

Research Article

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Nonzero Positive Solutions of Elliptic Systems with Gradient Dependence and Functional BCs

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Abstract: We discuss, by topological methods, the solvability of systems of second-order elliptic differential equations subject to functional boundary conditions under the presence of gradient terms in the nonlinearities. We prove the existence of nonnegative solutions and provide a non-existence result. We present some examples to illustrate the applicability of the existence and non-existence results.

Keywords: Positive Solution, Elliptic System, Gradient Terms, Functional Boundary Condition, Cone, Fixed Point Index

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1 Introduction

In this paper we study the solvability of a system of second-order elliptic differential equations subject to functional boundary conditions (BCs for short). Namely, we investigate parametric systems of the type

$$\begin{cases} \mathcal{L}_k u_k = \lambda_k f_k(x, u_1, \dots, u_m, \nabla u_1, \dots, \nabla u_m) & \text{in } \mathcal{O} \quad (k = 1, 2, \dots, m), \\ u_k(x) = \eta_k \zeta_k(x) h_k[u_1, \dots, u_m] & \text{for } x \in \partial \mathcal{O} \quad (k = 1, 2, \dots, m), \end{cases} \quad (1.1)$$

where $m \geq 1$ is a fixed natural number, $\mathcal{O} \subseteq \mathbb{R}^n$ is a bounded and connected open set of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, and $\lambda_k, \eta_k, k = 1, \dots, m$, are nonnegative real parameters. Moreover, $\mathcal{L}_1, \dots, \mathcal{L}_m$ are uniformly elliptic, second-order linear partial differential operators (PDOs) in divergence form on \mathcal{O} . That is,

$$\mathcal{L}_k u := - \sum_{i,j=1}^n \partial_{x_i} (a_{i,j}^{(k)}(x) \partial_{x_j} u + b_i^{(k)}(x) u) + \sum_{i=1}^n c_i^{(k)}(x) \partial_{x_i} u + d^{(k)}(x) u, \quad k = 1, \dots, m,$$

where

- the coefficient functions of \mathcal{L}_k belong to $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$,
- the matrix $A^{(k)}(x) := (a_{i,j}^{(k)}(x))_{i,j}$ is symmetric for every $x \in \mathcal{O}$,
- \mathcal{L}_k is uniformly elliptic in \mathcal{O} , i.e., there exists $\Lambda_k > 0$ such that

$$\frac{1}{\Lambda_k} \|\xi\|^2 \leq \sum_{i,j=1}^n a_{i,j}^{(k)}(x) \xi_i \xi_j \leq \Lambda_k \|\xi\|^2 \quad \text{for any } x \in \mathcal{O} \text{ and } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\|\xi\|$ stands for the Euclidean norm of $\xi \in \mathbb{R}^n$,

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- for every nonnegative function $\varphi \in C_0^\infty(\mathcal{O}, \mathbb{R})$ one has

$$\int_{\mathcal{O}} \left(d^{(k)} \varphi + \sum_{i=1}^n b_i^{(k)} \partial_{x_i} \varphi \right) dx \geq 0, \quad \int_{\mathcal{O}} \left(d^{(k)} \varphi + \sum_{i=1}^n c_i^{(k)} \partial_{x_i} \varphi \right) dx \geq 0.$$

Furthermore, for every fixed $k = 1, \dots, m$ we also assume that:

- f_k is a real-valued continuous function defined on $\overline{\mathcal{O}} \times \mathbb{R}^m \times \mathbb{R}^{nm}$,
- h_k is a real-valued continuous functional defined on the space $C^1(\overline{\mathcal{O}}, \mathbb{R}^m)$,
- $\zeta_k \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ and $\zeta_k \geq 0$ on \mathcal{O} .

System (1.1) is quite general, and includes, for example, as a particular case a Dirichlet boundary value problem for elliptic systems with gradient dependence of the form

$$\begin{cases} -\Delta u_1 = \lambda_1 f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \mathcal{O}, \\ -\Delta u_2 = \lambda_2 f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \mathcal{O}, \\ u_1|_{\partial\mathcal{O}} = 0 = u_2|_{\partial\mathcal{O}}. \end{cases} \tag{1.2}$$

Systems of nonlinear PDEs of this kind are widely studied in view of applications: in fact, the nonlinearities in (1.2) may depend also on the gradient of the solution, and thus represent convection terms. These problems, in general, are not easily dealt with by means of variational methods. Different approaches in the study of PDEs with gradient terms have been proposed: for example sub- and super-solutions, topological degree theory, mountain pass techniques. We mention, for instance, the pioneering works of Amann and Crandall [3], Brézis and Turner [4], Mawhin and Schmitt [21, 22], Pokhozhaev [23] and the more recent contributions [1, 7, 9, 12, 24, 26, 27, 29]. See also the very recent survey [8] and references therein.

In this paper we adopt a topological approach, based on the classical notion of fixed point index (see e.g. [15]) for the existence result, Theorem 3.3 below, whereas we prove a non-existence result via an elementary argument. In some sense we follow a path established by Amman [2, 3] and successfully used by many authors in different contexts. We point out that our approach applies not only to Dirichlet BCs but permits to consider (possibly nonlinear) functional BCs, including the special cases of *linear (multi-point or integral) BCs* of the form

$$h_k[u] = \sum_{j=1}^m \sum_{i=1}^N \left(\hat{\alpha}_{ijk} u_j(\omega_i) + \sum_{l=1}^n \hat{\beta}_{ijkl} \partial_{x_l} u_j(\tau_i) \right) \tag{1.3}$$

or

$$h_k[u] = \sum_{j=1}^m \left(\int_{\Omega} \hat{\alpha}_{jk}(x) u_j(x) dx + \sum_{l=1}^n \int_{\Omega} \hat{\beta}_{jkl}(x) \partial_{x_l} u_j(x) dx \right), \tag{1.4}$$

where, in (1.3), $\hat{\alpha}_{ijk}, \hat{\beta}_{ijkl}$ are nonnegative coefficients and $\omega_i, \tau_i \in \mathcal{O}$ while, in (1.4), $\hat{\alpha}_{jk}, \hat{\beta}_{jkl}$ are nonnegative continuous functions on $\overline{\mathcal{O}}$. In particular, we observe that nonlinear, nonlocal BCs have seen recently attention in the framework of elliptic equations: we refer the reader to the papers [5, 6, 13, 14, 16, 17, 25] and references therein.

We wish to point out that an advantage of our setting, with respect to the theory developed in [5, 6, 13, 14, 16, 25], is the possibility to allow also gradient dependence within the functionals occurring in the BCs. This follows the approach used recently in [18, 19] within the setting of ODEs.

Note that functional BCs that involve gradient terms may occur in applications. For example, consider a particular case of (1.1) for $m = 1$ and $n = 2$, namely

$$\begin{cases} -\Delta u(x) = f(x, u(x), \nabla u(x)) & x \in B, \\ u(x) = \eta_0 u(0) + \eta_1 \|\nabla u(0)\| & x \in \partial B, \end{cases} \tag{1.5}$$

where B is the Euclidean ball in \mathbb{R}^2 centered at 0 with radius 1, $\|\cdot\|$ is the Euclidian norm and η_i are nonnegative coefficients. The BVP (1.5) can be used as a model for the steady states of the temperature of a heated disk of radius 1, where a controller located in the border of the disk adds or removes heat according to the value of the temperature and to its variation, both registered by a sensor located in the center of the disk. In the context of ODEs, a good reference for this kind of thermostat problems is the recent paper [28].

As already pointed out, a peculiarity of system (1.1) is the dependence on the gradient of the solutions, both in the nonlinearity and in the functionals occurring in the BCs, and this represents the main technical difficulty that we have to deal with in this paper. For this purpose, we have to perform a preliminary study of the Green’s function of the partial differential operators which occur in (1.1). In Section 2 we collect some properties and estimates on Green’s function, which are probably known to the experts in the field, nevertheless we include them for the sake of completeness. Roughly speaking, these estimates yield the a priori bounds needed to compute the fixed point index in suitable cones of nonnegative functions.

Section 3 contains our main results, while the final Section 4 includes some examples illustrating our results. In particular, we fix $m = 2$ and $n = 3$, and, taking into account the parameters $\lambda_1, \lambda_2, \eta_1, \eta_2$, we provide existence and non-existence results in some concrete situations.

2 Preliminaries on Divergence-Form Elliptic Operators

In this section we present, mostly without proof, several results concerning divergence-form operators which shall play a central role in the forthcoming sections. We refer the reader to, e.g., [10, 11] for a detailed treatment of this topic.

To being with, let $\mathcal{O} \subseteq \mathbb{R}^n$ be a fixed open set and let \mathcal{L} be a second-order linear PDO on \mathcal{O} of the following divergence form:

$$\mathcal{L}u := - \sum_{i,j=1}^n \partial_{x_i}(a_{i,j}(x)\partial_{x_j}u + b_i(x)u) + \sum_{i=1}^n c_i(x)\partial_{x_i}u + d(x)u = -\operatorname{div}(A(x)\nabla u + \mathbf{b}u) + \langle \mathbf{c}, \nabla u \rangle + du \quad (2.1)$$

(here $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$). Throughout the sequel, we shall suppose that the following “structural assumptions” on \mathcal{O} and \mathcal{L} are satisfied:

(H0) \mathcal{O} is bounded, connected and of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$,

(H1) the coefficient functions of \mathcal{L} are Hölder-continuous of exponent α up to $\partial\mathcal{O}$, i.e.,

$$a_{i,j}, b_i, c_i, d \in C^\alpha(\overline{\mathcal{O}}, \mathbb{R}) \quad \text{for every } i, j \in \{1, \dots, n\},$$

(H2) the matrix $A(x) := (a_{i,j}(x))_{i,j}$ is symmetric in \mathcal{O} , i.e.,

$$a_{i,j}(x) = a_{j,i}(x) \quad \text{for every } x \in \overline{\mathcal{O}} \text{ and every } i, j \in \{1, \dots, n\},$$

(H3) \mathcal{L} is uniformly elliptic in \mathcal{O} , i.e., there exists $\Lambda > 0$ such that

$$\frac{1}{\Lambda} \|\xi\|^2 \leq \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \leq \Lambda \|\xi\|^2 \quad \text{for any } x \in \overline{\mathcal{O}} \text{ and any } \xi \in \mathbb{R}^n,$$

(H4) the inequalities $d - \operatorname{div}(\mathbf{b}) \geq 0$ and $d - \operatorname{div}(\mathbf{c}) \geq 0$ hold in the weak sense of distributions on \mathcal{O} , i.e., for every $\varphi \in C_0^\infty(\mathcal{O}, \mathbb{R})$ such that $\varphi \geq 0$ on \mathcal{O} , one has

$$\int_{\mathcal{O}} \left(d\varphi + \sum_{i=1}^n b_i \partial_{x_i} \varphi \right) dx \geq 0 \quad \text{and} \quad \int_{\mathcal{O}} \left(d\varphi + \sum_{i=1}^n c_i \partial_{x_i} \varphi \right) dx \geq 0.$$

It should be noticed that, since the coefficient functions of \mathcal{L} are assumed to be just Hölder-continuous on $\overline{\mathcal{O}}$, it is not possible to compute $\mathcal{L}u$ in a point-wise sense (even if u is smooth on \mathcal{O}); for this reason, the following definition is plainly justified.

Definition 2.1. Let assumptions (H0)–(H4) be in force, and let $f \in L^2(\mathcal{O})$. We say that a function $u : \mathcal{O} \rightarrow \mathbb{R}$ is a *solution of the equation*

$$\mathcal{L}u = f \quad \text{in } \mathcal{O} \quad (2.2)$$

if $u \in W^{1,2}(\mathcal{O})$ and if, for every test function $\phi \in C_0^\infty(\mathcal{O}, \mathbb{R})$, one has

$$\int_{\mathcal{O}} (\langle A(x)\nabla u + \mathbf{b}u, \nabla \phi \rangle + \langle \mathbf{c}, \nabla u \rangle \phi + du\phi) dx = \int_{\mathcal{O}} f\phi dx.$$

Given $g \in W^{1,2}(\mathcal{O})$, we say that u is a *solution of the Poisson problem*

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = g, \end{cases} \quad (2.3)$$

if u is a solution of (2.2) and, furthermore, $u - g \in W_0^{1,2}(\mathcal{O})$.

Now, as a consequence of the “sign assumption” (H4) it is possible to prove that a suitable form of the Weak Maximum Principle holds for \mathcal{L} (see, e.g., [11, Theorem 8.1]); from this, one can straightforwardly deduce Lemma 2.2 below (see [11, Corollary 8.2]), ensuring that the Poisson problem (2.3) possesses at most one solution.

