqueness and asymptotic behaviour are not discussed. See [29, 31] for related previous results in the slow diffusion range of exponents p_i. These papers contain thus parallel, non-overlapping information with respect to our present results that deal with fast diffusion. Let us finally point out that the existence of fundamental solutions for anisotropic elliptic equations is a different issue; it has been studied by several authors like [25].

2 Self-Similar Solutions

We start our study by taking a closer look at the possible class of self-similar solutions. This section follows closely the arguments of [34] for the anisotropic porous medium equation, but they lead to a quite different algebra; hence a careful analysis is needed. The common type of self-similar solutions of equation (1.1) takes into account the anisotropy in the form

$$B(x,t) = t^{-\alpha} F(t^{-a_1} x_1, \dots, t^{-a_N} x_N),$$

with constants $\alpha > 0$, $a_1, \dots, a_n \geq 0$ to be chosen below by algebraic considerations. Indeed, if we substitute this formula into equation (1.1) and write $y = (y_1, \dots, y_N)$ and $y_i = x_i t^{-a_i}$, equation (1.1) becomes

$$-t^{-\alpha-1}\Big[\alpha F(y) + \sum_{i=1}^{N} a_i y_i F_{y_i}\Big] = \sum_{i=1}^{N} t^{-[\alpha(p_i-1)+p_i a_i]} (|F_{y_i}|^{p_i-2} F_{y_i})_{y_i}.$$

We see that time is eliminated as a factor in the resulting equation on the condition that

$$\alpha(p_i - 1) + p_i a_i = \alpha + 1 \quad \text{for all } i = 1, 2, \dots, N.$$

We also look for integrable solutions that will enjoy the mass conservation property, and this implies that $\alpha = \sum_{i=1}^{N} a_i$. Imposing both conditions and putting $a_i = \sigma_i \alpha$, we get unique values for α and σ_i,

$$\alpha = \frac{N}{N\bar{p} - 2N + \bar{p}}, \tag{2.1}$$

$$\sigma_i = \frac{1}{p_i}\frac{(N+1)\bar{p}}{N} - 1, \quad \text{i.e.,} \quad \sigma_i - \frac{1}{N} = \frac{(N+1)}{N}\frac{(\bar{p} - p_i)}{p_i}, \tag{2.2}$$

so that $\sum_{i=1}^{N} \sigma_i = 1$. This is a delicate calculation that produces the special value $\bar{p}$.

Observe that condition (H2) is required to ensure that $\alpha > 0$ so that the self-similar solution will decay in time in maximum value like a power of time. This is a crucial condition for the self-similar solution to exist and play its role as asymptotic attractor since the existence theory we present contains the maximum principle; hence the sup norm of the constructed solutions cannot increase in time.

As for the σ_i exponents that control the rate of spatial spread in each coordinate direction, we know that $\sum_{i=1}^{N} \sigma_i = 1$, and in particular, $\sigma_i = \frac{1}{N}$ in the homogeneous case. Condition (H3) on the p_i ensures that $\sigma_i > 0$. This means that the self-similar solution expands as time passes (or at least, it does not contract), along any of the coordinate directions.

To fix ideas, we present in Section 12 a graphic analysis of assumptions (H1), (H2), (H3) for general exponents $p_i > 1$ in dimension $N = 2$. We also compare this analysis with the predictions made in [34] for the APME.

With these choices, the *profile function* $F(y)$ must satisfy the following nonlinear anisotropic stationary equation in $\mathbb{R}^N$:

$$\sum_{i=1}^{N} [(|F_{y_i}|^{p_i-2} F_{y_i})_{y_i} + \alpha\sigma_i (y_i F)_{y_i}] = 0. \tag{2.3}$$

Conservation of mass must also hold, $\int B(x,t)\,dx = \int F(y)\,dy = M < \infty$ for all $t > 0$. It is our purpose to prove that there exists a suitable solution of this elliptic equation, which is the anisotropic version of the equation of the Barenblatt profiles in the standard p-Laplacian; cf. [61].

Examples. (1) *The isotropic case*. It is well known that the source-type self-similar solution is indeed explicit in the isotropic case

$$u_t = \sum_{i=1}^{N} (|\nabla u|^{p-2} u_{x_i})_{x_i}.$$

Of course, for $p = 2$, we obtain the Gaussian kernel of the heat equation, $F(y) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}}$. In the nonlinear cases, we get two different but related formulas.

For $p_c < p < 2$,

$$F(y) = \left(C_0 + \frac{2-p}{p} \lambda^{-\frac{1}{p-1}} |y|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{2-p}}.$$

When $p > 2$, we get

$$F(y) = \left(C_0 - \frac{p-2}{p} \lambda^{-\frac{1}{p-1}} |y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}},$$

with $\lambda = N(p-2) + p$, and $C_0 > 0$ is an arbitrary constant such that it can be determined in terms of the initial mass M. They are called the Barenblatt solutions [7].

For $1 < p \le 2$, the profile F is everywhere positive; moreover, for $p_c < p < 2$, the profile F belongs to $L^1(\mathbb{R}^N)$ and has a decay with a characteristic power rate. On the contrary, for $p > 2$, the profile F has compact support and exhibits a free boundary. Free boundaries are important objects for slow diffusion, but they will appear in this paper only in passing.

(2) *The orthotropic case*. We have found a rather similar explicit formula for F when $p_i = p$ for all i so that $\bar{p} = p$. In that case, we have, if $p_c < p < 2$,

$$F(y) = \left(C_0 + \frac{2-p}{p} \lambda^{-\frac{1}{p-1}} \sum_{i=1}^{N} |y_i|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{2-p}}, \tag{2.4}$$

with $C_0 > 0$ and $\lambda = N(p-2) + p$ as above. It is a solution to (2.3) because it solves

$$|F_{y_i}|^{p-2} F_{y_i} + \frac{\alpha}{N} y_i F = 0 \quad \text{in } \mathbb{R}^N \quad \text{for all } i.$$

Moreover, the condition $p_c < p$ guarantees that $F \in L^1(\mathbb{R}^N)$. Note that the constant $C_0 > 0$ is arbitrary and allows fixing the mass $M > 0$ at will.

As a complement, we state the case $p > 2$,

$$F(y) = \left(C_0 - \frac{p-2}{p} \lambda^{-\frac{1}{p-1}} \sum_{i=1}^{N} |y_i|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{2.5}$$

with $C_0 > 0$ and same λ. To our best knowledge, the explicit formulas (2.4) and (2.5) are new, as well as the formulas for V below.

In order to fix the mass of F given by (2.4) or (2.5), we use the transformation $\mathcal{T}_k[F(y)] = kF(k^{\frac{2-p}{p}} y)$ that changes solutions into new solutions of the stationary equation (2.3) with $p_i = p$ and changes the mass according to the rule

$$\int \mathcal{T}_k[F(y)]\, dy = k^{N+1-\frac{2N}{p}} \int F(z)\, dz.$$

(3) Putting $C_0 = 0$ in (2.4), we get for $p_c < p < 2$ the following parabolic solution:

$$V(x,t) = k_1 t^{\frac{1}{2-p}} \left(\sum_{i=1}^{N} |x_i|^{\frac{p}{p-1}}\right)^{-\frac{p-1}{2-p}} \quad \text{for suitable } k_1 > 0.$$

This is called a *very singular solution* since it contains a singularity with infinite integral at $x = 0$. A much more singular solution can be obtained by separating the variables,

$$V(x,t) = k_2 t^{\frac{1}{2-p}} \left(\sum_{i=1}^{N} |x_i|^{-\frac{p}{2-p}}\right) \quad \text{for suitable } k_2 > 0.$$

(4) We will not get any explicit formula for F in the general anisotropic case, but we will have existence of self-similar solutions and suitable estimates, in particular decay.

2.1 Self-Similar Variables

In several instances in the sequel, it will be convenient to pass to self-similar variables, by zooming the original solution according to the self-similar exponents (2.1)–(2.2). More precisely, the change is done via the formulas

$$v(y,\tau)=(t+t_0)^{\alpha}u(x,t),\quad \tau=\log(t+t_0),\quad y_i=x_i(t+t_0)^{-\sigma_i\alpha},\quad i=1,\dots,N, \tag{2.6}$$

with α and σ_i as before. We recall that all of these exponents are positive. There is a free time parameter $t_0\geq 0$ (a time shift).

Lemma 2.1. *If $u(x,t)$ is a solution (resp. super-solution, sub-solution) of* (1.1)*, then $v(y,\tau)$ is a solution (resp. super-solution, sub-solution) of*

$$v_\tau=\sum_{i=1}^{N}[(|v_{y_i}|^{p_i-2}v_{y_i})_{y_i}+\alpha\sigma_i(y_iv)_{y_i}],\quad \mathbb{R}^N\times(\tau_0,+\infty). \tag{2.7}$$

This equation will be a key tool in our study. Note that the rescaled equation does not change with the time shift t_0, but the initial value in the new time does, $\tau_0=\log(t_0)$. Thus, if $t_0=1$, then $\tau_0=0$. If $t_0=0$, then $\tau_0=-\infty$, and the v equation is defined for $\tau\in\mathbb{R}$.

We stress that this change of variables preserves the L^1 norm. The mass of the v solution at new time $\tau\geq\tau_0$ equals that of the u at the corresponding time $t\geq 0$.

This equation enjoys a scaling transformation $\mathcal{T}_k$ that changes the mass,

$$\mathcal{T}_k[v(y,\tau)]=kv(k^{\beta_1}y_1,\dots,k^{\beta_N}y_N,\tau),\quad \beta_i=\frac{2-p_i}{p_i}, \tag{2.8}$$

with scaling parameter $k>0$. Working out the new mass, we get

$$\int_{\mathbb{R}^N}\mathcal{T}_k[v(y,\tau)]\,dy=\int_{\mathbb{R}^N}v(y,\tau)\,dy$$

with $\mu=1-\sum_i\beta_i=N+1-\sum_i(\frac{2}{p_i})=(N+1)-(\frac{2N}{\bar p})$. We have $\mu>0$ since $\bar p>p_c$.

3 Basic Theory, Variational Setting

The theory of the anisotropic p-Laplacian operator (1.1) shares a number of basic features with its best known relative, the standard isotropic p-Laplacian Δ_p. These common traits have been already mentioned in the literature in the case of anisotropy with same powers, but we will see here that the similarities extend to the general form. The only assumption we make in this setting is that $p_i>1$ for all $i=1,\dots,N$. We denote by $X^{\vec p}$ the anisotropic Banach space

$$X^{\vec p}=\{u\in L^2(\mathbb{R}^N):u_{x_i}\in L^{p_i}(\mathbb{R}^N)\text{ for all }i=1,\dots,N\}$$

endowed with the norm

$$\|u\|_{X^{\vec p}}=\|u\|_{L^2}+\sum_{i=1}^{N}\|u_{x_i}\|_{L^{p_i}}.$$

It is easy to see that $C_c^\infty(\mathbb{R}^N)$ is dense in $X^{\vec p}$ and that $X^{\vec p}$ reduces to $H^1(\mathbb{R}^N)$ when $p=2$.

Let us consider the anisotropic operator

$$\mathcal{A}(u):=-\sum_{i=1}^{N}(|u_{x_i}|^{p_i-2}u_{x_i})_{x_i}, \tag{3.1}$$

defined on the domain

$$D(\mathcal{A})=\{u\in X^{\vec p}:\mathcal{A}(u)\in L^2(\mathbb{R}^N)\}.$$

It is easy to see that $\mathcal{A}\colon D(\mathcal{A}) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the subdifferential of the convex functional

$$\mathcal{J}(u) = \begin{cases} \displaystyle\sum_{i=1}^{N} \frac{1}{p_i} \int_{\mathbb{R}^N} |u_{x_i}(x)|^{p_i}\,dx & \text{if } u \in X^{\vec{p}}, \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus X^{\vec{p}}, \end{cases} \tag{3.2}$$

whenever $p_i > 1$ for all i. Then we have that the domain of $\mathcal{J}$ is $D(\mathcal{J}) = X^{\vec{p}}$. Now we use the theory of maximal monotone operators of [19] (see also the monograph [6] and [62, Chapter 10] for a summary and its application to the porous medium equation). Let us prove some important facts, which follow from classical variational arguments. Thus we can solve the nonlinear elliptic equation

$$\lambda \mathcal{A} u + u = f \tag{3.3}$$

in a unique way for all $f \in L^2(\mathbb{R}^N)$ and all $\lambda > 0$, with solutions $u \in D(\mathcal{A})$. Solutions with such regularity are called strong solutions in the elliptic theory (see Definition 3.1 for the evolution problem).

Proposition 3.1. *For all $\lambda > 0$ and $f \in L^2(\mathbb{R}^N)$, there exists a unique strong solution $u \in X^{\vec{p}}$ of* (3.3). *Moreover, the T-contractivity holds: if $f_1, f_2 \in L^2(\mathbb{R}^N)$ and u_1, u_2 solve* (3.3) *with datum f_1, f_2 respectively, we have*

$$\int_{\mathbb{R}^N} (u_1 - u_2)_+^2\,dx \le \int_{\mathbb{R}^N} (f_1 - f_2)_+^2\,dx, \tag{3.4}$$

where $(f)_+ = \max\{f(x), 0\}$. *Finally, a comparison principle applies in the sense that $f_1 \ge f_2$ a.e. in $\mathbb{R}^N$ implies $u_1 \ge u_2$ a.e. in $\mathbb{R}^N$.*

Proof. Let us define the functional

$$\mathrm{J}(u) = \lambda \sum_{i=1}^{N} \frac{1}{p_i} \int_{\mathbb{R}^N} |u_{x_i}|^{p_i}\,dx + \frac{1}{2} \int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} fu\,dx$$

for any $u \in X^{\vec{p}}$. It is clear that J is strictly convex; thus, if a minimizer exists, it is the unique weak solution to (3.3). Let us prove that J is bounded from below. For any $u \in X^{\vec{p}}$, we have, by Young's inequality,

$$\mathrm{J}(u) \ge \lambda \sum_{i=1}^{N} \frac{1}{p_i} \int_{\mathbb{R}^N} |u_{x_i}|^{p_i}\,dx + \Big(\frac{1}{2} - \varepsilon\Big) \int_{\mathbb{R}^N} u^2\,dx - C(\varepsilon) \int_{\mathbb{R}^N} f^2\,dx.$$

Hence, choosing $\varepsilon < \frac{1}{2}$,

$$\mathrm{J}(u) \ge -C(\varepsilon) \int_{\mathbb{R}^N} f^2\,dx.$$

Now, if $\{u_n\} \subset X^{\vec{p}}$ is a minimizing sequence of J, it easily follows that

$$\|u_n\|_{L^2(\mathbb{R}^N)}^2 \le 2\mathrm{J}(u_n) + 2 \int_{\mathbb{R}^N} fu_n\,dx.$$

Then Young's inequality again provides

$$(1 - 2\varepsilon)\|u_n\|_{L^2(\mathbb{R}^N)}^2 \le 2\mathrm{J}(u_n) + C(\varepsilon) \int_{\mathbb{R}^N} f^2\,dx.$$

Then, by uniform boundedness of $\mathrm{J}(u_n)$, the sequence $\{u_n\} \subset X^{\vec{p}}$ is bounded in $L^2(\mathbb{R}^N)$. Thus it admits a subsequence, which we still label $\{u_n\}$, weakly converging to some $u \in L^2(\mathbb{R}^N)$. Now we observe that

$$\lambda \frac{1}{p_i} \int_{\mathbb{R}^N} |\partial_{x_i} u_n|^{p_i}\,dx \le \mathrm{J}(u_n) + \int_{\mathbb{R}^N} fu_n\,dx \quad \text{for every } i = 1, \dots, N,$$

and since $\mathrm{J}(u_n)$ is uniformly bounded and $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$, we have that $\{\partial_{x_i}u_n\}$ is bounded in $L^{p_i}(\mathbb{R}^N)$ for all $i = 1, \dots, N$. Thus, up to subsequences, it follows $\partial_{x_i}u_n \rightharpoonup g_i$ weakly in $L^{p_i}(\mathbb{R}^N)$ for each $i = 1, \dots, N$. Since u_n converges weakly in $L^2(\mathbb{R}^N)$ to u, we find $g_i = \partial_{x_i}u$ for all $i = 1, \dots, N$. By the lower semi-continuity of the $L^q(\mathbb{R}^N)$ norms, we then obtain

$$
\begin{aligned}
\liminf_{n\to\infty} \mathrm{J}(u_n) &= \liminf_{n\to\infty}\Big(\lambda\sum_{i=1}^{N}\frac{1}{p_i}\int_{\mathbb{R}^N}|\partial_{x_i}u_n|^{p_i}\,dx + \frac{1}{2}\int_{\mathbb{R}^N}u_n^2\,dx - \int_{\mathbb{R}^N}fu_n\,dx\Big)\\
&\geq \lambda\sum_{i=1}^{N}\frac{1}{p_i}\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\partial_{x_i}u_n|^{p_i}\,dx + \frac{1}{2}\liminf_{n\to\infty}\int_{\mathbb{R}^N}u_n^2\,dx - \int_{\mathbb{R}^N}fu\,dx\\
&\geq \lambda\sum_{i=1}^{N}\frac{1}{p_i}\int_{\mathbb{R}^N}|\partial_{x_i}u|^{p_i}\,dx + \frac{1}{2}\int_{\mathbb{R}^N}u^2\,dx - \int_{\mathbb{R}^N}fu\,dx = \mathrm{J}(u);
\end{aligned}
$$

therefore, u is the unique minimizer of J. In order to prove the T contraction, as usual, we multiply by $(u_1 - u_2)_+$ the difference of the equations related to data f_1 and f_2 and integrate in space. We are able to conclude using monotonicity of $\mathcal{A}$. Note that $\mathcal{A}(u) = \frac{f-u}{\lambda}$, so we have $u \in D(\mathcal{A})$. The solution is therefore a strong solution. □

Remark 3.2. Proposition 3.1 holds if f belongs to the dual space of $X^{\vec{p}}$, where the dual norm replaces the L^2 norm at the right-hand side of (3.4).

