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Approximation by Convolution Polyanalytic Operators in the Complex and Quaternionic Compact Unit Balls

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Abstract

In this paper, by using the convolution method, we obtain quantitative results in terms of various moduli of smoothness for approximation of polyanalytic functions by polyanalytic polynomials in the complex unit disc. Then, by introducing the polyanalytic Gauss–Weierstrass operators of a complex variable, we prove that they form a contraction semigroup on the space of polyanalytic functions defined on the compact unit disk. The quantitative approximation results in terms of moduli of smoothness are then extended to the case of slice p-polyanalytic functions on the quaternionic unit ball. Moreover, we show that also in the quaternionic case the Gauss–Weierstrass operators of a quaternionic variable form a contraction semigroup on the space of polyanalytic functions defined on the compact unit ball.

Keywords Complex polyanalytic functions · Complex polyanalytic polynomials · Slice quaternionic polyanalytic functions · Slice quaternionic polyanalytic polynomials · Convolution with trigonometric kernels · Quantitative estimates · Moduli of smoothness · Polyanalytic Gauss–Weierstrass operators · Contraction semigroup of operators

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1 Introduction and Preliminaries

Given a natural number p, a complex-valued function f of a complex variable is called a *p-analytic* or *polyanalytic of order p*, in an open set $G \subset \mathbb{C}$, if $\overline{\partial}^p(f) = 0$ in G, where $\overline{\partial}^p$ is the p-power of the Cauchy-Riemann operator, i.e. $\overline{\partial} = \partial/\partial\overline{z}$. It can be proved that f necessarily has the representation

$$f(z) = f_0(z) + \overline{z} f_1(z) + \cdots + \overline{z}^{p-1} f_{p-1}(z), \qquad z \in G, \tag{1}$$

where $f_0, \ldots, f_{p-1}$ are analytic (holomorphic) in G. If all $f_0, \ldots, f_{p-1}$ are polynomials, then f is called p-analytic polynomial and the degree $\deg(f)$ (with respect to z) of a p-analytic polynomial f, is defined as $\max\{\deg(f_j); j = 0, \ldots, p-1\}$. For simplicity, everywhere in the paper we assume that the degree of f is considered with respect to z.

The concept of a polyanalytic function was introduced in 1908 by Kolossov, see [34–37], to study elasticity problems. This stream of research was later on continued by his student Muskhelishvili, see the book [42].

It is also worth mentioning the early paper by Pompeiu [47] and, one decade later, the work of Burgatti, see [12], and in the thirties Teodorescu's doctoral dissertation, see [48]. However, a systematic study of polyanalytic functions was done by the Russian school under the supervision of Balk, see his book [10].

Although the representation (1) suggests that the building blocks of polyanalytic functions are holomorphic functions, the class of polyanalytic functions presents deep differences from the class of holomorphic functions, see [10] for more information.

The lines of the current research on polyanalytic functions are various: the problem of the uniform approximation by p-analytic polynomials, see, e.g., Fedorovskiy [18–21], Carmona–Fedorovskiy [13,14], Carmona–Paramonov–Fedorovskiy [15], Baranov–Carmona–Fedorovskiy [11], Mazalov [39,40], Mazalov–Paramonov –Fedorovskiy [41], Verdera [50], the study of wavelets and Gabor frames, see e.g., Abreu–Gröchenig [5], Abreu [2,3], the time-frequency analysis, see, e.g., Abreu–Feichtinger [4], the sampling and interpolation in function spaces, see, e.g. [1], the image and signal processing, see, e.g. Abreu [1]. Other contributions in this field can be found in Pascali's works [43–46].

For functions p-analytic in G and continuous in G, the available results on approximation using p-analytic polynomials are of qualitative type.

Thus, the first goal of the present paper is to obtain, in Sect. 2, quantitative uniform approximation results in terms of various moduli of smoothness in the particular case when $G = \mathbb{D}$—the open unit disk in $\mathbb{C}$. Section 3 introduces the polyanalytic Gauss–Weierstrass complex operators, for which one proves that they form a contraction semigroup on the space of polyanalytic complex functions in the unit disk. We then move to the quaternionic case, and in Sect. 3 we consider the particular case when $G = \mathbb{B}$ is the open unit ball and we obtain quantitative results, similar to those ones in the complex case, in uniform approximation by slice quaternionic polyanalytic polynomials. Finally, Sect. 5 deals with similar properties for the polyanalytic Gauss–Weierstrass quaternionic operators. The quaternionic cases in Sects. 4 and 5 are motivated by the recent introduction of the class of polyanalytic functions in the quaternionic framework, see Alpay–Diki–Sabadini [7–9], Alpay–Colombo–Diki–Sabadini [6].

To obtain our results, we use the classical method of convolution with various even trigonometric kernels and with the Gauss–Weierstrass kernels, successfully used by us in the past, see, e.g., Gal [22–24], Gal–Sabadini [27–31], Diki–Gal–Sabadini [17].