Lemma 2.2. *Let the assumptions (H0)–(H4) be in force, and let $u \in W_0^{1,2}(\mathcal{O})$ be such that $\mathcal{L}u = 0$ or $\mathcal{L}^T u = 0$ in \mathcal{O} . Then $u \equiv 0$ almost everywhere on \mathcal{O} .*

2.1 The Poisson Problem for \mathcal{L}

A first group of results we aim to present is about existence and regularity of solutions for the Poisson problem (2.3) for \mathcal{L} . In order to do this, we first introduce the following Banach spaces:

- $X = (C(\overline{\mathcal{O}}, \mathbb{R}), \|\cdot\|_\infty)$, where

$$\|f\|_\infty := \max_{x \in \overline{\mathcal{O}}} |f(x)|, \quad (2.4)$$

- $X = (C^1(\overline{\mathcal{O}}, \mathbb{R}), \|\cdot\|_{C^1(\overline{\mathcal{O}}, \mathbb{R})})$, where

$$\|f\|_{C^1(\overline{\mathcal{O}}, \mathbb{R})} := \max_{j=1, \dots, n} \{\|f\|_\infty, \|\partial_j f\|_\infty : j = 1, \dots, n\}, \quad (2.5)$$

- $X = C^{1,\theta}(\overline{\mathcal{O}}, \mathbb{R})$ (for some $\theta \in (0, 1)$), where

$$\|u\|_{C^{1,\theta}(\overline{\mathcal{O}}, \mathbb{R})} := \max_{j=1, \dots, n} \left\{ \|u\|_\infty, \|\partial_j u\|_\infty, \sup_{x, y \in \overline{\mathcal{O}}} \frac{|\partial_j u(x) - \partial_j u(y)|}{\|x - y\|^\theta} \right\}.$$

Given $f \in C^1(\overline{\mathcal{O}}, \mathbb{R})$, it will be also convenient to define, with abuse of notation,

$$\|\nabla f\|_\infty := \max_{j=1, \dots, n} \{\|\partial_j f\|_\infty : j = 1, \dots, n\},$$

so that, clearly, $\|f\|_{C^1(\overline{\mathcal{O}}, \mathbb{R})} = \max\{\|f\|_\infty, \|\nabla f\|_\infty\}$.

Now, by exploiting assumptions (H3)–(H4), Lemma 2.2 and the Fredholm alternative, one can establish the following basic theorem (for a proof, see [11, Theorem 8.3]).

Theorem 2.3. *Let assumptions (H0)–(H4) be in force. Then, for every $f \in L^2(\mathcal{O})$ and every $g \in W^{1,2}(\mathcal{O})$, there exists a unique solution $u_{f,g} \in W^{1,2}(\mathcal{O})$ of (2.3).*

Throughout the sequel, we indicate by $u_{f,g}$ the unique solution in $W^{1,2}(\mathcal{O})$ of (2.3) (for fixed $f \in L^2(\mathcal{O})$ and $g \in W^{1,2}(\mathcal{O})$), whose existence is guaranteed by Theorem 2.3. In the particular case when $g \equiv 0$, we simply write u_f instead of $u_{f,0}$.

Remark 2.4. Theorem 2.3 holds under more general hypotheses: in fact, it suffices to assume that \mathcal{O} is bounded and that the coefficient functions of \mathcal{L} are in $L^\infty(\mathcal{O})$.

Remark 2.5. Let $f_1, f_2 \in L^2(\mathcal{O})$ and, for $i = 1, 2$, let $u_i = u_{f_i} \in W_0^{1,2}(\mathcal{O})$ be the unique solution of (2.3) with $f = f_i$ (and $g \equiv 0$). Since, obviously, it holds that

$$\mathcal{L}(u_{f_1} + u_{f_2}) = u_{f_1} + u_{f_2} \quad \text{and} \quad u_{f_1} + u_{f_2} \in W_0^{1,2}(\mathcal{O}),$$

we conclude that the unique solution of (2.3) with $f = f_1 + f_2$ and $g \equiv 0$ is $u_{f_1} + u_{f_2}$.

Since we aim to apply suitable fixed-point techniques to operators acting on spaces of C^1 -functions, we are interested in solving (2.3) for continuous f and regular g . In this context, the unique solution $u_{f,g}$ of (2.3)

turns out to be much more regular than $W^{1,2}$; in fact, we have the following crucial result (for a proof, see [11, Theorems 8.16, 8.33 and 8.34]).

Theorem 2.6. *Let assumptions (H0)–(H4) be in force, and let \mathcal{L} be as in (2.1). Moreover, let $f \in C(\overline{\mathcal{O}}, \mathbb{R})$ and let $g \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$. Then the following facts hold true:*

(i) *There exists a unique $\hat{u}_{f,g} \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ such that*

$$\hat{u}_{f,g} \equiv u_{f,g} \quad \text{a.e. on } \mathcal{O}.$$

In particular, $\hat{u}_{f,g}$ solves (2.2) and $\hat{u}_{f,g} \equiv g$ point-wise on $\partial\Omega$.

(ii) *There exists a constant $C > 0$, only depending on n, Λ and \mathcal{O} , such that*

$$\|\hat{u}_{f,g}\|_{C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})} \leq C(\|f\|_{C(\overline{\mathcal{O}}, \mathbb{R})} + \|g\|_{C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})}).$$

(iii) *If $f \geq 0$ on $\overline{\mathcal{O}}$ and $g \geq 0$ on $\partial\mathcal{O}$, then $\hat{u}_{f,g} \geq 0$ on $\overline{\mathcal{O}}$.*

Now, in view of Theorem 2.6 (i), we can define a linear operator as follows

$$\mathcal{G}_{\mathcal{L}} : C(\overline{\mathcal{O}}, \mathbb{R}) \rightarrow C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R}), \quad \mathcal{G}_{\mathcal{L}}(f) := \hat{u}_f, \tag{2.6}$$

where $\hat{u}_f = \hat{u}_{f,0} \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ is the unique solution of (2.3) with $g \equiv 0$. We shall call $\mathcal{G}_{\mathcal{L}}$ the *Green operator* for \mathcal{L} . By exploiting assertions (ii)–(iii) of Theorem 2.6, it is possible to deduce some continuous-compactness properties of $\mathcal{G}_{\mathcal{L}}$ which shall play a central role in the next sections; to be more precise, we have the following proposition.

Proposition 2.7. *Let assumptions (H0)–(H4) be in force, and let $\mathcal{G}_{\mathcal{L}}$ be the operator defined in (2.6). Then the following facts hold:*

(i) $\mathcal{G}_{\mathcal{L}}$ *is continuous from* $C(\overline{\mathcal{O}}, \mathbb{R})$ *to* $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$.

(ii) $\mathcal{G}_{\mathcal{L}}$ *is compact from* $C(\overline{\mathcal{O}}, \mathbb{R})$ *to* $C^1(\overline{\mathcal{O}}, \mathbb{R}) \supseteq C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$.

(iii) *If* $V_0 := C(\overline{\mathcal{O}}, \mathbb{R}^+) \subseteq C(\overline{\mathcal{O}}, \mathbb{R})$ *denotes the (convex) cone of the nonnegative continuous functions on* $\overline{\mathcal{O}}$, *it holds that* $\mathcal{G}_{\mathcal{L}}(V_0) \subseteq V_0$.

Proof. (i) On account of Theorem 2.6 (ii), for every $f \in C(\overline{\mathcal{O}}, \mathbb{R})$ one has

$$\|\mathcal{G}_{\mathcal{L}}(f)\|_{C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})} \leq C\|f\|_{\infty} \tag{2.7}$$

(here $C > 0$ is a constant independent of f). Since $\mathcal{G}_{\mathcal{L}}$ is linear (see Remark 2.5), from (2.7) we immediately deduce that $\mathcal{G}_{\mathcal{L}}$ is continuous from $C(\overline{\mathcal{O}}, \mathbb{R})$ to $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$.

(ii) Let $\{f_j\}_j$ be a bounded sequence in $C(\overline{\mathcal{O}}, \mathbb{R})$. On account of (2.7), we see that the sequence $\{\mathcal{G}_{\mathcal{L}}(f_j)\}_j$ is bounded in $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$; as a consequence, a standard application of Arzelà–Ascoli’s Theorem implies the existence of $u_0, \dots, u_n \in C(\overline{\mathcal{O}}, \mathbb{R})$ such that

(a) $\|\mathcal{G}_{\mathcal{L}}(f_{j_k}) - u_0\|_{\infty} \rightarrow 0$ as $k \rightarrow \infty$,

(b) $\|\partial_i(\mathcal{G}_{\mathcal{L}}(f_{j_k})) - u_i\|_{\infty} \rightarrow 0$ as $k \rightarrow \infty$ (for every $i = 1, \dots, n$),

where $\{f_{j_k}\}_k$ is a suitable sub-sequence of $\{f_j\}_j$. By combining (a) and (b), we deduce that $u_0 \in C^1(\overline{\mathcal{O}}, \mathbb{R})$ and that $\nabla u_0 = (u_1, \dots, u_n)$; moreover, one has

$$\|\mathcal{G}_{\mathcal{L}}(f_{j_k}) - u_0\|_{C^1(\overline{\mathcal{O}}, \mathbb{R})} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and this proves that $\mathcal{G}_{\mathcal{L}}$ is compact from $C(\overline{\mathcal{O}}, \mathbb{R})$ to $C^1(\overline{\mathcal{O}}, \mathbb{R})$, as desired.

(iii) Let $f \in V_0$ be fixed. Since, by Theorem 2.6 (iii), we know that $\mathcal{G}_{\mathcal{L}}(f) = \hat{u}_{f,0} \geq 0$ throughout $\overline{\mathcal{O}}$, we immediately conclude that $\mathcal{G}_{\mathcal{L}}(f) \in V_0 \cap C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$, as desired. \square

2.2 Green’s Function for \mathcal{L}

Now we have established Proposition 2.7, we turn to present a second group of results: this is about the existence of a Green’s function for \mathcal{L} allowing to obtain an integral representation formula for $\mathcal{G}_{\mathcal{L}}$.

To begin with, we demonstrate the following key theorem.

Theorem 2.8. *Let assumptions (H0)–(H4) be in force, and let \mathcal{L} be as in (2.1). Then there exists a function $g_{\mathcal{L}} : \mathcal{O} \times \mathcal{O} \rightarrow [0, \infty)$ such that*

- (a) $g_{\mathcal{L}}(\cdot; x) \in L^1(\mathcal{O})$ for almost every $x \in \mathcal{O}$,
- (b) for every $f \in C(\overline{\mathcal{O}}, \mathbb{R})$ one has

$$\mathfrak{G}_{\mathcal{L}}(f)(x) = \int_{\mathcal{O}} g_{\mathcal{L}}(y; x)f(y) \, dy \quad \text{for a.e. } x \in \mathcal{O}. \tag{2.8}$$

Furthermore, $g_{\mathcal{L}}$ enjoys the following properties:

- (I) there exists a constant $c_0 > 0$ such that, for a.e. $x, y \in \mathcal{O}$, one has

$$0 \leq g_{\mathcal{L}}(y; x) \leq c_0 \|x - y\|^{2-n}, \tag{2.9}$$

- (II) $g_{\mathcal{L}}(\cdot; x) \in W_0^{1,p}(\mathcal{O})$ for a.e. $x \in \mathcal{O}$ and every $1 \leq p < \frac{n}{n-1}$,
- (III) $g_{\mathcal{L}}(y; \cdot) \in W_0^{1,p}(\mathcal{O})$ for a.e. $y \in \mathcal{O}$ and every $1 \leq p < \frac{n}{n-1}$,
- (IV) there exists a constant $c_1 > 0$ such that, for a.e. $x, y \in \mathcal{O}$, one has

$$\|\nabla_y g_{\mathcal{L}}(y; x)\| \leq c_1 \|x - y\|^{1-n} \quad \text{and} \quad \|\nabla_x g_{\mathcal{L}}(y; x)\| \leq c_1 \|x - y\|^{1-n}. \tag{2.10}$$

Finally, $g_{\mathcal{L}}$ is unique in the following sense: if $\tilde{g} : \mathcal{O} \times \mathcal{O} \rightarrow [0, \infty)$ is another function satisfying (a)–(b), then $g_{\mathcal{L}}(\cdot; x) = \tilde{g}(\cdot; x)$ in $L^1(\mathcal{O})$ for a.e. $x \in \mathcal{O}$.

Throughout the sequel, we shall refer to the function $g_{\mathcal{L}}$ in Theorem 2.8 as the *Green’s function for the operator $\mathfrak{G}_{\mathcal{L}}$* (and related to the open set \mathcal{O}).

Proof. We begin by proving the existence part of the theorem. In order to do this, we make pivotal use of several results established in the very recent paper [20].