Note that this also applies for the problem posed in a bounded domain Ω, and then the natural boundary condition is $u(x) \to 0$ as $|x| \to \partial\Omega$.

By Proposition 3.1, we have that $R(I + \lambda\mathcal{A}) = L^2(\mathbb{R}^N)$, and the resolvent operator

$$R_\lambda(\mathcal{A}) = (I + \lambda\mathcal{A})^{-1} : L^2(\mathbb{R}^N) \to D(\mathcal{A})$$

is onto and a contraction for all $\lambda > 0$. Hence [19, Proposition 2.2] implies that $\mathcal{A}$ is a maximal monotone operator in $L^2(\mathbb{R}^N)$ (in other words, $\mathcal{A}$ is maximal dissipative).

Recall that $\mathcal{A}$ is the subdifferential of the convex functional $\mathcal{J}(u)$, where $\mathcal{J}$ is lower semi-continuous on $L^2(\mathbb{R}^N)$ (indeed, it can be easily proven that its sublevel sets are strongly closed in $L^2(\mathbb{R}^N)$, following some arguments of Proposition 3.1). Hence it follows from [19, Theorem 3.1, Theorem 3.2] that we can solve the evolution equation

$$u_t = -\mathcal{A}(u) \tag{3.5}$$

for all initial data $u_0 \in L^2(\mathbb{R}^N)$. We observe that $D(\mathcal{A})$ is dense in $L^2(\mathbb{R}^N)$; in other words, we can construct the gradient flow in all of $L^2(\mathbb{R}^N)$ corresponding to the functional $\mathcal{J}$. In particular, the solution $u : [0, +\infty) \to L^2(\mathbb{R}^N)$ is such that $u(t) \in D(\mathcal{A})$ for all $t > 0$; this map is Lipschitz in time; it solves equation (3.5) pointwise on $\mathbb{R}^N$ for a.e. $t > 0$ and $u(0) = u_0$. Moreover, the semigroup maps $S_t^{\mathcal{A}} : u_0 \mapsto u(t)$ form a continuous semigroup of contractions in $L^2(\mathbb{R}^N)$. Comparison principle and T-contractivity hold in the sense that

$$\int_{\mathbb{R}^N}(u_1(t) - u_2(t))_+^2\,dx \leq \int_{\mathbb{R}^N}(u_{0,1} - u_{0,2})_+^2\,dx.$$

We call $S_t^{\mathcal{A}}$ the semigroup generated by $\mathcal{J}$, and the corresponding function $u(\cdot\,, t) = S_t^{\mathcal{A}}(u_0)$ is called the semigroup solution of the evolution problem (or more precisely the L^2 semigroup solution). In particular, u solves the partial differential equation (3.5) in the sense of *strong solutions* in $L^2(\mathbb{R}^N)$, i.e., it agrees with the following definition.

Definition 3.1. If X is a Banach space, a function $u \in C((0, T); X)$ is called a strong solution of the abstract ODE $u_t = -\mathcal{A}u$ if it is absolutely differentiable as an X-valued function of time for a.e. $t > 0$, and moreover, $u(t) \in D(\mathcal{A})$ and $u_t = -\mathcal{A}u$ for almost all times.

The theory says that, when X is a Hilbert space and $\mathcal{A}$ is a subdifferential, then the semigroup solution is a strong solution and $u(t) \in D(\mathcal{A})$ for all $t > 0$. When $u_0 \in L^2(\mathbb{R}^N)$, since $D(\mathcal{A})$ is dense $L^2(\mathbb{R}^N)$, we can use this theory to get strong solutions for every initial datum in that class.

The semigroup solution has extra regularity in anisotropic Sobolev spaces by virtue of the following two computations; see [19, Theorem 3.2]:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = -\langle \mathcal{A}u(t), u(t)\rangle_{L^2} = -\sum_{i=1}^{N}\int_{\mathbb{R}^N} |u_{x_i}(x)|^{p_i}\,dx \le -(\min_i p_i)\mathcal{J}(u(t)). \tag{3.6}$$

Moreover, we have the following entropy-entropy dissipation identity:

$$\frac{d}{dt}\mathcal{J}(u(t)) = \langle \mathcal{A}u(t), u_t(t)\rangle = -\|u_t(t)\|_2^2, \tag{3.7}$$

where the norms are taken in $\mathbb{R}^N$. It follows that both $\|u(t)\|_2$ and $\mathcal{J}(u(t))$ are decreasing in time. Then, from (3.6), integrating on $(0, t)$, we get the estimate

$$\mathcal{J}(u(t)) \le \frac{C\|u_0\|_2^2}{t} \quad \text{for every } t > 0, \tag{3.8}$$

and from (3.7), integrating on (t_1, t_2),

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} u_t^2(x, t)\,dx\,dt \le \mathcal{J}(u(t_1)). \tag{3.9}$$

This Sobolev regularity gives the compactness for times $t \ge \tau > 0$ that we will need in Subsection 10.3.

In this work, we will also need an important extra property of the L^2 semigroup which is the property of generating a contraction semigroup with respect to the norm of $L^q(\mathbb{R}^N)$ for all $q \ge 1$, in particular for $q = 1$. The q-semigroup in such a norm is defined first by restriction of the data to $L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, and then it is extended to $L^q(\mathbb{R}^N)$ by the technique of continuous extension of bounded operators. We leave the details to the reader since it is well-known theory, but see the next section.

We will concentrate in the sequel on the semigroup solutions corresponding to data $u_0 \in L^1(\mathbb{R}^N)$, which we may call L^1 semigroup solutions. Apart from existence, uniqueness and comparison, we will need three extra properties: boundedness for positive times and comparison with super- and subsolutions defined in a suitable way.

For future reference, let us state a general decay result.

Proposition 3.3. *If $u_0 \in L^q(\mathbb{R}^N)$ for $q \in [1, +\infty]$, then the L^q norms $\|u(t)\|_q$ are nonincreasing in time.*

Two reminders about related results. First the variational theory applies in bounded domains with suitable boundary data.

Remark 3.4. The semigroup theory applies to Dirichlet boundary problem defined in a bounded domain Ω as well with zero boundary data.

We can also consider equations with a right-hand side.

Remark 3.5. The complete evolution equation $u_t + \mathcal{A}(u) = f$ including a forcing term can also be treated with the same maximal monotone theory when $f \in L^2(0, T : L^2(\mathbb{R}^N))$ or $f \in L^2(0, T : L^2(\Omega))$.

We will not need such developments here. In the last case, we do not get a semigroup but a more complicated object $u = u(x, t; u_0, f)$.

4 The L^1 Theory

In this section, we will extend to the framework of the $L^1(\mathbb{R}^N)$ space the existence result for solutions to the Cauchy problem for the full anisotropic equation (1.1). This amounts in practice to extending the contraction semigroup defined in $L^2(\mathbb{R}^N)$ in the previous section to a contraction semigroup in $L^1(\mathbb{R}^N)$, an issue that has been studied in some detail in the literature on linear and nonlinear semigroups; see [26, 28, 32, 47, 54]. We will work for simplicity under assumptions (H1)–(H2) (but see Remark 4.3).

For the reader's benefit, we will present the most important details. Experts may skip this section. The extension will be done by means of nonlinear semigroup theory in Banach spaces and using the results of the previous section in Hilbert spaces. We will provide the existence of a *mild* solution by solving the *implicit time discretization scheme* (ITDS for short). Since the ITDS, as we see below, is based on the existence and uniqueness of solutions to the stationary elliptic problem with a zero-order term, we will first recollect briefly some information concerning the problem

$$\begin{cases} -\sum_{i=1}^{N}(|u_{x_i}|^{p_i-2}u_{x_i})_{x_i}+\mu u=f & \text{in } \mathbb{R}^N, \\ u(x)\to 0 & \text{as } |x|\to\infty, \end{cases} \tag{4.1}$$

for arbitrary constant $\mu > 0$.

Theorem 4.1. *Assume $f \in L^1(\mathbb{R}^N)$ and $\mu > 0$. Then there is a unique strong solution $u \in L^1(\mathbb{R}^N)$ to (4.1). Moreover, the following L^1 contraction principle holds: if $f_1, f_2 \in L^1(\mathbb{R}^N)$ and u_1, u_2 are the corresponding solutions, we have*

$$\int_{\mathbb{R}^N}(u_1-u_2)_+\,dx \le \int_{\mathbb{R}^N}(f_1-f_2)_+\,dx. \tag{4.2}$$

In particular, if $f_1 \le f_2$, we have $u_1 \le u_2$ a.e.

Proof. We can proceed by approximation. Let us denote $T_k(s) := \min\{|s|, |k|\}\operatorname{sign}(s)$, and let us take

$$f_k = T_k(f) \in L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$$

such that $f_k \to f$ in $L^1(\mathbb{R}^N)$ and $\|f_k\|_{L^1(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)}$ as a datum in (4.1).

(i) Let u_k^1 and u_k^2 be two solutions of the approximate problems with, respectively, data f_k^1 and f_k^2 in $L^2(\mathbb{R}^N)$. Following [62, Proposition 9.1], let $p(s)$ be a smooth approximation of the positive part of the sign function $\operatorname{sign}(s)$, with $p(s) = 0$ for $s \le 0$, $0 \le p(s) \le 1$ for all $s \in \mathbb{R}$ and $p'(s) \ge 0$ for all $s \ge 0$. Take any cutoff function $\zeta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \zeta \le 1$, $\zeta(x) = 1$ for $|x| \le 1$, $\zeta(x) = 0$ for $|x| \ge 2$, and set $\zeta_n(x) = \zeta(\frac{x}{n})$ for $n \ge 1$ so that $\zeta_n \uparrow 1$ as $n \to \infty$. Using $p(u_k^1 - u_k^2)\zeta_n(x)$ as test function in the difference of equations and letting p tend to sign^+, we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\mathbb{R}^N}&(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)_{x_i}\operatorname{sign}^+(u_k^1-u_k^2)\zeta_n(x)\,dx\\ &+\mu\int_{\mathbb{R}^N}(u_k^1-u_k^2)\operatorname{sign}^+(u_k^1-u_k^2)\zeta_n(x)\,dx=\int_{\mathbb{R}^N}(f_k^1-f_k^2)\operatorname{sign}^+(u_k^1-u_k^2)\zeta_n(x)\,dx.\end{aligned}$$

Now the monotonicity of the operator gives

$$\begin{aligned}\mu\int_{\mathbb{R}^N}&(u_k^1-u_k^2)\operatorname{sign}^+(u_k^1-u_k^2)\zeta_n(x)\,dx\\ &\le\int_{\mathbb{R}^N}(f_k^1-f_k^2)_+\zeta_n(x)\,dx-\sum_{i=1}^{N}\int_{\mathbb{R}^N}(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)\operatorname{sign}^+(u_k^1-u_k^2)\partial_{x_i}\zeta_n(x)\,dx.\end{aligned}$$

We let now $n \to \infty$ to obtain

$$\int_{\mathbb{R}^N}(u_k^1-u_k^2)_+\,dx \le \int_{\mathbb{R}^N}(f_k^1-f_k^2)_+\,dx \tag{4.3}$$

since the right-hand side goes to zero. Indeed, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\mathbb{R}^N}&(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)\operatorname{sign}^+(u_k^1-u_k^2)\partial_{x_i}\zeta_n(x)\,dx\\ &\le\sum_{i=1}^{N}\Big(\int_{\mathbb{R}^N}(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)^{p_i'}\,dx\Big)^{\frac{1}{p_i'}}\frac{1}{n}\Big(\int_{\mathbb{R}^N}\partial_{x_i}\zeta_n^{p_i}(x)\,dx\Big)^{\frac{1}{p_i}}\\ &\le\sum_{i=1}^{N}\Big(\int_{\mathbb{R}^N}(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)^{p_i'}\,dx\Big)^{\frac{1}{p_i'}}\frac{1}{n}\|\partial_{x_i}\zeta_n\|_\infty\Big(\int_{n<|x|<2n}dx\Big)^{\frac{1}{p_i}}\end{aligned}$$

and that $(|\partial_{x_i}u_k^1|^{p_i-2}\partial_{x_i}u_k^1-|\partial_{x_i}u_k^2|^{p_i-2}\partial_{x_i}u_k^2)^{p_i'} \in L^1(\mathbb{R}^N)$.

(ii) By (4.3), it follows that $\{u_k^j\}$ is a Cauchy sequence in $L^1(\mathbb{R}^N)$; then $u_k^j \to u^j$ in $L^1(\mathbb{R}^N)$ for $j = 1, 2$, and we can pass to the limit in (4.3) obtaining (4.2).

(iii) Using $T_m(u_k)$ as test function in the problem with datum f_k, we get the following a priori estimate:

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} |(T_m(u_k))_{x_i}|^{p_i}\, dx + \mu \int_{\mathbb{R}^N} (T_m(u_k))^2\, dx \le mC(N, p_1, \dots, p_N, \|f\|_{L^1(\mathbb{R}^N)})$$

for every $m > 0$. By an anisotropic version of [11, Lemmas 4.1 and 4.2], we have

$$\sum_{i=1}^{N} \|(u_k)_{x_i}\|_{M^{s_i}(\mathbb{R}^N)} \le C(N, p_1, \dots, p_N, \mu, \|f\|_{L^1(\mathbb{R}^N)}), \tag{4.4}$$

where M^{s_i} denote the Marcinkiewicz (or weak-L^{s_i}) spaces and $s_i = \frac{N'}{\bar{p}'} p_i$ for $i = 1, \dots, N$.

