THE SEMI-CLASSICAL LIMIT WITH A DELTA-PRIME POTENTIAL

CLAUDIO CACCIAPUOTI, DAVIDE FERMI, AND ANDREA POSILICANO

ABSTRACT. We consider the quantum evolution $e^{-i\frac{t}{\hbar}H_\beta}\psi_\xi^\hbar$ of a Gaussian coherent state $\psi_\xi^\hbar \in L^2(\mathbb{R})$ localized close to the classical state $\xi \equiv (q, p) \in \mathbb{R}^2$, where H_β denotes a self-adjoint realization of the formal Hamiltonian $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \beta\delta_0'$, with δ_0' the derivative of Dirac's delta distribution at $x = 0$ and β a real parameter. We show that in the semi-classical limit such a quantum evolution can be approximated (w.r.t. the $L^2(\mathbb{R})$-norm, uniformly for any $t \in \mathbb{R}$ away from the collision time) by $e^{\frac{i}{\hbar}A_t}e^{itL_B}\phi_x^\hbar$, where $A_t = \frac{p^2t}{2m}$, $\phi_x^\hbar(\xi) := \psi_\xi^\hbar(x)$ and L_B is a suitable self-adjoint extension of the restriction to $\mathcal{C}_c^\infty(\mathcal{M}_0)$, $\mathcal{M}_0 := \{(q, p) \in \mathbb{R}^2 \mid q \neq 0\}$, of $(-i$ times$)$ the generator of the free classical dynamics. While the operator L_B here utilized is similar to the one appearing in our previous work [2] regarding the semi-classical limit with a delta potential, in the present case the approximation gives a smaller error: it is of order $\hbar^{7/2-\lambda}$, $0 < \lambda < 1/2$, whereas it turns out to be of order $\hbar^{3/2-\lambda}$, $0 < \lambda < 3/2$, for the delta potential. We also provide similar approximation results for both the wave and scattering operators.

1. INTRODUCTION

The close relation between coherent states and semi-classical analysis is well known and it goes back to the early days of Quantum Mechanics, see, e.g., [4] and references therein for a modern mathematical treatment. By Fourier transform, the classical-quantum correspondence is exact in the case of a free particle: defining, for any $\sigma \in \mathbb{C}$, the Gaussian coherent wave packet $\psi_{\sigma,\xi}^\hbar : \mathbb{R} \to \mathbb{C}$ centered at the classical phase space point $\xi \equiv (q, p) \in \mathbb{R}^2$ by

$$(1.1) \qquad \psi_{\sigma,\xi}^\hbar(x) := \frac{1}{(2\pi\hbar)^{1/4}\sqrt{\sigma}} \exp\left(-\frac{1}{4\hbar\sigma_0\sigma}(x-q)^2 + \frac{i}{\hbar}p(x-q)\right)$$

and, for any $x \in \mathbb{R}$, the phase space function $\phi_{\sigma,x}^\hbar : \mathbb{R}^2 \to \mathbb{C}$ by

$$(1.2) \qquad \phi_{\sigma,x}^\hbar(\xi) := \psi_{\sigma,\xi}^\hbar(x),$$

one has the relation

$$(1.3) \qquad \left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^\hbar\right)(x) = e^{\frac{i}{\hbar}A_t}\left(e^{itL_0}\phi_{\sigma_t,x}^\hbar\right)(\xi).$$

Here

$$H_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad H_0 := -\frac{\hbar^2}{2m}\frac{d^2}{dx^2},$$

is the self-adjoint operator for a free quantum particle,

$$\sigma_t = \sigma_0 + \frac{it}{2m\sigma_0}, \qquad A_t = \frac{p^2t}{2m},$$

and e^{itL_0} is the realization in $L^\infty(\mathbb{R}^2)$ of the strongly continuous (in $L^2(\mathbb{R}^2)$) group of evolution generated by the self-adjoint operator

$$L_0 := -\,i\,X_0 \cdot \nabla\,, \qquad X_0(q,p) := \left(\frac{p}{m},0\right),$$

corresponding to the Hamiltonian vector field of a free classical particle, i.e.,

$$e^{itL_0}f(q,p) = f(q + \frac{p}{m}\,t,p)\,.$$

Such an exact quantum-classical correspondence still holds for quadratic Hamiltonians and, for more general regular (at least C^2) potentials, an approximate relation, up to an error of order $\hbar^{1/2-\lambda}$, $0 < \lambda < 1/2$, is valid (see, e.g., [6]). In a previous paper, see [2], we considered the semiclassical limit for a potential which is far from being a regular function, i.e. the case of the Dirac delta distribution. Here we consider a still more singular case, that is we consider the case where the potential is given by the (distributional) derivative δ_0' of Dirac's delta. Similarly to the case with a delta potential, the self-adjoint realization H_β, $\beta \in (\mathbb{R}\backslash\{0\}) \cup \infty$, of the formal Hamiltonian $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \beta\delta_0'$ is described as a self-adjoint extension of the symmetric operator given by the restriction of the free Hamiltonian H_0 to the set $C_c^\infty(\mathbb{R}\backslash\{0\})$ (see, e.g., [1, Ch. I.4] and Section 3 below for more details). This fact, together with the free case relation (1.3), suggest (as in the case examined in [2]) how to proceede: since H_β is a self-adjoint extension of the symmetric operator $H_0^\circ := H_0\!\restriction C_c^\infty(\mathbb{R}\backslash\{0\})$, one could try to approximate $e^{-i\frac{t}{\hbar}H_\beta}\,\psi_{\sigma_0,\xi}^\hbar$ by replacing L_0 with L_B, a suitable self-adjoint extension of $L_0^\circ := L_0\!\restriction C_c^\infty(\mathcal{M}_0)$, $\mathcal{M}_0 := \mathbb{R}^2 \backslash \{(0,p)\,|\,p\in\mathbb{R}\}$, and transforming $\phi_{\sigma_t,x}^\hbar$ using the realization in $L^\infty(\mathbb{R}^2)$, if any, of e^{itL_B}. Following the same reasonings as in [2, Sec. 2], in Section 2 we provide the construction of L_B. In this introduction we content ourselves with giving the corresponding unitary group of evolution: for any $f \in L^2(\mathbb{R}^2)$ one has

$$(1.4) \qquad \left(e^{itL_B}f\right)(q,p) = \left(e^{itL_0}f\right)(q,p) - \frac{\theta(-tqp)\,\theta\left(\frac{|pt|}{m} - |q|\right)}{1 - \mathrm{sgn}(t)\,\frac{2i|p|}{mB(p)}}\left(e^{itL_0}f_{odd}\right)(q,p)\,;$$

here θ denotes the Heaviside function (namely, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$), $B(p) = bp^2$, $b \in (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$, and $f_{odd}(\xi) := f(\xi) - f(-\xi)$. This shows that e^{itL_B} is a group of evolution in $L^\infty(\mathbb{R}^2)$. Notice that the case $b = \infty$ corresponds to complete reflection due to the infinite barrier at the origin, while the case $b \in \mathbb{R}\backslash\{0\}$ allows transmission ($b = 0$ gives the free generator L_0) thus introducing "extra" classical paths going beyond the singularity.

In Subsection 4.1 we prove the following

Theorem 1.1. *Let $B(p) := -\,(2\beta/\hbar^3)\,p^2$. Then, there exists a constant $C > 0$ such that, for any $\eta \in (0,1)$, for any $t \in \mathbb{R}$ and for any $\xi \equiv (q,p) \in \mathbb{R}^2$ with $qp \neq 0$, there holds*

$$\left\|e^{-i\frac{t}{\hbar}H_\beta}\,\psi_{\sigma_0,\xi}^\hbar - e^{\frac{i}{\hbar}A_t}\left(e^{itL_B}\phi_{\sigma_t,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$

$$(1.5) \qquad \leq C\left[\frac{\eta}{1-\eta}\left(\frac{\hbar^3}{m|\beta p|}\right) + e^{-\eta^2\frac{\sigma_0^2 p^2}{2\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}}\right.$$

$$\left. + \left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4}e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right) + e^{-\frac{q_t^2}{4\hbar|\sigma_t|^2}}\right].$$

Thus, whenever t is not too close to the collision time $t_{coll}(\xi) := -\frac{mq}{p}$, $\xi \equiv (q,p)$ (look at the last term in the above estimate), our approximation provides the following result (see Subsection 4.2):

Corollary 1.2. *Let* $B(p) := -(2\beta/\hbar^3)\, p^2$. *Then, for any* $0 < \lambda < 1/2$ *there exit constants* $0 < h_* < 1$ *and* $C_*, c_0 > 0$ *such that*

$$\underline{h} := \max\left\{\frac{\hbar\sigma_0^2}{q^2}, \frac{\hbar}{\sigma_0^2 p^2}, \frac{\hbar}{(m|\beta p|)^{1/3}}\right\} < h_*$$

implies

$$\left\|e^{-i\frac{t}{\hbar}H_\beta}\,\psi_{\sigma_0,\xi}^\hbar - e^{\frac{i}{\hbar}A_t}\left(e^{itL_B}\phi_{\sigma_t,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C_*\,\underline{h}^{\frac{7}{2}-\lambda},$$

for any $t \in \mathbb{R}$, $\xi \equiv (q,p) \in \mathbb{R}^2$ *with* $qp \neq 0$, *such that*

$$(1.6) \qquad \left|t - t_{coll}(\xi)\right| \geq c_0 \left|t_{coll}(\xi)\right| \sqrt{\left(\frac{7}{2} - \lambda\right)\underline{h}\,|\ln \underline{h}|}\,.$$

Moreover, the constraint $t \neq t_{coll}$ does not affect the semi-classical approximation for large times. Indeed, see Theorem 1.3 below, we can handle the approximation of the wave operators: denoting with $\Omega_\beta^\pm$ the wave operators defined, as usual, by the limits in $L^2(\mathbb{R}^2)$

$$\Omega_\beta^\pm f := \lim_{t \to \pm\infty} e^{i\frac{t}{\hbar}H_\beta}\, e^{-i\frac{t}{\hbar}H_0} f$$

and by $W_B^\pm$ the corresponding classical objects (compare with [11, Def. 3.4.4], see also [2, Rem. 2.7])

$$W_B^\pm f := \lim_{t \to \pm\infty} e^{itL_0}\, e^{-itL_B} f$$

(here the limits hold both pointwise in $\mathbb{R}^2$ and, if $f = \psi_{\sigma,\xi}^\hbar$ is a coherent state of the form (1.1), in $L^2(\mathbb{R}, dx)$, see Proposition 2.6 below), one has the following (see Subsection 5.1 for the proof)

Theorem 1.3. *Let* $B(p) := -(2\beta/\hbar^3)\, p^2$. *Then, for any* $\psi_{\sigma_0,\xi}^\hbar \in L^2(\mathbb{R})$ *of the form* (4.4) *with* $qp \neq 0$ *and for any* $\eta \in (0,1)$, *there exists a constant* $C > 0$ *such that*

$$(1.7) \quad \left\|\Omega_\beta^\pm\,\psi_{\sigma_0,\xi}^\hbar - \left(W_B^\pm\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C\left[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + e^{-\eta^2\frac{\sigma_0^2 p^2}{2\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}}\right].$$

Similarly, for the scattering operators $S_\beta := (\Omega_\beta^+)^*\Omega_\beta^-$ *and* $S_B^{cl} := (W_B^+)^* W_B^-$ *there holds*

$$(1.8) \quad \left\|S_\beta\,\psi_{\sigma_0,\xi}^\hbar - \left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C\left[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + e^{-\eta^2\frac{\sigma_0^2 p^2}{2\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}}\right.$$

$$\left. + \left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right)\right].$$

Corollary 1.4. *For any* $0 < \lambda < 1/2$ *there exit constants* $0 < h_* < 1$ *and* $C_* > 0$ *such that*

$$\left\|\Omega_\beta^\pm\,\psi_{\sigma_0,\xi}^\hbar - \left(W_B^\pm\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C_*\,\underline{h}^{7/2-\lambda},$$

$$\left\|S_\beta\,\psi_{\sigma_0,\xi}^\hbar - \left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C_*\,\underline{h}^{7/2-\lambda}.$$

Remark 1.5. Theorems 1.1 and 1.3 (and the relative Corollaries) parallel the analogous ones in [2] (see Theorems 1.1 and 1.3 therein) which provide semi-classical approximations for the quantum evolutions, wave and scattering operators for the operator H_α providing a self-adjoint realization of the formal Hamiltonian $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \alpha\,\delta_0$ (see, e.g., [1, Ch. I.3]). The classical approximating

self-adjoint operator L_β used there (see [2, Sec. 2]) is not too much different form the operator L_B used here: the group of evolution generated by L_β is given by (compare with (1.4))

$$\left(e^{itL_\beta}f\right)(q,p) = \left(e^{itL_0}f\right)(q,p) - \frac{\theta(-tqp)\,\theta\left(\frac{|pt|}{m} - |q|\right)}{1 - \mathrm{sgn}(t)\,\frac{2i|p|}{m\beta}}\left(e^{itL_0}f_{ev}\right)(q,p)\,,$$

where (see [2, Prop. 2.4]) $f_{ev}(\xi) := f(\xi) + f(-\xi)$. However, the mentioned results in [2] give an error of different order: $\underline{\underline{h}}^{3/2-\lambda}$ for $0 < \lambda < 3/2$, where

$$\underline{\underline{h}} := \max\left\{\frac{\hbar\sigma_0^2}{q^2}\,,\,\frac{\hbar}{\sigma_0^2 p^2}\,,\,\frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}}\right\}\,.$$

By techniques similar to the ones used here and in [2], analogous semiclassical estimates can also be obtained for the case of a quantum evolution on graphs, see [3].

2. Singular perturbations of the free classical dynamics

By the same kind of reasonings as in [2, Sec. 2], in this section we introduce a suitable self-adjoint extension of the restriction to functions vanishing on the line $\{(q,p) \in \mathbb{R}^2 : q = 0\}$ of the self-adjoint operator $-i\frac{p}{m}\partial_q$. At variance with the self-adjoint operator $L_\beta \equiv L_{\Pi,\beta}$ provided in [2], the operator $L_B \equiv L_{\Pi',B}$ here defined corresponds to different choices of both the extension parameters: in [2] Π is the projector onto the subspace of (p-dependent) even functions and the operator β identifies with the multiplication by the constant $\beta \in (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$, while here we use the projection Π' onto the subspace of odd functions and the operator B identifies with the multiplication by the function $B(p) := bp^2$, $b \in (\mathbb{R}\backslash\{0\})\cup\{\infty\}$. Notwithstanding such differences, the proofs of the results presented in this section follow almost verbatim the ones of the corresponding results in [2, Sec. 2] and therefore are not reproduced here.