2 Approximation by Polyanalytic Polynomials

For $p \in \mathbb{N}$ and $\mathbb{D}$ the open unit disk in $\mathbb{C}$, let us denote by $H_p(\overline{\mathbb{D}})$ the space of all p-analytic functions in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, endowed with the uniform norm $\|\cdot\|$.

Definition 2.1 Let $K_n(v)$ be an even trigonometric polynomial of degree $d_n \in \mathbb{N}$, with $K_n(v) \geq 0$, for all $v \in [0, 2\pi]$ and $n \in \mathbb{N}$.

For $i = \sqrt{-1}$, $f \in H_p(\overline{\mathbb{D}})$ and $n \in \mathbb{N}$, let us define the convolution operator

$$L_n(f)(z) = \frac{1}{c_n} \cdot \int_0^{2\pi} f(ze^{iv}) K_n(v) dv = \frac{1}{c_n} \cdot \int_{-\pi}^{\pi} f(ze^{iv}) K_n(v) dv, \tag{2}$$

where

$$c_n = \int_0^{2\pi} K_n(v) dv. \tag{3}$$

By the formula in (1), it is immediate that $L_n(f)(z)$ can be written in the form

$$L_n(f)(z) = \sum_{j=0}^{p-1} \overline{z}^j \cdot \frac{1}{c_n} \cdot \int_0^{2\pi} f_j(ze^{iv}) \cdot e^{-ijv} \cdot K_n(v) dv, \qquad n \in \mathbb{N},\ z \in \overline{\mathbb{D}}. \tag{4}$$

Let us set

$$\omega_1(f; \delta)_{\overline{\mathbb{D}}} = \sup\{|f(z_1) - f(z_2)|; |z_1 - z_2| \leq \delta, z_1, z_2 \in \overline{\mathbb{D}}\}.$$

The first main result is the following.

Theorem 2.2 *For $f \in H_p(\overline{\mathbb{D}})$, and each $n \in \mathbb{N}$, $L_n(f)(z)$ is a p-analytic polynomial of degree $d_n + p - 1$. In addition, if there exists a constant $M > 0$ (independent of n) and $\alpha_n \to +\infty$, such that*

$$\frac{1}{c_n} \cdot \int_0^{\pi} v K_n(v) dv \leq \frac{M}{\alpha_n} < +\infty$$

for all $n \in \mathbb{N}$, then

$$|f(z) - L_n(f)(z)| \leq 2(1 + M)\omega_1\left(f; \frac{1}{\alpha_n}\right)_{\overline{\mathbb{D}}}, \qquad n \in \mathbb{N},\ z \in \overline{\mathbb{D}}. \tag{5}$$

That is, since $\lim_{n\to\infty}\alpha_n = +\infty$, *it follows that* $L_n(f) \to f$ *uniformly on* $\overline{\mathbb{D}}$.

Proof Taking $z = re^{ix} \in \overline{\mathbb{D}}$, $0 \le r \le 1$, we can write

$$L_n(f)(z) = \frac{1}{c_n}\int_0^{2\pi} f(re^{i(v+x)})K_n(v)dv = \frac{1}{c_n}\int_0^{2\pi+x} f(re^{it})K_n(t-x)dt$$
$$= \frac{1}{c_n}\int_0^{2\pi} f(re^{it})K_n(t-x)dt,$$

so that $L_n(f)(z)$ is a convolution type operator.

Since $K_n(v)$ is a trigonometric polynomial of degree d_n, we have the representation

$$K_n(v) = \sum_{q=0}^{d_n}(A_q e^{iqv} + \overline{A}_q e^{-iqv}), \qquad A_q \in \mathbb{C},\ q = 0, \dots, d_n.$$

Since $f \in H_p(\overline{\mathbb{D}})$, we have

$$f(ze^{iv}) = \sum_{j=0}^{p-1} \overline{z}^j e^{-ijv} f_j(ze^{iv}),$$

where for each $j = 0, \dots, p-1$, we can write

$$f_j(z) = \sum_{l=0}^{\infty} c_l^{(j)} z^l, \quad z \in \mathbb{D},$$

with $c_l^{(j)} \in \mathbb{C}$.