When $s_i > 1$ for all i, estimate (4.4) yields that the sequence $\{\partial_{x_i} u_k\}$ is bounded in $L^{q_i}_{\mathrm{loc}}(\mathbb{R}^N)$ with $1 < q_i < \frac{N'}{\bar{p}'} p_i$. Then (up to a subsequence) $\partial_{x_i} u_k \to \partial_{x_i} u$ weakly in $L^{q_i}_{\mathrm{loc}}(\mathbb{R}^N)$ and $u \in L^1(\mathbb{R}^N) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ is a distributional solution to (4.1). Moreover, we get $u_{x_i} \in M^{\frac{N'}{\bar{p}'} p_i}(\mathbb{R}^N)$ and $u \in M^{\frac{N(\bar{p}-1)}{N-\bar{p}}}(\mathbb{R}^N)$ because

$$\|u_k\|_{M^{\frac{N(\bar{p}-1)}{N-\bar{p}}}(\mathbb{R}^N)} \le C(N, p_1, \dots, p_N, \mu, \|f\|_{L^1(\mathbb{R}^N)}). \tag{4.5}$$

When at least one $s_i \le 1$ and $\bar{p} > p_c$, we have to consider a different notion of solution; see e.g. [11] for an entropy solution's one. Following [11], there exists a unique entropy solution and $\frac{\partial}{\partial x_i} T_m(u) \in L^{p_i}(\mathbb{R}^N)$ and $u \in L^1(\mathbb{R}^N) \cap M^{\frac{N(\bar{p}-1)}{N-\bar{p}}}(\mathbb{R}^N)$ by (4.5). □

In order to obtain the existence of solutions to the nonlinear parabolic problem, we use the Crandall–Liggett theorem [27] (see also [62, Chapter 10]), which we briefly recall here in the abstract framework. Let X be a Banach space and $\mathcal{A}\colon D(\mathcal{A}) \subset X \to X$ a nonlinear operator defined on a suitable subset of X. We start from the abstract Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}(u) = f, & t > 0, \\ u(0) = u_0, \end{cases} \tag{4.6}$$

where $u_0 \in X$ and $f \in L^1(0, T; X)$ for some $T > 0$. We first take a partition of the interval, say, $t_k = kh$ for $k = 0, 1, \dots, n$ and $h = \frac{T}{n}$, and then we solve the ITDS, made by the system of difference relations

$$\frac{u_{h,k} - u_{h,k-1}}{h} + \mathcal{A}(u_{h,k}) = f_k^{(h)}$$

for $k = 0, 1, \dots, n$, where we set $u_{h,0} = u_0$. The data set $\{f_k^{(h)} : k = 1, \dots, n\}$ is supposed to be a discretization of the source term f, satisfying the relation $\|f^{(h)} - f\|_{L^1(0,T;X)} \to 0$ as $h \to 0$. The discretization scheme is then rephrased in the form $u_{h,k} = J_h(u_{h,k-1} + hf_k^{(h)})$, where $J_\lambda = (I + \lambda\mathcal{A})^{-1}$, $\lambda > 0$, is called the *resolvent operator*, I being the identity operator. When the ITDS is solved, we construct a *discrete approximate solution* $\{u_{h,k}\}_k$, which is the piecewise constant function $u_h(t)$, defined (for instance) by means of $u_h(t) = u_{h,k}$ if $t \in [(k-1)h, kh]$. If the operator $\mathcal{A}$ is m-accretive, we have that, for all $u_0 \in \overline{D(\mathcal{A})}$, the abstract problem (4.6) has a unique *mild solution* u, i.e., a function $u \in C([0, T); X)$ which is obtained as uniform limit of approximate solutions of the type u_h as $h \to 0$, where the initial datum is taken in the sense that $u(t)$ is continuous in $t = 0$ and $u(t) \to u_0$ as $t \to 0$. We have then, as $h \to 0$, $u(t) := \lim_{h\to 0} u_h(t)$, and the limit is always uniform in compact subsets of $[0, \infty)$. Then we can prove the following parabolic existence-uniqueness result.

Theorem 4.2. *Let $0 < T \le +\infty$ and $Q_T := \mathbb{R}^N \times (0, T)$. For any $u_0 \in L^1(\mathbb{R}^N)$ and any $f \in L^1(Q)$, there is a unique mild solution to the Cauchy problem*

$$\begin{cases} u_t - \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f & \text{in } Q, \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.7}$$

Moreover, for every two solutions u_1 and u_2 to (1.1) with, respectively, initial data $u_{0,1}$ and $u_{0,2}$ in $L^1(\mathbb{R}^N)$ and source terms $f_1, f_2 \in L^1(Q_T)$, we have, for any $0 \le s \le t < T$,

$$\int_{\mathbb{R}^N} (u_1(t) - u_2(t))_+\, dx \le \int_{\mathbb{R}^N} (u_1(s) - u_2(s))_+\, dx + \int_s^t [u_1(\tau) - u_2(\tau), f_1(\tau) - f_2(\tau)]_+\, d\tau, \tag{4.8}$$

with the Sato bracket notation

$$[v, w]_+ = \inf_{\lambda>0} \frac{\|(v+\lambda w)_+\|_{L^1} - \|w_+\|_{L^1}}{\lambda}.$$

In particular, if $u_{0,1} \le u_{0,2}$ and $f_1 \le f_2$ a.e., then, for every $t > 0$, we have $u_1(t) \le u_2(t)$ a.e.

Proof. In order to apply the abstract theory recalled above, we introduce the nonlinear operator

$$\mathcal{A}\colon D(\mathcal{A}) \subset L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N),$$

defined by (3.1) with domain

$$D(\mathcal{A}) := \{v \in L^1(\mathbb{R}^N) : v_{x_i} \in M^{s_i}(\mathbb{R}^N),\ \mathcal{A}(v) \in L^1(\mathbb{R}^N)\},$$

where we recall that $s_i = \frac{N'}{\bar{p}'} p_i$. By Theorem 4.1, we see that this operator is T-accretive on the space $X = L^1(\mathbb{R}^N)$. Therefore, we have that there is a unique mild solution u to (4.7), obtained as a limit of discrete approximate solutions by the ITDS scheme. Moreover, inequality (4.8) follows. □

Remark 4.3. This section also holds under assumption (H2) and $p_i > 1$ making minor changes in the proof of Theorem 4.1.

5 Symmetrization, New Comparison Results

In this section, we assume that (H2) holds. We want to prove a comparison result based on Schwarz symmetrization. We start by considering the simpler setting of nonlinear elliptic equations posed in a bounded open set of $\mathbb{R}^N$ with Dirichlet boundary condition following the classical paper [56]. In our case, it is known that if u solves the following stationary anisotropic problem in a bounded domain Ω:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

then rearrangement methods allow to obtain a pointwise comparison result for u with respect to the solution of the suitable radially symmetric problem in the case of energy solutions when the datum f belongs to the dual space. In [22], it is proved that if $\Omega^\sharp$ is the ball centred in the origin such that $|\Omega^\sharp| = |\Omega|$ and if $u^\sharp$ is the symmetric decreasing rearrangement of a solution u to problem (5.1), then the following inequality holds:

$$u^\# \le U \quad \text{in } \Omega^\#, \tag{5.2}$$

where U is the radially symmetric solution to the following isotropic problem:

$$\begin{cases} \Lambda \Delta_{\bar{p}} U = f^\#(x) & \text{in } \Omega^\#, \\ U = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{5.3}$$

where $\bar{p}$ is the harmonic mean of exponents $p_1, \dots, p_N$, given by formula (1.5), while $f^\#$ is the symmetric decreasing rearrangement of f. The result needs a constant $\Lambda > 0$ that has been determined as

$$\Lambda = \frac{2^{\bar{p}}(\bar{p}-1)^{\bar{p}-1}}{\bar{p}^{\bar{p}}} \left[\frac{\prod_{i=1}^{N} p_i^{\frac{1}{p_i}} (p_i')^{\frac{1}{p_i'}} \Gamma(1+\frac{1}{p_i'})}{\omega_N \Gamma(1+\frac{N}{\bar{p}'})} \right]^{\frac{\bar{p}}{N}} \tag{5.4}$$

with ω_N the measure of the N-dimensional unit ball, Γ the Gamma function and $p_i' = \frac{p_i}{p_i-1}$.

We stress that, in contrast to the isotropic p-Laplacian equation, not only the space domain and the data of problem (5.1) are symmetrized with respect to the space variable, but also the ellipticity condition is subject to an appropriate symmetrization. Indeed, the diffusion operator in problem (5.3) is the standard isotropic $\bar{p}$-Laplacian.

5.1 Main Ideas of the Parabolic Symmetrization

Now it is well known that the pointwise comparison (5.2) need not hold for nonlinear parabolic equations, not even for the heat equation, and has to be replaced by a comparison of integrals known in the literature as concentration comparison, which reads (see [5, 58–60])

$$\int_0^s u^*(\sigma, t)\,d\sigma \le \int_0^s U^*(\sigma, t)\,d\sigma \quad \text{in } (0, |\Omega|), \tag{5.5}$$

valid for all fixed $t \in (0, T)$. In [1], (5.5) is proved when * is the one-dimensional, decreasing rearrangement with respect to the space variable of the weak energy solution u to the following problem:

$$\begin{cases} u_t - \sum_{i=1}^N (|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} = f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

the datum belongs to the dual space, and U^* is the same type of rearrangement of the solution U to the following isotropic "symmetrized" problem:

$$\begin{cases} U_t - \Lambda\Delta_{\bar p} U = f^\#(x,t) & \text{in } \Omega^\# \times (0,T), \\ U(x,t) = 0 & \text{on } \partial\Omega^\# \times (0,T), \\ U(x,0) = u_0^\#(x) & \text{in } \Omega^\#, \end{cases}$$

respectively, with Λ defined in (5.4), $u_0^\#$ the symmetric decreasing rearrangement of u_0 and $f^\#(x,t)$ the symmetric decreasing rearrangement of f with respect to x for t fixed.

Let u be a measurable function on $\mathbb{R}^N$ (if u is defined on a bounded domain Ω, we extend u by 0 outside Ω) fulfilling

$$\big|\{x \in \mathbb{R}^N : |u(x)| > t\}\big| < +\infty \quad \text{for every } t > 0.$$

The (Hardy–Littlewood) one-dimensional *decreasing rearrangement* u^* of u is defined as

$$u^*(s) = \sup\{t > 0 : \big|\{x \in \mathbb{R}^N : |u(x)| > t\}\big| > s\} \quad \text{for } s \ge 0,$$

and the *symmetric decreasing rearrangement* of u is the function $u^\# : \mathbb{R}^N \to [0, +\infty[$ given by

$$u^\#(x) = u^*(\omega_N |x|^N) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

In what follows, we need the following order relationship, taken from [58]. Given two radially symmetric functions $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we say that f is more concentrated than g, $f \succ g$ if, for every $R > 0$,

$$\int_{B_R(0)} f(x)\,dx \ge \int_{B_R(0)} g(x)\,dx.$$

5.2 Comparison Result for Stationary Problems in the Whole Space with a Lower-Order Term

A lack of pointwise comparison already arises in elliptic equations with lower-order terms, which have a close relationship with parabolic equations (see [60] where the isotropic case is treated). Indeed, by the Crandall–Liggett implicit discretization scheme [27] (see below or [62]), the parabolic comparison can be obtained from a similar comparison result for the following stationary problem with a lower-order term:

$$\begin{cases} \sum_{i=1}^N (|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} + \mu u = f & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{5.6}$$

for arbitrary $\mu > 0$.

Theorem 5.1. *Let u be the solution of problem (5.6) with $f \in L^1(\mathbb{R}^N)$, and let U be the solution of the following isotropic problem:*

$$\begin{cases} -\Lambda\Delta_{\bar{p}} U + \mu U = g & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

with $g = g^\# \in L^1(\mathbb{R}^N)$. If $f^\# \prec g$, then we have $u^\# \prec U$.

Proof. We can argue as in [1, Theorem 3.6], but considering the problem defined in whole space $\mathbb{R}^N$ and with a smooth datum. In order to obtain the result when the datum is in $L^1(\mathbb{R}^N)$, we argue by approximation (see Section 4), and we pass to the limit in the concentration estimate, recalling that the rearrangement application $u \to u^*$ is a contraction in $L^r(\mathbb{R}^N)$ for any $r \geq 1$ (see [38]). □

5.3 Statement and Proof of the Parabolic Comparison Result

Now we are in position to state a comparison result for problem (4.7). We set $Q := \mathbb{R}^N \times (0, \infty)$.

Theorem 5.2. *Let u be the mild solution of problem (4.7) with initial data $u_0 \in L^1(\mathbb{R}^N)$ and $f \in L^1(Q)$. Let U be the mild solution to the isotropic parabolic problem*

$$\begin{cases} U_t - \Lambda\Delta_{\bar{p}} U = g & \text{in } Q, \\ U(x,0) = U_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.7}$$

with a nonnegative rearranged initial datum $U_0 \in L^1(\mathbb{R}^N)$ and nonnegative source $g \in L^1(Q)$ which is rearranged with respect to $x \in \mathbb{R}^N$. Assume moreover that

(i) $u_0^\# \prec U_0$,

(ii) $f^\#(\cdot, t) \prec g(\cdot, t)$ for every $t \geq 0$.

Then, for every $t \geq 0$,

$$u^\#(\cdot, t) \prec U(\cdot, t).$$

In particular, for every $q \in [1, \infty]$, we have the comparison of L^q norms

$$\|u(\cdot, t)\|_q \leq \|U(\cdot, t)\|_q \tag{5.8}$$

Note that the norms of (5.8) can also be infinite for some or all values of q.

Proof. According to what was explained in Theorem 4.2, we use the implicit time discretization scheme to obtain the mild solutions to the parabolic problems. For each time $T > 0$, we divide the time interval $[0, T]$ in n subintervals $(t_{k-1}, t_k]$, where $t_k = kh$ and $h = \frac{T}{n}$, and we perform a discretization of f and g adapted to the time mesh $t_k = kh$; let us call them $\{f_k^{(h)}\}$, $\{g_k^{(h)}\}$ so that the piecewise constant (or linear in time) interpolations of this sequences give the functions $f^{(h)}(x,t)$, $g^{(h)}(x,t)$ such that $\|f - f^{(h)}\|_1 \to 0$ and $\|g - g^{(h)}\|_1 \to 0$ as $h \to 0$. We can define $f_k^{(h)}$, $g_k^{(h)}$ in this way:

$$f_k^{(h)}(x) = \frac{1}{h} \int_{(k-1)h}^{kh} f(x,t)\,dt, \quad g_k^{(h)}(x) = \frac{1}{h} \int_{(k-1)h}^{kh} g(x,t)\,dt.$$

Now we construct the function u_h, which is piecewise constant in each interval $(t_{k-1}, t_k]$, by

$$u_h(x,t) = \begin{cases} u_{h,1}(x) & \text{if } t \in [0, t_1], \\ u_{h,2}(x) & \text{if } t \in (t_1, t_2], \\ \vdots & \\ u_{h,n}(x) & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

where $u_{h,k}$ solves the equation

$$h\mathcal{A}(u_{h,k}) + u_{h,k} = u_{h,k-1} + f_k^{(h)} \tag{5.9}$$

with the initial value $u_{h,0} = u_0$. Similarly, concerning the symmetrized problem (5.7), we define the piecewise constant function U_h by

$$U_h(x,t) = \begin{cases} U_{h,1}(x) & \text{if } t \in [0, t_1], \\ U_{h,2}(x) & \text{if } t \in (t_1, t_2], \\ \vdots \\ U_{h,n}(x) & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

where $U_{h,k}(x)$ solves the equation

$$-h\Delta_{\bar{p}} U_{h,k} + U_{h,k} = U_{h,k-1} + g_k^{(h)} \tag{5.10}$$

with the initial value $U_{h,0} = U_0$. Our goal is now to compare the solution $u_{h,k}$ to (5.9) with solution (5.10) by means of mass concentration comparison. We proceed by induction. Using Theorem 5.1, we get $u_{h,1}^{\#} \prec U_{h,1}$. If we assume by induction that $u_{h,k-1}^{\#} \prec U_{h,k-1}$ and call $\tilde{u}_{h,k}$ the (radially decreasing) solution to the equation

$$h\mathcal{A}(\tilde{u}_{h,k}) + \tilde{u}_{h,k} = u_{h,k-1}^{\#} + (f_k^{(h)})^{\#},$$

Theorem 5.1 again implies

$$u_{h,k}^{\#} \prec \tilde{u}_{h,k} \prec U_{h,k}; \tag{5.11}$$

hence (5.11) holds for all $k = 1, \dots, n$. Hence the definitions of u_h and U_h immediately imply

$$u_h(\cdot, t)^{\#} \prec U_h(\cdot, t)) \tag{5.12}$$

for all times t. Since we have $u_h \to u$, $U_h \to U$ uniformly, passing to the limit in (5.12), we get the result. □

6 Boundedness of Solutions

In this section, we assume conditions (H2) and (H3). The following result is usually known as the L^1-L^∞ smoothing effect.

Theorem 6.1. *If $u_0 \in L^1(\mathbb{R}^N)$, then the mild solution to* (1.1) *with initial condition* (1.4) *satisfies the L^∞ bound*

$$\|u(t)\|_\infty \le Ct^{-\alpha}\|u_0\|_1^{\frac{\bar{p}\alpha}{N}} \quad \textit{for all } t > 0, \tag{6.1}$$

where the exponent α is just the one defined in (2.1) *and $C = C(N, \bar{p})$.*

Proof. It is clear that the worst case with respect to the symmetrization and concentration comparison in the class of solutions with the same initial mass M is just the Barenblatt solution B of the isotropic $\bar{p}$-Laplacian with Dirac mass initial data, i.e., $u_0(x) = M\delta(x)$. We are thus reduced to calculate the L^∞ norm of B,

$$\|B\|_\infty = C(N, \bar{p})t^{-\alpha}\|u_0\|_1^{\frac{\bar{p}\alpha}{N}}.$$

Actually, there is a difficulty in taking B as a worst case in the comparison, namely that $B(x, 0)$ is not a function but a Dirac mass. We overcome the difficulty by approximation. We take first a solution with bounded initial data, $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We then replace $B(x, t)$ by a slightly delayed function $B(x, t + \tau)$, which is a solution with initial data $B(x, \tau)$, bounded but converging to $M\delta(x)$ as $\tau \to 0$. It is then clear that, for a small $\tau > 0$, such a solution is more concentrated than u_0. From the comparison theorem, we get

$$|u(x,t)| \le \|B(\cdot, t + \tau)\|_\infty = C(N, \bar{p})M^{\frac{\bar{p}\alpha}{N}}(t + \tau)^{-\alpha}$$

which of course implies (6.1). The result for general L^1 data follows by approximation and density once it is proved for bounded L^1 functions. □

Remarks. (1) Our proof relies on symmetrization. The result was proved in [50] using a different approach; see also [49] and previously for the orthotropic case in [36].