Let $X_0(q,p) = (p/m, 0)$ be the Hamiltonian vector field of a classical free particle in $\mathbb{R}$ and let

$$L_0 : \mathrm{dom}(L_0) \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\,, \qquad L_0 f = -i\,X_0\cdot\nabla f\,,$$

defined on the maximal domain $\mathrm{dom}(L_0) := \{f \in L^2(\mathbb{R}^2) \mid X_0\cdot\nabla f \in L^2(\mathbb{R}^2)\}$, be the corresponding self-adjoint operator in $L^2(\mathbb{R}^2)$; one has $\sigma(L_0) = \sigma_{ac}(L_0) = \mathbb{R}$.

The linear map $(\gamma f)(p) := f(0,p)$ extends to a bounded operator $\gamma : \mathrm{dom}(L_0) \to L^2(\mathbb{R}, |p|\,dp)$ (here $\mathrm{dom}(L_0) \subset L^2(\mathbb{R})$ is endowed with the graph norm) such that $\ker(\gamma)$ is dense in $L^2(\mathbb{R}^2)$ (see [2, Lem. 2.1]).

Denoting by $R_z^0 := (L_0 - z)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$ the resolvent of L_0, one gets

$$(R_z^0 f)(q,p) = \int_\mathbb{R} dq'\, g_z(q - q', p)\, f(q', p)\,,$$

where

$$g_z(q,p) = \theta(q\,p\,\mathrm{Im}\,z)\,\mathrm{sgn}(\mathrm{Im}\,z)\,\frac{i\,m}{|p|}\,e^{imzq/p}$$

(recall that θ indicates the Heaviside step function). For any $z \in \mathbb{C}\backslash\mathbb{R}$, we define the bounded linear map

$$G_z : L^2(\mathbb{R}, |p|^{-1}dp) \to L^2(\mathbb{R}^2)\,, \qquad (G_z\phi)(q,p) := ((\gamma R_z^0)^*\phi)(q,p) \equiv g_z(q,p)\,\phi(p)\,;$$

here $L^2(\mathbb{R}, |p|^{-1}dp)$ and $L^2(\mathbb{R}, |p|dp)$ are considered as a dual couple with respect to the duality induced by the scalar product in $L^2(\mathbb{R})$.

For any $b \in (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$ we define the function

$$(2.1) \qquad B : \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \qquad B(p) := \begin{cases} bp^2 & b \in \mathbb{R}\backslash\{0\} \\ \infty & b = \infty \end{cases}$$

and then, for any $z \in \mathbb{C}\backslash\mathbb{R}$, we define the bounded linear map

$$\Lambda_z^B : L^2(\mathbb{R}, |p|dp) \to L^2(\mathbb{R}, |p|^{-1}dp), \quad (\Lambda_z^B \phi)(p) := \frac{\phi(p)}{\frac{1}{B(p)} - \mathrm{sgn}(\mathrm{Im}\,z)\frac{\mathrm{i}\,m}{2|p|}}$$

(here we set $\frac{1}{\infty} := 0$). Finally, we introduce the projector on odd functions (here either $\rho(p) = |p|$ or $\rho(p) = |p|^{-1}$)

$$\Pi' : L^2(\mathbb{R}, \rho dp) \to L^2(\mathbb{R}, \rho dp), \qquad (\Pi'f)(p) := \frac{1}{2}\left(f(p) - f(-p)\right)$$

and notice that

$$\Pi' \Lambda_z^B = \Lambda_z^B \Pi'.$$

Then, by [10, Thm 2.1] here employed with $\tau := \Pi'\gamma$, we obtain the following (compare with [2, Thm. 2.2])

Theorem 2.1. *For any* $b \in (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$, *and* B *defined as in* (2.1), *the linear bounded operator*

$$(2.2) \qquad R_z^B := R_z^0 + G_z \Pi' \Lambda_z^B \Pi' G_{\bar{z}}^* \quad \text{with } z \in \mathbb{C}\backslash\mathbb{R},$$

is the resolvent of a self-adjoint extension L_B *of the densely defined, closed symmetric operator* $L_0 \upharpoonright \ker(\gamma)$. *It acts on its domain*

$$\mathrm{dom}(L_B) := \left\{ f \in L^2(\mathbb{R}^2) \mid f = f_z + G_z \Lambda_z^B \Pi' \gamma f_z, \ f_z \in \mathrm{dom}(L_0) \right\},$$

by

$$(L_B - z)f = (L_0 - z)f_z.$$

Remark 2.2. Notice that the functions $f = f_z + G_z\phi$, $\phi \in \mathrm{ran}(\Pi')$, belonging to $\mathrm{dom}(L_B)$ fulfill the boundary condition

$$\phi(p) = B(p)(\Pi'\widehat{\gamma}f)(p).$$

where $\widehat{\gamma}$ is the extension of the trace map γ defined as

$$(\widehat{\gamma}f)(p) := \frac{1}{2}\left(f(0_+, p) + f(0_-, p)\right).$$

Moreover, on account of the basic identity $(-\mathrm{i}\,X_0 \cdot \nabla - z)G_z\phi = \phi\,\delta_{\Sigma_0}$, where $\phi\delta_{\Sigma_0}$ is the distribution supported on the line $\Sigma_0 = \{(q, p) \in \mathbb{R}^2 \mid q = 0\}$ defined by

$$\left(\phi\,\delta_{\Sigma_0}\right)(\varphi) := \int_{\Sigma_0} dp\,\phi(p)\,\varphi(0, p), \quad \text{for any } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2),$$

from Theorem 2.1 one can readily infer that

$$L_B f = -\mathrm{i}\,X_0 \cdot \nabla f - \phi\,\delta_{\Sigma_0} \equiv -\mathrm{i}\,X_0 \cdot \nabla f - (B\Pi'\widehat{\gamma}f)\,\delta_{\Sigma_0}.$$

By functional calculus and by (2.2), the action of the unitary group $e^{-\mathrm{i}tL_B}$ ($t \in \mathbb{R}$) describing the dynamics induced by L_B can be explicitly characterized (the proof coincides with the one for [2, Prop. 2.4] by noticing that all the integrals appearing there regard the q-variable only):

Proposition 2.3. *Let L_B be as in Theorem 2.1 and $f \in L^2(\mathbb{R}^2)$. Then*

$$(2.3) \qquad \left(e^{-itL_B}f\right)(q, p) = \left(e^{-itL_0}f\right)(q, p) - \frac{\theta(t\, q\, p)\, \theta\left(\frac{|p\, t|}{m} - |q|\right)}{1 + \text{sgn}(t)\frac{2i\,|p|}{mB(p)}} \left(e^{-itL_0}f_{\text{odd}}\right)(q, p),$$

where $f_{\text{odd}}(q, p) := f(q, p) - f(-q, -p)$ and e^{-itL_0} denotes the free unitary group

$$\left(e^{-itL_0}f\right)(q, p) = f\left(q - \frac{pt}{m}, p\right).$$

Remark 2.4. Formula (2.3) shows that e^{-itL_B} defines a group of evolution in $L^\infty(\mathbb{R}^2)$.

Remark 2.5. Notice that while the free operator e^{-itL_0} maps real-valued functions into real-valued functions, the same is not true for e^{-itL_B}, unless $b = \infty$, which corresponds to a complete reflection. In this particular case, Eq. (2.3) reduces to

$$\left(e^{-itL_B}f\right)(q, p)$$
$$\overset{(b\,=\,\infty)}{=} \left[1 - \theta(t\, q\, p)\, \theta\left(\frac{|p\, t|}{m} - |q|\right)\right] f\left(q - \frac{pt}{m}, p\right) + \theta(t\, q\, p)\, \theta\left(\frac{|p\, t|}{m} - |q|\right) f\left(-q + \frac{pt}{m}, -p\right).$$

Defining the classical wave operators by

$$(2.4) \qquad\qquad W_B^\pm f := \lim_{t \to \pm\infty} e^{itL_0} e^{-itL_B}f,$$

one then has the following (compare with [2, Prop. 2.8])

Proposition 2.6. *The limits in (2.4) exist pointwise for any $\xi \equiv (q, p) \in \mathbb{R}^2$ with $qp \neq 0$ and in $L^2(\mathbb{R}^2)$ for any $f \in L^2(\mathbb{R}^2)$:*

$$(2.5) \qquad\qquad \left(W_B^\pm f\right)(q, p) = f(q, p) - \frac{\theta(\mp qp)}{1 \pm \frac{2i\,|p|}{mB(p)}} f_{\text{odd}}(q, p).$$

Furthermore, the classical scattering operator $S_B^{\text{cl}} := (W_B^+)^* W_B^-$ is given by

$$(2.6) \qquad\qquad \left(S_B^{\text{cl}}f\right)(q, p) = f(q, p) - \frac{f_{\text{odd}}(q, p)}{1 - \frac{2i\,|p|}{mB(p)}}.$$

Remark 2.7. On account of Eq. (2.5), it is easy to check that

$$W_B^+ W_B^- f = W_B^- W_B^+ f.$$

Moreover, from the identity

$$\frac{1}{1 + \frac{2i\,|p|}{mB(p)}} + \frac{1}{1 - \frac{2i\,|p|}{mB(p)}} = \frac{2}{\left|1 + \frac{2i\,|p|}{mB(p)}\right|^2}$$

and a straightforward calculation it follows that

$$W_B^\pm (W_B^\pm)^* f = (W_B^\pm)^* W_B^\pm f = f.$$

Hence, in particular, $S_B^{\text{cl}} W_B^+ f = W_B^- f$.

Remark 2.8. By arguments similar to those used in the proof of Proposition 2.6, one gets that the limits

$$\breve{W}_B^\pm f := \lim_{t \to \pm\infty} e^{itL_B} e^{-itL_0}f$$

exist in $L^2(\mathbb{R}^2)$ for any $f \in L^2(\mathbb{R}^2)$ and

$$\left(\check{W}_B^\pm f\right)(p, q) = f(q, p) - \frac{\theta(\mp q\,p)}{1 \mp \frac{2i\,|p|}{mB(p)}}\, f_{odd}(q, p)\,.$$

Therefore, by [7, Ch. X, Thm. 3.5], both $W_B^\pm$ and $\check{W}_B^\pm$ are complete, and the absolutely continuous part of L_B is unitarily equivalent to the absolutely continuous part of L_0, i.e., to L_0 itself; thus

$$\sigma_{ac}(L_B) = \sigma_{ac}(L_0) = \mathbb{R}$$

and L_B is unitarily equivalent to L_0.

3. The quantum Hamiltonian with a delta-prime potential

Here we recall the definition and main properties of the operator H_β, $\beta \in \mathbb{R} \cup \infty$, defined as a self-adjoint extension of the symmetric operator given by the restriction of the free Hamiltonian

$$H_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})\,, \quad H_0 := -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,,$$

to the set $\{\psi \in H^2(\mathbb{R}) : \psi(0) = 0\}$, where $H^2(\mathbb{R})$ denotes the usual Sobolev space of order two, namely $H^2(\mathbb{R}) := \{\psi \in L^2(\mathbb{R}) \,|\, \psi'' \in L^2(\mathbb{R})\}$. In more detail, one has (see [1, Thms. 4.2 and 4.3])

$$\mathrm{dom}(H_\beta) := \left\{\psi \in H^2(\mathbb{R}\backslash\{0\}) \,\middle|\, \psi'(0^+) = \psi'(0^-) = \psi'(0),\ \psi(0^+) - \psi(0^-) = \frac{2m\beta}{\hbar^2}\,\psi'(0)\right\},$$

$$H_\beta\psi = -\frac{\hbar^2}{2m}\,\psi'' + \beta\,\psi'(0)\,\delta_0'\,.$$

Moreover,

$$\sigma_{ac}(H_\beta) = [0, +\infty)\,, \qquad \sigma_{sc}(H_\beta) = \varnothing\,,$$

$$\sigma_p(H_\beta) = \begin{cases} \varnothing & \text{if } \beta \geqslant 0 \text{ or } \beta = \infty\,, \\ \left\{\lambda_\beta \equiv -\dfrac{\hbar^6}{2m^3\,\beta^2}\right\} & \text{if } \beta < 0\,. \end{cases}$$

The normalized eigenfunction associated to the negative eigenvalue for $\beta < 0$ reads

$$\varphi_\beta(x) = \frac{\hbar}{\sqrt{m\,|\beta|}}\,\mathrm{sgn}(x)\,e^{-\frac{\hbar^2}{m\,|\beta|}\,|x|}\,.$$

Remark 3.1. The possible eigenvalue approaches the absolutely continuous spectrum from below in the semiclassical limit, *i.e.*, $\lambda_\beta \to 0^-$ for $\hbar \to 0^+$; correspondingly, the associated eigenfunction vanishes almost everywhere, namely $\|\varphi_\beta\|_{L^\infty(\mathbb{R})} \to 0$. This marks a noteworthy difference with respect to the case of a delta potential discussed in [2], where the possible eigenvalue moves away from the absolutely continuous part of the spectrum and the associated eigenfunction becomes sharply peaked at one point for $\hbar \to 0^+$. As a consequence, many of the arguments employed in [2] cannot be implemented in the present setting.