First of all, by [20, Proposition 5.3] there exists a function $g_{\mathcal{L}} : \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ such that

- (i) $g_{\mathcal{L}}(\cdot; x) \in W^{1,p}(\mathcal{O})$ for a.e. $x \in \mathcal{O}$ and every $1 \leq p < \frac{n}{n-1}$,
- (ii) for every fixed $f \in C(\overline{\mathcal{O}}, \mathbb{R})$ one has

$$\mathfrak{G}_{\mathcal{L}}(f)(x) = \int_{\mathcal{O}} g_{\mathcal{L}}(y; x)f(y) \, dy \quad \text{for a.e. } x \in \mathcal{O}.$$

Moreover, by [20, Theorem 6.10] we also have that

$$0 \leq g_{\mathcal{L}}(y; x) \leq c_0 \|x - y\|^{2-n} \quad \text{for a.e. } x, y \in \mathcal{O} \text{ with } x \neq y,$$

where $c_0 > 0$ is a suitable constant. In view of these facts, to complete the demonstration we are left to prove assertion (III) and the point-wise estimates in (2.10).

To this end, let us introduce the so-called (formal) adjoint \mathcal{L}^T of \mathcal{L} : this is the linear differential operator defined on \mathcal{O} in the following way:

$$\mathcal{L}^T v := - \sum_{i,j=1}^n \partial_{x_i}(a_{i,j}(x)\partial_{x_j} v + c_i(x)v) + \sum_{i=1}^n b_i(x)\partial_{x_i} v + d(x)v = -\text{div}(A(x)\nabla v + \mathbf{c}v) + \langle \mathbf{b}, \nabla v \rangle + dv. \tag{2.11}$$

Clearly, \mathcal{L}^T takes the same divergence-form of \mathcal{L} in (2.1) (with \mathbf{b} and \mathbf{c} interchanged); furthermore, due to the “symmetry” in assumption (H4), it is readily seen that \mathcal{L}^T satisfies the “structural assumptions” (H1)–(H4).

As a consequence, all the results established so far do apply to \mathcal{L}^T . In particular, for every fixed $g \in C(\overline{\mathcal{O}}, \mathbb{R})$ there exists a unique function $\mathcal{T}(g) \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ such that

$$\mathcal{L}^T \mathcal{T}(g) = g \quad \text{in } \mathcal{O} \quad \text{and} \quad \mathcal{T}(g) \equiv 0 \quad \text{on } \partial\mathcal{O}.$$

Now, by [20, Theorem 6.12] there exists a function $G : \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{R}$ such that

- (iii) $G(\cdot; y) \in W^{1,p}(\mathcal{O})$ for a.e. $y \in \mathcal{O}$ and every $1 \leq p < \frac{n}{n-1}$,
- (iv) for every fixed $g \in C(\overline{\mathcal{O}}, \mathbb{R})$ one has

$$\mathcal{T}(g)(y) = \int_{\mathcal{O}} G(x; y)g(x) \, dx \quad \text{for a.e. } y \in \mathcal{O}.$$

On the other hand, since [20, Proposition 6.13] shows that

$$G(x; y) = g_{\mathcal{L}}(y; x) \quad \text{for a.e. } x, y \in \mathcal{O} \text{ with } x \neq y, \tag{2.12}$$

from (iii) we infer that $g_{\mathcal{L}}(y; \cdot) = G(\cdot; y) \in W^{1,p}(\mathcal{O})$ for almost every $y \in \mathcal{O}$ and every exponent $p \in [1, \frac{n}{n-1})$. This is exactly assertion (III).

Finally, we prove the point-wise estimates in assertion (IV). First of all, since \mathcal{L} satisfies assumptions (H1)–(H4), we are entitled to apply [20, Theorem 8.1], ensuring that

$$\|\nabla_x G(x; y)\| \leq c'_1 \|x - y\|^{1-n} \quad \text{for a.e. } x, y \in \mathcal{O} \text{ with } x \neq y, \tag{2.13}$$

where $c'_1 > 0$ is a suitable constant. Moreover, since also \mathcal{L}^T satisfies assumptions (H1)–(H4), another application of [20, Theorem 8.1] gives

$$\|\nabla_y g_{\mathcal{L}}(y; x)\| \leq c''_1 \|x - y\|^{1-n} \quad \text{for a.e. } x, y \in \mathcal{O} \text{ with } x \neq y, \tag{2.14}$$

where $c''_1 > 0$ is another suitable constant. Gathering together (2.14), (2.13) and (2.12) we immediately obtain the desired (2.10) (with $c_1 := \max\{c'_1, c''_1\}$).

As for the uniqueness part of the theorem, suppose that there exists another function $\tilde{g} : \mathcal{O} \times \mathcal{O} \rightarrow [0, \infty)$ satisfying (a)–(b). In particular, for every $\phi \in C_0^\infty(\mathcal{O}, \mathbb{R})$ one has

$$\int_{\mathcal{O}} (g_{\mathcal{L}}(y; x) - \tilde{g}(y; x))\phi(y) dy = 0 \quad \text{for a.e. } x \in \mathcal{O}. \tag{2.15}$$

The space $C_0^\infty(\mathcal{O}, \mathbb{R})$ being separable (with its usual LF-topology), there exists a countable set $\mathcal{F} \subseteq C_0^\infty(\mathcal{O}, \mathbb{R})$ which is dense; moreover, by (2.15), for every $\phi \in \mathcal{F}$ there exists a set $E(\phi) \subseteq \mathcal{O}$, with zero-Lebesgue measure, such that

$$\int_{\mathcal{O}} (g_{\mathcal{L}}(y; x) - \tilde{g}(y; x))\phi(y) dy = 0 \quad \text{for all } x \in E(\phi).$$

We then define $E := \bigcup_{\phi \in \mathcal{F}} E(\phi)$. Since \mathcal{F} is countable and $E(\phi)$ has zero-Lebesgue measure for every ϕ , we see that E has measure zero; moreover, for every $x \in \mathcal{O} \setminus E$ we have

$$\int_{\mathcal{O}} (g_{\mathcal{L}}(y; x) - \tilde{g}(y; x))\phi(y) dy = 0 \quad \text{for all } \phi \in \mathcal{F}.$$

This proves that, for every $x \in \mathcal{O} \setminus E$, the distribution $g_{\mathcal{L}}(\cdot; x) - \tilde{g}(\cdot; x)$ vanishes on \mathcal{F} ; the latter being dense, we then conclude that $g_{\mathcal{L}}(\cdot; x) = \tilde{g}(\cdot; x)$ in $L^1(\mathcal{O})$ for a.e. $x, y \in \mathcal{O}$.

This ends the proof. □

Remark 2.9. The approach adopted for the proof of Theorem 2.8 shows the reason why we have assumed that $d - \operatorname{div}(\mathbf{b}) \geq 0$ and $d - \operatorname{div}(\mathbf{c}) \geq 0$ in the sense of distributions.

In fact, under this assumption, all the mentioned results in [20] hold both for \mathcal{L} and for its transpose \mathcal{L}^T ; in particular, this allows us to obtain point-wise estimates both for

$$\nabla_x g_{\mathcal{L}}(y; x) = \nabla_x G(x; y) \quad \text{and} \quad \nabla_y g_{\mathcal{L}}(y; x).$$

Remark 2.10. It is contained in the proof of Theorem 2.8 the following fact: if \mathcal{L} is of the form (2.1) and if $\mathbf{b} \equiv \mathbf{c}$ on \mathcal{O} , then the Green's function for $\mathcal{G}_{\mathcal{L}}$ is symmetric, that is,

$$g_{\mathcal{L}}(y; x) = g_{\mathcal{L}}(x; y) \quad \text{for a.e. } x, y \in \mathcal{O}.$$

In fact, if $\mathbf{b} \equiv \mathbf{c}$ on \mathcal{O} , then the adjoint operator \mathcal{L}^T coincides with \mathcal{L} (see (2.11)); thus, following the notation in the proof of Theorem 2.8, we have

$$g_{\mathcal{L}}(x; y) = G(x; y) = g_{\mathcal{L}}(y; x).$$

Remark 2.11. By carefully scrutinizing the proofs of the existence results for $g_{\mathcal{L}}$ contained in [20, Proposition 5.3], one can recognize that the following properties hold:

- (a) for a.e. $x \in \mathcal{O}$ and every $\epsilon > 0$, we have $g_{\mathcal{L}}(\cdot; x) \in W^{1,2}(\mathcal{O} \setminus B(x, \epsilon))$,
- (b) $g_{\mathcal{L}}(\cdot; x)$ is a solution of $\mathcal{L}^T u = 0$ in $\mathcal{O} \setminus B(x, \epsilon)$, where \mathcal{L}^T is as in (2.11).

Analogously, an inspection to the proof of [20, Theorem 6.12] shows that

- (a') for a.e. $y \in \mathcal{O}$ and every $\epsilon > 0$, we have $G(\cdot; y) = g_{\mathcal{L}}(y; \cdot) \in W^{1,2}(\mathcal{O} \setminus B(y, \epsilon))$,
- (b') $G(\cdot; y) = g_{\mathcal{L}}(y; \cdot)$ is a solution of $\mathcal{L} u = 0$ in $\mathcal{O} \setminus B(y, \epsilon)$.

Gathering together all these facts, from the classical elliptic regularity theory (see, e.g., [11, Corollary 8.36]) we deduce that $g_{\mathcal{L}}$ is of class $C^{1,\alpha}$ out of the diagonal of $\mathcal{O} \times \mathcal{O}$.

We now use the point-wise estimates in (2.9)–(2.10) to prove the following lemma.

Lemma 2.12. *Let assumptions (H0)–(H4) be in force, and let $g_{\mathcal{L}}$ be the Green’s function for $\mathcal{G}_{\mathcal{L}}$. Moreover, let $\rho := \text{diam}(\mathcal{O})$. Then the following estimates hold:*

$$\int_{\mathcal{O}} g_{\mathcal{L}}(y; x) \, dy \leq c_0 \cdot \frac{n \omega_n \rho^2}{2} \quad \text{for a.e. } x \in \mathcal{O}, \tag{2.16}$$

$$\int_{\mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \, dy \leq c_1 \cdot n \omega_n \rho \quad \text{for a.e. } x \in \mathcal{O}. \tag{2.17}$$

Here ω_n is the Lebesgue measure of the unit ball $B(0, 1) \subseteq \mathbb{R}^n$.

Proof. We begin by proving (2.16). To this end we first notice that, if $x \in \mathcal{O}$ is arbitrary, then $\mathcal{O} \subseteq \overline{B(x, \rho)}$; as a consequence, by crucially exploiting estimate (2.9) we get

$$\begin{aligned} \int_{\mathcal{O}} g_{\mathcal{L}}(y; x) \, dy &\leq c_0 \int_{\mathcal{O}} \|x - y\|^{2-n} \, dy \leq c_0 \int_{\overline{B(x, \rho)}} \|x - y\|^{2-n} \, dy \\ &= c_0 \int_{\overline{B(0, \rho)}} \|y\|^{2-n} \, dy = c_0 \int_0^\rho t^{2-n} \mathcal{H}^{n-1}(\partial B(0, t)) \, dt \\ &= c_0 n \omega_n \int_0^\rho t \, dt = c_0 \cdot \frac{n \omega_n \rho^2}{2}, \end{aligned}$$

which is exactly the desired (2.16). As for the proof of (2.17), we argue essentially in the same way: by crucially exploiting estimate (2.10) we get

$$\begin{aligned} \int_{\mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \, dy &\leq c_1 \int_{\mathcal{O}} \|x - y\|^{1-n} \, dy \leq c_1 \int_{\overline{B(x, \rho)}} \|x - y\|^{1-n} \, dy \\ &= c_1 \int_{\overline{B(0, \rho)}} \|y\|^{1-n} \, dy = c_1 \int_0^\rho t^{1-n} \mathcal{H}^{n-1}(\partial B(0, t)) \, dt \\ &= c_1 n \omega_n \int_0^\rho dt = c_1 \cdot n \omega_n \rho, \end{aligned}$$

and this is precisely the desired inequality (2.17). □

Remark 2.13. We explicitly observe that, by combining estimate (2.16) in Lemma 2.12 with the representation formula (2.8), for a.e. $x \in \mathcal{O}$ we obtain

$$0 \leq \mathcal{G}_{\mathcal{L}}(\hat{1})(x) = \int_{\mathcal{O}} g(y; x) \, dy \leq c_0 \cdot \frac{n \omega_n \rho^2}{2},$$

where $\rho := \text{diam}(\mathcal{O})$ and $\hat{1}$ denotes the constant function equal to 1 on \mathcal{O} . As a consequence, since we have

$\mathcal{G}_{\mathcal{L}}(\hat{1}) \in C(\overline{\mathcal{O}}, \mathbb{R})$, we get

$$\|\mathcal{G}_{\mathcal{L}}(\hat{1})\|_{\infty} \leq c_0 \cdot \frac{n \omega_n \rho^2}{2}.$$

We conclude this part of the section by deducing from (2.8) an integral representation for the x_i -derivatives of $\mathcal{G}_{\mathcal{L}}(f)$. To this end we first observe that, if $f \in C(\overline{\mathcal{O}}, \mathbb{R})$, Lemma 2.12 ensures that the following “potential-type” functions are well-defined:

$$\mathcal{P}^{(i)}f(x) := \int_{\mathcal{O}} \partial_{x_i} g_{\mathcal{L}}(y; x) f(y) \, dy \quad (\text{for } i = 1, \dots, n). \tag{2.18}$$

In fact, by estimate (2.17) in Lemma 2.12 we have (for $i = 1, \dots, n$)

$$\int_{\mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \cdot |f(y)| \, dy \leq \|f\|_{\infty} \cdot \int_{\mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \, dy \leq \|f\|_{\infty} \cdot c_1 n \omega_n \text{diam}(\mathcal{O}) \quad (\text{for a.e. } x \in \mathcal{O}).$$

Moreover, from the above computation we also infer that (again for $i = 1, \dots, n$)

$$\mathcal{P}^{(i)}f \in L^{\infty}(\mathcal{O}) \quad \text{and} \quad \|\mathcal{P}^{(i)}f\|_{L^{\infty}(\mathcal{O})} \leq \|f\|_{\infty} \cdot c_1 n \omega_n \text{diam}(\mathcal{O}).$$

We are then ready to prove the following proposition.