(2) From Proposition 3.3 and Theorem 6.1, we have that, for $u_0 \in L^1 \cap L^\infty$, the rescaled evolution solution v (2.6) is uniformly bounded in time.

7 Anisotropic Upper Barrier Construction

The construction of an upper barrier in an outer domain will play a key role in the proof of existence of the fundamental solution in Section 8. From now on, we assume (H1) and (H2) hold as in the introduction.

Proposition 7.1. *The function*

$$\overline{F}(y) = \Big(\sum_{i=1}^{N} \gamma_i |y_i|^{\frac{p_i}{2-p_i}}\Big)^{-1} \tag{7.1}$$

with

$$\gamma_i \leq \Big[\frac{\alpha}{N}\Big(\min_i\Big\{\sigma_i \frac{p_i}{2-p_i}\Big\} - 1\Big)\frac{1}{2(p_i-1)}\Big(\frac{p_i}{2-p_i}\Big)^{-p_i}\Big]^{\frac{1}{2-p_i}} \tag{7.2}$$

is a weak supersolution to (2.3) *in* $\mathbb{R}^N \setminus B_R(0)$ *and a classical supersolution in* $\mathbb{R}^N \setminus \{0\}$, *with* $B_R(0)$ *being a ball of radius* $R > 0$. *Moreover,* $\overline{F} \in L^1(\mathbb{R}^N \setminus B_R(0))$.

Proof. We observe that, from our hypotheses, $1 < p_i < 2$ and (H2) and the value of α and σ_i guarantee that

$$\frac{2-p_i}{p_i} < \sigma_i, \tag{7.3}$$

which gives the summability outside a ball centred in the origin (see [55, Lemma 2.2]). Note that $\frac{p_i}{2-p_i} \geq 1$. Let γ_i be some positive constants that we will choose later. Denoting

$$X = \sum_{j=1}^{N} \gamma_j |y_j|^{\frac{p_j}{2-p_j}} \quad \text{for } y \in \mathbb{R}^N \setminus \bigcup_{i=1}^{N} \{y \in \mathbb{R}^N : y_i = 0\},$$

we have

$$\begin{aligned}
I &:= \sum_{i=1}^{N} [(|\overline{F}_{y_i}|^{p_i-2}\overline{F}_{y_i})_{y_i} + a_i(y_i\overline{F})_{y_i}] \\
&\leq \sum_{i=1}^{N} 2(p_i-1)\Big(\frac{p_i\gamma_i}{2-p_i}\Big)^{p_i} X^{-2p_i+1}|y_i|^{2p_i\frac{p_i-1}{2-p_i}} + \alpha X^{-1} - X^{-2}\sum_{i=1}^{N} \alpha\sigma_i\gamma_i \frac{p_i}{2-p_i}|y_i|^{\frac{p_i}{2-p_i}} \\
&= X^{-1}\Big[\sum_{i=1}^{N} 2(p_i-1)\Big(\frac{p_i\gamma_i}{2-p_i}\Big)^{p_i} X^{-2p_i+2}|y_i|^{2p_i\frac{p_i-1}{2-p_i}} + \alpha - X^{-1}\sum_{i=1}^{N} \alpha\sigma_i\gamma_i\frac{p_i}{2-p_i}|y_i|^{\frac{p_i}{2-p_i}}\Big] \\
&\leq X^{-1}\Big[\sum_{i=1}^{N} 2(p_i-1)\Big(\frac{p_i\gamma_i}{2-p_i}\Big)^{p_i} X^{-2(p_i-1)}|y_i|^{2p_i\frac{p_i-1}{2-p_i}} + \alpha\Big(1 - \min_i\Big\{\sigma_i\frac{p_i}{2-p_i}\Big\}\Big)\Big].
\end{aligned}$$

Since, for every i, we have

$$\gamma_i|y_i|^{\frac{p_i}{2-p_i}} \leq \sum_{j=1}^{N} \gamma_j|y_j|^{\frac{p_j}{2-p_j}} = X,$$

it follows that

$$X^{-2(p_i-1)} \leq \gamma_i^{-2(p_i-1)}|y_i|^{2p_i\frac{1-p_i}{2-p_i}}.$$

Then

$$I \leq X^{-1}\sum_{i=1}^{N}\Big[2(p_i-1)\Big(\frac{p_i}{2-p_i}\Big)^{p_i}\gamma_i^{2-p_i} + \frac{\alpha}{N}\Big(1 - \min_i\Big\{\sigma_i\frac{p_i}{2-p_i}\Big\}\Big)\Big],$$

where $1 - \min_i\{\sigma_i\frac{p_i}{2-p_i}\} < 0$ by (7.3). In order to conclude that $I \leq 0$, it is enough to show that

$$2(p_i-1)\Big(\frac{p_i}{2-p_i}\Big)^{p_i}\gamma_i^{2-p_i} + \frac{\alpha}{N}\Big(1 - \min_i\Big\{\sigma_i\frac{p_i}{2-p_i}\Big\}\Big) \leq 0$$

for every $i = 1, \ldots, N$, i.e., (7.2). It is easy to check that computations work for $y \in \mathbb{R}^N \setminus \{0\}$. Finally, we stress that $\overline{F}_{y_i} \in L^{p_i}(\mathbb{R}^N \setminus B_R(0))$ with $R > 0$, and then we can easy conclude that $\overline{F}$ is a weak super-solution as well. □

Remark 7.2. We stress that $\overline{F}$ is a weak supersolution to (2.3) in $\mathbb{R}^N \setminus \{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} \le \rho\}$ and belongs to $L^1(\mathbb{R}^N \setminus \{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} \le \rho\})$ for any $\rho > 0$. Moreover, if F_* is the value of $\overline{F}$ on $\{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} = \frac{1}{F_*}\}$, then $\min\{\overline{F}, F_*\}$ agrees with $\overline{F}$ on $\{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} \ge \frac{1}{F_*}\}$ and with F_* on $\{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} < \frac{1}{F_*}\}$.

We are ready to prove a comparison theorem that is needed in the proof of existence of the self-similar fundamental solution. We set as a barrier the truncation of the supersolution $\overline{F}(y)$ given in (7.1). The proof is similar to [34, Theorem 3.2], but for the sake of completeness, we include here the details.

Theorem 7.3 (Barrier Comparison). *For any $M > 0$ and $L_1 > 0$, there exists F_* such that, if $v_0(y) \ge 0$ is an L^1 bounded function such that imposing*
(i) $v_0(y) \le L_1$ *a.e. in* $\mathbb{R}^N$,
(ii) $\int v_0(y)\,dy \le M$,
(iii) $v_0(y) \le G_{M,L_1}(y)$ *a.e. in* $\mathbb{R}^N$,
where $G_{M,L_1} = \min\{\overline{F}, F_*\}$ is the truncation of $\overline{F}(y)$ given in (7.1) *at level F_*, then*

$$v(y,\tau) \le G_{M,L_1}(y) \quad \text{for a.e. } y \in \mathbb{R}^N,\ \tau > \tau_0. \tag{7.4}$$

where $v(y,\tau)$ solves (2.7) *with initial datum $v_0(y)$.*

Proof. (i) Let us pick some $\tau_1 > 0$. Starting from initial mass $M > 0$, from the smoothing effect (6.1) and the scaling transformation (2.6) (we put $t_0 = 1$ and then $\tau_0 = 0$), we know that

$$v(y,\tau) = (t+1)^\alpha u(x,t) \le C_1 M^{\frac{\bar{p}\alpha}{N}} \Big(\frac{t+1}{t}\Big)^\alpha = C_1 M^{\frac{\bar{p}\alpha}{N}} (1 - e^{-\tau})^{-\alpha}, \tag{7.5}$$

where C_1 is a universal constant as in (6.1). Since $\tau = \log(t+1)$, we have $\|v(\tau)\|_\infty \le F_*$ for all $\tau \ge \tau_1$ if F_* is such that

$$C_1 M^{\frac{\bar{p}\alpha}{N}} (1 - e^{-\tau_1})^{-\alpha} \le F_*. \tag{7.6}$$

(ii) For $0 \le \tau < \tau_1$, we argue as follows: from $v_0(y) \le L_1$ a.e., we get $u_0(x) \le L_1$ a.e., so $u(x,t) \le L_1$ a.e.; therefore,

$$\|v(\tau)\|_\infty \le L_1 (t+1)^\alpha = L_1 e^{\alpha\tau} \quad \text{a.e.}$$

We now impose F_* is such that

$$L_1 e^{\alpha\tau_1} \le F_*. \tag{7.7}$$

Then we choose F_* such that (7.6) and (7.7) hold.

(iii) Under these choices, we get $\|v(\tau)\|_\infty \le F_*$ for every $\tau > 0$, which gives a comparison between $v(y,\tau)$ with $G_{M,L_1}(y)$ in the complement of the exterior cylinder $Q_o = \Omega \times (0,\infty)$, where $\Omega = \{y : \overline{F} \le F_*\}$, i.e., $\{\sum_{j=1}^N \gamma_j |y_j|^{\frac{p_j}{2-p_j}} \ge \frac{1}{F_*}\}$. By the comparison in Proposition 11.1 for solutions in Q_o, we conclude that

$$v(y,\tau) \le G_{M,L_1}(y) \quad \text{for a.e. } y \in \Omega,\ \tau > 0,$$

The comparison for $y \notin \Omega$ has been already proved, hence the result (7.4). □

As a consequence of mass conservation and the existence of the upper barrier, we obtain a positivity lemma for certain solutions of the equation. This is the uniform positivity that is needed in the proof of existence of self-similar solutions, and it avoids the fixed point from being trivial.

Lemma 7.1 (A Quantitative Positivity Lemma). *Let v be the solution of the rescaled equation* (2.7) *with integrable initial data v_0 such that v_0 is an SSNI, bounded, nonnegative function with support in the ball of radius R, $\int v_0(y)\,dy = M > 0$ and $v_0 \le G_{M,L_1}$ a.e., where G_{M,L_1} is as in Theorem 7.3. Then there is a continuous nonnegative function $\zeta(y)$, positive in a ball of radius $r_0 > 0$, such that $v(y,\tau) \ge \zeta(y)$ for a.e. $y \in \mathbb{R}^N$, $\tau > 0$. In particular, we may take $\zeta(y) \ge c_1 > 0$ a.e. in $B_{r_0}(0)$ for suitable r_0 and $c_1 > 0$. The function ζ will depend on the choice of M and $\|v_0\|_\infty$.*

We will recall the denomination SSNI stands for separately symmetric and nonincreasing. It was introduced in [34]. The proof of Lemma 7.1 runs as [34, Lemma 5.1].

8 Existence of a Self-Similar Fundamental Solution

Now we are ready to prove the main theorem of this section, dealing with the difficult problem of finding a self-similar fundamental solution to (1.1), enjoying good symmetry properties and the expected decay rate at infinity.

Theorem 8.1. *For any mass $M > 0$, there is a self-similar fundamental solution of equation* (1.1) *with mass M. The profile F_M of such solution is an SSNI nonnegative function. Moreover, $F_M(y) \le \overline{F}(y)$ for a.e. y such that $|y|$ is big enough, where $\overline{F}(y)$ is given in* (7.1).

Remark. Therefore, we get an upper bound for the behaviour of F at infinity. It has a clean form in every coordinate direction, $F(y) \le O(|y_i|^{-\frac{p_i}{2-p_i}})$ as $|y_i| \to \infty$.

The basic idea for proving existence with self-similarity is contained in [34, Theorem 6.1]. The full existence includes self-similarity and will be established next.

8.1 Proof of Existence of a Self-Similar Solution

We will proceed in a number of steps.

(i) Let $\phi \ge 0$ be bounded, symmetric decreasing with respect to x_i, supported in a ball of radius 1 centred at 0, with total mass M (we ask for such specific properties for convenience). We consider the solution u_1 such that $u_1(x, 1) = \phi$, which is bounded and integrable for all $t > 1$, and denote

$$u_k(x, t) = \mathcal{T}_k u_1(x, t) = k^\alpha u_1(k^{\sigma_1\alpha} x_1, \dots, k^{\sigma_N\alpha} x_N, kt)$$

for every $k > 1$. We want to let $k \to \infty$. In terms of rescaled variables (2.6) (with $t_0 = 0$), we have

$$\begin{aligned} v_k(y, \tau) &= e^{\alpha\tau} u_k(y_1 e^{\alpha\sigma_1\tau}, \dots, y_N e^{\alpha\sigma_N\tau}, e^\tau) \\ &= e^{\alpha\tau} k^\alpha u_1(k^{\sigma_1\alpha} y_1 e^{\tau\sigma_1\alpha}, \dots, k^{\sigma_n\alpha} x_N e^{\tau\sigma_N\alpha}, ke^\tau), \end{aligned}$$

where $t = e^\tau$, $\tau > 0$. Put $k = e^h$ so that $k^{\sigma_i\alpha} e^{\tau\sigma_i\alpha} = e^{(\tau+h)\sigma_i\alpha}$. Then

$$v_k(y, \tau) = e^{(\tau+h)\alpha} u_1(y_1 e^{(\tau+h)\sigma_1\alpha}, \dots, y_N e^{(\tau+h)\sigma_N\alpha}, e^{(\tau+h)}).$$

Putting $v_1(y', \tau') = t^\alpha u_1(x, t)$ with $y_i' = x_i t^{-\alpha\sigma_i}$, $\tau' = \log t$, then

$$v_k(y, \tau) = e^{(\tau+h-\tau')\alpha} v_1(y_1 e^{(\tau+h-\tau')\sigma_1\alpha}, \dots, y_N e^{(\tau+h-\tau')\sigma_N\alpha}, \tau + h).$$

Setting $\tau' = \tau + h$, we get $v_k(y, \tau) = v_1(y, \tau + h)$. This means that the transformation $\mathcal{T}_k$ becomes a forward time shift in the rescaled variables that we call $\mathcal{S}_h$ with $h = \log k$.

(ii) Next, we prove the existence of periodic orbits with the following setup. We take $X = L^1(\mathbb{R}^N)$ as ambient space and consider an important subset of X defined as follows. For any $L_1 > 0$, we define the set $K = K(L_1)$ as the set of all $\phi \in L^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

(a) $\int \phi(y)\, dy = 1$,
(b) ϕ is SSNI (separately symmetric and nonincreasing with respect to all coordinates),
(c) ϕ is a.e. bounded above by $G_{L_1}(y)$, where $G_{L_1}(y) = \min\{\overline{F}, F_*\}$ is a fixed barrier, with F_* conveniently large and $\overline{F}(y)$ defined in (7.1),
(d) ϕ is uniformly bounded above by $L_1 > 0$.

Observe that $G_{L_1}(y)$ is obtained in Theorem 7.3 by truncating $\overline{F}(y)$ at a convenient level F_*; this gives that $G_{L_1}(y)$ is a barrier for solutions to (2.7) with mass $M = 1$ and initial data verifying the assumption of Theorem 7.3.

By the previous considerations, it is easy to see that $K(L_1)$ is a non-empty, convex, closed and bounded subset with respect to the norm of the Banach space X.

Now, for all $\phi \in K(L_1)$, we consider the solution $v(y,\tau)$ to equation (2.7) starting at $\tau = 0$ with data $v(y,0) = \phi(y)$, and we consider for all small $h > 0$ the semigroup map $S_h\colon X \to X$ defined by $\mathcal{S}_h(\phi) = v(y,h)$. The following lemma collects some facts we need.

Lemma 8.1. *Given $h > 0$, there exists $L_1 = L_1(h)$ such that $S_h(K(L_1(h)) \subset K(L_1(h))$. Moreover, $S_h(K(L_1(h)))$ is relatively compact in X. Finally, for every $\phi \in K(L_1(h))$,*

$$S_h(\phi)(y) \geq \zeta_h(y) \quad \text{for a.e. } y \in \mathbb{R}^N,\ \tau > 0, \tag{8.1}$$

where ζ_h is a fixed function as in Lemma 7.1. It only depends on h.