A complete set of generalized eigenfunctions associated to the absolutely continuous part of the spectrum is given by (compare with [1, Eq. (4.23)])

$$(3.1) \qquad \varphi_k^\pm(x) := \frac{e^{ikx}}{\sqrt{2\pi}} + R_\pm(k)\,\mathrm{sgn}(x)\,\frac{e^{\mp i|k|\,|x|}}{\sqrt{2\pi}} \qquad (k \in \mathbb{R}\backslash\{0\})\,,$$

$$(3.2) \qquad R_\pm(k) := \frac{\frac{im\beta k}{\hbar^2}}{1 \pm \frac{im\beta\,|k|}{\hbar^2}} = \pm\frac{k}{|k| \mp \frac{i\hbar^2}{m\beta}}\,.$$

Notice that

$$(3.3) \qquad R_+(k) - R_-(k) = 2 \, \text{sgn}(k) \, |R_+(k)|^2 \,.$$

For any $\beta \in \mathbb{R}$, taking into account the above spectral decomposition of H_β, let us consider the bounded operators

$$\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \,, \qquad (\mathscr{F}\psi)(k) := \int_{\mathbb{R}} dx \, \frac{e^{-ikx}}{\sqrt{2\pi}} \, \psi(x) \,,$$

$$(3.4) \qquad \mathscr{F}_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \,, \qquad (\mathscr{F}_\pm \psi)(k) := \int_{\mathbb{R}} dx \, \overline{\varphi_k^\pm(x)} \, \psi(x) \,.$$

Correspondingly, we introduce the orthogonal projectors

$$(3.5) \qquad P_{ac} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \,, \qquad (P_{ac}\psi)(x) := \int_{\mathbb{R}} dk \, \varphi_k^+(x) \, (\mathscr{F}_+ \psi)(k) \,,$$

$$(3.6) \qquad P_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \,, \qquad (P_\beta\psi)(x) := \theta(-\beta) \, \varphi_\beta(x) \int_{\mathbb{R}} dy \, \varphi_\beta(y) \, \psi(y) \,;$$

these are such that

$$(3.7) \qquad P_{ac} + P_\beta = 1 \,.$$

Eqs. (3.5) and (3.6) reduce to

$$P_{ac} = 1 \,, \quad P_\beta = 0 \qquad \text{for } \beta \geqslant 0 \,.$$

For any $\beta \in \mathbb{R}$ the time evolution of any state $\psi \in L^2(\mathbb{R})$ induced by the unitary group $e^{-i\frac{t}{\hbar}H_\beta}$ can be characterized as

$$(3.8) \qquad \left(e^{-i\frac{t}{\hbar}H_\beta} \psi\right)(x) = \int_{\mathbb{R}} dk \, e^{-i\frac{t}{\hbar}\frac{\hbar^2 k^2}{2m}} \, \varphi_k^+(x) \, (\mathscr{F}_+ \psi)(k) + e^{-i\frac{t}{\hbar}\lambda_\beta} \, (P_\beta\psi)(x) \,.$$

In the definition of P_{ac} and in Eq. (3.8), one could equivalently use the generalized eigenfunctions φ_k^- and the bounded operator $\mathscr{F}_-$, respectively in place of φ_k^+ and $\mathscr{F}_+$.

Since $(H_\beta - z)^{-1} - (H_0 - z)^{-1}$, $z \in \mathbb{C}\backslash\mathbb{R}$, is a rank-one operator (see [1, Thm. 4.1]) existence and completeness of the wave operators

$$\Omega_\beta^\pm := \underset{t \to \pm\infty}{\text{s-lim}} \, e^{i\frac{t}{\hbar}H_\beta} e^{-i\frac{t}{\hbar}H_0}$$

follows from [7, p. 550, Thm. 4.12]; in particular, $(\Omega_\beta^\pm)^* \Omega_\beta^\pm = P_{ac}$. The corresponding scattering operator is defined, as usual, by

$$S_\beta := (\Omega_\beta^+)^* \Omega_\beta^- \,.$$

Moreover, one has

$$(3.9) \qquad \Omega_\beta^\pm = \mathscr{F}_\pm^* \mathscr{F} \,.$$

Relation (3.9) is well known in the case of perturbations by regular potentials and can also be proved, by essentially the same kind of proof, in the case of a singular perturbation (see the proof of [8, Thm. 5.5]).

4. Convergence of the dynamics

We focus our attention on coherent states of the form

$$(4.1) \qquad \psi^\hbar(x) \equiv \psi^\hbar(\sigma, \breve{\sigma}, q, p; x) = \frac{1}{(2\pi\hbar)^{1/4}\sqrt{\sigma}} \, e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(x-q)^2 + i\frac{p}{\hbar}(x-q)} \qquad (x \in \mathbb{R}),$$

where $(q, p) \in \mathbb{R}^2$ and $\sigma, \breve{\sigma} \in \mathbb{C}$ are such that

$$(4.2) \qquad \operatorname{Re}\sigma > 0, \qquad \operatorname{Re}\breve{\sigma} > 0, \qquad \operatorname{Re}\big[\overline{\sigma}\,\breve{\sigma}\big] = 1.$$

The Fourier transform with respect to x of any state $\psi^\hbar$ of the form (4.1) reads

$$(4.3) \qquad \big(\mathscr{F}\psi^\hbar\big)(k) \equiv \widehat{\psi}^\hbar(\sigma, \breve{\sigma}, q, p; k) = \frac{1}{\sqrt{\breve{\sigma}}}\left(\frac{2\hbar}{\pi}\right)^{1/4} e^{-\frac{\hbar\sigma}{\breve{\sigma}}(k-p/\hbar)^2 - ikq}.$$

From now on we fix $\sigma_0 > 0$, $\breve{\sigma} = \sigma_0^{-1}$ and define, for any $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \sigma_0$, $\xi \equiv (q, p) \in \mathbb{R}^2$, the state $\psi^\hbar_{\sigma,\xi}$ as in Eq. (1.1); notice that $\psi^\hbar_{\sigma,\xi} \equiv \psi^\hbar(\sigma, \sigma_0^{-1}, q, p)$.

In the sequel we analyze the time evolution, generated by the unitary group $e^{-i\frac{t}{\hbar}H_\beta}$ ($t \in \mathbb{R}$), of an initial state of the form

$$(4.4) \qquad \psi^\hbar_{\sigma_0,\xi}(x) = \psi^\hbar(\sigma_0, \sigma_0^{-1}, q, p; x) \qquad \big(\xi = (q, p)\big).$$

Proposition 4.1. *For any $\psi^\hbar \in L^2(\mathbb{R})$ of the form* (4.1) *with* $qp \neq 0$*, there holds*

$$\big(e^{-i\frac{t}{\hbar}H_\beta}\psi^\hbar\big)(x) = \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar\big)(x) + \theta(qp)\operatorname{sgn}(x)\,F^\hbar_{+,t}\big(-\operatorname{sgn}(q)|x|\big) + \theta(-qp)\operatorname{sgn}(x)\,F^\hbar_{-,t}\big(-\operatorname{sgn}(q)|x|\big)$$

$$(4.5) \qquad\qquad + E^\hbar_{1,t}(x) + E^\hbar_{2,t}(x) + E^\hbar_{\beta,t}(x),$$

where we set

$$(4.6) \qquad F^\hbar_{\pm,t}(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2} e^{ikx}\,R_\pm(k)\,\widehat{\psi}^\hbar(k),$$

$$E^\hbar_{1,t}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\Big(\operatorname{sgn}(x)\, e^{-i|k||x|}\,|R_+(k)|^2 - e^{ikx}\,R_-(k)\Big) \times$$

$$(4.7) \qquad\qquad \times \int_{\mathbb{R}} dy\,\big(\operatorname{sgn}(y)\,e^{i|k||y|} - \operatorname{sgn}(q)\,e^{i\operatorname{sgn}(q)|k|y}\big)\,\psi^\hbar(y),$$

$$(4.8) \qquad E^\hbar_{2,t}(x) := \frac{\operatorname{sgn}(qpx)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2} e^{-i\operatorname{sgn}(q)k|x|}\,\theta\big(-\operatorname{sgn}(p)k\big)\big[R_-(k) - R_+(k)\big]\,\widehat{\psi}^\hbar(k),$$

and

$$E^\hbar_{\beta,t}(x) := e^{-i\frac{t}{\hbar}\lambda_\beta}\,(P_\beta\psi^\hbar)(x).$$

Proof. Firstly recall the definitions (3.1) for $\varphi^\pm_k$ and (3.2) for $R_\pm(k)$. Besides, notice that $\overline{R_\pm(k)} = -R_\mp(k)$. Taking as well into account the results of Section 3 (see, in particular, Eq. (3.8)), for any $\psi^\hbar \in L^2(\mathbb{R})$ of the form (4.1) with $qp \neq 0$, we obtain

$$\big(e^{-i\frac{t}{\hbar}H_\beta}\,\psi^\hbar\big)(x) = \big(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\big)(x) - \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{ikx}\,R_-(k)\int_{\mathbb{R}} dy\,\operatorname{sgn}(y)\,e^{i|k||y|}\,\psi^\hbar(y)$$

$$(4.9) \qquad + \frac{\operatorname{sgn}(x)}{2\pi}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i|k||x|}\,R_+(k)\int_{\mathbb{R}} dy\, e^{-iky}\,\psi^\hbar(y)$$

$$+ \frac{\operatorname{sgn}(x)}{2\pi}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i|k||x|}\,|R_+(k)|^2\int_{\mathbb{R}} dy\,\operatorname{sgn}(y)\,e^{i|k||y|}\,\psi^\hbar(y)$$

$$+ e^{-i\frac{t}{\hbar}\lambda_\beta}\,(P_\beta\psi^\hbar)(x).$$

Using the elementary identity

$$\int_{\mathbb{R}} dy \; \mathrm{sgn}(y) \, e^{i|k||y|} \, \psi^{\hbar}(y)$$
$$= \mathrm{sgn}(q) \int_{\mathbb{R}} dy \; e^{i \, \mathrm{sgn}(q)|k|y} \, \psi^{\hbar}(y) + \int_{\mathbb{R}} dy \, \big(\mathrm{sgn}(y) \, e^{i|k||y|} - \mathrm{sgn}(q) \, e^{i \, \mathrm{sgn}(q)|k|y} \big) \, \psi^{\hbar}(y) \,,$$

Eq. (4.9) can be reformulated as follows:

$$\big(e^{-i\frac{t}{\hbar}H_{\beta}} \, \psi^{\hbar} \big)(x) = \big(e^{-i\frac{t}{\hbar}H_{0}} \, \psi^{\hbar} \big)(x) - \frac{\mathrm{sgn}(q)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{ikx} \, R_{-}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big)$$
$$+ \frac{\mathrm{sgn}(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, R_{+}(k) \, \widehat{\psi}^{\hbar}(k)$$
$$+ \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, |R_{+}(k)|^2 \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big) + E_{1,t}^{\hbar}(x) + E_{\beta,t}^{\hbar}(x) \,.$$

Noting that $R_{\pm}(s\,k) = s\,R_{\pm}(k)$ for $s \in \{\pm 1\}$, by elementary changes of the integration variables we obtain the following identities:

$$\frac{\mathrm{sgn}(q)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{ikx} \, R_{-}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big)$$
$$= - \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} e^{-ik|x|} \, R_{-}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big) \,;$$

$$\frac{\mathrm{sgn}(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, R_{+}(k) \, \widehat{\psi}^{\hbar}(k)$$
$$= - \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, R_{+}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)k\big) \,;$$

$$\frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, |R_{+}(k)|^2 \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big)$$
$$= \frac{2\,\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{0}^{\infty} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-ik|x|} \, |R_{+}(k)|^2 \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)k\big) \,.$$

From the above relations we infer

$$\big(e^{-i\frac{t}{\hbar}H_{\beta}} \, \psi^{\hbar} \big)(x) = \big(e^{-i\frac{t}{\hbar}H_{0}} \, \psi^{\hbar} \big)(x) + \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-ik|x|} \, R_{-}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)|k|\big)$$
$$- \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-i|k||x|} \, R_{+}(k) \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)k\big)$$
$$+ \frac{2\,\mathrm{sgn}(qx)}{\sqrt{2\pi}} \int_{0}^{\infty} dk \; e^{-i\frac{\hbar t}{2m}k^2} \, e^{-ik|x|} \, |R_{+}(k)|^2 \, \widehat{\psi}^{\hbar}\big(-\mathrm{sgn}(q)k\big) + E_{1,t}^{\hbar}(x) + E_{\beta,t}^{\hbar}(x) \,.$$

Hence, taking into account the basic Identity (3.3) we obtain

$$
\begin{aligned}
\left(e^{-i\frac{t}{\hbar}H_\beta}\,\psi^\hbar\right)(x) &= \left(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right)(x) + \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_{-\infty}^0 dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-ik|x|}\, R_-(k)\,\widehat{\psi}^\hbar\!\left(-\mathrm{sgn}(q)|k|\right) \\
&\quad - \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_{-\infty}^0 dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i|k||x|}\, R_+(k)\,\widehat{\psi}^\hbar\!\left(-\mathrm{sgn}(q)k\right) + E_{1,t}^\hbar(x) + E_{\beta,t}^\hbar(x) \\
&= \left(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right)(x) - \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_0^\infty dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{ik|x|}\, R_-(k)\,\widehat{\psi}^\hbar\!\left(-\mathrm{sgn}(q)k\right) \\
&\quad + \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_0^\infty dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-ik|x|}\, R_+(k)\,\widehat{\psi}^\hbar\!\left(\mathrm{sgn}(q)k\right) + E_{1,t}^\hbar(x) + E_{\beta,t}^\hbar(x)\,.
\end{aligned}
$$

Recalling once more that $R_\pm(s\,k) = s\,R_\pm(k)$ for $s \in \{\pm1\}$ and using the basic identity

$$
\theta\!\left(\pm\,\mathrm{sgn}(q)k\right) = \theta(\pm q) \mp \mathrm{sgn}(q)\,\theta\!\left(-k\right) = \theta(\pm qp) \mp \mathrm{sgn}(qp)\,\theta\!\left(-\mathrm{sgn}(p)k\right),
$$

by a few elementary manipulations we obtain

$$
\begin{aligned}
\left(e^{-i\frac{t}{\hbar}H_\beta}\,\psi^\hbar\right)(x) &= \left(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right)(x) - \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{ik|x|}\,\theta(k)\, R_-(k)\,\widehat{\psi}^\hbar\!\left(-\mathrm{sgn}(q)k\right) \\
&\quad + \frac{\mathrm{sgn}(qx)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-ik|x|}\,\theta(k)\, R_+(k)\,\widehat{\psi}^\hbar\!\left(\mathrm{sgn}(q)k\right) + E_{1,t}^\hbar(x) + E_{\beta,t}^\hbar(x) \\
&= \left(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right)(x) + \frac{\mathrm{sgn}(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i\,\mathrm{sgn}(q)k|x|}\,\theta\!\left(-\mathrm{sgn}(q)k\right) R_-(k)\,\widehat{\psi}^\hbar(k) \\
&\quad + \frac{\mathrm{sgn}(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i\,\mathrm{sgn}(q)k|x|}\,\theta\!\left(\mathrm{sgn}(q)k\right) R_+(k)\,\widehat{\psi}^\hbar(k) + E_{1,t}^\hbar(x) + E_{\beta,t}^\hbar(x) \\
&= \left(e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right)(x) + \frac{\theta(qp)\,\mathrm{sgn}(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i\,\mathrm{sgn}(q)k|x|}\, R_+(k)\,\widehat{\psi}^\hbar(k) \\
&\quad + \frac{\theta(-qp)\,\mathrm{sgn}(x)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i\,\mathrm{sgn}(q)k|x|}\, R_-(k)\,\widehat{\psi}^\hbar(k) \\
&\quad - \frac{\mathrm{sgn}(qpx)}{\sqrt{2\pi}}\int_{\mathbb{R}} dk\, e^{-i\frac{\hbar t}{2m}k^2}\, e^{-i\,\mathrm{sgn}(q)k|x|}\,\theta\!\left(-\mathrm{sgn}(p)k\right)\left[R_+(k) - R_-(k)\right]\widehat{\psi}^\hbar(k) \\
&\quad + E_{1,t}^\hbar(x) + E_{\beta,t}^\hbar(x)\,.
\end{aligned}
$$