Proposition 2.14. *Let assumptions (H0)–(H4) be in force, and let $f \in C(\overline{\mathcal{O}}, \mathbb{R})$. Moreover, let $i \in \{1, \dots, n\}$ be fixed, and let $\mathcal{P}^{(i)}f$ be as in (2.18). Then we have*

$$\partial_{x_i} \mathcal{G}_{\mathcal{L}}(f)(x) = \mathcal{P}^{(i)}f(x) = \int_{\mathcal{O}} \partial_{x_i} g_{\mathcal{L}}(y; x) f(y) \, dy \quad \text{for a.e. } x \in \mathcal{O}. \tag{2.19}$$

Proof. We first notice, since $\mathcal{G}_{\mathcal{L}}(f) \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$, the identity (2.19) follows if we show that the L^{∞} -function $\mathcal{P}^{(i)}f$ is the weak derivative (in $L^1(\mathcal{O})$) of $\mathcal{G}_{\mathcal{L}}(f)$. To prove this fact, we argue as follows: firstly, if $\phi \in C_0^{\infty}(\mathcal{O}, \mathbb{R})$, by estimate (2.16) in Lemma 2.12 we get

$$\begin{aligned} \int_{\mathcal{O} \times \mathcal{O}} g_{\mathcal{L}}(y; x) \cdot |f(y)| \cdot |\partial_{x_i} \phi(x)| \, dx \, dy &\leq \|f\|_{C(\overline{\mathcal{O}}, \mathbb{R})} \cdot \|\partial_i \phi\|_{\infty} \cdot \int_{\mathcal{O}} \left(\int_{\mathcal{O}} g_{\mathcal{L}}(y; x) \, dy \right) dx \\ &\leq \|f\|_{C(\overline{\mathcal{O}}, \mathbb{R})} \cdot \|\partial_i \phi\|_{\infty} \cdot c_0 \cdot \frac{n \omega_n \text{diam}(\mathcal{O})^2}{2} \cdot |\mathcal{O}|; \end{aligned}$$

we are then entitled to apply Fubini’s Theorem, obtaining

$$\begin{aligned} \int_{\mathcal{O}} \mathcal{G}_{\mathcal{L}}(f)(x) \partial_{x_i} \phi(x) \, dx &= \int_{\mathcal{O}} \left(\int_{\mathcal{O}} g_{\mathcal{L}}(y; x) f(y) \, dy \right) \partial_{x_i} \phi(x) \, dx \\ &= \int_{\mathcal{O}} \left(\int_{\mathcal{O}} g_{\mathcal{L}}(y; x) \partial_{x_i} \phi(x) \, dx \right) f(y) \, dy \quad (\text{since } g_{\mathcal{L}}(y; \cdot) \in W_0^{1,1}(\mathcal{O}), \text{ see Theorem 2.8 (III)}) \\ &= - \int_{\mathcal{O}} \left(\int_{\mathcal{O}} \partial_{x_i} g_{\mathcal{L}}(y; x) \phi(x) \, dx \right) f(y) \, dy =: (*). \end{aligned}$$

On the other hand, since the estimate (2.17) in Lemma 2.12 implies that

$$\begin{aligned} \int_{\mathcal{O} \times \mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \cdot |f(y)| \cdot |\phi(x)| \, dx \, dy &\leq \|f\|_{\infty} \cdot \|\phi\|_{\infty} \cdot \int_{\mathcal{O}} \left(\int_{\mathcal{O}} |\partial_{x_i} g_{\mathcal{L}}(y; x)| \, dy \right) dx \\ &\leq \|f\|_{\infty} \cdot \|\phi\|_{\infty} \cdot c_1 \cdot n \omega_n \text{diam}(\mathcal{O}) \cdot |\mathcal{O}|, \end{aligned}$$

another application of Fubini’s Theorem is legitimate, and we get

$$(*) = - \int_{\mathcal{O}} \left(\int_{\mathcal{O}} \partial_{x_i} g_{\mathcal{L}}(y; x) f(y) \, dy \right) \phi(x) \, dx \stackrel{(2.18)}{=} - \int_{\mathcal{O}} \mathcal{P}^{(i)}f(x) \phi(x) \, dx.$$

Due to the arbitrariness of $\phi \in C_0^{\infty}(\overline{\mathcal{O}}, \mathbb{R})$, we then conclude that $\mathcal{P}^{(i)}f$ is the weak derivative of $\mathcal{G}_{\mathcal{L}}(f)$ in $L^1(\mathcal{O})$, and the proof is complete. \square

Remark 2.15. By using the regularity of $g_{\mathcal{L}}$ described in Remark 2.11, it is quite standard to recognize that, for a fixed $f \in C(\overline{\mathcal{O}}, \mathbb{R})$, the functions

$$\mathcal{O} \ni x \mapsto \int_{\mathcal{O}} g_{\mathcal{L}}(y; x) f(y) \, dy \quad \text{and} \quad \mathcal{P}^{(1)}f, \dots, \mathcal{P}^{(n)}f$$

are continuous on \mathcal{O} . As a consequence, the representation formulas (2.8) and (2.19) actually hold true for every $x \in \mathcal{O}$ (not only almost everywhere).

2.3 Spectral Properties of $\mathcal{G}_{\mathcal{L}}$

We conclude this section by briefly turning our attention to the spectral properties of the Green’s operator $\mathcal{G}_{\mathcal{L}}$.

To begin with, we remind the following theorem (see, e.g., [11, Theorem 8.6]).

Theorem 2.16. *Let assumptions (H0)–(H4) be in force. Then there exists a countable and discrete set $\Sigma \subseteq (0, \infty)$ with the following property: for every $\sigma \in \Sigma$ the subspace of solutions of the homogeneous problem*

$$\begin{cases} \mathcal{L}u = \sigma u & \text{in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = 0, \end{cases}$$

has positive finite dimension (as a subspace of $W^{1,2}(\mathcal{O})$).

By making use of Theorem 2.16, we can prove the Proposition 2.17 below.

Proposition 2.17. *Let assumptions (H0)–(H4) be in force, and let $\mathcal{G}_{\mathcal{L}}$ be the Green’s operator for \mathcal{L} (thought of as an operator from $C(\overline{\mathcal{O}}, \mathbb{R})$ into itself). Then the following facts hold true:*

- (i) *The spectral radius $r(\mathcal{G}_{\mathcal{L}})$ of $\mathcal{G}_{\mathcal{L}}$ is strictly positive.*
- (ii) *There exists a nonnegative $u_0 \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R}) \setminus \{0\}$ such that*

$$\mathcal{G}_{\mathcal{L}}(u_0) = r(\mathcal{G}_{\mathcal{L}})u_0.$$

Proof. (i) On account of Theorem 2.16, we can find a real number $\sigma > 0$ and a function $u_{\sigma} \in W^{1,2}(\mathcal{O}) \setminus \{0\}$ such that

$$\begin{cases} \mathcal{L}u = \sigma u & \text{in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = 0. \end{cases}$$

On the other hand, by applying the classical Elliptic Regularity Theory to $\mathcal{L}_{\sigma} := \mathcal{L} - \sigma$ (see, e.g., [11, Corollary 8.35]), one can find a function $\hat{u}_{\sigma} \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ such that

$$\hat{u}_{\sigma} \equiv u_{\sigma} \quad \text{a.e. on } \mathcal{O};$$

as a consequence, by the very definition of $\mathcal{G}_{\mathcal{L}}$ we infer that

$$\mathcal{G}_{\mathcal{L}}(\hat{u}_{\sigma}) = \frac{1}{\sigma} \hat{u}_{\sigma}.$$

This proves that $\lambda := \frac{1}{\sigma} > 0$ lays in the (point-wise) spectrum of $\mathcal{G}_{\mathcal{L}}$ (thought of as an operator from $C(\overline{\mathcal{O}}, \mathbb{R})$ into itself), and thus $r(\mathcal{G}_{\mathcal{L}}) > 0$.

(ii) First of all, since $C^1(\overline{\mathcal{O}}, \mathbb{R})$ is continuously embedded in $C(\overline{\mathcal{O}}, \mathbb{R})$, we straightforwardly derive from Proposition 2.7 (ii) that $\mathcal{G}_{\mathcal{L}}$ is compact from $C(\overline{\mathcal{O}}, \mathbb{R})$ into itself; moreover, if we denote by V_0 the convex cone in $C(\overline{\mathcal{O}}, \mathbb{R})$ defined as

$$V_0 := C(\overline{\mathcal{O}}, \mathbb{R}^+) = \{u \in C(\overline{\mathcal{O}}, \mathbb{R}) : u \geq 0 \text{ on } \mathcal{O}\},$$

we know from Proposition 2.7 (iii) that $\mathcal{G}_{\mathcal{L}}(V_0) \subseteq V_0$. Since, obviously, $V_0 - V_0$ is dense in $C(\overline{\mathcal{O}}, \mathbb{R})$ and since, by statement (i), the spectral radius $r(\mathcal{G}_{\mathcal{L}})$ of $\mathcal{G}_{\mathcal{L}}$ is strictly positive, we are entitled to apply Krein–Rutman’s Theorem, ensuring that $r(\mathcal{G}_{\mathcal{L}})$ is an eigenvalue of $\mathcal{G}_{\mathcal{L}}$ with positive eigenvector: this means that there exists $u_0 \in V_0 \setminus \{0\}$ such that

$$\mathcal{G}_{\mathcal{L}}(u_0) = r(\mathcal{G}_{\mathcal{L}})u_0 \iff u_0 = \frac{1}{r(\mathcal{G}_{\mathcal{L}})} \mathcal{G}_{\mathcal{L}}(u_0).$$

Now, since $u_0 \in V_0 \setminus \{0\}$, we have $u_0 \geq 0$ and $u \neq 0$ on \mathcal{O} ; moreover, reminding that $\mathcal{G}_{\mathcal{L}}$ maps $C(\overline{\mathcal{O}}, \mathbb{R})$ into $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ (see (2.6)), we derive that $u_0 \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$. Gathering together all these facts, we conclude that $u_0 \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R}) \setminus \{0\}$ and that $u \geq 0$ on $\overline{\mathcal{O}}$, as desired. \square

3 Existence and Non-existence Results

In this section we study the solvability of the following system of second order elliptic differential equations subject to functional BCs

$$\begin{cases} \mathcal{L}_k u_k = \lambda_k f_k(x, u_1, \dots, u_m, \nabla u_1, \dots, \nabla u_m) & \text{in } \mathcal{O} & (k = 1, 2, \dots, m), \\ u_k(x) = \eta_k \zeta_k(x) h_k[u_1, \dots, u_m] & \text{for } x \in \partial\mathcal{O} & (k = 1, 2, \dots, m), \end{cases} \quad (3.1)$$

where, as in the Introduction, $m \geq 1$ is a fixed natural number, $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set and $\mathcal{L}_1, \dots, \mathcal{L}_m$ are uniformly elliptic PDOs on \mathcal{O} as in Section 2. To be more precise, we suppose that

- (I) \mathcal{O} is bounded, connected and of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$,
- (II) for every fixed $k = 1, \dots, m$, the differential operator \mathcal{L}_k satisfies assumptions (H1)–(H3) introduced in Section 2, that is,

- \mathcal{L}_k takes the divergence form (2.1), i.e.,

$$\mathcal{L}_k u := - \sum_{i,j=1}^n \partial_{x_i} \left(a_{i,j}^{(k)}(x) \partial_{x_j} u + b_i^{(k)}(x) u \right) + \sum_{i=1}^n c_i^{(k)}(x) \partial_{x_i} u + d^{(k)}(x) u,$$

- the coefficient functions of \mathcal{L}_k belong to $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$,
- the matrix $A^{(k)}(x) := (a_{i,j}^{(k)}(x))_{i,j}$ is symmetric for any $x \in \mathcal{O}$,
- \mathcal{L}_k is uniformly elliptic in \mathcal{O} , i.e., there exists $\Lambda_k > 0$ such that

$$\frac{1}{\Lambda_k} \|\xi\|^2 \leq \sum_{i,j=1}^n a_{i,j}^{(k)}(x) \xi_i \xi_j \leq \Lambda_k \|\xi\|^2 \quad \text{for any } x \in \mathcal{O} \text{ and } \xi \in \mathbb{R}^n \setminus \{0\},$$

- for every nonnegative function $\varphi \in C_0^\infty(\mathcal{O}, \mathbb{R})$ one has

$$\int_{\mathcal{O}} \left(d^{(k)} \varphi + \sum_{i=1}^n b_i^{(k)} \partial_{x_i} \varphi \right) dx \geq 0, \quad \int_{\mathcal{O}} \left(d^{(k)} \varphi + \sum_{i=1}^n c_i^{(k)} \partial_{x_i} \varphi \right) dx \geq 0.$$

Furthermore, for every fixed $k = 1, \dots, m$ we also assume that

- (III) f_k is a real-valued function defined on $\overline{\mathcal{O}} \times \mathbb{R}^m \times \mathbb{R}^{nm}$,
- (IV) h_k is a real-valued operator defined on the space $C^1(\overline{\mathcal{O}}, \mathbb{R}^m)$,
- (V) $\zeta_k \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ and $\zeta_k \geq 0$ on \mathcal{O} ,
- (VI) λ_k, η_k are nonnegative real parameters.