Proof. Fix a small $h > 0$, and let $L_1 = L_1(h)$ such that

$$L_1 \geq C_1 M^{\frac{\bar{p}\alpha}{N}}(1 - e^{-h})^{-\alpha}, \tag{8.2}$$

where C_1 is the constant in the smoothing effect (6.1). We take $\tau_1 = h$ in the proof of Theorem (7.3) and choose $F_* = F_*(h)$ such that (7.7) holds, that is, $L_1 e^{\alpha h} \leq F_*$. Then we have in particular that (7.6) is satisfied, namely

$$C_1 M^{\frac{\bar{p}\alpha}{N}}(1 - e^{-h})^{-\alpha} \leq F_*.$$

This ensures the existence of a barrier $G_{L_1(h)}(y)$ (a truncation of $\overline{F}$ defined in (7.1)) such that, for $\phi \in K(L_1(h))$ and any $\tau > 0$, we have $\mathcal{S}_h(\phi) \leq G_{L_1(h)}(y)$ a.e. Then $\mathcal{S}_h(\phi)$ obviously verifies (c), while (a) is a consequence of mass conservation, and (b) follows by Proposition 11.3. Moreover, (8.2) ensures that, from (7.5), we immediately find $\mathcal{S}_h(\phi) \leq L_1$ a.e., that is, property (d). The relative compactness comes from known regularity theory. The last estimate (8.1) comes from Lemma 7.1, which holds once a fixed barrier is determined. □

It now follows from the Schauder fixed point theorem (cf. [33, Theorem 3, Section 9.2.2]) that there exists at least a fixed point $\phi_h \in K(L_1(h))$, i.e., $\mathcal{S}_h(\phi_h) = \phi_h$. Set $\mathcal{S}_\tau(\phi_h) =: v_h(y,\tau)$; thus, in particular, $v_h(y,0) = \phi_h(y)$. The fixed point is in K, so it is not trivial because it has mass 1, and moreover, it satisfies the lower bound (8.1). Iterating the equality, we get periodicity for the orbit $v_h(y,\tau)$ starting at $\tau = 0$,

$$v_h(y, \tau + kh) = v_h(y,\tau) \quad \text{for all } \tau > 0, \tag{8.3}$$

which is valid for all integers $k \geq 1$.

(iii) Once the periodic orbit is obtained, we may examine the family of periodic orbits $\{v_h : h > 0\}$ as a way to obtain a stationary solution in the limit $h \to 0$. Prior to that, let us derive a uniform boundedness property of this family based on the rough idea that periodic solutions enjoy special properties. Indeed, the smoothing effect implies that any solution with mass $M \leq 1$ will be bounded by $C_1 t^{-\alpha}$ (see (6.1)) in terms of the u variable; hence $v(y,\tau)$ will be bounded uniformly in y for all large τ when written in the v variable. Since our functions v_h are periodic, this asymptotic property actually implies that each v_h is a bounded function, uniformly in y and t. On close inspection, we see that the bound is also uniform in h, $v_h \leq C_1$ a.e. That is quite handy since then we can also get a positive lower bound ζ valid for all times using uniform upper bounds in L^∞, L^1 and the upper barrier $\overline{F}$. Then we have that the family v_h is uniformly bounded in $L^1 \cap L^\infty$; thus the family v_h is equi-integrable. Moreover, v_h is tight because the mass confinement holds; indeed, since $v_h \leq \overline{F}$ a.e. uniformly with respect to h, for a large $R > 0$, it follows that

$$\int_{|y|>R} v_h \, dy < \int_{|y|>R} \overline{F}(y)\, dy;$$

thus (recall that $\overline{F} \in L^1(\mathbb{R}^N \setminus B_R(0))$)

$$\lim_{R\to\infty} \int_{|y|>R} v_h \, dy = 0.$$

Then the Dunford–Pettis theorem implies that, up to subsequences, $v_h(\tau) \rightharpoonup \widehat{v}(\tau)$ weakly in $L^1(\mathbb{R}^N)$ for some $\widehat{v}(y,\tau)$. In particular, this gives $\|\widehat{v}(\tau)\|_{L^1} = 1$. Moreover, the a priori estimates (3.6), (3.8),(3.9) and the smoothing effect (6.1) allow to employ the usual compactness argument and find that $\widehat{v}$ solves the rescaled equation (2.7) in the limit.

(iv) We can now take the dyadic sequence $h_n = 2^{-n}$ and $k_n = k' 2^{n-m}$ with $n, m, k' \in \mathbb{N}$ and $m \leq n$ in this collection of periodic orbits v_h. Inserting these values in (8.3) and passing to the limit (along such subsequence) as $n \to \infty$, we find the equality

$$\widehat{v}(y, \tau + k' 2^{-m}) = \widehat{v}(y, \tau) \quad \text{for all } \tau > 0$$

holds for all integers $m, k' \geq 1$. By continuity of the orbit in L^1_{loc}, $\widehat{v}$ must be stationary in time. Passing to the limit, we conclude that $\widehat{v}(y) \leq C$, and moreover, $\widehat{v}(y) \leq \overline{F}$, which gives in particular the required asymptotic behaviour at infinity with the correct rate. Going back to the original variables, this means that the corresponding function $\widehat{u}(x, t)$ is a self-similar solution of equation (1.1). Hence its initial data must be a non-zero Dirac mass. Now we choose any mass $M > 0$. If $M = 1$, then $\widehat{u}$ is the self-similar solution we looked for. If $M \neq 1$, we apply the mass changing scaling transformation (2.8). □

Remark 8.2 (Local Positivity). We know from the proof that $\widehat{v}(y) \leq C$ and $\widehat{v}(y) \leq \overline{F}$; then Theorem 7.3 and Lemma 7.1 ensure that $\widehat{v}(y) \geq \zeta(y)$ for some positive function ζ. Hence $\widehat{v}$ is locally positive.

We have a further property of the self-similar solutions that we will use later.

Proposition 8.3. *Any nonnegative self-similar solution $B(x, t)$ with finite mass is SSNI.*

Proof. We use two general ideas: (i) SSNI is an asymptotic property of many solutions, and (ii) self-similar solutions necessarily verify asymptotic properties for all times.

Let us consider a nonnegative self-similar solution $B(x, t)$. The issue is to prove it has the SSNI property. This is done by approximation and rescaling. We begin with approximating B at time $t = 1$ with a sequence of bounded, compactly supported functions $u_n(x, 1)$ with increasing supports and converging to $B(x, 1)$ in $L^1(\mathbb{R}^N)$. We consider the corresponding solutions $u_n(x, t)$ to (1.1) for $t \geq 1$.

The Aleksandrov principle says that these functions $u_n(\cdot, t)$ have, as $t \to \infty$, an approximate version of the SSNI properties as follows. If the initial support at $t = 1$ is contained in ball of radius $R > 0$, then, for all $t > 1$ and for every $x, \widetilde{x} \in \mathbb{R}^N$, $|x|, |\widetilde{x}| \geq 2R$, we have $u(x, t) \geq u(\widetilde{x}, t)$ on the condition that $|\widetilde{x}^i| \geq |x^i| + 2R$ for every $i = 1, \ldots, N$. A convenient reference can be found in [20] or [62, Proposition 14.27].

The last step is to translate these asymptotic approximate properties into exact properties. This is better done in the v formulation, introduced with formulas (2.6) and (2.7). We first observe that u_n converges to some $\widetilde{B}$; thus, by the contraction principle, for $t \geq 1$,

$$\|u_n(t) - B(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_n(1) - B(1)\|_{L^1(\mathbb{R}^N)},$$

and passing to the limit as $n \to \infty$, we have $u_n(t)$ L^1-converges to some $B(t)$ for $t \geq 1$. This implies that the sequence $v_n(y, \tau)$ of rescaled solutions converges to the self-similar profile $F(x) = B(x, 1)$ at $\tau \geq 0$ (i.e., $t \geq 1$). On the other hand, the definition of the rescaled variables $y_i = x_i t^{-a_i}$ implies that the monotonicity properties derived for u_n by Aleksandrov keep being valid in terms of $(y_1, \ldots, y_N)$ with the reformulation

$$v_n(y, \tau) \geq v_n(\widetilde{y}, \tau) \tag{8.4}$$

on the condition that $|\widetilde{y}_i| \geq |y_i| + 2Rt^{-a_i}$. Similarly, symmetry comparisons are true up to a displacement Rt^{-a_i}. Passing to the limit in (8.4) as $n \to \infty$, we find $F(y) \geq F(\widetilde{y})$ provided $|\widetilde{y}_i| \geq |y_i| + 2Rt^{-a_i}$. Since t can be chosen arbitrarily large, the same property holds for $|\widetilde{y}_i| \geq |y_i|$. Thus F is symmetric with respect to each y_i, and the full SSNI applies to F, hence to the original B. □

9 Lower Barrier Construction and Global Positivity

Now we get a lower barrier that looks a bit like the upper barrier of Section 7.

Proposition 9.1. *Let us take $\gamma > 0$, and let $0 < \vartheta_i \leq 1$ be chosen such that*

$$\frac{1}{\gamma \vartheta_i} < \frac{2 - p_i}{p_i} (< \sigma_i). \tag{9.1}$$

Then

$$\underline{F}(y) = \Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big)^{-\gamma} \in L^1(\mathbb{R}^N)$$

is a weak sub-solution in $\mathbb{R}^N$ *and a classical sub-solution to the stationary equation* (2.3) *in*

$$\mathbb{R}^N \setminus \bigcup_{i=1}^{N} \{y \in \mathbb{R}^N : y_i = 0\} \quad \text{for } A > A_0,$$

where

$$A_0 := \max_{i=1,\dots,N} \Big(\frac{N\gamma^{p_i-1}(p_i-1)(\gamma+1)\vartheta_i^{p_i}}{\alpha(\gamma \max_i\{\sigma_i\vartheta_i\} - 1)}\Big)^{\frac{1}{\gamma-\gamma(p_i-1)-p_i/\vartheta_i}}.$$

Proof. Since $\vartheta_i \le 1$, we get

$$\begin{aligned}
I &:= \sum_{i=1}^{N} [(|\underline{F}_{y_i}|^{p_i-2}\underline{F}_{y_i})_{y_i} + \alpha\sigma_i(y_i\underline{F})_{y_i}] \\
&\ge \sum_{i=1}^{N} \Big(A + \sum_{j=1}^{N} |\eta_j|^{\vartheta_j}\Big)^{-(\gamma+1)(p_i-1)-1} \gamma^{p_i-1}(p_i-1)(\gamma+1)\vartheta_i^{p_i}|y_i|^{p_i(\vartheta_i-1)} \\
&\quad + \alpha\Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big)^{-\gamma} - \gamma\alpha \max_i\{\sigma_i\vartheta_i\}\Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big)^{-\gamma-1} \sum_{i=1}^{N} |y_i|^{\vartheta_i} \\
&\ge \sum_{i=1}^{N} \Big(A + \sum_{j=1}^{N} |\eta_j|^{\vartheta_j}\Big)^{-(\gamma+1)(p_i-1)-1} \gamma^{p_i-1}(p_i-1)(\gamma+1)\vartheta_i^{p_i}\Big(A + \sum_{j=1}^{N} |y_j|^{\vartheta_j}\Big)^{p_i(1-\frac{1}{\vartheta_i})} \\
&\quad + \alpha\Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big)^{-\gamma} - \gamma\alpha \max_i\{\sigma_i\vartheta_i\}\Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big)^{-\gamma-1}\Big(A + \sum_{i=1}^{N} |y_i|^{\vartheta_i}\Big).
\end{aligned}$$

Denoting $X = A + \sum_{j=1}^{N} |\eta_j|^{\vartheta_j}$, we obtain

$$I \ge \sum_{i=1}^{N} X^{-\gamma(p_i-1)-\frac{p_i}{\vartheta_i}}\Big[\gamma^{p_i-1}(p_i-1)(\gamma+1)\vartheta_i^{p_i} + X^{-\gamma+\gamma(p_i-1)+\frac{p_i}{\vartheta_i}} \frac{\alpha}{N}(1 - \gamma \max_i\{\sigma_i\vartheta_i\})\Big].$$

We stress that (9.1) yields $1 - \gamma \max\{\sigma_i\vartheta_i\} \le 0$ and $-\gamma + \gamma(p_i-1) + \frac{p_i}{\vartheta_i} < 0$. In order to have $I \ge 0$, we have to require $X \ge A_0$. Choosing $A > A_0$, it follows that $\underline{F}$ is a sub-solution to equation (2.3) in $\mathbb{R}^N \setminus \{0\}$. It is easy to check that $\underline{F} \in L^1(\mathbb{R}^N)$ and $\underline{F}_{y_i} \in L^{p_i}(\mathbb{R}^N)$ for all i. In order to prove that it is a weak solution in all $\mathbb{R}^N$, we have to multiply by a test function $\psi \in \mathcal{D}(\mathbb{R}^N)$, to integrate in

$$\mathbb{R}^N \setminus \bigcup_{i=1}^{N} \{y : |y_i| < \varepsilon\} \quad \text{for } \varepsilon > 0$$

and finally to estimate the boundary terms. We observe that, for every $i = 1, \dots, N$,

$$\begin{aligned}
\Big|\int_{\partial\{[-\varepsilon,\varepsilon]^N\}} \underline{F}y_i\partial_{y_i}\psi \, d\sigma\Big| &\le A^{-\gamma}\|\psi_{y_i}\|_\infty C(N)\varepsilon^{N+1}, \\
\Big|\int_{\partial\{[-\varepsilon,\varepsilon]^N\}} |\partial_{y_i}\underline{F}|^{p_i-2}\partial_{y_i}\underline{F}\partial_{y_i}\psi \, d\sigma\Big| &\le A^{-(\gamma+1)(p_i-1)}\|\psi_{y_i}\|_\infty C(N)\varepsilon^{N+(\vartheta_i-1)(p_i-1)},
\end{aligned}$$

where $N + (\vartheta_i - 1)(p_i - 1) > 0$ under our assumptions. Similar computations work for the other boundary terms. It is clear that all boundary terms go to zero when $\varepsilon \to 0$. □

Remark 9.2. Under the assumption of Proposition 9.1,

$$\underline{U}(x,t) = t^{-\alpha}\underline{F}(t^{-\alpha\sigma_i}x_1, \dots, t^{-\alpha\sigma_i}x_N) \tag{9.2}$$

is a weak sub-solution to (1.1) in $\mathbb{R}^N \times [0,\infty)$ such that $\underline{U}(x,t) \to \|\underline{F}\|_{L^1}\delta_0(x)$ as $t \to 0$ in distributional sense. In particular, for every $x \neq 0$, we have

$$\lim_{t\to 0} \underline{U}(x,t) = 0. \tag{9.3}$$

We prove a comparison result from below. We take as comparison the following two functions:
(i) the self-similar solution in original variables (with $t_0 = 1$ for simplicity),

$$B(x,t) = (t+1)^{-\alpha} F\big(x_1(t+1)^{-\alpha\sigma_1}, \dots, x_N(t+1)^{-\alpha\sigma_N}\big),$$

with α and σ_i as prescribed in (2.1) and (2.2), and
(ii) the function $\underline{U}(x,t)$ stated in (9.2), that depends on the parameter A.

Theorem 9.3 (Lower Barrier Comparison). *There is a time $\bar{t} > 0$, a radius $R > 0$ and a constant A large enough such that, for every $|x| \geq R$, $0 \leq t \leq \bar{t}$, we have*

$$\underline{U}(x,t) \leq B(x,t). \tag{9.4}$$

The proof of the previous theorem is a simple comparison in an outer cylinder that runs as [34, Theorem 7.4] since the limit (9.3) is uniform in x as long as $|x| \geq R > 0$ for $t > 0$ small enough.

From Theorem 9.3, we derive the positivity for small times of the self-similar fundamental solution determined in Theorem 8.1. Furthermore, we have the following result.

Corollary 9.4. *If F is the profile of a self-similar solution, there are constants $c_1, c_2 > 0$ such that*

$$F(x) \geq c_1 \underline{F}(x_1 c_2^{\alpha\sigma_1}, \dots, x_N c_2^{\alpha\sigma_N})$$

for every $|x| \geq R$ if $R > 0$ and A_2 is large enough. In particular, the profile F decays at most like $O(|x_i|^{-\vartheta_i \gamma})$ in any coordinate direction.

To prove the previous corollary, it is enough to evaluate (9.4) at $t = \bar{t}$.

We can pass from the positivity of just the fundamental solution to the strict positivity for general solutions. This uses a variation of [34, Theorem 7.6] together with the positivity result for the solutions of the fractional p-Laplacian equation, which has been proved in [64, Section 6].