The proof is concluded noting that the latter identity is equivalent to Eq. (4.5). $\qquad\square$

Lemma 4.2. *There exists a constant $C > 0$ such that, for any $\psi^\hbar \in L^2(\mathbb{R})$ of the form (4.1) with $qp \neq 0$, for all $t \in \mathbb{R}$ and for all $\eta \in (0,1)$, there holds*

$$
(4.10) \qquad \left\|F_{\pm,t}^\hbar - R_\pm(p/\hbar)\, e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right\|_{L^2(\mathbb{R})} \leq C\left[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + e^{-\eta^2\frac{p^2}{2\hbar|\check{\sigma}|^2}}\right].
$$

Proof. We essentially retrace the same arguments described in [2, Proof of Lem. 3.3]. Firstly, notice that by unitarity of the Fourier transform we have

$$
\left\|F_{\pm,t}^\hbar - R_\pm(p/\hbar)\, e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} dk\, \left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \left|\widehat{\psi}^\hbar(k)\right|^2\right)^{1/2}.
$$

Recalling the definition of $R_\pm(k)$ given in Eq. (3.2), by explicit computations we obtain

$$\left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 = \frac{\left(\frac{m\beta}{\hbar^2}\right)^2 \left((k - p/\hbar)^2 + \left(\frac{m\beta}{\hbar^2}\right)^2 (k|p|/\hbar - |k|\,p/\hbar)^2\right)}{\left(1 + \left(\frac{m\beta}{\hbar^2}\right)^2 (p/\hbar)^2\right)\left(1 + \left(\frac{m\beta}{\hbar^2}\right)^2 k^2\right)}$$

$$= \frac{\left(\frac{m\beta p}{\hbar^3}\right)^2 \left((\hbar k/p - 1)^2 + (1 - \mathrm{sgn}(k\,p))^2 \left(\frac{m\beta p}{\hbar^3}\right)^2 (\hbar\,k/p)^2\right)}{\left(1 + \left(\frac{m\beta p}{\hbar^3}\right)^2\right)\left(1 + \left(\frac{m\beta p}{\hbar^3}\right)^2 (\hbar k/p)^2\right)}\,.$$

Starting from here, one obtains the following for any $\eta \in (0,1)$:[1]

$$(4.11) \qquad\qquad \left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \le 5\,, \qquad \text{for all } k \in \mathbb{R}\,;$$

$$(4.12) \qquad \left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \le \frac{1}{(1-\eta)^2}\left(\frac{\hbar^3}{m\beta p}\right)^2 (\hbar k/p - 1)^2\,, \qquad \text{for } |\hbar k/p - 1| \le \eta\,.$$

So, let us fix $\eta \in (0,1)$ and note that

$$\left\|F^\hbar_{\pm,t} - R_\pm(p/\hbar)\,e^{-i\frac{t}{\hbar}H_0}\,\psi^\hbar\right\|_{L^2(\mathbb{R})} \le \left(\int_{\{|\hbar k/p - 1|\le\eta\}} dk\,\left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \left|\widehat{\psi}^\hbar(k)\right|^2\right)^{1/2}$$

$$+ \left(\int_{\{|\hbar k/p - 1|\ge\eta\}} dk\,\left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \left|\widehat{\psi}^\hbar(k)\right|^2\right)^{1/2}\,.$$

Taking into account the inequalities (4.11) and (4.12), we infer respectively

$$\int_{\{|\hbar k/p - 1|\ge\eta\}} dk\,\left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \left|\widehat{\psi}^\hbar(k)\right|^2 \le \frac{5}{|\breve{\sigma}|}\sqrt{\frac{2\hbar}{\pi}}\int_{\{|\hbar k/p - 1|\ge\eta\}} dk\,e^{-\frac{2\hbar}{|\breve{\sigma}|^2}(k - p/\hbar)^2}$$

$$\le \frac{5}{|\breve{\sigma}|}\sqrt{\frac{2\hbar}{\pi}}\,e^{-\eta^2\frac{p^2}{\hbar|\breve{\sigma}|^2}}\int_\mathbb{R} dk\,e^{-\frac{\hbar}{|\breve{\sigma}|^2}(k - p/\hbar)^2} = 5\sqrt{2}\,e^{-\eta^2\frac{p^2}{\hbar|\breve{\sigma}|^2}}\,,$$

$$\int_{\{|\hbar k/p - 1|\le\eta\}} dk\,\left|R_\pm(k) - R_\pm(p/\hbar)\right|^2 \left|\widehat{\psi}^\hbar(k)\right|^2 \le \frac{1}{(1-\eta)^2}\left(\frac{\hbar^3}{m\beta p}\right)^2 \int_{\{|\hbar k/p - 1|\le\eta\}} dk\,(\hbar k/p - 1)^2 \left|\widehat{\psi}^\hbar(k)\right|^2$$

$$\le \frac{\eta^2}{(1-\eta)^2}\left(\frac{\hbar^3}{m\beta p}\right)^2 \int_\mathbb{R} dk\,\left|\widehat{\psi}^\hbar(k)\right|^2 = \frac{\eta^2}{(1-\eta)^2}\left(\frac{\hbar^3}{m\beta p}\right)^2\,.$$

Summing up, the above arguments and the basic relation $\sqrt{a^2 + b^2} \le a + b$ for $a, b \ge 0$ yield the bound (4.10). $\qquad\qquad\square$

---

[1] On one hand notice that, for all $b \in \mathbb{R}$, there holds

$$\sup_{\xi\in\mathbb{R}} \frac{b^2\left((\xi-1)^2 + (1-\mathrm{sgn}\,\xi)^2\,b^2\xi^2\right)}{(1+b^2)(1+b^2\xi^2)} \le \frac{b^2}{1+b^2}\sup_{\xi\in\mathbb{R}}\left(\frac{(\xi-1)^2}{1+b^2\xi^2} + \frac{4\,b^2\xi^2}{1+b^2\xi^2}\right) \le 4 + \frac{b^2}{1+b^2}\max_{\xi\in\mathbb{R}}\frac{(\xi-1)^2}{1+b^2\xi^2} = 4 + \frac{b^2(1+b^{-2})}{1+b^2} = 5\,.$$

On the other hand, for any given $b \in \mathbb{R}$, $\eta \in (0,1)$ and for all $|\xi - 1| \le \eta$ (which ensures $\xi > 0$) we have

$$\frac{b^2\left((\xi-1)^2 + (1-\mathrm{sgn}\,\xi)^2\,b^2\xi^2\right)}{(1+b^2)(1+b^2\xi^2)} = \frac{(\xi-1)^2}{(1+1/b^2)(1+b^2\xi^2)} \le \frac{(\xi-1)^2}{b^2\xi^2} \le \frac{(\xi-1)^2}{(1-\eta)^2 b^2}\,.$$

Lemma 4.3. *Let P_{ac} and P_β be defined, respectively, as in Eqs. (3.5) and (3.6). Then, there exists a constant $\mathsf{C} > 0$ such that, for any $\psi^\hbar \in L^2(\mathbb{R})$ of the form (4.1) with $\mathsf{qp} \neq 0$, there holds*

$$\left\| \mathsf{P}_\beta \psi^\hbar \right\|_{L^2(\mathbb{R})} = \left\| \mathsf{P}_{\mathrm{ac}} \psi^\hbar - \psi^\hbar \right\|_{L^2(\mathbb{R})}$$

$$\leq \mathsf{C} \left(\frac{\hbar^5 |\sigma|^2}{m^2 \beta^2} \right)^{1/4} e^{\frac{\hbar^5 |\sigma|^2}{m^2 \beta^2}} \left(\frac{1}{\sqrt{|\sigma \, \breve{\sigma}|}} \, e^{-\frac{p^2}{\hbar |\breve{\sigma}|^2} - \frac{\hbar^2}{m |\beta|} \left(|q| + \frac{2 \, \mathrm{sgn}(qp) \, |p| \, \mathrm{Im}[\breve{\sigma}\overline{\sigma}]}{|\breve{\sigma}|^2} \right) - \frac{\hbar^5 |\sigma|^2}{m^2 \beta^2} \left(1 - \frac{1}{|\sigma \, \breve{\sigma}|^2} \right)} + e^{-\frac{q^2}{4 \hbar |\sigma|^2}} \right).$$

In particular, for any $\psi^\hbar = \psi^\hbar_{\sigma_0, \xi}$ of the form (4.4) with $\mathsf{qp} \neq 0$ there holds

$$(4.13) \qquad \left\| \mathsf{P}_\beta \psi^\hbar \right\|_{L^2(\mathbb{R})} = \left\| \mathsf{P}_{\mathrm{ac}} \psi^\hbar - \psi^\hbar \right\|_{L^2(\mathbb{R})} \leq \mathsf{C} \left(\frac{\hbar^5 \sigma_0^2}{m^2 \beta^2} \right)^{1/4} e^{\frac{\hbar^5 \sigma_0^2}{m^2 \beta^2}} \left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4 \hbar \sigma_0^2}} \right).$$

Proof. Recalling that φ_β is a normalized eigenfunction such that $\|\varphi_\beta\|_{L^2(\mathbb{R})} = 1$, from Eq. (3.6) it readily follows

$$\left\| \mathsf{P}_\beta \psi^\hbar \right\|_{L^2(\mathbb{R})} = \left| \int_{\mathbb{R}} dx \, \varphi_\beta(x) \, \psi^\hbar(x) \right|.$$

By direct computations we obtain

$$\int_{\mathbb{R}} dx \, \varphi_\beta(x) \, \psi^\hbar(x) = \frac{\hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, \sigma}} \int_{\mathbb{R}} dx \, \mathrm{sgn}(x) \, e^{-\frac{\hbar^2}{m |\beta|} |x|} \, e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(x-q)^2 + i\frac{p}{\hbar}(x-q)}$$

$$= \frac{\mathrm{sgn}(q) \, \hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, \sigma}} \int_{\mathbb{R}} dy \, \mathrm{sgn}(y) \, e^{-\frac{\hbar^2}{m |\beta|} |y|} \, e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(y-|q|)^2 + i\frac{\mathrm{sgn}(q) \, p}{\hbar}(y-|q|)}$$

$$= \frac{\mathrm{sgn}(q) \, \hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, \sigma}} \left[\int_{\mathbb{R}} dy \, e^{-\frac{\hbar^2}{m |\beta|} y} \, e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(y-|q|)^2 + i\frac{\mathrm{sgn}(q) \, p}{\hbar}(y-|q|)} \right.$$

$$\left. + \int_{\mathbb{R}} dy \left(\mathrm{sgn}(y) \, e^{-\frac{\hbar^2}{m |\beta|} |y|} - e^{-\frac{\hbar^2}{m |\beta|} y} \right) e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(y-|q|)^2 + i\frac{\mathrm{sgn}(q) \, p}{\hbar}(y-|q|)} \right].$$

On the one hand, keeping in mind our assumptions about the covariance parameters $\sigma, \breve{\sigma}$ and evaluating explicitly the Gaussian integral we get

$$\left| \frac{\mathrm{sgn}(q) \, \hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, \sigma}} \int_{\mathbb{R}} dy \, e^{-\frac{\hbar^2}{m |\beta|} y} \, e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(y-|q|)^2 + i\frac{\mathrm{sgn}(q) \, p}{\hbar}(y-|q|)} \right|$$

$$= \left(\frac{8\pi \, \hbar^5}{m^2 \, \beta^2 \, |\breve{\sigma}|^2} \right)^{1/4} e^{-\frac{p^2}{\hbar |\breve{\sigma}|^2} - \frac{\hbar^2}{m |\beta|} \left(|q| + \frac{2 \, \mathrm{sgn}(qp) \, |p| \, \mathrm{Im}[\breve{\sigma}\overline{\sigma}]}{|\breve{\sigma}|^2} \right) + \frac{\hbar^5}{m^2 \beta^2 |\breve{\sigma}|^2}}.$$

On the other hand we have

$$\left| \frac{\mathrm{sgn}(q) \, \hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, \sigma}} \int_{\mathbb{R}} dy \left(\mathrm{sgn}(y) \, e^{-\frac{\hbar^2}{m |\beta|} |y|} - e^{-\frac{\hbar^2}{m |\beta|} y} \right) e^{-\frac{\breve{\sigma}}{4\hbar\sigma}(y-|q|)^2 + i\frac{\mathrm{sgn}(q) \, p}{\hbar}(y-|q|)} \right|$$

$$\leq \frac{\hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, |\sigma|}} \int_{-\infty}^{0} dy \left(e^{\frac{\hbar^2}{m |\beta|} y} + e^{-\frac{\hbar^2}{m |\beta|} y} \right) e^{-\frac{1}{4\hbar|\sigma|^2}(y-|q|)^2}$$

$$\leq \frac{2 \, \hbar}{(2\pi\hbar)^{1/4} \sqrt{m |\beta| \, |\sigma|}} \, e^{-\frac{q^2}{4\hbar|\sigma|^2}} \int_{\mathbb{R}} dy \, e^{-\frac{1}{4\hbar|\sigma|^2} y^2 + \frac{\hbar^2}{m |\beta|} y} = \left(\frac{128 \, \pi \, \hbar^5 |\sigma|^2}{m^2 \, \beta^2} \right)^{1/4} e^{-\frac{q^2}{4\hbar|\sigma|^2} + \frac{\hbar^5 |\sigma|^2}{m^2 \beta^2}}.$$

The above arguments suffice to infer the thesis. $\qquad\qquad\square$

Remark 4.4. If $p = 0$, recalling the definition of error function and the asymptotic expansions of the latter (see, *e.g.*, [9, Ch. 7]), it can be shown by explicit computations that in the semiclassical limit there holds

$$\left\| P_\beta \psi^\hbar \right\|_{L^2(\mathbb{R})} = \left\| P_{ac} \psi^\hbar - \psi^\hbar \right\|_{L^2(\mathbb{R})} = O\left(\left(\frac{\hbar^5 \sigma_0^2}{m^2 \beta^2} \right)^{1/4} \right).$$