Throughout the sequel, if u_1, \dots, u_m are real-valued functions defined on \mathcal{O} , we set

$$\mathbf{u}(x) := (u_1(x), \dots, u_m(x)) \quad (x \in \mathcal{O}).$$

If, in addition, $\mathbf{u} \in C^1(\mathcal{O}, \mathbb{R}^m)$ (that is, $u_1, \dots, u_m \in C^1(\mathcal{O}, \mathbb{R})$), we define

$$D\mathbf{u}(x) := (\nabla u_1(x), \dots, \nabla u_m(x)) \quad (x \in \mathcal{O}).$$

Now, in view of assumptions (I)–(II), all the results presented in Section 2 can be applied to each operator \mathcal{L}_k (for a fixed $k = 1, \dots, m$); in particular, for every $f \in C(\overline{\mathcal{O}}, \mathbb{R})$ there exists a unique solution $u_f \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ of the Poisson problem

$$\begin{cases} \mathcal{L}_k u = f & \text{in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = 0. \end{cases} \quad (3.2)$$

Furthermore, since the function ζ_k belongs to $C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ (see assumption (V)), there exists a unique solution $\gamma_k \in C^{1,\alpha}(\overline{\mathcal{O}}, \mathbb{R})$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_k u = 0 & \text{in } \mathcal{O}, \\ u|_{\partial\mathcal{O}} = \zeta_k. \end{cases} \quad (3.3)$$

We then denote by \mathcal{G}_k the Green’s operator $\mathcal{G}_{\mathcal{L}_k}$ for \mathcal{L}_k defined in (2.6), and we indicate by g_k the Green’s function $g_{\mathcal{L}_k}$ for the operator \mathcal{G}_k defined through Theorem 2.8. We remind that, if $f \in C(\bar{\mathcal{O}}, \mathbb{R})$ is arbitrary fixed, $\mathcal{G}_k(f)$ is the unique solution in $C^{1,\alpha}(\bar{\mathcal{O}}, \mathbb{R})$ of the Poisson problem (3.2); moreover, we have the representation formulas

$$\mathcal{G}_k(f)(x) = \int_{\mathcal{O}} g_k(y; x)f(y) \, dy \quad \text{and} \quad \partial_{x_i} \mathcal{G}_k(f)(x) = \int_{\mathcal{O}} \partial_{x_i} g_k(y; x)f(y) \, dy,$$

holding true for a.e. $x \in \mathcal{O}$ and any $i = 1, \dots, n$ (see Theorem 2.8 and Proposition 2.14).

Finally, according to Proposition 2.17, we denote by $r_k = r(\mathcal{G}_k) > 0$ the spectral radius of the operator \mathcal{G}_k (thought of as an operator from $C^1(\bar{\mathcal{O}}, \mathbb{R})$ into itself) and we fix once and for all a function $\varphi_k \in C^{1,\alpha}(\bar{\mathcal{O}}, \mathbb{R}) \setminus \{0\}$ such that (setting $\mu_k := \frac{1}{r_k}$)

$$\varphi_k = \mu_k \mathcal{G}_k(\varphi_k) \quad \text{and} \quad \varphi_k \geq 0 \quad \text{on } \mathcal{O}. \tag{3.4}$$

Now that we have properly introduced all the “mathematical objects” appearing in problem (3.1), it is opportune to define what we mean by a *solution* of this problem.

To this end, we first fix some notation. Assume for the moment that f_1, \dots, f_m are continuous on $\bar{\mathcal{O}} \times \mathbb{R}^m \times \mathbb{R}^{nm}$ (as we shall see in Theorem 3.3, this assumption can be relaxed by requiring that the functions f_i are continuous on a suitable subset of their domain). Then, for every index $k \in \{1, \dots, m\}$, we denote by \mathcal{F}_k the so-called *superposition (Nemytskii)* operator associated with f_k , that is,

$$\mathcal{F}_k : C^1(\bar{\mathcal{O}}, \mathbb{R}^m) \rightarrow C(\bar{\mathcal{O}}, \mathbb{R}), \quad \mathcal{F}_k(\mathbf{u}) := f_k(x, \mathbf{u}, D\mathbf{u}).$$

Moreover, we consider the operators $\mathcal{T}, \Gamma : C^1(\bar{\mathcal{O}}, \mathbb{R}^m) \rightarrow C^1(\bar{\mathcal{O}}, \mathbb{R}^m)$ defined by

$$\mathcal{T}(\mathbf{u}) = (\lambda_k (\mathcal{G}_k \circ \mathcal{F}_k)(\mathbf{u}))_{k=1,\dots,m} \quad \text{and} \quad \Gamma(\mathbf{u}) := (\eta_k \gamma_k(x) h_k[\mathbf{u}])_{k=1,\dots,m}.$$

We can now give the definition of *solution* of problem (3.1).

Definition 3.1. We say that a function $\mathbf{u} \in C^1(\bar{\mathcal{O}}, \mathbb{R}^m)$ is a *weak solution* of system (3.1) if \mathbf{u} is a fixed point of the operator $\mathcal{T} + \Gamma$, that is,

$$\mathbf{u} = \mathcal{T}(\mathbf{u}) + \Gamma(\mathbf{u}) = (\lambda_k (\mathcal{G}_k \circ \mathcal{F}_k)(\mathbf{u}) + \eta_k \gamma_k(x) h_k[\mathbf{u}])_{k=1,\dots,m}.$$

If, in addition, the components of \mathbf{u} are nonnegative and $u_j \neq 0$ for some j , we say that \mathbf{u} is a *nonzero positive solution* of system (3.1).

For our existence result, we make use of the following proposition that states the main properties of the classical fixed point index, for more details see [2, 15]. In what follows the closure and the boundary of subsets of a cone \hat{P} are understood to be relative to \hat{P} .

Proposition 3.2. *Let X be a real Banach space and let $\hat{P} \subset X$ be a cone. Let D be an open bounded set of X with $0 \in D \cap \hat{P}$ and $D \cap \hat{P} \neq \hat{P}$. Assume that $T : D \cap \hat{P} \rightarrow \hat{P}$ is a compact operator such that $x \neq T(x)$ for $x \in \partial(D \cap \hat{P})$. Then the fixed point index $i_{\hat{P}}(T, D \cap \hat{P})$ has the following properties:*

- (i) *If there exists $e \in \hat{P} \setminus \{0\}$ such that $x \neq T(x) + \sigma e$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 0$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 0$.*
- (ii) *If $T(x) \neq \sigma x$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 1$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 1$.*
- (iii) *Let D^1 be open bounded in X such that $(D^1 \cap \hat{P}) \subset (D \cap \hat{P})$. If $i_{\hat{P}}(T, D \cap \hat{P}) = 1$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 0$, then T has a fixed point in $(D \cap \hat{P}) \setminus (D^1 \cap \hat{P})$. The same holds if $i_{\hat{P}}(T, D \cap \hat{P}) = 0$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 1$.*

We can now state a result regarding the existence of positive solutions for system (3.1).

In the sequel, we will consider on the space \mathbb{R}^s (where s will be either m, n or mn) the maximum norm

$$|\mathbf{v}| := \max_{i=1,\dots,s} |v_i| \quad (\text{if } \mathbf{v} = (v_1, \dots, v_s)). \tag{3.5}$$

We will work in the Banach space $C(\bar{\mathcal{O}}, \mathbb{R}^m)$ endowed with the norm

$$\|\mathbf{z}\|_{\infty} = \max_{x \in \bar{\mathcal{O}}} |\mathbf{z}(x)| := \max\{\|z_1\|_{\infty}, \dots, \|z_m\|_{\infty}\},$$

where $\mathbf{z} = (z_1, \dots, z_m) \in C(\bar{\mathcal{O}}, \mathbb{R}^m)$, compare also with (2.4). Moreover, we will consider the Banach space $C^1(\bar{\mathcal{O}}, \mathbb{R}^m)$ endowed with the norm

$$\begin{aligned} \|\mathbf{u}\|_{C^1(\bar{\mathcal{O}}, \mathbb{R}^m)} &:= \max\left\{\max_{k=1,2,\dots,m} \|u_k\|_\infty, \max_{k=1,2,\dots,m} \|\nabla u_k\|_\infty\right\}, \\ &= \max\{\|u_k\|_\infty, \|\partial_{x_l} u_k\|_\infty : k = 1, \dots, m \text{ and } l = 1, \dots, n\}; \end{aligned} \tag{3.6}$$

notice that (3.6) reduces to (2.5) when $m = 1$. Given a finite sequence $\varrho = \{\rho_k\}_{k=1}^m \subseteq (0, +\infty)$, we define

$$I(\varrho) = \prod_{k=1}^m [0, \rho_k] \quad \text{and} \quad R(\varrho) = \prod_{k=1}^m R_{\rho_k}, \tag{3.7}$$

where $R_\rho = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v}| \leq \rho\}$ (for $t > 0$); we also introduce, with abuse of notation, the sets

$$\begin{aligned} P &:= \{\mathbf{u} \in C^1(\bar{\mathcal{O}}, \mathbb{R}^m) : u_k \geq 0 \text{ on } \mathcal{O} \text{ for every } k = 1, \dots, m\}, \\ P(\varrho) &= \{\mathbf{u} \in C^1(\bar{\mathcal{O}}, \mathbb{R}^m) : \mathbf{u}(x) \in I(\varrho) \text{ and } D\mathbf{u}(x) \in R(\varrho) \text{ for all } x \in \bar{\mathcal{O}}\} \subseteq P. \end{aligned}$$

Theorem 3.3. *Let assumptions (I)–(VI) be in force. Moreover, let us suppose that one can find a finite sequence $\varrho = \{\rho_k\}_{k=1}^m \subseteq (0, \infty)$ satisfying the following hypotheses:*

- (a) *For every $k = 1, \dots, m$, one has that*
 - (a)₁ f_k *continuous and nonnegative on $\bar{\mathcal{O}} \times I(\varrho) \times R(\varrho)$,*
 - (a)₂ h_k *continuous, nonnegative and bounded on $P(\varrho)$.*
- (b) *There exist $\delta \in (0, +\infty)$, $k_0 \in \{1, 2, \dots, m\}$ and $\rho_0 \in (0, \min_{k=1,\dots,m} \rho_k)$ such that*

$$f_{k_0}(x, \mathbf{z}, \mathbf{w}) \geq \delta z_{k_0} \quad \text{for every } (x, \mathbf{z}, \mathbf{w}) \in \bar{\mathcal{O}} \times I_0 \times R_0, \tag{3.8}$$

where $I_0 := \prod_{i=1}^m [0, \rho_0]$ and $R_0 := \prod_{k=1}^m R_{\rho_0}$.