Theorem 9.5 (Infinite Propagation of Positivity). *Any integrable solution with continuous and nonnegative initial data and positive mass is strictly positive a.e. in $\mathbb{R}^N \times (0,\infty)$.*

Proof. (i) Arguing as in the proof of [34, Theorem 7.6], we obtain the infinite propagation of positivity of u when the initial datum u_0 is SSNI, continuous and compactly supported.

(ii) Take now a continuous initial datum $u_0 \geq 0$. We can put below u_0 a smaller SSNI continuous compactly supported initial datum $\tilde{u}(x)$ as in point (i) around some point x_0, and in particular, $u_0(x) \geq \tilde{u}(x)$ in $\mathbb{R}^N$. If $u_1(x,t)$ is the solution of the Cauchy problem with data $\tilde{u}$, we use the comparison principle to obtain that $u(x,t) \geq u_1(x,t) > 0$ a.e. in $\mathbb{R}^N$ for every $t > 0$. Hence u is strictly positive in $\mathbb{R}^N$ in the sense of measure theory, $t_0 - \varepsilon < t < t_0 + t_2 - \varepsilon$. After checking that t_2 does not depend on ε, we conclude that $u(x,t_0) > 0$. □

10 The Orthotropic Case

In this section, we consider equation (1.1) in the *orthotropic* case, namely when all exponents are equal, $p_1 = \cdots = p_N = p < 2$, i.e.,

$$u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i} \quad \text{posed in } Q := \mathbb{R}^N \times (0,+\infty). \tag{10.1}$$

We have to restrict ourselves to this case to prove a uniqueness result for SSNI fundamental solutions because we need some solution regularity that has not yet been proved (to our knowledge) in the general anisotropic case.

10.1 Continuity of Solutions

This subsection is devoted to proving the continuity of mild solutions to the Cauchy problem for equation (1.1) in the orthotropic case. We first recall from Section 3 that the operator $L_{p,h}$ defined in (1.3) generates an L^2 semigroup that can be extended to L^q for any $q \geq 1$ by the technique of continuous extensions of bounded operators. Indeed, the functional $\mathcal{J}$, defined in (3.2), is a Dirichlet form on L^2 (see for instance [24, Theorem 3.6, Theorem 4.1]). As a consequence, due to the fact that $L_{p,h}$ is positively homogeneous, for a given nonnegative datum $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we can apply [12, Theorem 1] and find, for all $q \geq 1$,

$$\|\partial_t u\|_q \leq C\frac{\|u_0\|_q}{t}.$$

If we take $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$, then, by the smoothing effect (6.1), we get, for any $\tau > 0$ and $t \geq 0$,

$$\|\partial_t u(t+\tau)\|_q \leq C\frac{\|u(\tau)\|_q}{t}.$$

Thus, if we combine this estimate with the smoothing effect (6.1), we obtain, for all $t \geq \tau$,

$$\|\partial_t u(t)\|_\infty \leq C\tau^{-\alpha-1}\|u_0\|_1^{\frac{p\alpha}{N}}. \tag{10.2}$$

Hence equation (10.1) can be viewed as the elliptic anisotropic equation

$$A_h(u) := -\sum_{i=1}^{N}(|u_{x_i}|^{p-2}u_{x_i})_{x_i} = f, \tag{10.3}$$

where $f := \partial_t u(\cdot, t)$ is a bounded source term. Then this equation fits into the Lipschitz regularity theory of [21], whose main result implies what follows.

Theorem 10.1. *Let $\frac{2N}{N+2} < p < 2$. There exists a universal constant $C > 0$ such that, for all*

$$u \in W^{1,1}(B_{2R}(x_0)) \cap L^\infty(B_{2R}(x_0))$$

such that $A_h(u) = f$ weakly in $B_{2R}(x_0)$, where $f \in L^\infty(B_{2R}(x_0))$, the following estimate holds:

$$\sup_{x\in B_R(x_0)} |\nabla u| \leq C\Big\{\int_{B_{2R}(x_0)}\Big[1 + \frac{1}{p}\sum|\partial_{x_i}u|^p + \|f\|_{L^\infty}|u|\Big]\,dx\Big\}^\alpha, \tag{10.4}$$

where $C = C(p, N, R, \|f\|_{L^\infty})$ and $\alpha = \alpha(p, N)$.

Then we are in position to prove the following result.

Theorem 10.2. *Assume that $\frac{2N}{N+2} < p < 2$, $u_0 \in L^1(\mathbb{R}^N)$, and let u be the mild solution to equation* (10.1)*, satisfying the initial condition* (1.4)*. Then, for all $\tau > 0$, $u \in L^\infty(\mathbb{R}^N \times [\tau, +\infty))$, and u is global Lipschitz continuous in $\mathbb{R}^N \times [\tau, \infty)$, with a bound*

$$\sup_{\mathbb{R}^N\times[\tau,\infty)} |\nabla_{x,t}u(x,t)| \leq C(N, p, M, \tau, u_0). \tag{10.5}$$

Proof. The fact that $u \in L^\infty(\mathbb{R}^N \times [\tau, +\infty))$ immediately follows from the L^1-L^∞ smoothing effect (6.1). Moreover, by estimate (10.2), we have that u is Lipschitz continuous in time for $t \geq \tau$. Finally, writing the parabolic equation as in (10.3), Theorem 10.1 yields global Lipschitz continuity in space; indeed, observe that, using (10.2), the Lipschitz estimate (10.4) implies (recall that $\nabla u(t) \in L^p(\mathbb{R}^N)$ for any $t > 0$ by Section 3)

$$|\nabla u(x_0, t)| \leq C(N, p, M, \tau, u_0)$$

for all $x_0 \in \mathbb{R}^N$. Then u is globally Lipschitz continuous in $\mathbb{R}^N \times [\tau, \infty)$. □

Remark 10.3. The local Lipschitz regularity in space in the range $p < 2$ descends from the main result in [48, Theorem 1]. For the case $p > 2$, gradient estimates for parabolic orthotropic equations have been recently established in [18].

10.2 Uniqueness of SSNI Fundamental Solutions

Now we give a uniqueness result for nonnegative SSNI fundamental solutions.

Theorem 10.4. *Let $p_c < p < 2$. The nonnegative self-similar fundamental solution of the orthotropic equation* (10.1) *with given mass $M > 0$, given by*

$$B(x,t) = t^{-\alpha} F(t^{-\frac{\alpha}{N}} x), \tag{10.6}$$

with the explicit profile F of mass M given by (2.4)*, is the unique fundamental SSNI solution of that equation with mass M.*

In particular, the explicit solution (2.4) is the unique solution of the stationary equation (2.3) with given mass $M > 0$.

Proof. (i) By contradiction, let us suppose there exists another SSNI fundamental solution B_1 to (10.1), with same mass M. We observe that B_1 satisfies the Lipschitz continuity stated in Theorem 10.2.

We shall really need the non-degeneracy properties of B, given by (10.6) with the explicit profile F (2.4). A key point in the argument is that two different solutions with the same mass must intersect. We define the maximum of the two solutions $B^* = \max\{B_1, B\}$ and the minimum $B_* = \min\{B_1, B\}$. Obviously, B^* and B_* are positive and Lipschitz continuous solutions (with respect to each variable) to (10.1). Under the assumption that the two functions B_1 and B are not the same, we define the open sets $\Omega_1 = \{(x,t) \in Q : B_1(x,t) < B(x,t)\}$ and $\Omega_2 = \{(x,t) \in Q : B_1(x,t) > B(x,t)\}$, where as usual $Q = \mathbb{R}^N \times (0,\infty)$. Then Ω_1 and Ω_2 are disjoint, and both are non-void open sets since the integrals of both functions over $Q_T = \mathbb{R}^N \times (0,T)$ are the same for all $T > 0$. In particular, neither of them can be dense in Q. Moreover, Ω_1 is the set where $B_* < B$ and Ω_2 is the set where $B^* > B$.

(ii) We now show that the situation $B_1 \neq B$ is not possible because of strong maximum principle arguments applied to the difference of the two equations concerning B^* and B. It is here that we use the fact that all the spatial derivatives of B are different from zero away from the set of points where a least one coordinate is zero, a set that we may call the *coordinate skeleton*. Its complement in Q is given by $\Omega = Q \setminus \bigcup_{i=1}^N \mathcal{A}_i$, where $\mathcal{A}_i = \{(x,t) \in Q : x_i = 0\}$ for $i = 1, \dots, N$. Moreover, Ω is an open set, the union of symmetric copies of $Q_i = \{(x,t) \in Q : x_i > 0 \text{ for all } i\}$. We will work in Ω to avoid the presence of degenerate points. We do as follows: we put $w(x,t) = B^*(x,t) - B(x,t)$; then w is nonnegative and continuous and satisfies (in the weak sense; recall that the stationary profiles are differentiable a.e.)

$$w_t = \sum_i (a_i(x,t) w_{x_i})_{x_i}, \tag{10.7}$$

where the coefficients are

$$a_i(x,t) = \frac{|B^*_{x_i}|^{p-2} B^*_{x_i} - |B_{x_i}|^{p-2} B_{x_i}}{B^*_{x_i} - B_{x_i}}.$$

Thus, by the locally Lipschitz continuity of the solutions given by Theorem 10.2, all the $a_i(x,t)$ are locally bounded below by $C_1 > 0$,

$$a_i(x,t) \geq \frac{C_p}{|B^*_{x_i}|^{2-p} + |B_{x_i}|^{2-p}} > C_1 > 0,$$

revealing that each $a_i(x,t)$ is of the order of $\xi^{p-2}(x,t)$ for ξ between $|B^*_{x_i}|$ and $|B_{x_i}|$. The problem is the bound from above, the equation might be not uniformly elliptic if we approach the skeleton.

(iii) Under our assumption $B_1 \neq B$, we know that $w > 0$ somewhere. By continuity, we will have $w \geq c > 0$ in a ball that does not intersect the skeleton, contained in Q_i. Then w cannot be zero everywhere in Ω. Now assume there is a point $P = (x,T)$ of intersection between B^* and B, having all the coordinate values nonzero, $x_i \neq 0$ for all i. Then $w(P) = 0$. For definiteness, let us be in Q_1. In such a case, $|B_{x_i}| > c_i$ is bounded in a neighbourhood of P for all i, and that means that all $a_i(x,t)$ are bounded above as announced in (ii). Indeed, arguing as in [13, Lemma 5.1], we can write

$$a_i(x,t) = \frac{|B_{x_i}|^{p-2} B_{x_i} - |B^*_{x_i}|^{p-2} B^*_{x_i}}{B_{x_i} - B^*_{x_i}} = (p-1) \int_0^1 |sB_{x_i} + (1-s) B^*_{x_i}|^{p-2}\, ds.$$

We use the algebraic inequality

$$\int_0^1 |a+sb|^{p-2}\,ds \le C_p\big(\max_{s\in[0,1]}|a+sb|\big)^{p-2},$$

valid for all $a, b \in \mathbb{R}$ such that $|a|+|b|>0$, with the choice $a = B^*_{x_i}$ and $b = B_{x_i} - B^*_{x_i}$ (so that $|a|+|b| > c_i$ in the neighbourhood of P); hence

$$a_i(x,t) \le C_p\big(\max_{s\in[0,1]}|sB_{x_i} + (1-s)B^*_{x_i}|\big)^{p-2} \le C.$$

Considering the parabolic equation (10.7) in a small cylinder $Q_{\varepsilon,\tau,T} = B_\varepsilon(x)\times(\tau,T)$, the linear parabolic Harnack inequality (see [43, 44]) applies to it, and we can conclude that necessarily w must vanish identically in $Q_{\varepsilon,\tau,T}$. By extension of the same principle, w must vanish in the whole Q_1, i.e., $B^* > B_1$ everywhere in Q_1. What is important is that this implies that Q_1 does not contain any point of Ω_1. We now use the symmetry with respect to the axes and invariance by translation with respect to any hyperplane $t = T$, and we arrive at the conclusion that Ω_1 does not contain any interior point of any quadrant. This is impossible. □

10.3 Asymptotic Behaviour

In the orthotropic case, once the unique SSNI self-similar fundamental solution B_M, given in Theorem (10.4), is determined for any mass $M>0$, it is natural to expect that this is a good candidate to be the attractor for solutions to the Cauchy problem for equation (10.1). Indeed, we have the following result.

Theorem 10.5. *Let $p_c < p < 2$. Let $u(x,t) \ge 0$ be the unique weak solution of the Cauchy problem of the orthotropic equation* (10.1) *with initial data $u_0 \in L^1(\mathbb{R}^N)$ of mass M. Let B_M the self-similar solution*

$$B_M(x,t) = t^{-\alpha}F(t^{-\frac{\alpha}{N}}x)$$

with F defined in (2.4) *having mass M. Then*

$$\lim_{t\to\infty}\|u(t) - B_M(t)\|_1 = 0. \tag{10.8}$$

The convergence holds in the L^∞ norm in the proper scale

$$\lim_{t\to\infty} t^{\alpha}\|u(t) - B_M(t)\|_\infty = 0, \tag{10.9}$$

where α is given by (2.1). *Weighted convergence in $L^q(\mathbb{R}^N)$, $1<q<\infty$, is obtained by interpolation.*

Proof. First let us observe that the smoothing effect estimate (6.1) implies in particular that $u(t) \in L^2(\mathbb{R}^N)$ for all $t \ge \tau$, for any $\tau > 0$, so that u is the solution of (10.1) for $t \ge \tau$ with datum in $L^2(\mathbb{R}^N)$. It follows from the theory that u is a *strong semigroup L^2* solution, as explained in Section 3, meaning that the first and the second energy estimate (3.6), (3.7) hold in any time interval (τ, T). Let us define now the family of rescaled solutions. For all $\lambda > 0$, we put $u_\lambda(x,t) = \lambda^\alpha u(\lambda^{\frac{\alpha}{N}}x, \lambda t)$. By the mass invariance, it follows that, for all $\lambda > 0$, $\|u_\lambda(\cdot,t)\|_1 = M = \|u(\cdot,t)\|_1$, and the smoothing estimate (6.1) yields, for any $\bar t > 0$,

$$\|u_\lambda(\cdot,\bar t)\|_\infty = \lambda^\alpha\|u_\lambda(\cdot,\lambda\bar t)\|_\infty \le C\bar t^{-\alpha}M^{\frac{p\alpha}{N}}. \tag{10.10}$$

Then, since the norms $\|u_\lambda(\cdot,\bar t)\|_1$ and $\|u_\lambda(\cdot,\bar t)\|_\infty$ are equibounded with respect to λ, we have by interpolation that the norms $\|u_\lambda(\cdot,\bar t)\|_p$ are equibounded for all $p\in[1,\infty]$. Now we fix $\bar t > 0$ so that, by the previous remark, $u(\bar t) \in L^2(\mathbb{R}^N)$, and we can use the first energy estimate (3.6) for $t \ge \bar t$,

$$\sum_{i=1}^N \int_{\bar t}^{t}\int_{\mathbb{R}^N} |u_{x_i}|^p\,dx\,d\tau \le \frac{1}{2}\|u(\bar t)\|_2^2.$$

Moreover, (3.8) and (3.9) provide

$$\int_{\bar t}^{t}\int_{\mathbb{R}^N} |u_t(x,\tau)|^2\,dx\,d\tau \le C\frac{\|u(\bar t)\|_2^2}{t}.$$

Then we have

$$\sum_{i=1}^{N}\int_{\bar t}^{t}\int_{\mathbb{R}^N} |\partial_{x_i}u_\lambda|^p\,dx\,d\tau \le C\lambda^\alpha\|u(\cdot,\lambda\bar t)\|_2^2 = C\|u_\lambda(\cdot,\bar t)\|_2^2, \tag{10.11}$$

and since $\|u_\lambda(\cdot,\bar t)\|_2$ is equibounded, we have that $\partial_{x_i}u_\lambda$ are equibounded in $L^p_{x,t}$ for $i = 1,\dots,N$, $t \ge \bar t$. Moreover, we have the following estimate of the time derivatives:

$$\int_{\bar t}^{t}\int_{\mathbb{R}^N} |\partial_t u_\lambda(x,\tau)|^2\,dx\,d\tau = \lambda^{\alpha+1}\int_{\lambda\bar t}^{\lambda t}\int_{\mathbb{R}^N} |\partial_t u(x,\tau)|^2\,dx\,d\tau \le C\lambda^\alpha\frac{\|u(\cdot,\lambda\bar t)\|_2^2}{\bar t} = \frac{C}{\bar t}\|u_\lambda(\cdot,\lambda\bar t)\|_2^2, \tag{10.12}$$

and this gives weak compactness of the time derivatives $\partial_t u_\lambda$ in $L^2_{x,t}$ for $t \ge \bar t$. Then estimates (10.10), (10.11) and (10.12) imply, for $t \ge \bar t$, $u_\lambda \in L^\infty_{x,t}$, $\partial_{x_i}u_\lambda \in L^p_{x,t}$ for every i, $\partial_t u_\lambda \in L^2_{x,t}$ with uniform bounds with respect to λ. Then the Rellich–Kondrachov theorem allows to say that the family u_λ is relatively locally compact in $L^1_{x,t}$. Therefore, up to subsequences, we have $\lim_{\lambda\to\infty} u_\lambda(x,t) = U(x,t)$ for some finite-mass function $U(x,t) \ge 0$, and the convergence holds in $L^1_{\mathrm{loc}}(Q)$. Then, arguing as in [62, Lemma 18.3], it is easy to show that U is a *weak* solution to (10.1) in the sense that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} U\phi_t\,dx\,dt - \sum_{i=1}^{N}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |\partial_{x_i}U|^{p-2}\partial_{x_i}U\partial_{x_i}\phi\,dx\,dt = 0$$

for all the test functions $\phi \in C_c^\infty(\mathbb{R}^N\times(0,\infty))$.