Lemma 4.5. *There exists a constant* $C > 0$ *such that, for any* $\psi^\hbar \in L^2(\mathbb{R})$ *of the form* (4.1) *with* $qp \neq 0$, *and for all* $t \in \mathbb{R}$, *there holds*

$$\left\| E_{1,t}^\hbar \right\|_{L^2(\mathbb{R})} \leq C\, e^{-\frac{q^2}{4\hbar|\sigma|^2}}.$$

Proof. Firstly, let us remark that the definition (4.7) of $E_{1,t}$ can be reformulated as follows, recalling that $R_\pm(s\,k) = s\, R_\pm(k)$ for $s \in \{\pm 1\}$ and using the basic Identity (3.3):

$$E_{1,t}^\hbar(x) = -\frac{\mathrm{sgn}(qx)}{2\pi} \int_\mathbb{R} dk\; e^{-i\frac{\hbar t}{2m} k^2} \left(e^{-i|k||x|}\, |R_+(k)|^2 + e^{-ik|x|}\, R_-(k) \right) \times$$

$$\times \int_\mathbb{R} dy\; \left(\mathrm{sgn}(y)\, e^{i|k||y|} + e^{-i|k|y} \right) \psi^\hbar\!\left(-\mathrm{sgn}(q)y\right)$$

$$= -\frac{\mathrm{sgn}(qx)}{2\pi} \int_0^\infty dk\; e^{-i\frac{\hbar t}{2m} k^2} \left(e^{-ik|x|}\, 2\, |R_+(k)|^2 + (e^{-ik|x|} - e^{ik|x|})\, R_-(k) \right) \times$$

$$\times \int_0^\infty dy\; \left(e^{iky} + e^{-iky} \right) \psi^\hbar\!\left(-\mathrm{sgn}(q)y\right)$$

$$= -\frac{\mathrm{sgn}(qx)}{\pi} \int_0^\infty dk\; e^{-i\frac{\hbar t}{2m} k^2} \left(e^{-ik|x|}\, R_+(k) - e^{ik|x|}\, R_-(k) \right) \int_0^\infty dy\; \cos(ky)\, \psi^\hbar\!\left(-\mathrm{sgn}(q)\, y\right).$$

To proceed, notice that

$$e^{-ik|x|}\, R_+(k) - e^{ik|x|}\, R_-(k) = \frac{k^2\, (e^{ik|x|} + e^{-ik|x|})}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} - \frac{i\,\hbar^2}{m\beta} \frac{k\, (e^{ik|x|} - e^{-ik|x|})}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} \qquad \text{for } k > 0,$$

and that the latter expression is an even function of k, for $k \in \mathbb{R}$. Notice also that the integral w.r.t. y gives an even function of k as well. Thus, by symmetry arguments we obtain

$$E_{1,t}^\hbar(x) = -\frac{\mathrm{sgn}(qx)}{2\pi} \int_\mathbb{R} dk\; e^{-i\frac{\hbar t}{2m} k^2} \left(\frac{k^2\, (e^{ik|x|} + e^{-ik|x|})}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} - \frac{i\,\hbar^2}{m\beta} \frac{k\, (e^{ik|x|} - e^{-ik|x|})}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} \right) \times$$

$$\times \int_0^\infty dy\; \cos(ky)\, \psi^\hbar\!\left(-\mathrm{sgn}(q)\, y\right)$$

$$= -\frac{\mathrm{sgn}(qx)}{\pi} \int_\mathbb{R} dk\; e^{-i\frac{\hbar t}{2m} k^2}\, e^{ik|x|}\, \frac{k^2 - \frac{i\,\hbar^2}{m\beta} k}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} \int_0^\infty dy\; \cos(ky)\, \psi^\hbar\!\left(-\mathrm{sgn}(q)\, y\right)$$

$$= \frac{\mathrm{sgn}(qx)}{\pi} \int_\mathbb{R} dk\; e^{-i\frac{\hbar t}{2m} k^2}\, e^{ik|x|}\, \frac{k - \frac{i\,\hbar^2}{m\beta}}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} \int_0^\infty dy\; \sin(ky)\, \partial_y \psi^\hbar\!\left(-\mathrm{sgn}(q)\, y\right).$$

where the last identity is easily derived integrating by parts w.r.t. y and noting that the boundary terms vanish.

Then, by the elementary inequality $\|\psi(|\cdot|)\|_{L^2(\mathbb{R})}^2 \le 2\,\|\psi\|_{L^2(\mathbb{R})}^2$ and by the unitarity of the Fourier transform it follows that

$$
\|E_{1,t}^{\hbar}\|_{L^2(\mathbb{R})}^2 \le 2 \left\| \sqrt{\frac{2}{\pi}}\, \frac{1}{k + \frac{i\hbar^2}{m\beta}} \int_0^\infty dy\ \sin(ky)\, \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)\,y) \right\|_{L^2(\mathbb{R},dk)}^2
$$

$$
= \frac{2}{\pi} \int_{\mathbb{R}} dk\, \frac{1}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} \int_0^\infty dy \int_0^\infty dy' \Big(\cos\big(k(y-y')\big) - \cos\big(k(y+y')\big) \Big) \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)\,y) \overline{\partial_{y'} \psi^{\hbar}(-\operatorname{sgn}(q)\,y')} .
$$

From here and from the identity (see [5, p. 424, Eq. 3.723.2])

$$
\int_{\mathbb{R}} dk\, \frac{\cos(k\xi)}{k^2 + \left(\frac{\hbar^2}{m\beta}\right)^2} = \frac{\pi\, m\beta}{\hbar^2}\, e^{-\frac{\hbar^2}{m\beta}|\xi|} ,
$$

it follows

(4.14)
$$
\|E_{1,t}^{\hbar}\|_{L^2(\mathbb{R})}^2 \le \mathcal{I}_1^{\hbar} + \mathcal{J}_1^{\hbar} ,
$$

where we put

$$
\mathcal{I}_1^{\hbar} := \frac{2m\beta}{\hbar^2} \int_0^\infty dy \int_0^\infty dy'\, e^{-\frac{\hbar^2}{m\beta}|y-y'|}\, \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)y)\, \overline{\partial_{y'} \psi^{\hbar}(-\operatorname{sgn}(q)y')} ,
$$

$$
\mathcal{J}_1^{\hbar} := -\frac{2m\beta}{\hbar^2} \int_0^\infty dy \int_0^\infty dy'\, e^{-\frac{\hbar^2}{m\beta}(y+y')}\, \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)y)\, \overline{\partial_{y'} \psi^{\hbar}(-\operatorname{sgn}(q)y')} .
$$

Via repeated integration by parts and a few elementary manipulations, the latter definitions can be rephrased as follows:

$$
\mathcal{I}_1^{\hbar} = \frac{2\,m\beta}{\hbar^2} \int_0^\infty dy\, \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)y) \left[e^{-\frac{\hbar^2}{m\beta}y} \int_0^y dy'\, e^{\frac{\hbar^2}{m\beta}y'}\, \overline{\partial_{y'} \psi^{\hbar}(-\operatorname{sgn}(q)y')} \right.
$$

$$
\left. + e^{\frac{\hbar^2}{m\beta}y} \int_y^\infty dy'\, e^{-\frac{\hbar^2}{m\beta}y'}\, \overline{\partial_{y'} \psi^{\hbar}(-\operatorname{sgn}(q)y')} \right]
$$

$$
= -2 \int_0^\infty dy\, \partial_y \psi^{\hbar}(-\operatorname{sgn}(q)y) \left[\frac{m\beta}{\hbar^2}\, e^{-\frac{\hbar^2}{m\beta}y}\, \overline{\psi^{\hbar}(0)} + e^{-\frac{\hbar^2}{m\beta}y} \int_0^y dy'\, e^{\frac{\hbar^2}{m\beta}y'}\, \overline{\psi^{\hbar}(-\operatorname{sgn}(q)y')} \right.
$$

$$
\left. - e^{\frac{\hbar^2}{m\beta}y} \int_y^\infty dy'\, e^{-\frac{\hbar^2}{m\beta}y'}\, \overline{\psi^{\hbar}(-\operatorname{sgn}(q)y')} \right]
$$

$$
= \frac{2m\beta}{\hbar^2} |\psi^{\hbar}(0)|^2 + 4 \int_0^\infty dy\, |\psi^{\hbar}(-\operatorname{sgn}(q)y)|^2
$$

$$
- 2 \int_0^\infty dy\, e^{-\frac{\hbar^2}{m\beta}y} \left[\overline{\psi^{\hbar}(0)}\, \psi^{\hbar}(-\operatorname{sgn}(q)y) + \psi^{\hbar}(0)\, \overline{\psi^{\hbar}(-\operatorname{sgn}(q)y)} \right]
$$

$$
- \frac{2\hbar^2}{m\beta} \int_0^\infty dy\, \psi^{\hbar}(-\operatorname{sgn}(q)y)\, e^{-\frac{\hbar^2}{m\beta}y} \int_0^y dy'\, e^{\frac{\hbar^2}{m\beta}y'}\, \overline{\psi^{\hbar}(-\operatorname{sgn}(q)y')}
$$

$$
- \frac{2\hbar^2}{m\beta} \int_0^\infty dy\, \psi^{\hbar}(-\operatorname{sgn}(q)y)\, e^{\frac{\hbar^2}{m\beta}y} \int_y^\infty dy'\, e^{-\frac{\hbar^2}{m\beta}y'}\, \overline{\psi^{\hbar}(-\operatorname{sgn}(q)y')} ;
$$

$$\mathcal{J}_1^\hbar = -\frac{2m\beta}{\hbar^2} \left| \int_0^\infty dy \; e^{-\frac{\hbar^2}{m\beta} y} \, \partial_y \psi^\hbar(-\operatorname{sgn}(q)y) \right|^2$$

$$= -\frac{2m\beta}{\hbar^2} \left| -\psi^\hbar(0) + \frac{\hbar^2}{m\beta} \int_0^\infty dy \; e^{-\frac{\hbar^2}{m\beta} y} \psi^\hbar(-\operatorname{sgn}(q)y) \right|^2$$

$$= -\frac{2m\beta}{\hbar^2} |\psi^\hbar(0)|^2 + 2 \int_0^\infty dy \; e^{-\frac{\hbar^2}{m\beta} y} \left(\overline{\psi^\hbar(0)} \, \psi^\hbar(-\operatorname{sgn}(q)y) + \psi^\hbar(0) \, \overline{\psi^\hbar(-\operatorname{sgn}(q)y)} \right)$$

$$- \frac{2\hbar^2}{m\beta} \int_0^\infty dy \int_0^\infty dy' \; e^{-\frac{\hbar^2}{m\beta} (y+y')} \psi^\hbar(-\operatorname{sgn}(q)y) \overline{\psi^\hbar(-\operatorname{sgn}(q)y')} \,.$$

Noting that cancellations occur, from the above relations and from Eq. (4.14) we infer

$$\| E_{1,t}^\hbar \|_{L^2(\mathbb{R})}^2 \leq \mathcal{U}_1^\hbar + \mathcal{V}_1^\hbar + \mathcal{W}_1^\hbar \,,$$

where we put

$$\mathcal{U}_1^\hbar := 4 \int_0^\infty dy \left| \psi^\hbar(-\operatorname{sgn}(q)y) \right|^2 \,,$$

$$\mathcal{V}_1^\hbar := -\frac{2\hbar^2}{m\beta} \int_0^\infty dy \int_0^\infty dy' \; e^{-\frac{\hbar^2}{m\beta} |y-y'|} \psi^\hbar(-\operatorname{sgn}(q)y) \, \overline{\psi^\hbar(-\operatorname{sgn}(q)y')} \,,$$

$$\mathcal{W}_1^\hbar := -\frac{2\hbar^2}{m\beta} \int_0^\infty dy \int_0^\infty dy' \; e^{-\frac{\hbar^2}{m\beta} (y+y')} \psi^\hbar(-\operatorname{sgn}(q)y) \, \overline{\psi^\hbar(-\operatorname{sgn}(q)y')} \,.$$

Now, keeping in mind the basic identity (cf. Eq. (4.1) and the related comments)

$$\left| \psi^\hbar(-\operatorname{sgn}(q)y) \right| = \frac{1}{(2\pi\hbar)^{1/4}\sqrt{|\sigma|}} \; e^{-\frac{(y+|q|)^2}{4\hbar|\sigma|^2}} \,,$$

by arguments similar to those described in the proof of [2, Lem. 3.5], we infer the following inequalities:

$$\left| \mathcal{U}_1^\hbar \right| = \frac{4}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^\infty dy \; e^{-\frac{(y+|q|)^2}{2\hbar|\sigma|^2}} \leq \frac{4\,e^{-\frac{|q|^2}{2\hbar|\sigma|^2}}}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^\infty dy \; e^{-\frac{y^2}{2\hbar|\sigma|^2}} = 2\,e^{-\frac{|q|^2}{2\hbar|\sigma|^2}} \,;$$

$$\left| \mathcal{V}_1^\hbar \right| \leq \frac{2\hbar^2}{m\beta} \int_0^\infty dy \left| \psi^\hbar(-\operatorname{sgn}(q)y) \right| \left[e^{-\frac{\hbar^2}{m\beta} y} \int_0^y dy' \, e^{\frac{\hbar^2}{m\beta} y'} \left| \psi^\hbar(-\operatorname{sgn}(q)y') \right| \right.$$

$$\left. + e^{\frac{\hbar^2}{m\beta} y} \int_y^\infty dy' \, e^{-\frac{\hbar^2}{m\beta} y'} \left| \psi^\hbar(-\operatorname{sgn}(q)y') \right| \right]$$

$$\leq \frac{2\hbar^2}{m\beta} \frac{e^{-\frac{|q|^2}{4\hbar|\sigma|^2}}}{(2\pi\hbar)^{1/4}\sqrt{|\sigma|}} \int_0^\infty dy \left| \psi^\hbar(-\operatorname{sgn}(q)y) \right| \left[e^{-\frac{\hbar^2}{m\beta} y} \int_0^y dy' \, e^{\frac{\hbar^2}{m\beta} y'} + e^{\frac{\hbar^2}{m\beta} y} \int_y^\infty dy' \, e^{-\frac{\hbar^2}{m\beta} y'} \right]$$

$$\leq \frac{4\,e^{-\frac{|q|^2}{4\hbar|\sigma|^2}}}{(2\pi\hbar)^{1/4}\sqrt{|\sigma|}} \int_0^\infty dy \left| \psi^\hbar(-\operatorname{sgn}(q)y) \right| \leq \frac{4\,e^{-\frac{|q|^2}{2\hbar|\sigma|^2}}}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^\infty dy \; e^{-\frac{y^2}{4\hbar|\sigma|^2}} = 2\sqrt{2}\,e^{-\frac{|q|^2}{2\hbar|\sigma|^2}} \,;$$

$$\left|\mathcal{W}_1^\hbar\right| = \frac{2\hbar^2}{m\beta} \left|\int_0^\infty \!\! dy \, e^{-\frac{\hbar^2}{m\beta}y} \, \psi^\hbar\big(-\operatorname{sgn}(q)y\big)\right|^2 \leq \frac{2\hbar^2}{m\beta} \left(\int_0^\infty \!\! dy \, e^{-\frac{2\hbar^2}{m\beta}y}\right) \left(\int_0^\infty \!\! dy \, \big|\psi^\hbar\big(-\operatorname{sgn}(q)y\big)\big|^2\right)$$

$$\leq \frac{e^{-\frac{|q|^2}{2\hbar|\sigma|^2}}}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^\infty \!\! dy \, e^{-\frac{y^2}{2\hbar|\sigma|^2}} = \frac{1}{2} \, e^{-\frac{|q|^2}{2\hbar|\sigma|^2}} \, .$$