- (c) *Setting, for every $k = 1, \dots, m$,*

$$\begin{aligned} M_k &:= \max\{f_k(x, \mathbf{z}, \mathbf{w}) : (x, \mathbf{z}, \mathbf{w}) \in \bar{\mathcal{O}} \times I(\varrho) \times R(\varrho)\}, \\ H_k &:= \sup_{\mathbf{u} \in P(\varrho)} h_k[\mathbf{u}], \end{aligned} \tag{3.9}$$

the following inequalities are satisfied:

- (c)₁ $\mu_{k_0} \leq \delta \lambda_{k_0}$,
- (c)₂ $\lambda_k M_k \|\mathcal{G}_k(\hat{1})\|_\infty + \eta_k H_k \|\gamma_k\|_\infty \leq \rho_k$,
- (c)₃ *for any $l = 1, \dots, n$ we have $\lambda_k M_k G_{k,l} + \eta_k H_k \|\partial_{x_l} \gamma_k\|_\infty \leq \rho_k$, where*

$$G_{k,l} := \sup_{x \in \bar{\mathcal{O}}} \int_{\mathcal{O}} |\partial_{x_l} g_k(y; x)| \, dy \quad (\text{see Lemma 2.12}). \tag{3.10}$$

Then system (3.1) has a nonzero positive weak solution $\mathbf{u} \in C^1(\bar{\mathcal{O}}, \mathbb{R}^m)$ such that

$$\|\mathbf{u}\|_{C^1(\bar{\mathcal{O}}, \mathbb{R}^m)} \geq \rho_0 \quad \text{and} \quad \|u_k\|_\infty \leq \rho_k \text{ for every } k = 1, \dots, m. \tag{3.11}$$

Proof. For the sake of readability, we split the proof into different steps.

Step I: We first prove that the operator $\mathcal{A} := \mathcal{T} + \Gamma$ maps $P(\varrho)$ into P . To this end, let $\mathbf{u} \in P(\varrho)$ and let $k \in \{1, \dots, m\}$ be fixed. Since $\mathbf{u} \in P(\varrho)$, from assumption (a)₂ we derive that $h_k[\mathbf{u}] \geq 0$; moreover, since $\gamma_k \geq 0$ on $\bar{\mathcal{O}}$ (see Proposition 2.7 (iii)) and since, by assumption (VI), $\eta_k \geq 0$, we get

$$\Gamma_k(\mathbf{u})(x) = \eta_k \gamma_k(x) h_k[\mathbf{u}] \geq 0 \quad \text{for all } x \in \bar{\mathcal{O}}. \tag{3.12}$$

On the other hand, since $\mathbf{u} \in P(\varrho)$, by assumption (a)₁ we also have that

$$\mathcal{F}_k(\mathbf{u})(x) = f_k(x, \mathbf{u}(x), D\mathbf{u}(x)) \geq 0 \quad \text{for all } x \in \bar{\mathcal{O}};$$

as a consequence, from Proposition 2.7 (iii) we derive that $\mathcal{G}_k(\mathcal{F}_k(\mathbf{u})) \geq 0$ on $\bar{\mathcal{O}}$. Finally, since $\lambda_k \geq 0$ (by assumption (IV)), we get

$$\mathcal{T}_k(\mathbf{u})(x) = \lambda_k \mathcal{G}_k(\mathcal{F}_k(\mathbf{u})(x)) \geq 0 \quad \text{for every } x \in \bar{\mathcal{O}}. \tag{3.13}$$

By (3.12), (3.13) and the arbitrariness of k , we conclude that $\mathcal{A}(P(\varrho)) \subseteq P$.

Step II: We now prove that $\mathcal{A} : P(\varrho) \rightarrow P$ is compact. To this end, let $\{\mathbf{u}_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $P(\varrho)$, and let $k \in \{1, \dots, m\}$ be fixed. Since h_k is nonnegative and bounded on $P(\varrho)$ (see assumption (a)₂), the sequence $\{h_k[\mathbf{u}_j]\}_j$ is bounded in $(0, \infty)$; as a consequence, there exists $\theta_0 \in [0, \infty)$ such that (up to a sub-sequence)

$$\lim_{j \rightarrow \infty} \Gamma_k(\mathbf{u}_j) = \eta_k \gamma_k(x) \theta_0 \quad \text{in } C^1(\overline{\mathcal{O}}, \mathbb{R}). \tag{3.14}$$

On the other hand, since $\{\mathbf{u}_j\}_j \subseteq P(\varrho)$ and since f_k is continuous on $\overline{\mathcal{O}} \times I(\varrho) \times R(\varrho)$ (see assumption (a)₁), we have (using the notation in (3.9))

$$\|\mathcal{F}(\mathbf{u}_j)\|_\infty \leq M_k \quad \text{for every } j \in \mathbb{N}.$$

As a consequence, since the operator \mathcal{G}_k is compact (as an operator from $C(\overline{\mathcal{O}}, \mathbb{R})$ into $C^1(\overline{\mathcal{O}}, \mathbb{R})$, see Proposition 2.7 (ii)), it is possible to find a function $w_k \in C^1(\overline{\mathcal{O}}, \mathbb{R})$ such that (again by possibly passing to a sub-sequence)

$$\lim_{j \rightarrow \infty} \mathcal{T}_k(\mathbf{u}_j) = \lim_{j \rightarrow \infty} (\lambda_k \mathcal{G}_k(\mathcal{F}_k(\mathbf{u}_j))) = \lambda_k w_k \quad \text{in } C^1(\overline{\mathcal{O}}, \mathbb{R}). \tag{3.15}$$

Gathering together (3.14), (3.15) and (3.6), we infer that (up to a suitable sub-sequence)

$$\lim_{j \rightarrow \infty} \mathcal{A}(\mathbf{u}_j) = (\lambda_k w_k + \eta_k \gamma_k \theta_0)_{k=1, \dots, m} =: \tilde{\mathbf{u}} \quad \text{in } C^1(\overline{\mathcal{O}}, \mathbb{R}^m).$$

Finally, since $\{\mathcal{A}(\mathbf{u}_j)\}_j \subseteq P$ (by Step I) and since P is closed, we conclude that $\tilde{\mathbf{u}} \in P$; this proves the compactness of \mathcal{A} (as an operator from $P(\varrho)$ to P).

To proceed further, we consider the set $P_0 \subseteq C^1(\overline{\mathcal{O}}, \mathbb{R}^m)$ defined as follows:

$$P_0 = \{\mathbf{u} \in C^1(\overline{\mathcal{O}}, \mathbb{R}^m) : \mathbf{u}(x) \in I_0 \text{ and } D\mathbf{u}(x) \in R_0 \text{ for all } x \in \overline{\mathcal{O}}\} \subseteq P(\varrho),$$

where I_0 and R_0 are as in assumption (b). Now, if the operator $\mathcal{A} = \mathcal{T} + \Gamma$ has a fixed point $\mathbf{u}_0 \in \partial P_0 \cup \partial P(\varrho)$ (where the boundaries are both relative to P), then \mathbf{u}_0 is a solution of problem (3.1) satisfying (3.11), and the theorem is proved.

If, instead, \mathcal{A} is fixed-point free on $\partial P_0 \cup \partial P(\varrho)$, the fixed-point indexes $i_P(\mathcal{A}, P_0 \setminus \partial P_0)$, $i_P(\mathcal{A}, P(\varrho) \setminus \partial P(\varrho))$ are well-defined. Assuming this last possibility, we consider the following steps.

Step III: In this step we prove the following fact:

$$i_P(\mathcal{A}, P(\varrho) \setminus \partial P(\varrho)) = 1. \tag{3.16}$$

According to Proposition 3.2 (ii), to prove (3.16) it suffices to show that

$$\mathcal{A}(\mathbf{u}) \neq \sigma \mathbf{u} \quad \text{for every } \mathbf{u} \in \partial P(\varrho) \text{ and every } \sigma > 1. \tag{3.17}$$

To establish (3.17), we argue by contradiction, and we suppose that there exist a function $\mathbf{u} \in \partial P(\varrho)$ and a real $\sigma > 1$ such that

$$\sigma \mathbf{u} = \mathcal{A}(\mathbf{u}) = \mathcal{T}(\mathbf{u}) + \Gamma(\mathbf{u}).$$

Since $\mathbf{u} \in \partial P(\varrho)$, there exists an index $k \in \{1, \dots, m\}$ such that either

$$\|u_k\|_\infty = \rho_k \quad \text{or} \quad \|\nabla u_k\|_\infty = \rho_k.$$

We then distinguish these two cases.

Case $\|u_k\|_\infty = \rho_k$. By exploiting assumption (a)₁ and (3.9), we have

$$0 \leq \mathcal{F}_k(\mathbf{u})(x) = f_k(x, \mathbf{u}(x), D\mathbf{u}(x)) \leq M_k \quad \text{for all } x \in \overline{\mathcal{O}}; \tag{3.18}$$

from this, we derive the following chain of inequalities:

$$\begin{aligned} \sigma u_k(x) &= \lambda_k \mathcal{G}_k(\mathcal{F}_k(\mathbf{u}))(x) + \eta_k \gamma_k(x) h_k[\mathbf{u}] \\ &\leq \lambda_k \mathcal{G}_k(M_k \hat{1})(x) + \eta_k \gamma_k(x) h_k[\mathbf{u}] \quad (\text{since } \mathcal{G}_k(M_k \hat{1} - \mathcal{F}_k(\mathbf{u})) \geq 0, \text{ see (3.18) and Proposition 2.7 (iii)}) \\ &\leq \|\lambda_k \mathcal{G}_k(M_k \hat{1})\|_\infty + \|\eta_k H_k \gamma_k\|_\infty \quad (\text{since } \mathbf{u} \in \partial P(\varrho) \subseteq P(\varrho), \text{ see (3.9)}) \\ &= \lambda_k M_k \|\mathcal{G}_k(\hat{1})\|_\infty + \eta_k H_k \|\gamma_k\|_\infty \\ &\leq \rho_k \quad (\text{see assumption (c)}_2). \end{aligned} \tag{3.19}$$

As a consequence, by taking the supremum for $x \in \bar{\mathcal{O}}$ in (3.19) (and by reminding that $\mathbf{u} \in \partial P(\rho) \subseteq P(\rho)$), we then obtain

$$\sup_{x \in \bar{\mathcal{O}}} |\sigma u_k(x)| \leq \sigma \rho_k \leq \rho_k,$$

which is clearly a contradiction (since $\sigma > 1$).

Case $\|\nabla u_k\|_\infty = \rho_k$. By the very definition of $\|\cdot\|_\infty$, there exists $l \in \{1, \dots, n\}$ such that $\|\partial_{x_l} u_k\|_\infty = \rho_k$. Moreover, by Proposition 2.14 we have

$$\sigma \partial_{x_l} u_k(x) = \lambda_k \int_{\mathcal{O}} \partial_{x_l} g_k(y; x) f_k(x, \mathbf{u}(y), D\mathbf{u}(y)) \, dy + \eta_k \partial_{x_l} \gamma_k(x) h_k[\mathbf{u}],$$

for a.e. $x \in \mathcal{O}$. By means of this representation formula, we then obtain

$$\begin{aligned} \sigma |\partial_{x_l} u_k(x)| &\leq \lambda_k \int_{\mathcal{O}} |\partial_{x_l} g_k(y; x) f_k(x, \mathbf{u}(y), D\mathbf{u}(y))| \, dy + \eta_k h_k[\mathbf{u}] |\partial_{x_l} \gamma_k(x)| \\ &\leq \lambda_k M_k \int_{\mathcal{O}} |\partial_{x_l} g_k(y; x)| \, dy + \eta_k H_k |\partial_{x_l} \gamma_k(x)| \quad (\text{since } \mathbf{u} \in \partial P(\rho) \subseteq P(\rho), \text{ see also (3.18)}) \\ &\leq \lambda_k M_k G_{k,l} + \eta_k H_k \|\partial_{x_l} \gamma_k\|_\infty \quad (\text{see (3.10)}). \end{aligned}$$

As a consequence, by taking the supremum for $x \in \bar{\mathcal{O}}$ in (3.19) (and by reminding that $\|\partial_{x_l} u_k\|_\infty = \rho_k$), from assumption (c)₃ we infer that

$$\sup_{x \in \bar{\mathcal{O}}} (\sigma |\partial_{x_l} u_k(x)|) = \sigma \rho_k \leq \lambda_k M_k G_{k,l} + \eta_k H_k \|\partial_{x_l} \gamma_k\|_\infty \leq \rho_k,$$

which is clearly a contradiction (as $\sigma > 1$).

This completes the demonstration of (3.17).