(ii) Assuming that u_0 is bounded and compactly supported in a ball B_R, we argue as in [62, Theorem 18.1]. We take a larger mass $M' > M$ and the self-similar solution $B_{M'}(x,t)$ such that $B_{M'}(x,1) \ge u_0(x)$. Then we clearly have

$$u_\lambda(x,0) = \lambda^\alpha u(\lambda^{\frac{\alpha}{N}}x,0) \le \lambda^\alpha B_{M'}(\lambda^{\frac{\alpha}{N}}x,1) = B_{M'}\Big(x,\frac{1}{\lambda}\Big).$$

Then the comparison principle gives

$$u_\lambda(x,t) \le B_{M'}\Big(x,t+\frac{1}{\lambda}\Big). \tag{10.13}$$

Since $u_\lambda \to U$ a.e. and $B_{M'}(x,t+\frac{1}{\lambda}) \to B_{M'}(x,t)$ as $\lambda\to\infty$, the mass invariance of $B_{M'}$ and (10.13) allows to apply the Lebesgue dominated convergence theorem and obtain (up to subsequence) $u_\lambda(t) \to U(t)$ in $L^1(\mathbb{R}^N)$, which means that the mass of U is equal to M at any positive time t. This gives that U is a fundamental solution with initial mass M; it is bounded for all $t > 0$, and the usual estimates apply. Moreover, observe that the rescaled sequence u_λ has initial data supported in a sequence of shrinking balls $B_{R/\lambda^{\frac{\alpha}{N}}}(0)$. The usual application of the Aleksandrov principle implies that $U(x,t)$ will have the properties of monotonicity along coordinate directions and also the property of symmetry with respect to coordinate hyperplanes. For more details, see [37, Theorem 3]. Then the uniqueness theorem, Theorem 10.4, applies, and we have $U = B_M$. Actually, we have that any subsequence of $u_\lambda(t)$ converges in $L^1(\mathbb{R}^N)$ to $B_M(t)$; thus the whole family of rescaled solutions $u_\lambda(t)$ converges to $B_M(t)$ in $L^1(\mathbb{R}^N)$.

In particular, we have $u_\lambda(x,1) \to B_M(x,1) = F(x)$ in $L^1(\mathbb{R}^N)$ with F defined in (2.4), which gives formula (10.8). The general case $u_0 \in L^1(\mathbb{R}^N)$ can be done by following the arguments in [62, Theorem 18.1].

(iv) Now we pass to achieve the uniform convergence (10.9). First of all, the equiboundedness of the family u_λ and the Lipschitz estimates (10.4) given by Theorem 10.2 allow the use of the Ascoli–Arzelá theorem, in order to obtain $u_\lambda \to B_M$ uniformly on compact sets of $Q = \mathbb{R}^N\times(0,\infty)$. In order to obtain the full convergence in $\mathbb{R}^N$ at time $t = 1$, we need a tail analysis at infinity, and we argue as in [62, Theorem 18.1]. Take any $\varepsilon > 0$; then the very definition of the rescaled solutions u_λ gives, for $\lambda > 1$ and $R > 1$,

$$\int_{|x|>\frac{R}{2}} u_\lambda(x,1)\,dx = \int_{|x|>\frac{R}{2}} [u_\lambda(x,1) - F(x)]\,dx + \int_{|x|>\frac{R}{2}} F(x)\,dx \le \int_{\mathbb{R}^N} [u(y,\lambda) - B_M(y,\lambda)]\,dx + \int_{|x|>\frac{R}{2}} F(x)\,dx.$$

Now (10.8) allows to select a sufficiently large λ such that

$$\int_{\mathbb{R}^N} |u(y,\lambda) - B_M(y,\lambda)|\,dy < \frac{\varepsilon}{2}.$$

Then, choosing a large $R \gg 1$ such that

$$\int_{|x|>\frac{R}{2}} F(x)\,dx < \frac{\varepsilon}{2},$$

we have, for λ large,

$$\int_{|x|>\frac{R}{2}} u_\lambda(x,1)\,dx < \varepsilon. \tag{10.14}$$

Let us take any x_0 such that $|x_0| > R$, so that $B_{\frac{R}{2}}(x_0) \subset \{|x| > \frac{R}{2}\}$. From the Gagliardo–Nirenberg inequality on bounded domains (see e.g. [35, 46]), we have

$$\|u_\lambda(\cdot,1)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le C_1 \|u_\lambda(\cdot,1)\|^{\tilde{\alpha}}_{L^1(B_{\frac{R}{2}}(x_0))} \|\nabla u_\lambda(\cdot,1)\|^{1-\tilde{\alpha}}_{L^\infty(B_{\frac{R}{2}}(x_0))} + C_2 \|u_\lambda(\cdot,1)\|_{L^1(B_{\frac{R}{2}}(x_0))},$$

where $\tilde{\alpha} = \frac{1}{N+1}$ and C_i, $i = 1, 2$, are constants depending on N, x_0 and R. Then, by (10.14) and the uniform bound of the gradient (10.5), we have, for λ large,

$$\|u_\lambda(x,1)\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \le C\varepsilon^{\tilde{\alpha}};$$

therefore, for all x_0 such that $|x_0| > R$,

$$u_\lambda(x_0,1) \le C\varepsilon^{\tilde{\alpha}}.$$

Thus the uniform convergence on compact sets implies that $u_\lambda(x,1) \to F(x)$ uniformly on $\mathbb{R}^N$ as $\lambda \to \infty$, which easily translates to (10.9). □

11 Complements on the Theory

11.1 A Comparison Theorem

First we prove a comparison for solutions to a Cauchy–Dirichlet problem associated to equation (1.1) posed on a domain U, where U can be bounded or unbounded. In the latter case, we will consider U either as an outer domain (i.e., the complement of a bounded domain) or a half-space. Let us consider the following Cauchy–Dirichlet problem:

$$\begin{cases} u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} & \text{in } U \times [0,\infty), \\ u(x,t) = h(x,t) \ge 0 & \text{in } \partial U \times [0,\infty), \\ u(x,0) = u_0(x) \ge 0 & \text{in } U, \end{cases} \tag{11.1}$$

where, in general, we take $u_0 \in L^1(U)$ and $h \in C(\partial U \times [0,\infty))$.

Proposition 11.1. *Let u_1 and u_2 be two nonnegative solutions of* (11.1) *with initial data $u_{0,1}, u_{0,2} \in L^1(U)$ and boundary data $h_1 \le h_2$ on $\partial U \times [0,\infty)$. Then we have*

$$\int_U (u_1(t) - u_2(t))_+\,dx \le \int_U (u_{0,1} - u_{0,2})_+\,dx.$$

In particular, if $u_{0,1} \le u_{0,2}$ for a.e. $x \in U$, then, for every $t > 0$, we have $u_1(t) \le u_2(t)$ a.e. in U.

Proof. We point out that the boundary conditions of u_1, u_2 on ∂U imply in particular that $u_1 \le u_2$ on ∂U and in particular $(u_1 - u_2)_+ = 0$ on ∂U. We follow the lines of the proof of (4.2) in Theorem 4.1. Indeed, using the same test function, by the monotonicity of the operator, we find

$$\begin{aligned}\frac{d}{dt}\int_U (u_1(t)-u_2(t))^+\zeta_n(x)\,dx &= \sum_{i=1}^N \int_U \partial_{x_i}(|\partial_{x_i}u_1|^{p_i-2}\partial_{x_i}u_1 - |\partial_{x_i}u_2|^{p_i-2}\partial_{x_i}u_2)(u_1-u_2)_+\zeta_n(x)\,dx \\ &\leq -\sum_{i=1}^N \int_U (|\partial_{x_i}u_1|^{p_i-2}\partial_{x_i}u_1 - |\partial_{x_i}u_2|^{p_i-2}\partial_{x_i}u_2)(u_1-u_2)_+\partial_{x_i}\zeta_n(x)\,dx \\ &\quad + \sum_{i=1}^N \int_{\partial U} (|\partial_{x_i}u_1|^{p_i-2}\partial_{x_i}u_1 - |\partial_{x_i}u_2|^{p_i-2}\partial_{x_i}u_2)(u_1-u_2)_+\zeta_n(x)\nu_i\,d\sigma \\ &= -\sum_{i=1}^N \int_U (|\partial_{x_i}u_1|^{p_i-2}\partial_{x_i}u_1 - |\partial_{x_i}u_2|^{p_i-2}\partial_{x_i}u_2)(u_1-u_2)_+\partial_{x_i}\zeta_n(x)\,dx.\end{aligned}$$

From now on, we argue as in (i) in the proof of Theorem 4.1. □

11.2 Aleksandrov's Reflection Principle

In this auxiliary section, we prove Aleksandrov's principle. Let $H_j^+ = \{x \in \mathbb{R}^N : x_j > 0\}$ be the positive half-space with respect to the x_j coordinate for any fixed $j \in \{1,\dots,N\}$. For any $j = 1,\dots,N$, the hyperplane $H_j = \{x_j = 0\}$ divides $\mathbb{R}^N$ into two half-spaces $H_j^+ = \{x_j > 0\}$ and $H_j^- = \{x_j < 0\}$. We denote by π_{H_j} the specular symmetry that maps a point $x \in H_j^+$ into $\pi_{H_j}(x) \in H_j^-$, its symmetric image with respect to H_j. We have the following important results.

Proposition 11.2. *Let u be a nonnegative solution of the Cauchy problem for* (1.1) *with nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$. If, for a given hyperplane H_j with $j = 1,\dots,N$, we have $u_0(\pi_{H_j}(x)) \leq u_0(x)$ for a.e. $x \in H_j{}^+$, then, for all t, $u(\pi_{H_j}(x), t) \leq u(x,t)$ for a.e. $(x,t) \in H_j^+ \times (0,\infty)$.*

Proposition 11.3. *Let u be a nonnegative solution of the Cauchy problem for* (1.1) *with nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$. If u_0 is a symmetric function in each variable x_i, and also a decreasing function in $|x_i|$ for all i a.e., then $u(x,t)$ is also symmetric and a nonincreasing function in $|x_i|$ for all i, for all t, a.e. in x (for short SSNI, meaning separately symmetric and nonincreasing).*

In order to prove the previous two propositions, we can argue as in [34]. In particular, Proposition 11.2 is a consequence of Proposition 11.1 and yields Proposition 11.3.

12 Control on the Anisotropy

In our analysis of existence of self-similar solutions for equation (APLE), we have found conditions (H2) and (H3). It is interesting to examine what these requirements mean for $N = 2$ and $p_1, p_2 > 1$. Condition (H2) means

$$\frac{p_1p_2}{p_1+p_2} > \frac{2}{3}, \quad \text{i.e.,} \quad \Big(p_1 - \frac{2}{3}\Big)\Big(p_2 - \frac{2}{3}\Big) > \frac{4}{9}.$$

The region is limited below in Figure 1 by a symmetric hyperbola which passes through the points $(2,1)$, $(\frac{4}{3},\frac{4}{3})$ and $(1,2)$. As for condition (H3), we have

$$p_i < \frac{3}{2}\bar{p} = \frac{3p_1p_2}{p_1+p_2},$$

which amounts to $p_1 < 2p_2$ (delimited by line r_2 in Figure 1) and symmetrically $p_2 < 2p_1$ (delimited by line r_1). We thus get a necessary "small anisotropy condition" which takes the form

$$\frac{1}{2} < \frac{p_1}{p_2} < 2,$$

and it is automatically satisfied for fast diffusion $1 < p_1, p_2 < 2$.

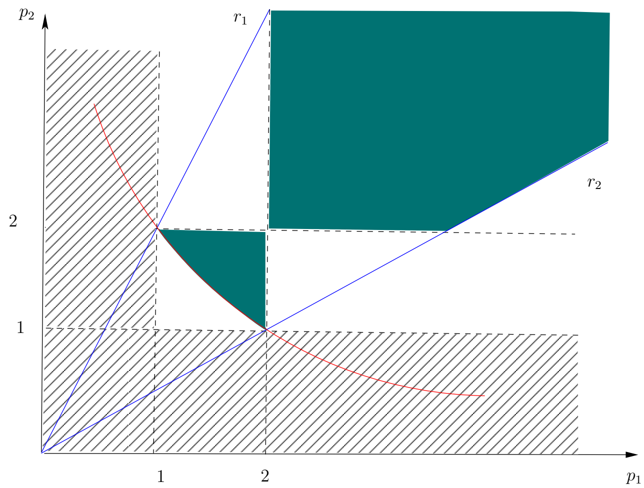

Figure 1: p_1, p_2 that verify conditions (H2)–(H3) when $p_1, p_2 \le 2$ or $p_1, p_2 \ge 2$

The analysis of the (APME) in [34] leads to a simpler algebra. According to the results of the paper, the analogue of condition (H2) becomes

$$\frac{1}{N}\sum_i m_i > \frac{N-2}{N},$$

which in dimension $N = 2$ reads $m_1 + m_2 > 0$. For $N \ge 3$, we get $m_1 + m_2 + \cdots + m_N > N - 2$. This is much simpler than the (APLE) condition. Otherwise, the anisotropy control, the analogue of (H3), reads

$$m_i < \bar{m} + \frac{2}{N},$$

where $\bar{m} = \frac{1}{N}\sum_{i=1}^N m_i$. For $N = 2$, this means $|m_1 - m_2| < 2$. This is automatically satisfied for fast diffusion $0 < m_1, m_2 < 1$, but is important when slow diffusion occurs in some coordinate direction.

13 Self-Similarity for Anisotropic Doubly Nonlinear Equations

We have studied two types of anisotropic evolution equations: the anisotropic equation of porous medium type (APME) treated in [34] and the model (APLE) involving anisotropic p-Laplacian type (1.1), studied here above. The similarities lead to consider a more general evolution equation with anisotropic nonlinearities involving powers of both the solution and its spatial derivatives

$$u_t = \sum_{i=1}^N (|(u^{m_i})_{x_i}|^{p_i-2}(u^{m_i})_{x_i})_{x_i}. \tag{13.1}$$

We will call it (ADNLE). We assume that $m_i > 0$ and $p_i > 1$. The isotropic case is well known; see [61, Section 11]. We describe next the self-similarity analysis applied to solutions plus the physical requirement of finite conserved mass.