Summing up, the above relations imply the thesis. $\qquad\square$

Lemma 4.6. *There exists a constant* $C > 0$ *such that, for any* $\psi^\hbar \in L^2(\mathbb{R})$ *of the form* (4.1) *with* $qp \neq 0$ *and for all* $t \in \mathbb{R}$, *there holds*

$$\big\|E_{2,t}^\hbar\big\|_{L^2(\mathbb{R})} \leq C \, e^{-\frac{p^2}{\hbar|\check\sigma|^2}} \, .$$

Proof. Recalling the definition of $E_{2,t}^\hbar$ (see Eq. (4.8)), by the elementary inequality $\|\psi(\pm|\cdot|)\|_{L^2(\mathbb{R})}^2 \leq 2\|\psi\|_{L^2(\mathbb{R})}^2$ and by unitarity of the Fourier transform, we have

$$\big\|E_{2,t}^\hbar\big\|_{L^2(\mathbb{R})}^2 \leq 2\left\|\frac{1}{\sqrt{2\pi}}\int_\mathbb{R} dk \, e^{-i\frac{\hbar t}{2m}k^2} \, e^{i\operatorname{sgn}(qp)kx} \, \theta(k)\Big[R_-(k) - R_+(k)\Big]\widehat\psi^\hbar\big(-\operatorname{sgn}(p)k\big)\right\|_{L^2(\mathbb{R})}^2$$

$$= 2\int_0^\infty dk \, \big|R_-(k) - R_+(k)\big|^2 \, \big|\widehat\psi^\hbar\big(-\operatorname{sgn}(p)k\big)\big|^2 \, .$$

Moreover, from Eqs. (3.2) and (3.3) it follows

$$\big|R_-(k) - R_+(k)\big|^2 = 4\big|R_+(k)\big|^4 = \frac{4\left(\frac{m\beta k}{\hbar^2}\right)^4}{\left(1 + \left(\frac{m\beta|k|}{\hbar^2}\right)^2\right)^2} \leq 4 \sup_{\xi \in \mathbb{R}} \frac{\xi^4}{(1 + \xi^2)^2} = 4 \, .$$

Thus, taking into account Identity (4.3) for $\widehat\psi^\hbar$, we infer

$$\big\|E_{2,t}^\hbar\big\|_{L^2(\mathbb{R})}^2 \leq \frac{8}{|\check\sigma|}\sqrt{\frac{2\hbar}{\pi}}\int_0^\infty dk \, e^{-\frac{2\hbar(k+|p|/\hbar)^2}{|\check\sigma|^2}} \leq \frac{8}{|\check\sigma|}\sqrt{\frac{2\hbar}{\pi}} \, e^{-\frac{2p^2}{\hbar|\check\sigma|^2}}\int_0^\infty dk \, e^{-\frac{2\hbar k^2}{|\check\sigma|^2}} = 8 \, e^{-\frac{2p^2}{\hbar|\check\sigma|^2}} \, ,$$

which yields the thesis. $\qquad\square$

In the next lemma we collect all the results of the previous lemmata.

Lemma 4.7. *There exists a constant* $C > 0$ *such that for any* $\psi^\hbar \in L^2(\mathbb{R})$ *of the form* (4.1) *with* $qp \neq 0$, *for all* $t \in \mathbb{R}$, *and for all* $\eta \in (0,1)$, *there holds*
(4.15)
$$\big\|e^{-i\frac{t}{\hbar}H_\beta}\psi^\hbar - \Upsilon_t^\hbar\big\|_{L^2(\mathbb{R})}$$

$$\leq C\Bigg[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + e^{-\eta^2\frac{p^2}{2\hbar|\check\sigma|^2}} + e^{-\frac{q^2}{4\hbar|\sigma|^2}} + e^{-\frac{p^2}{\hbar|\check\sigma|^2}}$$

$$+ \left(\frac{\hbar^5|\sigma|^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5|\sigma|^2}{m^2\beta^2}}\left(\frac{1}{\sqrt{|\sigma\,\check\sigma|}} \, e^{-\frac{p^2}{\hbar|\check\sigma|^2} - \frac{\hbar^2}{m|\beta|}\left(|q| + \frac{2\operatorname{sgn}(qp)|p|\operatorname{Im}[\check\sigma\overline\sigma]}{|\check\sigma|^2}\right) - \frac{\hbar^5|\sigma|^2}{m^2\beta^2}\left(1 - \frac{1}{|\sigma\,\check\sigma|^2}\right)} + e^{-\frac{q^2}{4\hbar|\sigma|^2}}\right)\Bigg] \, ,$$

where

$$(4.16) \quad \begin{aligned} \Upsilon_t^\hbar(x) &:= \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar\big)(x) + \theta(qp) \, R_+(p/\hbar) \, \operatorname{sgn}(x) \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar\big)\big(-\operatorname{sgn}(q)|x|\big) \\ &\quad + \theta(-qp) \, R_-(p/\hbar) \, \operatorname{sgn}(x) \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar\big)\big(-\operatorname{sgn}(q)|x|\big) \, . \end{aligned}$$

Proof. The claim (4.15) follows immediately from Eq. (4.5), together with the expansions of the terms $F^\hbar_{\pm,t}$ in Lemma 4.2, and the bounds on the remainders $E^\hbar_{\beta,t}$, $E^\hbar_{1,t}$, $E^\hbar_{2,t}$ in Lemmata 4.3, 4.5, 4.6. $\qquad\square$

Lemma 4.8. *For any $\psi^\hbar \in L^2(\mathbb{R})$ of the form* (4.1) *with* $qp \neq 0$, *there holds*

(4.17)
$$\left\| \mathrm{sgn}(\cdot)\,\psi^\hbar\big(\sigma, \breve{\sigma}, q, p; \mathrm{sgn}(q)\,|\cdot|\big) - \mathrm{sgn}(q)\big(\psi^\hbar(\sigma, \breve{\sigma}, q, p; \cdot) - \psi^\hbar(\sigma, \breve{\sigma}, -q, -p; \cdot)\big) \right\|_{L^2(\mathbb{R})} \leq e^{-\frac{q^2}{4\hbar|\sigma|^2}},$$

(4.18)
$$\left\| \psi^\hbar\big(\sigma, \breve{\sigma}, q, p; -\mathrm{sgn}(q)\,|\cdot|\big) \right\|_{L^2(\mathbb{R})} \leq e^{-\frac{q^2}{4\hbar|\sigma|^2}}.$$

Proof. Taking into account that $\psi^\hbar(\sigma, \breve{\sigma}, -q, -p; x) = \psi^\hbar(\sigma, \breve{\sigma}, q, p; -x) \equiv \psi^\hbar(-x)$, using the elementary identities $\mathrm{sgn}(q)\,|x| = \mathrm{sgn}(qx)\,x$, $1 = \theta(qx) + \theta(-qx)$ and $\mathrm{sgn}(qx) = \theta(qx) - \theta(-qx)$, by direct computations we get

$$\left\| \mathrm{sgn}(\cdot)\,\psi^\hbar\big(\sigma, \breve{\sigma}, q, p; \mathrm{sgn}(q)\,|\cdot|\big) - \mathrm{sgn}(q)\big(\psi^\hbar(\sigma, \breve{\sigma}, q, p; \cdot) - \psi^\hbar(\sigma, \breve{\sigma}, -q, -p; \cdot)\big) \right\|^2_{L^2(\mathbb{R})}$$

$$= \int_{\mathbb{R}} dx \left| \psi^\hbar\big(\sigma, \breve{\sigma}, q, p; \mathrm{sgn}(qx)\,x\big) - \mathrm{sgn}(qx)\big(\psi^\hbar(\sigma, \breve{\sigma}, q, p; x) - \psi^\hbar(\sigma, \breve{\sigma}, q, p; -x)\big) \right|^2$$

$$= \int_{\mathbb{R}} dx \left| \theta(q\,x)\,\psi^\hbar(\sigma, \breve{\sigma}, q, p; -x) + \theta(-q\,x)\,\psi^\hbar(\sigma, \breve{\sigma}, q, p; x) \right|^2$$

$$= 2\int_{\mathbb{R}} dx\,\theta(q\,x)\,\big|\psi^\hbar(\sigma, \breve{\sigma}, q, p; -x)\big|^2 = 2\int_0^{+\infty} dy\,\left| \psi^\hbar\big(\sigma, \breve{\sigma}, q, p; -\mathrm{sgn}(q)\,y\big) \right|^2.$$

From here, noting the identity $\mathrm{Re}(\breve{\sigma}/\sigma) = |\sigma|^{-2}$ (see Eq. (4.2)) and using the inequality $e^{-(a+b)^2} \leq e^{-a^2 - b^2}$ for $a, b \geq 0$, we infer

$$\left\| \mathrm{sgn}(\cdot)\,\psi^\hbar\big(\sigma, \breve{\sigma}, q, p; \mathrm{sgn}(q)\,|\cdot|\big) - \mathrm{sgn}(q)\big(\psi^\hbar(\sigma, \breve{\sigma}, q, p; \cdot) - \psi^\hbar(\sigma, \breve{\sigma}, -q, -p; \cdot)\big) \right\|^2_{L^2(\mathbb{R})}$$

$$= \frac{2}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^{+\infty} dy\,e^{-\frac{(y+|q|)^2}{2\hbar|\sigma|^2}} \leq \frac{2\,e^{-\frac{q^2}{2\hbar|\sigma|^2}}}{\sqrt{2\pi\hbar}\,|\sigma|} \int_0^{+\infty} dy\,e^{-\frac{y^2}{2\hbar|\sigma|^2}} = e^{-\frac{q^2}{2\hbar|\sigma|^2}},$$

which proves Eq. (4.17). Eq. (4.18) can be derived by similar arguments (cf. [2, Lem. 3.8]). $\qquad\square$

4.1. **Proof of Theorem 1.1.** At first, in the following proposition we give an explicit formula for the semiclassical limit evolution of a coherent state.

Proposition 4.9. *Let* $B(p) := -(2\beta/\hbar^3)\,p^2$. *Then, under the assumptions of Theorem 1.1 there holds*

$$e^{\frac{i}{\hbar}A_t}\big(e^{itL_B}\phi^\hbar_{\sigma_t,x}\big)(\xi) = \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)$$

$$- \mathrm{sgn}(q)\,\theta(-qp)\,\theta\left(t + \frac{mq}{p}\right) R_-(p/\hbar)\left(\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x) - \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(-x)\right)$$

$$- \mathrm{sgn}(q)\,\theta(qp)\,\theta\left(-t - \frac{mq}{p}\right) R_+(p/\hbar)\left(\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x) - \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(-x)\right).$$

Proof. Recall that $\xi = (q, p)$. We start by noticing that, by Eq. (2.3),

$$\big(e^{itL_B}\phi^\hbar_{\sigma_t,x}\big)(\xi) = \big(e^{itL_0}\phi^\hbar_{\sigma_t,x}\big)(q, p) - \frac{\theta(-t\,q\,p)\,\theta\big(\frac{|p\,t|}{m} - |q|\big)}{1 - \mathrm{sgn}(t)\,\frac{2i\,|p|}{m\,B(p)}}\left(\big(e^{itL_0}\phi^\hbar_{\sigma_t,x}\big)(q, p) - \big(e^{itL_0}\phi^\hbar_{\sigma_t,x}\big)(-q, -p)\right),$$

hence, on account of Identity (1.3), we infer

$$e^{\frac{i}{\hbar}A_t}\big(e^{itL_B}\phi^\hbar_{\sigma_t,x}\big)(\xi) = \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)$$
$$-\frac{\theta(-t\,q\,p)\,\theta\big(\frac{|p\,t|}{m}-|q|\big)}{1-\mathrm{sgn}(t)\,\frac{2i\,|p|}{m\,B(p)}}\left(\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)-\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,-\xi}\big)(x)\right).$$

We note that

$$\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,-\xi}\big)(x) = e^{\frac{i}{\hbar}A_t}\psi^\hbar\Big(\sigma_0+\frac{it}{2m\sigma_0},\sigma_0^{-1},-q-\frac{pt}{m},-p;x\Big)$$
$$= e^{\frac{i}{\hbar}A_t}\psi^\hbar\Big(\sigma_0+\frac{it}{2m\sigma_0},\sigma_0^{-1},q+\frac{pt}{m},p;-x\Big) = \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(-x),$$

whence,

$$e^{\frac{i}{\hbar}A_t}\big(e^{itL_B}\phi^\hbar_{\sigma_t,x}\big)(\xi) = \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)$$
$$-\frac{\theta(-t\,q\,p)\,\theta\big(\frac{|p\,t|}{m}-|q|\big)}{1-\mathrm{sgn}(t)\,\frac{2i\,|p|}{m\,B(p)}}\left(\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)-\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(-x)\right).$$