Step IV: In this last step we prove the following fact:

$$i_P(A, P_0 \setminus \partial P_0) = 0. \tag{3.20}$$

According to Proposition 3.2 (i), to prove (3.20) it suffices to show that there exists a suitable function $e \in P \setminus \{0\}$ satisfying the property

$$A(\mathbf{u}) + \sigma e \neq \mathbf{u} \quad \text{for every } \mathbf{u} \in \partial P_0 \text{ and every } \sigma > 0. \tag{3.21}$$

To establish (3.21), we let $e := (\varphi_1, \dots, \varphi_m)$ (where $\varphi_1, \dots, \varphi_m$ are as in (3.4)) and we argue by contradiction: we thus suppose that there exist $\mathbf{u} \in \partial P_0$ and $\sigma > 0$ such that

$$\mathbf{u} = A(\mathbf{u}) + \sigma e = \mathcal{J}(\mathbf{u}) + \Gamma(\mathbf{u}) + \sigma e.$$

Since $\mathbf{u} \in \partial P_0 \subseteq P_0 \subseteq P(\rho)$ (by the definition of P_0 , see assumption (b)), we know from Step I that $A(\mathbf{u}) \in P$; as a consequence, if k_0 is as in assumption (b), we have

$$u_{k_0} = A(\mathbf{u})_{k_0} + \sigma \varphi_{k_0} \geq \sigma \varphi_{k_0} \quad \text{on } \bar{\mathcal{O}}.$$

Furthermore, by exploiting once again assumption (b) we get

$$\mathcal{F}_{k_0}(\mathbf{u}) = f_{k_0}(x, \mathbf{u}(x), D\mathbf{u}(x)) \geq \delta u_{k_0}(x) \geq \delta \sigma \varphi_{k_0}(x) \quad \text{for all } x \in \bar{\mathcal{O}}. \tag{3.22}$$

Gathering together all these facts, for every $x \in \bar{\mathcal{O}}$ we have

$$\begin{aligned} u_{k_0}(x) &= \lambda_{k_0} \mathcal{G}_{k_0}(\mathcal{F}_{k_0}(\mathbf{u}))(x) + \eta_{k_0} \gamma_{k_0}(x) h_{k_0}[\mathbf{u}] + \sigma \varphi_{k_0}(x) \\ &\geq \lambda_{k_0} \mathcal{G}_{k_0}(\delta \sigma \varphi_{k_0})(x) + \sigma \varphi_{k_0}(x) \quad (\text{since } \mathcal{G}_{k_0}(\mathcal{F}_{k_0}(\mathbf{u}) - \delta \sigma \varphi_{k_0}) \geq 0, \text{ see (3.22) and Proposition 2.7 (iii)}) \\ &= \frac{\delta \lambda_{k_0}}{\mu_{k_0}} \cdot \sigma \varphi_{k_0}(x) + \sigma \varphi_{k_0}(x) \quad (\text{since } \varphi_{k_0} \text{ is an eigenfunction of } \mathcal{G}_{k_0}, \text{ see (3.4)}) \\ &\geq 2\sigma \varphi_{k_0}(x) \quad (\text{see assumption (c)}_1). \end{aligned}$$

By iterating the above argument, for every $x \in \bar{\Omega}$ we get

$$u_{k_0}(x) \geq p\sigma\varphi_{k_0}(x) \quad \text{for every } p \in \mathbb{N},$$

but this is contradiction with the boundedness of $u_{k_0} \in C^1(\bar{\Omega}, \mathbb{R})$ (as $\varphi_{k_0} \not\equiv 0$).

We are now ready to conclude the proof of the theorem: in fact, by combining (3.16), (3.20) and Proposition 3.2 (iii), we infer the existence of a fixed point

$$\mathbf{u}_0 \in (P(\varrho) \setminus \partial P(\varrho)) \setminus P_0$$

of $\mathcal{A} = \mathcal{T} + \Gamma$; thus, \mathbf{u}_0 is a solution of (3.1) satisfying (3.11). □

Remark 3.4. Let the assumption and the notation of Theorem 3.3 do apply. We have already pointed out that, since $\zeta_1, \dots, \zeta_m \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R})$ (see assumption (V)), one has

$$\gamma_k \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}) \quad \text{for every } k = 1, \dots, m.$$

As a consequence, the operator Γ maps $C^1(\bar{\Omega}, \mathbb{R}^m)$ into $C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^m)$. On the other hand, since the operators $\mathcal{G}_1, \dots, \mathcal{G}_m$ map $C(\bar{\Omega}, \mathbb{R})$ into $C^{1,\alpha}(\bar{\Omega}, \mathbb{R})$, we also have that

$$\mathcal{T}(C^1(\bar{\Omega}, \mathbb{R}^m)) \subseteq C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^m).$$

Gathering together all these facts, we conclude that any weak solution of (3.1) (i.e., any fixed point of $\mathcal{A} = \mathcal{T} + \Gamma$ in $C^1(\bar{\Omega}, \mathbb{R}^m)$) is actually of class $C^{1,\alpha}$ on $\bar{\Omega}$.

An elementary argument yields the following non-existence result.

Theorem 3.5. *Let assumptions (I)–(IV) be in force. Moreover, let us suppose that there exists a finite sequence $\varrho = \{\rho_k\}_{k=1}^m \subseteq (0, \infty)$ such that, for every $k = 1, \dots, m$, the following conditions hold:*

(a) f_k is continuous on $\bar{\Omega} \times I(\varrho) \times R(\varrho)$, and there exist $\tau_k \in (0, +\infty)$ such that

$$0 \leq f_k(x, \mathbf{z}, \mathbf{w}) \leq \tau_k z_k \quad \text{for every } (x, \mathbf{z}, \mathbf{w}) \in \bar{\Omega} \times I(\varrho) \times R(\varrho).$$

(b) h_k is continuous and nonnegative on $P(\varrho)$; moreover, there exist $\xi_k \in (0, +\infty)$ such that

$$h_k[\mathbf{u}] \leq \xi_k \cdot \|\mathbf{u}\|_\infty \quad \text{for every } u \in P(\varrho).$$

(c) The following inequality holds:

$$\lambda_k \tau_k \|\mathcal{G}_k(\hat{1})\|_\infty + \eta_k \xi_k \|\gamma_k\|_\infty < 1. \tag{3.23}$$

Then system (3.1) has at most the zero solution in $P(\varrho)$.

Proof. We argue by contradiction and we assume that (3.1) has a solution $\mathbf{u} \in P(\varrho) \setminus \{0\}$. According to Definition 3.1, this means that \mathbf{u} is a fixed point of the operator $\mathcal{A} = \mathcal{T} + \Gamma$. Setting $\rho := \|\mathbf{u}\|_\infty > 0$, we let $j \in \{1, 2, \dots, m\}$ such that

$$\|u_j\|_\infty = \rho. \tag{3.24}$$

For every $x \in \bar{\Omega}$, we then have

$$0 \leq \mathcal{F}_j(\mathbf{u})(x) = f_j(x, \mathbf{u}(x), D\mathbf{u}(x)) \leq \tau_j u_j(x) \leq \tau_j \rho; \tag{3.25}$$

from this, we obtain

$$\begin{aligned} u_j(x) &= \lambda_j \mathcal{G}_j(\mathcal{F}_j(\mathbf{u}))(x) + \eta_j \gamma_j(x) h_j[\mathbf{u}] \\ &\leq \lambda_j \mathcal{G}_j(\tau_j \rho \hat{1})(x) + \eta_j \gamma_j(x) h_j[\mathbf{u}] \quad (\text{since } \mathcal{G}_j(\tau_j \rho \cdot \hat{1} - \mathcal{F}_j(\mathbf{u})) \geq 0, \text{ see (3.25) and Proposition 2.7 (iii)}) \\ &\leq \|\lambda_j \mathcal{G}_j(\tau_j \rho \hat{1})\|_\infty + \|\eta_j \xi_j \rho \gamma_j\|_\infty \quad (\text{by assumption (b) and since } \|\mathbf{u}\|_\infty = \rho) \\ &= (\lambda_j \tau_j \|\mathcal{G}_k(\hat{1})\|_\infty + \eta_j \xi_j \|\gamma_j\|_\infty) \rho. \end{aligned} \tag{3.26}$$

By taking the supremum in (3.26) for $x \in \bar{\Omega}$, from (3.23) and (3.24) we finally obtain

$$\rho = \sup_{x \in \bar{\Omega}} u_j(x) \leq (\lambda_j \tau_j \|\mathcal{G}_k(\hat{1})\|_\infty + \eta_j \xi_j \|\gamma_j\|_\infty) \rho < \rho,$$

a contradiction. Thus, problem (3.1) cannot have nonzero solutions in $P(\varrho)$. □

4 Examples

In this last section we present a couple of concrete examples illustrating the applicability of our main results, namely Theorems 3.3 and 3.5.

Example 4.1. On Euclidean space \mathbb{R}^3 , let us consider the following BVP:

$$\begin{cases} -\Delta u_1 = \lambda_1 e^{u_1} (1 + |\nabla u_2|^2) & \text{in } B, \\ -\Delta u_2 = \lambda_2 (16 - u_2^2) \cos(\langle \nabla u_1, \nabla u_2 \rangle) & \text{in } B, \\ u_1|_{\partial B} = \eta_1 (u_1(0) + u_2(0)), \\ u_2|_{\partial B} = \eta_2 \int_{\partial \bar{B}_1} u_1 (1 - |\nabla u_2|^2) d\sigma, \end{cases} \tag{4.1}$$

where B is the Euclidean ball centered at 0 with radius 1, and $|\cdot|$ is the max norm in \mathbb{R}^3 , as in (3.5).

Obviously, this problem takes the form (3.1) with (here and throughout, we denote the points of \mathbb{R}^6 by $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$, with $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$)

- (i) $\mathcal{O} := B$,
- (ii) $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$,
- (iii) $f_1 : \bar{B} \times \mathbb{R}^2 \times \mathbb{R}^6 \rightarrow \mathbb{R}, f_1(x, \mathbf{z}, \mathbf{w}) = e^{z_1} (1 + |\mathbf{w}_2|^2)$,
- (iv) $f_2 : \bar{B} \times \mathbb{R}^2 \times \mathbb{R}^6 \rightarrow \mathbb{R}, f_2(x, \mathbf{z}, \mathbf{w}) = (16 - z_2^2) \cos(\langle \mathbf{w}_1, \mathbf{w}_2 \rangle)$,
- (v) $h_1 : C^1(\bar{B}, \mathbb{R}^2) \rightarrow \mathbb{R}, h_1[u_1, u_2] := u_1(0) + u_2(0)$,
- (vi) $h_2 : C^1(\bar{B}, \mathbb{R}^2) \rightarrow \mathbb{R}, h_2[u_1, u_2] := \int_{\partial B} u_1^2 (1 - |\nabla u_2|^2) d\sigma$,
- (vii) $\zeta_1 \equiv \zeta_2 \equiv 1$.

Furthermore, it is straightforward to check that all the structural assumptions (I)–(VI) listed at the beginning of Section 3 are satisfied (for every $\alpha \in (0, 1)$). We now aim to show that, in this case, also assumptions (a)–(c) in statement of Theorem 3.3 are fulfilled.

Assumption (a). To begin with, we consider the finite sequence

$$\varrho = \{\rho_1, \rho_2\}, \quad \text{where } \rho_1 = \rho_2 = \sqrt{\frac{\pi}{6}}. \tag{4.2}$$

Clearly, the function f_1 is continuous and nonnegative on $\bar{B} \times I(\varrho) \times R(\varrho)$ (see (3.7) for the definition of $I(\varrho)$ and $R(\varrho)$); moreover, since $\rho_1, \rho_2 \leq 4$ and since, by Cauchy–Schwarz inequality, we have (remind the definition of $|\cdot|$ in (3.5))

$$|\langle \mathbf{w}_1, \mathbf{w}_2 \rangle| \leq 3 |\mathbf{w}_1| \cdot |\mathbf{w}_2| \leq \frac{\pi}{2} \quad \text{for any } \mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in R(\varrho),$$

we easily deduce that also f_2 is (continuous and) nonnegative on $\bar{B} \times I(\varrho) \times R(\varrho)$.

As for the operators h_1, h_2 , it is immediate to check that they are (continuous and) nonnegative when restricted to the cone $P(\varrho)$ (note that, if $\mathbf{u} \in P(\varrho)$, we have $|\nabla u_2| \leq \rho_2 < 1$); furthermore, since $\mathbf{u} = (u_1, u_2) \in P(\varrho)$ implies that $0 \leq u_1, u_2 \leq \sqrt{\frac{\pi}{6}}$, we have

$$h_1[\mathbf{u}] = h_1[u_1, u_2] \leq 2\sqrt{\frac{\pi}{6}} \quad \text{and} \quad h_2[\mathbf{u}] = h_2[u_1, u_2] \leq \frac{\pi}{6} |\partial B| = \frac{2\pi^2}{3}. \tag{4.3}$$

Thus, h_1, h_2 are bounded on $P(\varrho)$, and this proves that assumption (a) is fulfilled.

Assumption (b). First of all we observe that, by definition, one has

$$f_1(x, \mathbf{z}, \mathbf{w}) \geq e^{z_1} \quad \text{for every } (x, \mathbf{z}, \mathbf{w}) \in \bar{B} \times \mathbb{R}^2 \times \mathbb{R}^6;$$

as a consequence, given any $\delta > 0$, it is possible to find a small $\rho_0 = \rho_0(\delta) \in (0, \sqrt{\frac{\pi}{6}})$ such that (here, $I_0 = [0, \rho_0] \times [0, \rho_0]$ and $R_0 := R_{\rho_0} \times R_{\rho_0}$, see (3.7))

$$f_1(x, \mathbf{z}, \mathbf{w}) \geq e^{z_1} \geq \delta z_1 \quad \text{for every } (x, \mathbf{z}, \mathbf{w}) \in \bar{B} \times I_0 \times R_0.$$

This proves that f_1 satisfies (3.8), and thus assumption (b) is fulfilled (with $k_0 = 1$).