The type of self-similar solutions of equation (1.1) has again the usual form

$$B(x,t) = t^{-\alpha}F(t^{-a_1}x_1, \dots, t^{-a_N}x_N)$$

with constants $\alpha > 0$, $a_1, \dots, a_n \ge 0$ to be chosen below. We substitute this formula into equation (13.1). Note that, writing $y = (y_i)$ with $y_i = x_i t^{-a_i}$, equation (13.1) becomes

$$-t^{-\alpha-1}\left[\alpha F(y) + \sum_{i=1}^N a_i y_i F_{y_i}\right] = \sum_{i=1}^N t^{-[\alpha m_i(p_i-1)+p_i a_i]}(|(F^{m_i})_{y_i}|^{p_i-2}(F^{m_i})_{y_i})_{y_i}.$$

Time is eliminated as a factor in the resulting equation on the condition that

$$\alpha(m_i(p_i-1)-1)+p_i a_i = 1 \quad \text{for all } i=1,2,\dots,N.$$

We also look for integrable solutions that will enjoy the mass conservation property, and this implies that $\alpha = \sum_{i=1}^N a_i$. Writing $a_i = \sigma_i\alpha$, we get the conditions $\sum_{i=1}^N \sigma_i = 1$ and

$$\alpha[m_i(p_i-1)-1+p_i\sigma_i] = 1 \quad \text{for all } i.$$

From this set of conditions, we can get the unique admissible values of α and σ_i. We proceed as follows. From the last displayed formula, we get

$$\sigma_i = \frac{1}{p_i}\Big(\frac{1}{\alpha}+1-m_i(p_i-1)\Big). \tag{13.2}$$

Then the condition $\sum_{i=1}^N \sigma_i = 1$ implies that

$$1 = \Big(\frac{1}{\alpha}+1\Big)\sum_{i=1}^N \frac{1}{p_i} - \sum_{i=1}^N m_i + \sum_{i=1}^N \frac{m_i}{p_i}.$$

At this moment, we introduce some suitable notation:

$$\frac{1}{N}\sum_{i=1}^N \frac{1}{p_i} = \bar{p}, \qquad \frac{1}{N}\sum_{i=1}^N m_i = \bar{m}, \qquad \frac{1}{N}\sum_{i=1}^N \frac{m_i}{p_i} = \frac{q}{\bar{p}}.$$

Using that, we get

$$\alpha = \frac{N}{N(\bar{m}\bar{p}-q-1)+\bar{p}}.$$

We want to work in a parameter range that ensures that $\alpha > 0$, and this means the condition

$$\bar{p}\bar{m} + \frac{\bar{p}}{N} > q+1,$$

which is the equivalence in this setting to condition (H2) in the introduction. Under this condition, the self-similar solution will decay in time in maximum value like a power of time. This is a crucial condition for the self-similar solution to exist and play its role since the suitable existence theory contains the maximum principle.

Once α is obtained, the σ_i are given by (13.2). These exponents control the rate of spatial spread in every coordinate direction; we know that $\sum_{i=1}^N \sigma_i = 1$, and in particular, $\sigma_i = \frac{1}{N}$ in the homogeneous case. The condition to ensure that $\sigma_i > 0$ is

$$m_i(p_i-1) < \frac{1}{\alpha}+1, \quad \text{i.e.,} \quad m_i(p_i-1) < \bar{p}\bar{m} + \frac{\bar{p}}{N} - q.$$

This means that the self-similar solution expands as time passes (or at least it does not contract), along any of the coordinate directions.

Note that the simple fast diffusion conditions $m_i < 1$ and $p_i < 2$ and $\alpha > 0$ ensure that $\sigma_i > 0$.

(1) Particular Cases.

(a) When all the m_i equal 1, we find the results of our present paper contained in Section 2 for equation (APLE). On the other hand, when $p_i = 2$, we find the results of the previous paper [34] for equation (APME).

(b) It is also interesting to look at cases where the m_i equal m, but not necessarily 1, and when $p_i = p$ but not necessarily 2. In the first case, $q = m$, while in the second case, we get $q = \bar{m}$. In both cases, α is given by the simpler formula

$$\alpha = \frac{N}{N(\bar{m}(\bar{p}-1)-1)+\bar{p}}$$

that looks very much like the isotropic case; see the Barenblatt solution, which is explicitly written in [61, Subsection 11.4.2].

(2) On the theory. With these choices, the profile function $F(y)$ must satisfy the following doubly-nonlinear anisotropic stationary equation in $\mathbb{R}^N$:

$$\sum_{i=1}^{N} \big[\big(|(F^{m_i})_{y_i}|^{p_i-2}(F^{m_i})_{y_i}\big)_{y_i} + \alpha\sigma_i (y_i F)_{y_i}\big] = 0.$$

Conservation of mass must also hold: $\int B(x,t)\,dx = \int F(y)\,dy = M < \infty$ for $t > 0$.

The next step would be to prove that there exists a suitable solution of this elliptic equation, which is the anisotropic version of the equation of the doubly nonlinear Barenblatt profiles in the standard m-p-Laplacian. The solution is indeed explicit in the isotropic case, as we have said.

14 Comments, Extensions and Open Problems

- We may replace the main equation (1.1) by

$$u_t = \sum_{i=1}^{N} (a_i |u_{x_i}|^{p_i-2} u_{x_i})_{x_i} \quad \text{in } Q := \mathbb{R}^N \times (0,+\infty)$$

  with all constants $a_i > 0$, and nothing changes in the theory. Inserting the constants may be needed in the applications. The case where the a_i depend on x appears in inhomogeneous media, and it is out of our scope. And we did not touch on the theory of equations like (1.1) where the exponents $p(x,t)$ are space-time dependent; see [3] in this respect.
- We may replace the main equation (1.1) by

$$u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} + \varepsilon\Delta_p(u) \quad \text{in } Q := \mathbb{R}^N \times (0,+\infty).$$

  At least in the case of homogeneous anisotropy, the same theory will work, and we have uniqueness of self-similar solutions, which are also explicit, and we can write them.
- The cases where some or all of the p_i are larger than 2 are not treated here in any systematic way. Notice that our general theory applies, as well as the symmetrization and boundedness. The upper barrier has to be changed into a barrier compatible with the compact support properties. In the orthotropic case, the existence theorem for self-similar Barenblatt solutions obtained in the paper [23] can be completed with the proof of uniqueness and the theorem of asymptotic behaviour as in Section 10 above.
- The limit cases where some $p_i = 2$ deserve attention.
- Symmetrization does not give sharp bounds probably when the p_i are not the same, but it implies the L^1-L^∞ bound where the constant is explicit. Can we compare our self-similar solutions with the isotropic Barenblatt solution by symmetrization?
- If we check the explicit self-similar solutions of the isotropic and orthotropic equations, they are comparable but for a constant.
- We have not discussed the Harnack or the Hölder regularity for this theory.
- Following the idea of [45], it is possible to prove a strong maximum principle in the *homogeneous case* where all exponents are equal, $p_1 = \cdots = p_N = p < 2$.

Theorem 14.1. *Let $T > 0$, Ω a bounded domain of $\mathbb{R}^N$, $u \in C^0([0,T)\times\Omega)$ satisfying $u_t - L_h u \ge 0$ with L_h defined as in* (1.3)*, $p < 2$ and data u_0 non-identically zero such that $u(\cdot,t) \ge 0$ on $\partial\Omega$ for all $t \ge 0$. If there exists some $x \in \Omega$ and $t > 0$ such that $u(x,t) = 0$, then $u(\cdot,t) \equiv 0$ on Ω.*

Acknowledgment: J. L. Vázquez is an Honorary Professor at Universidad Complutense de Madrid. The authors wish to warmly thank L. Brasco for fruitful discussions and valuable suggestions. The referee made useful suggestions.

Funding: J. L. Vázquez was funded by grant PGC2018-098440-B-I00 from MICINN (the Spanish Government).

References

[1] A. Alberico, G. di Blasio and F. Feo, Comparison results for nonlinear anisotropic parabolic problems, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 2, 305–322.

[2] S. Antontsev and M. Chipot, Anisotropic equations: Uniqueness and existence results, *Differential Integral Equations* **21** (2008), no. 5–6, 401–419.

[3] S. Antontsev and S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions. Existence, Uniqueness, Localization, Blow-Up*, Atlantis Stud. Differ. Equ. 4, Atlantis Press, Paris, 2015.

[4] A. Baernstein, II, *Symmetrization in Analysis*, New Math. Monogr. 36, Cambridge University, Cambridge, 2019.

[5] C. Bandle, *Isoperimetric Inequalities and Applications*, Monogr. Stud. Math. 7, Pitman, Boston, 1980.

[6] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monogr. Math., Springer, New York, 2010.

[7] G. I. Barenblatt, On some unsteady motions of a liquid and gas in a porous medium, *Akad. Nauk SSSR. Prikl. Mat. Meh.* **16** (1952), 67–78.

[8] G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge Texts Appl. Math. 14, Cambridge University, Cambridge, 1996.

[9] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Paper No. 62.

[10] M. Belloni and B. Kawohl, The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$, *ESAIM Control Optim. Calc. Var.* **10** (2004), no. 1, 28–52.

[11] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **22** (1995), no. 2, 241–273.

[12] P. Bénilan and M. G. Crandall, Regularizing effects of homogeneous evolution equations, in: *Contributions to Analysis and Geometry* (Baltimore 1980), Johns Hopkins University, Baltimore (1981), 23–39.

[13] V. Bobkov and P. Takáč, On maximum and comparison principles for parabolic problems with the p-Laplacian, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), no. 2, 1141–1158.

[14] L. Boccardo, T. Gallouët and P. Marcellini, Anisotropic equations in L^1, *Differential Integral Equations* **9** (1996), no. 1, 209–212.

[15] V. Bögelein, F. Duzaar and P. Marcellini, Parabolic equations with p, q-growth, *J. Math. Pures Appl. (9)* **100** (2013), no. 4, 535–563.

[16] P. Bousquet and L. Brasco, Lipschitz regularity for orthotropic functionals with nonstandard growth conditions, *Rev. Mat. Iberoam.* **36** (2020), no. 7, 1989–2032.

[17] P. Bousquet, L. Brasco, C. Leone and A. Verde, On the Lipschitz character of orthotropic p-harmonic functions, *Calc. Var. Partial Differential Equations* **57** (2018), no. 3, Paper No. 88.

[18] P. Bousquet, L. Brasco, C. Leone and A. Verde, Gradient estimates for an orthotropic nonlinear diffusion equation, preprint (2021), https://cvgmt.sns.it/paper/5120/.

[19] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[20] L. A. Caffarelli, J. L. Vázquez and N. I. Wolanski, Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation, *Indiana Univ. Math. J.* **36** (1987), no. 2, 373–401.

[21] P. Celada, G. Cupini and M. Guidorzi, Existence and regularity of minimizers of nonconvex integrals with $p - q$ growth, *ESAIM Control Optim. Calc. Var.* **13** (2007), no. 2, 343–358.

[22] A. Cianchi, Symmetrization in anisotropic elliptic problems, *Comm. Partial Differential Equations* **32** (2007), no. 4–6, 693–717.

[23] S. Ciani and V. Vespri, An introduction to Barenblatt solutions for anisotropic p-Laplace equations, in: *Anomalies in Partial Differential Equations*, Springer INdAM Ser. 43, Springer, Cham (2021), 99–125.

[24] F. Cipriani and G. Grillo, Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces, *J. Reine Angew. Math.* **562** (2003), 201–235.

[25] F. C. Cîrstea and J. Vétois, Fundamental solutions for anisotropic elliptic equations: Existence and a priori estimates, *Comm. Partial Differential Equations* **40** (2015), no. 4, 727–765.

[26] M. G. Crandall, Nonlinear semigroups and evolution governed by accretive operators, in: *Nonlinear Functional Analysis and its Applications. Part 1* (Berkeley 1983), Proc. Sympos. Pure Math. 45, American Mathematical Society, Providence (1986), 305–337.

[27] M. G. Crandall and T. M. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, *Amer. J. Math.* **93** (1971), 265–298.

[28] M. G. Crandall and A. Pazy, Nonlinear evolution equations in Banach spaces, *Israel J. Math.* **11** (1972), 57–94.

[29] S. P. Degtyarev and A. F. Tedeev, Bounds for solutions of the Cauchy problem for an anisotropic degenerate doubly nonlinear parabolic equation with growing initial data, *Dokl. Math.* **76** (2007), 824–827.

[30] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer, New York, 1993.

[31] F. G. Düzgün, S. Mosconi and V. Vespri, Anisotropic Sobolev embeddings and the speed of propagation for parabolic equations, *J. Evol. Equ.* **19** (2019), no. 3, 845–882.
[32] L. C. Evans, Application of nonlinear semigroup theory to certain partial differential equations, in: *Nonlinear Evolution Equations* (Madison 1977), Publ. Math. Res. Center Univ. Wisconsin 40, Academic Press, New York (1978), 163–188.
[33] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, American Mathematical Society, Providence, 1998.
[34] F. Feo, J. L. Vázquez and B. Volzone, Anisotropic fast diffusion equations, preprint (2020), https://arxiv.org/abs/2007.00122.
[35] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, *Ricerche Mat.* **8** (1959), 24–51.
[36] M. A. Herrero and J. L. Vázquez, Asymptotic behaviour of the solutions of a strongly nonlinear parabolic problem, *Ann. Fac. Sci. Toulouse Math. (5)* **3** (1981), no. 2, 113–127.
[37] S. Kamin and J. L. Vázquez, Fundamental solutions and asymptotic behaviour for the p-Laplacian equation, *Rev. Mat. Iberoam.* **4** (1988), no. 2, 339–354.
[38] S. Kesavan, *Symmetrization & Applications*, Ser. Anal. 3, World Scientific, Hackensack, 2006.
[39] F. Li and H. Zhao, Anisotropic parabolic equations with measure data, *J. Partial Differential Equations* **14** (2001), no. 1, 21–30.
[40] P. Lindqvist, *Notes on the Stationary p-Laplace Equation*, Springer Briefs Math., Springer, Cham, 2019.
[41] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
[42] P. Marcellini, A variational approach to parabolic equations under general and p, q-growth conditions, *Nonlinear Anal.* **194** (2020), Article ID 111456.
[43] J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.* **17** (1964), 101–134.
[44] J. Moser, Correction to: "A Harnack inequality for parabolic differential equations", *Comm. Pure Appl. Math.* **20** (1967), 231–236.
[45] B. Nazaret, Principe de maximum strict pour un opérateur quasi linéaire, *C. R. Math. Acad. Sci. Paris Sér. I* **333** (2001), no. 2, 97–102.
[46] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (3)* **13** (1959), 115–162.
[47] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, New York, 1983.
[48] G. Pisante and A. Verde, Regularity results for non smooth parabolic problems, *Adv. Differential Equations* **13** (2008), no. 3–4, 367–398.
[49] M. M. Porzio, L^∞-regularity for degenerate and singular anisotropic parabolic equations, *Boll. Unione Mat. Ital. A (7)* **11** (1997), no. 3, 697–707.
[50] M. M. Porzio, On decay estimates, *J. Evol. Equ.* **9** (2009), no. 3, 561–591.
[51] P. A. Raviart, Sur la résolution et l'approximation de certaines équations paraboliques non linéaires dégénérées, *Arch. Ration. Mech. Anal.* **25** (1967), 64–80.
[52] P. A. Raviart, Sur la résolution de certaines équations paraboliques non linéaires, *J. Funct. Anal.* **5** (1970), 299–328.
[53] M. Sango, On a doubly degenerate quasilinear anisotropic parabolic equation, *Analysis (Munich)* **23** (2003), no. 3, 249–260.
[54] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surveys Monogr. 49, American Mathematical Society, Providence, 1997.
[55] B. H. Song and H. Y. Jian, Fundamental solution of the anisotropic porous medium equation, *Acta Math. Sin. (Engl. Ser.)* **21** (2005), no. 5, 1183–1190.
[56] G. Talenti, Elliptic equations and rearrangements, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **3** (1976), no. 4, 697–718.
[57] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), no. 1, 126–150.
[58] J. L. Vázquez, Symétrisation pour $u_1 = \Delta\varphi(u)$ et applications, *C. R. Math. Acad. Sci. Paris Sér. I* **295** (1982), no. 2, 71–74.
[59] J. L. Vázquez, Symmetrization in nonlinear parabolic equations, *Portugal. Math.* **41** (1982), no. 1–4, 339–346.
[60] J. L. Vázquez, Symmetrization and mass comparison for degenerate nonlinear parabolic and related elliptic equations, *Adv. Nonlinear Stud.* **5** (2005), no. 1, 87–131.
[61] J. L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*, Oxford Lecture Ser. Math. Appl. 33, Oxford University, Oxford, 2006.
[62] J. L. Vázquez, *The Porous Medium Equation. Mathematical Theory*, Oxford Math. Monogr., The Clarendon Press, Oxford, 2007.
[63] J. L. Vázquez, The mathematical theories of diffusion: Nonlinear and fractional diffusion, in: *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions*, Lecture Notes in Math. 2186, Springer, Cham (2017), 205–278.
[64] J. L. Vázquez, The evolution fractional p-Laplacian equation in $\mathbb{R}^N$ in the sublinear case, preprint (2020), https://arxiv.org/abs/2011.01521; to appear in *Calc. Var. Partial Differential Equations*.
[65] I. M. Vishik, Sur la résolutions des problèmes aux limites pour des équations paraboliques quasi-linéaires d'ordre quelconque, *Mat. Sb.* **59** (1962), 289–325.
[66] M. I. Višik, Quasi-linear strongly elliptic systems of differential equations of divergence form (in Russian), *Trudy Moskov. Mat. Obšč.* **12** (1963), 125–184; translation in *Trans. Moscow. Math. Soc.* **12** (1963) 140–208.