To conclude the proof we observe that

$$\frac{\theta(-tqp)\,\theta\big(\frac{|p|t}{m}-|q|\big)}{1-\mathrm{sgn}(t)\,\frac{2i\,|p|}{m\,B(p)}} = \frac{\theta(t)\,\theta(-qp)\,\theta\big(\frac{|p|t}{m}-|q|\big)}{1-\frac{2i\,|p|}{m\,B(p)}}+\frac{\theta(-t)\,\theta(qp)\,\theta\big(-\frac{|p|t}{m}-|q|\big)}{1+\frac{2i\,|p|}{m\,B(p)}}$$
$$=\frac{\theta(-qp)\,\theta\big(t-\frac{m|q|}{|p|}\big)}{1-\frac{2i\,|p|}{m\,B(p)}}+\frac{\theta(qp)\,\theta\big(-t-\frac{m|q|}{|p|}\big)}{1+\frac{2i\,|p|}{m\,B(p)}}$$
$$=\frac{\theta(-qp)\,\theta\big(t+\frac{mq}{p}\big)}{1-\frac{2i\,|p|}{m\,B(p)}}+\frac{\theta(qp)\,\theta\big(-t-\frac{mq}{p}\big)}{1+\frac{2i\,|p|}{m\,B(p)}}\ .$$

Notably, setting $B(p):=-(2\beta/\hbar^3)\,p^2$ and recalling the definition (3.2) of $R_\pm(k)$ we obtain

$$\frac{\theta(\pm qp)}{1\pm\frac{2i|p|}{m\,B(p)}} = \theta(\pm qp)\,\Big(\pm\,\mathrm{sgn}(p)\,R_\pm(p/\hbar)\Big) = \mathrm{sgn}(q)\,\theta(\pm qp)\,R_\pm(p/\hbar)\,.$$

Summing up, the arguments described above imply the thesis. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we use Lemma 4.7 to approximate the state $e^{-i\frac{t}{\hbar}H_\beta}\psi^\hbar_{\sigma_0,\xi}$ with $\Upsilon^\hbar_{\sigma_0,\xi,t}$ defined according to Eq. (4.16) by

$$\Upsilon^\hbar_{\sigma_0,\xi,t}(x) := \big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)(x)+\theta(qp)\,R_+(p/\hbar)\,\mathrm{sgn}(x)\,\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)\big(-\mathrm{sgn}(q)\,|x|\big)$$
$$+\theta(-qp)\,R_-(p/\hbar)\,\mathrm{sgn}(x)\,\big(e^{-i\frac{t}{\hbar}H_0}\psi^\hbar_{\sigma_0,\xi}\big)\big(-\mathrm{sgn}(q)\,|x|\big)\,.$$

Next, we compare $\Upsilon^\hbar_{\sigma_0,\xi,t}$ with the expression for $e^{\frac{i}{\hbar}A_t}\,e^{itL_B}\phi^\hbar_{\sigma_t,x}(\xi)$ from Proposition 4.9. Retracing the arguments described in [2, Proof of Thm. 1.1)], we infer

$$\Upsilon_{\sigma_0,\xi,t}^{\hbar}(x) = \left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)(x) + \theta\left(t+\frac{mq}{p}\right)\theta(qp)\,R_+(p/\hbar)\,\mathrm{sgn}(x)\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(-\mathrm{sgn}(q_t)\,|x|\right)$$

$$+\,\theta\left(t+\frac{mq}{p}\right)\theta(-qp)\,R_-(p/\hbar)\,\mathrm{sgn}(x)\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(\mathrm{sgn}(q_t)\,|x|\right)$$

$$+\,\theta\left(-t-\frac{mq}{p}\right)\theta(qp)\,R_+(p/\hbar)\,\mathrm{sgn}(x)\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(\mathrm{sgn}(q_t)\,|x|\right)$$

$$+\,\theta\left(-t-\frac{mq}{p}\right)\theta(-qp)\,R_-(p/\hbar)\,\mathrm{sgn}(x)\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(-\mathrm{sgn}(q_t)\,|x|\right).$$

Recall that

$$\left(e^{-i\frac{t}{\hbar}H_0}\,\psi_{\sigma_0,\xi}^{\hbar}\right)(x) = e^{\frac{i}{\hbar}A_t}\,\psi^{\hbar}\left(\sigma_t,\sigma_0^{-1},q_t,p;x\right),$$

with $\sigma_t = \sigma_0 + \frac{it}{2m\sigma_0}$ and $q_t = q + \frac{pt}{m}$. Hence, by Lemma 4.8, we deduce:

$$\left\|\mathrm{sgn}(\cdot)\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(\mathrm{sgn}(q_t)|\cdot|\right) - \mathrm{sgn}(q_t)\left(\left(e^{-i\frac{t}{\hbar}H_0}\,\psi_{\sigma_0,\xi}^{\hbar}\right)(x) - \left(e^{-i\frac{t}{\hbar}H_0}\,\psi_{\sigma_0,\xi}^{\hbar}\right)(-x)\right)\right\|_{L^2(\mathbb{R})} \le e^{-\frac{q_t^2}{4\hbar|\sigma_t|^2}};$$

$$\left\|\left(e^{-i\frac{t}{\hbar}H_0}\psi_{\sigma_0,\xi}^{\hbar}\right)\left(-\mathrm{sgn}(q_t)|\cdot|\right)\right\|_{L^2(\mathbb{R})} \le e^{-\frac{q_t^2}{4\hbar|\sigma_t|^2}}.$$

Noting that

$$\theta\left(\pm t\pm\frac{mq}{p}\right)\theta(\mp qp)\,\mathrm{sgn}(q_t) = \pm\theta\left(\pm t\pm\frac{mq}{p}\right)\theta(qp)\,\mathrm{sgn}(p) = -\theta\left(\pm t\pm\frac{mq}{p}\right)\theta(\mp qp)\,\mathrm{sgn}(q),$$

the previous bounds, together with Proposition 4.9, imply

$$\left\|\Upsilon_{\sigma_0,\xi,t}^{\hbar}(x) - e^{\frac{i}{\hbar}A_t}\left(e^{itL_B}\phi_{\sigma_t,(\cdot)}^{\hbar}\right)(\xi)\right\|_{L^2(\mathbb{R})} \le 4\,e^{-\frac{q_t^2}{4\hbar|\sigma_t|^2}}.$$

Besides the exponential $e^{-\frac{q_t^2}{4\hbar|\sigma_t|^2}}$, the remaining terms on the r.h.s. of Eq. (1.5) are a consequence of the fact that we approximated $e^{-i\frac{t}{\hbar}H_\beta}\,\psi_{\sigma_0,\xi}^{\hbar}$ with $\Upsilon_{\sigma_0,\xi,t}^{\hbar}$ and of Lemma 4.7 (here employed with $\sigma = \sigma_0$, $\breve{\sigma} = 1/\sigma_0$, and using as well the basic inequality $e^{-\frac{\hbar^2|q|}{m|\beta|}} \le 1$). $\qquad\square$

4.2. **Proof of Corollary 1.2.**

Proof. Fix $\eta = \underline{h}^{1/2-\lambda}$ in Theorem 1.1. Then, for $0 < \lambda < 1/2$ and fixed $(q,p) \in \mathbb{R}^2$ with $qp \ne 0$, the time-independent part on the r.h.s. of inequality (1.5) is bounded by[2]

$$C\left[\frac{\underline{h}^{7/2-\lambda}}{1-\underline{h}^{1/2-\lambda}} + e^{-\frac{1}{2\underline{h}^{2\lambda}}} + e^{-\frac{1}{4\underline{h}}} + e^{-\frac{1}{\underline{h}}} + \underline{h}^{3/2}\left(e^{-\frac{1}{4\underline{h}}} + e^{-\frac{1}{16\underline{h}}}\right)\right] \le C_*\,\underline{h}^{7/2-\lambda},$$

for some $C_* > 0$ and for all $\underline{h} < h_*$ with h_* small enough.

---

[2]Especially, let us mention that we used the upper bound descending from the following chain of inequalities:

$$\left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right) = \left(\frac{\hbar}{(m|\beta p|)^{1/3}}\right)^{3/2}\left(\frac{\sigma_0^2 p^2}{\hbar}\right)^{1/4} e^{\left(\frac{\hbar}{(m|\beta p|)^{1/3}}\right)^6\frac{\sigma_0^2 p^2}{\hbar}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right)$$

$$\le \underline{h}^{3/2}\left(\frac{\sigma_0^2 p^2}{\hbar}\right)^{1/4}\left[e^{-(1-\underline{h}^6)\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\sigma_0^2 p^2}\left(1-\frac{4\sigma_0^4 p^2}{q^2}\underline{h}^6\right)\frac{\sigma_0^2 p^2}{\hbar}}\right] \le \frac{16}{e}\max\left\{1,\frac{4\sigma_0^4 p^2}{q^2}\right\}\underline{h}^{3/2}\left(e^{-\frac{1}{4\underline{h}}} + e^{-\frac{1}{16\underline{h}}}\right),$$

where the last inequality follows noting that $\min\left\{1-\underline{h}^6, 1-\frac{4\sigma_0^4 p^2}{q^2}\underline{h}^6\right\} \ge 1/2$ for $\underline{h}$ small enough, and that $\sup_{\{\xi>0\}}\left(\xi^{-1/4}e^{-\mu/\xi}\right) \le \left.\xi^{-1/4}e^{-\mu/\xi}\right|_{\xi=\mu} = (4/e)\,\mu^{-1}$ for any $\mu > 0$.

On the other hand, to take into account the time-dependent term on the r.h.s. of inequality (1.5) it is enough to show that if $|t - t_{coll}(\xi)| \geq c_0 \, |t_{coll}(\xi)| \, \sqrt{(7/2 - \lambda) \, \underline{h} \, |\ln \underline{h}|}$ (for some $c_0 > 0$), then $q_t^2/(4\hbar|\sigma_t|^2) \geq (7/2 - \lambda)\,|\ln \underline{h}|$. Setting $y = 1 - t/t_{coll}(\xi)$, $a = \frac{4\hbar\sigma_0^2}{q^2}\,(7/2 - \lambda)\,|\ln \underline{h}|$ and $b = \frac{\hbar}{\sigma_0^2 p^2}\,(7/2 - \lambda)\,|\ln \underline{h}|$, the latter relation can be rephrased as $y^2/(a + b\,(1 - y)^2) \geq 1$; a simple calculation shows that this inequality is fulfilled if

$$(4.19) \qquad\qquad a, b \in (0, 1) \qquad \text{and} \qquad |y| \geq \frac{b + \sqrt{a + b - ab}}{1 - b}.$$

Taking into account that $a + b - ab \leq a + b$, $a \leq 4(7/2 - \lambda)\,\underline{h}\,|\ln \underline{h}|$ and $b \leq (7/2 - \lambda)\,\underline{h}\,|\ln \underline{h}|$, it is easy to convince oneself that when $\underline{h}$ is small enough Eq. (4.19) holds true as soon as $|y| > c_0 \sqrt{(7/2 - \lambda)\,\underline{h}\,|\ln \underline{h}|}$ for some $c_0 > \sqrt{5}$, which proves Eq. (1.6). $\qquad\square$

5. CONVERGENCE OF THE WAVE AND SCATTERING OPERATORS

Lemma 5.1. *For any $\psi^\hbar \in L^2(\mathbb{R})$ of the form (4.1) with $qp \neq 0$, there holds*

$$(5.1)$$
$$\big(\Omega_\beta^\pm \psi^\hbar\big)(x) = \psi^\hbar(x) + \text{sgn}(x)\left[\theta(qp)\,F_{\pm,0}^\hbar\big(\mp\,\text{sgn}(q)\,|x|\big) + \theta(-qp)\,F_{\pm,0}^\hbar\big(\pm\,\text{sgn}(q)\,|x|\big)\right] + E_{3,\pm}^\hbar(x),$$

where $F_{\pm,0}^\hbar \equiv F_{\pm,t=0}^\hbar$, see Eq. (4.6), and

$$E_{3,\pm}^\hbar(x) := \pm\,\frac{\text{sgn}(q\,x)}{\sqrt{2\pi}}\int_\mathbb{R} dk\,\Big(e^{i\,\text{sgn}(qp)\,k\,|x|} - e^{-i\,\text{sgn}(qp)\,k\,|x|}\Big)\,\theta(k)\,R_\pm(k)\,\widehat{\psi}^\hbar\big(-\,\text{sgn}(p)\,k\big).$$

Proof. First notice that, from (3.9), (3.1) and (3.4) it follows

$$\big(\Omega_\beta^\pm \psi^\hbar\big)(x) = \psi^\hbar(x) + \frac{\text{sgn}(x)}{\sqrt{2\pi}}\int_\mathbb{R} dk\, e^{\mp i\,|k|\,|x|}\,R_\pm(k)\,\widehat{\psi}^\hbar(k).$$

From here and from the identities $\theta\big(\pm\,\text{sgn}(q)k\big) + \theta\big(\mp\,\text{sgn}(q)k\big) = 1$ and $\theta\big(\pm\,\text{sgn}(q)k\big) = \theta(\pm qp) \mp \text{sgn}(qp)\,\theta\big(-\,\text{sgn}(p)k\big)$, we obtain

$$\big(\Omega_\beta^\pm \psi^\hbar\big)(x) = \psi^\hbar(x) + \frac{\text{sgn}(x)}{\sqrt{2\pi}}\int_\mathbb{R} dk\, e^{\mp i\,|k|\,|x|}\Big[\theta\big(\pm\,\text{sgn}(q)k\big) + \theta\big(\mp\,\text{sgn}(q)k\big)\Big]\,R_\pm(k)\,\widehat{\psi}^\hbar(k)$$

$$= \psi^\hbar(x) + \frac{\text{sgn}(x)}{\sqrt{2\pi}}\int_\mathbb{R} dk\,\Big[\theta\big(\pm\,\text{sgn}(q)k\big)\,e^{-i\,\text{sgn}(q)k|x|} + \theta\big(\mp\,\text{sgn}(q)k\big)\,e^{i\,\text{sgn}(q)k|x|}\Big]\,R_\pm(k)\,\widehat{\psi}^\hbar(k)$$

$$= \psi^\hbar(x) + \frac{\text{sgn}(x)}{\sqrt{2\pi}}\int_\mathbb{R} dk\,\Big[\theta(\pm qp)\,e^{-i\,\text{sgn}(q)k|x|} + \theta(\mp qp)\,e^{i\,\text{sgn}(q)k|x|}\Big]\,R_\pm(k)\,\widehat{\psi}^\hbar(k)$$

$$\mp\,\frac{\text{sgn}(x)\,\text{sgn}(qp)}{\sqrt{2\pi}}\int_\mathbb{R} dk\,\Big(e^{-i\,\text{sgn}(q)k|x|} - e^{i\,\text{sgn}(q)k|x|}\Big)\,\theta\big(-\,\text{sgn}(p)k\big)\,R_\pm(k)\,\widehat{\psi}^\hbar(k),$$

which, noting that $R_\pm\big(-\,\text{sgn}(p)k\big) = -\,\text{sgn}(p)\,R_\pm(k)$ (see Eq. (3.2)), is equivalent to Eq. (5.1). $\quad\square$