Assumption (c). We begin by explicitly computing the quantities appearing in (3.9). On the one hand, by the very definition of f_1, f_2 we have

$$M_1 = \max_{\bar{B} \times I(\varrho) \times R(\varrho)} f_1 = e^{\sqrt{\frac{\pi}{6}}} \left(1 + \frac{\pi}{6}\right) \quad \text{and} \quad M_2 = \max_{\bar{B} \times I(\varrho) \times R(\varrho)} f_2 = 16. \tag{4.4}$$

On the other hand, on account of (4.3), we have (notice that the constant function defined on \bar{B} by $\mathbf{u} := (\sqrt{\frac{\pi}{6}}, 0)$ certainly belongs to $P(\varrho)$)

$$H_1 = \sup_{\mathbf{u} \in P(\varrho)} h_1[\mathbf{u}] = 2\sqrt{\frac{\pi}{6}} \quad \text{and} \quad H_2 = \sup_{\mathbf{u} \in P(\varrho)} h_2[\mathbf{u}] = \frac{2\pi^2}{3}. \tag{4.5}$$

We now observe that, since $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ (and taking into account the very definition of Green operator, see (2.6)), one obviously has

$$\mathcal{G}_1(\hat{1}) = \mathcal{G}_{\mathcal{L}_1}(\hat{1}) = \mathcal{G}_{(-\Delta)}(\hat{1}) \quad \text{and} \quad \mathcal{G}_2(\hat{1}) = \mathcal{G}_{\mathcal{L}_2}(\hat{1}) = \mathcal{G}_{(-\Delta)}(\hat{1}),$$

where $\mathcal{G}_{(-\Delta)}(\hat{1})$ is the unique solution of

$$\begin{cases} -\Delta u = 1 & \text{in } B, \\ u|_{\partial B} = 0. \end{cases}$$

As a consequence, since a direct computation gives $\mathcal{G}_{(-\Delta)}(\hat{1}) = \frac{1}{6}(1 - \|x\|^2)$, we get

$$\|\mathcal{G}_1(\hat{1})\|_\infty = \|\mathcal{G}_2(\hat{1})\|_\infty = \frac{1}{6}. \tag{4.6}$$

Analogously, since $\zeta_1 \equiv \zeta_2 \equiv 1$ (and again since $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$), from (3.3) we deduce that $\gamma_1 = \gamma_2 = \hat{\gamma}$, where $\hat{\gamma}$ is the unique solution of

$$\begin{cases} \Delta u = 0 & \text{in } B, \\ u|_{\partial B} = 1. \end{cases}$$

As a consequence, since $\hat{\gamma} \equiv 1$ clearly solves the above problem, we get

$$\|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1. \tag{4.7}$$

Finally, according to (3.10), we turn to provide an explicit estimate for

$$\sup_{x \in B} \int_B |\partial_{x_l} g_{(-\Delta)}(y; x)| \, dy \quad (\text{with } l = 1, 2, 3),$$

where $g_{(-\Delta)}$ is the Green function for $(-\Delta)$ (and related to B). To this end, we make crucial use of the *explicit* expression of $g_{(-\Delta)}$ (see, e.g., [10, Section 2.2.4 (c)])

$$g_{(-\Delta)}(y; x) = \frac{1}{4\pi} \left(\|x - y\|^{-1} - (1 + \|x\|^2 \|y\|^2 - 2\langle x, y \rangle)^{-\frac{1}{2}} \right) \tag{4.8}$$

where $\|\cdot\|$ is the usual Euclidean norm in \mathbb{R}^3 . Starting from (4.8), a direct yet tedious computation shows that (for every $x, y \in B$ with $x \neq y$)

$$|\partial_{x_l} g_{(-\Delta)}(y; x)| \leq \frac{1}{2\pi \|x - y\|^2};$$

as a consequence, for every $x \in B$ we have

$$\begin{aligned} \int_B |\partial_{x_l} g_{(-\Delta)}(y; x)| \, dy &\leq \frac{1}{2\pi} \int_B \|x - y\|^{-2} \, dy \leq \frac{1}{2\pi} \int_{\{\|x-y\|<2\}} \|x - y\|^{-2} \, dy \\ &= \frac{1}{2\pi} \int_{\{\|y\|<2\}} \|y\|^{-2} \, dy = \frac{1}{2\pi} |\partial B| \int_0^2 d\rho = 4. \end{aligned}$$

Thus, taking into account that $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, we obtain

$$\mathcal{G}_{1,l} = \mathcal{G}_{2,l} = \sup_{x \in B} \int_B |\partial_{x_l} g_{(-\Delta)}(y; x)| \, dy \leq 4 \quad \text{for every } l = 1, 2, 3. \tag{4.9}$$

By gathering together (4.2), (4.4), (4.5), (4.6), (4.7) and (4.9), we are finally entitled to apply Theorem 3.3: for any $\lambda_1 > 0$ and any $\lambda_2, \eta_1, \eta_2 \geq 0$ satisfying

$$\begin{aligned} \frac{\lambda_1}{6} e^{\sqrt{\frac{\pi}{6}}} \left(1 + \frac{\pi}{6}\right) + 2 \eta_1 \sqrt{\frac{\pi}{6}} &\leq \sqrt{\frac{\pi}{6}} \quad (\text{see assumption (c)}_2), \\ \frac{8}{3} \lambda_2 + \frac{2\pi^2}{3} \eta_2 &\leq \sqrt{\frac{\pi}{6}} \quad (\text{see assumption (c)}_2), \\ \max\left\{4\lambda_1 e^{\sqrt{\frac{\pi}{6}}} \left(1 + \frac{\pi}{6}\right), 64\lambda_2\right\} &\leq \sqrt{\frac{\pi}{6}} \quad (\text{see assumption (c)}_3), \end{aligned}$$

there exists at least one solution $\mathbf{u} = (u_1, u_2) \in C^1(\bar{B}, \mathbb{R}^2)$ of (4.1) such that

$$\|u_1\|_\infty, \|u_2\|_\infty \leq \sqrt{\frac{\pi}{6}} \quad \text{and} \quad \|\mathbf{u}\|_{C^1(\bar{B}, \mathbb{R}^2)} \geq \rho_0.$$

Here, $\rho_0 = \rho_0(\delta) > 0$ is as in assumption (b) and $\delta > 0$ is such that $\mu_1 \leq \delta\lambda_1$ (see assumption (c)₁ and remind that $\mu_1 > 0$ denotes the inverse of the spectral radius of $\mathcal{L}_1 = -\Delta$, see (3.4)). It should be noticed that, since (3.8) holds for any given $\delta > 0$ (by accordingly choosing $\rho_0 = \rho_0(\delta) > 0$), there is no need to have an explicit knowledge of μ_1 .

Example 4.2. On Euclidean space \mathbb{R}^3 , let us consider the following BVP

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1^2(1 - e^{-|\nabla u_2|}) & \text{in } B, \\ -\Delta u_2 = \lambda_2 \sin(u_2)(u_1^3 + |\langle \nabla u_1, \nabla u_2 \rangle|) & \text{in } B, \\ u_1|_{\partial B} = \eta_1 \int_B u_2^2 dx, \\ u_2|_{\partial B} = \eta_2 \max_{\partial B} u_1, \end{cases} \quad (4.10)$$

where B is the Euclidean ball with center 0 and radius 1 and we adopt the same notation of Example 4.1.

Obviously, this problem takes the form (3.1) with

- (i) $\mathcal{O} := B$,
- (ii) $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$,
- (iii) $f_1 : \bar{B} \times \mathbb{R}^2 \times \mathbb{R}^6 \rightarrow \mathbb{R}, f_1(x, \mathbf{z}, \mathbf{w}) = z_1^2(1 - e^{-|\mathbf{w}_2|})$,
- (iv) $f_2 : \bar{B} \times \mathbb{R}^2 \times \mathbb{R}^6 \rightarrow \mathbb{R}, f_2(x, \mathbf{z}, \mathbf{w}) = \sin(z_2)(z_1^3 + |\langle \mathbf{w}_1, \mathbf{w}_2 \rangle|)$,
- (v) $h_1 : C^1(\bar{B}, \mathbb{R}^2) \rightarrow \mathbb{R}, h_1[u_1, u_2] := \int_B u_2^2 dx$,
- (vi) $h_2 : C^1(\bar{B}, \mathbb{R}^2) \rightarrow \mathbb{R}, h_2[u_1, u_2] := \max_{\partial B} u_1$,
- (vii) $\zeta_1 \equiv \zeta_2 \equiv 1$.

Furthermore, it is straightforward to check that all the structural assumptions (I)–(VI) listed at the beginning of Section 3 are satisfied (for every $\alpha \in (0, 1)$). We now aim to show that, in this case, assumptions (a)–(c) in statement of Theorem 3.5 are fulfilled.

Assumption (a). To begin with, we consider the finite sequence

$$\varrho = \{\rho_1, \rho_2\}, \quad \text{where } \rho_1 = \rho_2 = 1. \quad (4.11)$$

Clearly, f_1 is continuous and nonnegative on $\bar{B} \times I(\varrho) \times R(\varrho)$; moreover, for every $(x, \mathbf{z}, \mathbf{w}) \in \bar{B} \times I(\varrho) \times R(\varrho)$ one has (notice that, if $z \in I(\varrho)$, then $0 \leq z_1 \leq 1$)

$$0 \leq f_1(x, \mathbf{z}, \mathbf{w}) = z_1 \cdot (z_1(1 - e^{-|\mathbf{w}_2|})) \leq z_1. \quad (4.12)$$

Thus, f_1 fulfills assumption (a) (with $\tau_1 = 1$).

As regards f_2 , we obviously have that also this function is continuous and nonnegative on $\bar{B} \times I(\varrho) \times R(\varrho)$; moreover, since $0 \leq \sin(t) \leq t$ for every $0 \leq t \leq 1$, we have for every $(x, \mathbf{z}, \mathbf{w}) \in \bar{B} \times I(\varrho) \times R(\varrho)$,

$$\begin{aligned} 0 \leq f_2(x, \mathbf{z}, \mathbf{w}) &\leq z_2(1 + |\langle \mathbf{w}_1, \mathbf{w}_2 \rangle|) \\ &\leq z_2(1 + 3|\mathbf{w}_1| \cdot |\mathbf{w}_2|) \quad (\text{by Cauchy–Schwarz inequality, see Example 4.1}) \\ &\leq 4z_2 \quad (\text{since } \mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in R(\varrho) \text{ implies that } |\mathbf{w}_1|, |\mathbf{w}_2| \leq 1). \end{aligned} \quad (4.13)$$

As a consequence, also f_2 satisfies assumption (a) (with $\tau_2 = 4$).

Assumption (b). First of all, it is very easy to check that both h_1 and h_2 are continuous and nonnegative when restricted to the cone $P(\varrho) \subseteq C^1(\overline{B}, \mathbb{R})$; moreover, since the condition $\mathbf{u} = (u_1, u_2) \in P(\varrho)$ implies that $0 \leq u_1, u_2 \leq 1$, we get

$$h_1[\mathbf{u}] = h_1[u_1, u_2] \leq \int_B u_2 \, dx \leq \left(\max_B u_2 \right) \cdot |B_1| \leq \frac{4\pi}{3} \|\mathbf{u}\|_\infty, \quad (4.14)$$

and this proves that h_1 fulfills assumption (b) (with $\xi_1 = \frac{4\pi}{3}$).

Finally, by exploiting the very definition of $\|\cdot\|_\infty$, we have

$$h_2[\mathbf{u}] = h_2[u_1, u_2] = \max_{\partial B} u_1 \leq \|\mathbf{u}\|_\infty, \quad (4.15)$$

and thus also h_2 satisfies assumption (b) (with $\xi_2 = 1$).

Assumption (c). By making use of all the computations already carried out in the previous Example 4.1, we know that (see, precisely, (4.6) and (4.7))

$$(a) \quad \|\mathcal{G}_1(\hat{1})\|_\infty = \|\mathcal{G}_2(\hat{1})\|_\infty = \frac{1}{6},$$

$$(b) \quad \|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1.$$

As a consequence, by gathering together (4.11), (4.12), (4.13), (4.14), (4.15) and the above (a)–(b), we are entitled to apply Theorem 3.5: for any $\lambda_1, \lambda_2, \eta_1, \eta_2 \geq 0$ satisfying

$$\frac{\lambda_1}{2} + \frac{4\pi}{3} \eta_1 < 1 \quad \text{and} \quad 2\lambda_2 + \eta_2 < 1,$$

the BVP (4.10) possesses only the zero solution (notice that $\mathbf{u} \equiv 0$ trivially solves (4.10)).

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