Lemma 5.2. *There exists a constant $C > 0$ such that, for any $\psi^\hbar \in L^2(\mathbb{R})$ of the form (4.1) with $qp \neq 0$, there holds*

$$\big\|E_{3,\pm}^\hbar\big\|_{L^2(\mathbb{R})} \leq C\,e^{-\frac{p^2}{\hbar|\sigma|^2}}.$$

Proof. By the elementary inequality $\|\psi(|\cdot|)\|^2_{L^2(\mathbb{R})} + \|\psi(-|\cdot|)\|^2_{L^2(\mathbb{R})} \le 4\,\|\psi\|^2_{L^2(\mathbb{R})}$, by unitarity of the Fourier transform, and by the basic bound $|R_\pm(k)| \le 1$, we infer that

$$
\big\|E^\hbar_{3,\pm}\big\|^2_{L^2(\mathbb{R})} \le 4 \left\|\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} dk\, e^{-ikx}\,\theta(k)\,R_\pm(k)\,\widehat{\psi}^\hbar\big(-\operatorname{sgn}(p)k\big)\right\|^2_{L^2(\mathbb{R})}
$$

$$
= 4 \int_0^\infty dk\, \big|R_\pm(k)\big|^2 \big|\widehat{\psi}^\hbar\big(-\operatorname{sgn}(p)k\big)\big|^2 \le 4 \int_0^\infty dk\, \big|\widehat{\psi}^\hbar\big(-\operatorname{sgn}(p)k\big)\big|^2
$$

$$
= \frac{4}{|\breve{\sigma}|}\left(\frac{2\hbar}{\pi}\right)^{1/2} \int_0^\infty dk\, e^{-2\frac{\hbar(k+|p|/\hbar)^2}{|\breve{\sigma}|^2}} \le \frac{4}{|\breve{\sigma}|}\left(\frac{2\hbar}{\pi}\right)^{1/2} e^{-\frac{2p^2}{\hbar|\breve{\sigma}|^2}} \int_0^\infty dk\, e^{-\frac{2\hbar k^2}{|\breve{\sigma}|^2}} = 2\, e^{-\frac{2p^2}{\hbar|\breve{\sigma}|^2}}\,,
$$

which yields the thesis. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.1. **Proof of Theorem 1.3.** We first prove claim (1.7).

Preliminarily we apply the classical wave operators $W^\pm_B$, with $B(p) := -(2\beta/\hbar^3)\,p^2$ to the state $\phi^\hbar_{\sigma_0,(\cdot)}(\xi)$, with $\xi = (q,p)$. Recalling the definition (3.2) of $R_\pm(k)$ and noting the basic identity $\mp\operatorname{sgn}(p)\theta(\mp qp) = \operatorname{sgn}(q)\,\theta(\mp qp)$, from Eq. (2.5) we infer:

$$
\big(W^\pm_B \phi^\hbar_{\sigma_0,x}\big)(\xi) = \phi^\hbar_{\sigma,x}(\xi) - \frac{\theta(\mp qp)}{1 \mp \frac{i\hbar^3}{m\beta|p|}}\big(\phi^\hbar_{\sigma,x}(\xi) - \phi^\hbar_{\sigma,x}(-\xi)\big)
$$

$$
= \phi^\hbar_{\sigma,x}(\xi) \mp \operatorname{sgn}(p)\,\theta(\mp qp)\,R_\pm(p/\hbar)\big(\phi^\hbar_{\sigma,x}(\xi) - \phi^\hbar_{\sigma,x}(-\xi)\big)
$$

$$
= \phi^\hbar_{\sigma,x}(q,p) + \operatorname{sgn}(q)\,\theta(\mp qp)\,R_\pm(p/\hbar)\big(\phi^\hbar_{\sigma,x}(\xi) - \phi^\hbar_{\sigma,x}(-\xi)\big)\,.
$$

On the other hand, recalling the Identity (5.1) established in Lemma 5.1, noting the basic inequality $\big|R_\pm(k)\big| \le 1$ and using the estimates reported in Lemmata 4.2, 4.8 and 5.2, we obtain the following for any $\eta \in (0,1)$:

$$
\left\|\Omega^\pm_\beta\,\psi^\hbar_{\sigma_0,\xi} - \Big[\psi^\hbar_{\sigma_0,\xi} + \operatorname{sgn}(q)\,\theta(\mp qp)\,R_\pm(p/\hbar)\big(\psi^\hbar_{\sigma_0,\xi} - \psi^\hbar_{\sigma_0,-\xi}\big)\Big]\right\|_{L^2(\mathbb{R})}
$$

$$
\le C\left[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + \left(1 + \frac{\hbar^3}{m|\beta p|}\right)e^{-\eta^2\frac{\sigma_0^2 p^2}{2\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}}\right].
$$

Then, the proof is concluded recalling that $\psi^\hbar_{\sigma_0,\xi}(x) = \phi^\hbar_{\sigma,x}(\xi)$ (see Eq. (1.2)).

Next we prove claim (1.8). To begin with, we apply the classical scattering operator with $B(p) := -(2\beta/\hbar^3)\,p^2$ to the state $\phi^\hbar_{\sigma_0,(\cdot)}(\xi)$. Recalling the definition (3.2) of $R_\pm(k)$, from Eq. (2.6) we obtain

$$
\big(S^{cl}_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi) = \phi^\hbar_{\sigma_0,(\cdot)}(\xi) + \operatorname{sgn}(p)\,R_-(p/\hbar)\big(\phi^\hbar_{\sigma_0,(\cdot)}(\xi) - \phi^\hbar_{\sigma_0,(\cdot)}(-\xi)\big)
$$

(5.2)
$$
= \psi^\hbar_{\sigma_0,\xi} + \operatorname{sgn}(p)\,R_-(p/\hbar)\big(\psi^\hbar_{\sigma_0,\xi} - \psi^\hbar_{\sigma_0,-\xi}\big)\,.
$$

On the other hand, recalling the basic Identity (3.7), by simple addition and subtraction arguments and by the triangular inequality we get

$$
\big\|(\Omega^+_\beta)^*\Omega^-_\beta\,\psi^\hbar_{\sigma_0,\xi} - \big(S^{cl}_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi)\big\|_{L^2(\mathbb{R})} \le \big\|(\Omega^+_\beta)^*\big(\Omega^-_\beta\,\psi^\hbar_{\sigma_0,\xi} - \big(W^-_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi)\big)\big\|_{L^2(\mathbb{R})}
$$

$$
+ \big\|(\Omega^+_\beta)^* P_{ac}\big(W^-_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi) - P_{ac}\big(S^{cl}_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi)\big\|_{L^2(\mathbb{R})}
$$

$$
+ \big\|(\Omega^+_\beta)^* P_\beta\big(W^-_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi)\big\|_{L^2(\mathbb{R})} + \big\|P_\beta\big(S^{cl}_B \phi^\hbar_{\sigma_0,(\cdot)}\big)(\xi)\big\|_{L^2(\mathbb{R})}\,.
$$

Firstly, from Identity (5.2) and Lemma 4.3 (see, in particular, Eq. (4.13)), noting once more the basic inequality $|R_{\pm}(p/\hbar)| \leq 1$ we infer

$$\left\|P_\beta\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq \left\|P_\beta\psi_{\sigma_0,\xi}^\hbar\right\|_{L^2(\mathbb{R})} + |R_-(p/\hbar)|\left(\left\|P_\beta\psi_{\sigma_0,\xi}^\hbar\right\|_{L^2(\mathbb{R})} + \left\|P_\beta\psi_{\sigma_0,-\xi}^\hbar\right\|_{L^2(\mathbb{R})}\right)$$

$$\text{(5.3)} \qquad\qquad \leq C\left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right).$$

Secondly, let us notice that $\|\Omega_\beta^+\psi\|_{L^2(\mathbb{R})} \leq \|\psi\|_{L^2(\mathbb{R})}$, since Ω_β^+ is the strong limit of operators with unit norm; thus, the same holds true for the adjoint $(\Omega_\beta^+)^*$. Hence, by arguments similar to those described above, in view of Eq. (2.5) and of Lemma 4.3, we have

$$\left\|(\Omega_\beta^+)^* P_\beta\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq \left\|P_\beta\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$

$$\leq \left\|P_\beta\psi_{\sigma_0,\xi}^\hbar\right\|_{L^2(\mathbb{R})} + \theta(qp)\,|R_-(p/\hbar)|\left(\left\|P_\beta\psi_{\sigma_0,\xi}^\hbar\right\|_{L^2(\mathbb{R})} + \left\|P_\beta\psi_{\sigma_0,-\xi}^\hbar\right\|_{L^2(\mathbb{R})}\right)$$

$$\text{(5.4)} \qquad\qquad \leq C\left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right).$$

To say more, again from the bound on $(\Omega_\beta^+)^*$, we infer

$$\left\|(\Omega_\beta^+)^*\left(\Omega_\beta^-\psi_{\sigma_0,\xi}^\hbar - \left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right)\right\|_{L^2(\mathbb{R})} \leq \left\|\Omega_\beta^-\psi_{\sigma_0,\xi}^\hbar - \left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})},$$

which is bounded by Eq. (1.7) (proven previously).
Finally, on account of the unitarity of Ω_β^+ on $\mathrm{ran}(P_{ac})$, we obtain

$$\left\|(\Omega_\beta^+)^* P_{ac}\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi) - P_{ac}\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$

$$= \left\|P_{ac}\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi) - \Omega_\beta^+ P_{ac}\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$

$$\leq \left\|\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi) - \Omega_\beta^+\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} + \left\|P_\beta\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} + \left\|\Omega_\beta^+ P_\beta\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}.$$

On the one hand, using once more arguments analogous to those described in the proof of the bounds (5.3) and (5.4), we get

$$\left\|P_\beta\left(W_B^-\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C\left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right)$$

and

$$\left\|\Omega_\beta^+ P_\beta\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq \left\|P_\beta\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} \leq C\left(\frac{\hbar^5\sigma_0^2}{m^2\beta^2}\right)^{1/4} e^{\frac{\hbar^5\sigma_0^2}{m^2\beta^2}}\left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}}\right).$$

On the other hand, since $\Omega_\beta^+ S_B^{cl} \phi_{\sigma_0,(\cdot)}^\hbar(\xi) = S_B^{cl} \Omega_\beta^+ \phi_{\sigma_0,(\cdot)}^\hbar(\xi)$ (due to the fact that the operators Ω_β^+ and S_B^{cl} act on different variables), on account of Remark 2.7 we infer

$$\left\|\left(W_B^- \phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi) - \Omega_\beta^+\left(S_B^{cl}\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$
$$= \left\|\left(W_B^- \phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi) - \left(S_B^{cl}\Omega_\beta^+\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$
$$= \left\|\left(S_B^{cl}(\Omega_\beta^+ - W_B^+)\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})}$$
$$\leq C\left(\left\|\left((\Omega_\beta^+ - W_B^+)\phi_{\sigma_0,(\cdot)}^\hbar\right)(\xi)\right\|_{L^2(\mathbb{R})} + \left\|\left((\Omega_\beta^+ - W_B^+)\phi_{\sigma_0,(\cdot)}^\hbar\right)(-\xi)\right\|_{L^2(\mathbb{R})}\right)$$
$$\leq C\left[\frac{\eta}{(1-\eta)}\left(\frac{\hbar^3}{m|\beta p|}\right) + \left(1 + \frac{\hbar^3}{m|\beta p|}\right)e^{-\eta^2\frac{\sigma_0^2 p^2}{2\hbar}} + e^{-\frac{q^2}{4\hbar\sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}}\right],$$

where in the last two inequalities we used $|(S_B^{cl}f)(q,p)| \leq C(|f(q,p)| + |f(-q,-p)|)$ (see Eq. (2.6) and note that $\left|1 - \frac{2i|p|}{m\,B(p)}\right|^{-1} \leq 1$) and the bound in Eq. (1.7).

Summing up, the above estimates imply Eq. (1.8).

The proof of Corollary 1.4 is similar to that of Corollary 1.2 and we omit it for brevity.

REFERENCES

[1] S. Albeverio, F. Gesztesy, R. Hōegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics.* Second edition. With an appendix by Pavel Exner. AMS Chelsea Publishing, Providence, RI (2005).

[2] C. Cacciapuoti, D. Fermi, A. Posilicano: The semi-classical limit with a delta potential. *Annali di Matematica Pura e Applicata* (2020). https://doi.org/10.1007/s10231-020-01002-4

[3] C. Cacciapuoti, D. Fermi, A. Posilicano: The semiclassical limit on a star-graph with Kirchhoff conditions. To be published in *Analysis and Mathematical Physics*.

[4] M. Combescure, D. Robert: *Coherent States and Applications in Mathematical Physics.* Springer Netherlands (2012).

[5] I. S. Gradshteyn, I. M. Ryzhik: *Table of integrals, series, and products.* Seventh edition. Elsevier/Academic Press, Amsterdam (2007).

[6] G. A. Hagedorn: Semiclassical Quantum Mechanics I. The $\hbar \to 0$ Limit for Coherent States. *Comm. Math. Phys.* **71** (1980), 77-93.

[7] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin (1976).

[8] A. Mantile, A. Posilicano, M. Sini: Limiting Absorption Principle, Generalized Eigenfunctions and Scattering Matrix for Laplace Operators with Boundary conditions on Hypersurfaces. *J. Spectr. Theory* **8** (2018), 1443-1486.

[9] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark: *NIST Handbook of mathematical functions.* Cambridge University Press, Cambridge (2010).

[10] A. Posilicano: A Kreĭn-like formula for singular perturbations of self-adjoint operators and applications. *J. Funct. Anal.* **183** (2001), 109-147.

[11] W. Thirring: *Classical Dynamical Systems. A Course in Mathematical Physics, vol. I.* Springer-Verlag (1978).

DiSAT, Sezione di Matematica, Università dell'Insubria, via Valleggio 11, I-22100 Como, Italy
Email address: claudio.cacciapuoti@uninsubria.it

Classe di Scienze, Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
Email address: fermidavide@gmail.com

DiSAT, Sezione di Matematica, Università dell'Insubria, via Valleggio 11, I-22100 Como, Italy
Email address: andrea.posilicano@uninsubria.it