Inspired by formula (4), we compute

$$\begin{aligned}
&f_j(ze^{iv}) \cdot e^{-ijv} \cdot K_n(v)\\
&= \left(\sum_{l=0}^{\infty} c_l^{(j)} z^l e^{ilv}\right)\left(\sum_{q=0}^{d_n}(A_q e^{iqv} + \overline{A}_q e^{-iqv})\right) e^{-ijv}\\
&= \left(\sum_{l=0}^{\infty}\sum_{q=0}^{d_n} c_l^{(j)} A_q e^{iv(l+q)} z^l + \sum_{l=0}^{\infty}\sum_{q=0}^{d_n} c_l^{(j)} \overline{A}_q e^{iv(l-q)} z^l\right) e^{-ijv}\\
&= \sum_{l=0}^{\infty}\sum_{q=0}^{d_n} c_l^{(j)} A_q e^{iv(l+q-j)} z^l + \sum_{l=0}^{\infty}\sum_{q=0}^{d_n} c_l^{(j)} \overline{A}_q e^{iv(l-q-j)} z^l\\
&:= S_1 + S_2.
\end{aligned}$$

Now, by integrating as in formula (4) and taking into account that

$$\int_0^{2\pi} e^{iv(\lambda+k)}dv = \begin{cases} 0 & \text{if } k+\lambda \neq 0 \\ 2\pi & \text{if } k+\lambda \neq 0 \end{cases},$$

it easily follows that for each fixed $j \in \{0, \dots, p-1\}$, $\int_0^{2\pi} S_1 dv$ reduces to a finite sum of powers of z^l (for those $l, q \geq 0$ with $l+q=j$) and $\int_0^{2\pi} S_2 dv$ reduces to a finite sum of powers of z^l (for those $l, q \geq 0$ with $l-q=j$). Moreover, it is clear that the maximum for l is obtained in the sum S_2 and it is given from the formula $l-q=j$, i.e. $l=q+j$, therefore is attained for for $j=p-1$.

In other words, this means that $L_n(f)(z)$ is a p-analytic polynomial of degree d_n+p-1 and this proves the first part of the theorem.

To prove the second part of the theorem, we use formula (2) and we have

$$\begin{aligned}
|f(z)-L_n(f)(z)| &\leq \frac{1}{c_n}\int_0^{2\pi} |f(z)-f(ze^{iv})|K_n(v)dv \\
&= \frac{1}{c_n}\int_{-\pi}^{\pi} |f(z)-f(ze^{iv})|K_n(v)dv \\
&\leq \frac{1}{c_n}\int_0^{2\pi} \omega_1(f;|z|\cdot|e^{iv}-1|)_{\overline{\mathbb{D}}}K_n(v)dv \\
&\leq \frac{1}{c_n}\cdot\int_0^{2\pi} \omega_1(f;2\cdot|\sin(v/2)|)_{\overline{\mathbb{D}}}K_n(v)dv \\
&= \frac{2}{c_n}\cdot\int_0^{\pi} \omega_1(f;2\cdot|\sin(v/2)|)_{\overline{\mathbb{D}}}K_n(v)dv \\
&\leq \frac{2}{c_n}\cdot\int_0^{\pi} \omega_1(f;\alpha_n|v|/\alpha_n)_{\overline{\mathbb{D}}}K_n(v)dv \\
&\leq 2\omega_1\left(f;\frac{1}{\alpha_n}\right)_{\overline{\mathbb{D}}}\cdot\frac{1}{c_n}\int_0^{\pi}(1+\alpha_n v)K_n(v)dv \\
&= 2\omega_1\left(f;\frac{1}{\alpha_n}\right)_{\overline{\mathbb{D}}}\cdot\left[1+\alpha_n\cdot\frac{1}{c_n}\int_0^{\pi} vK_n(v)dv\right] \\
&\leq 2(1+M)\cdot\omega_1\left(f;\frac{1}{\alpha_n}\right)_{\overline{\mathbb{D}}},
\end{aligned}$$

and the proof is complete. □

Remark 2.3 By taking in Theorem 2.2 as $K_n(v)$ other approximate units, we will get various other approximation results. For example, if we choose as $K_n(v)$ the so-called Jackson's kernel, then we deduce the following result on the p-analytic polynomials $L_n(f)$ given by (2):

Corollary 2.4 *If*

$$K_n(v) = \left(\frac{\sin \frac{n'v}{2}}{\sin \frac{v}{2}} \right)^4,$$

where $n' = [n/2] + 1$, $n \in \mathbb{N}$ *and therefore*

$$c_n = \frac{3}{2\pi n'[2(n')^2 + 1]}$$

(*see* [22]), *then,* $L_n(f)(z)$, $n \in \mathbb{N}$ *are* p*-analytic polynomials of degree* $n + p - 1$*, which satisfy the quantitative estimate*

$$|f(z) - L_n(f)(z)| \leq 2(1 + M)\omega_1 \left(f; \frac{1}{n} \right)_{\overline{\mathbb{D}}}, \qquad n \in \mathbb{N},\ z \in \overline{\mathbb{D}}.$$

Proof In this case, since by relation (5) in Lorentz [38, p. 57], the case $r = 2$, we have

$$\frac{1}{c_n} \cdot \int_0^{\pi} v K_n(v) dv \leq \frac{M}{n},$$

it follows that we can choose $\alpha_n = n$, for all $n \in \mathbb{N}$, which by the estimate (5) leads to

$$|f(z) - L_n(f)(z)| \leq 2(1 + M)\omega_1 \left(f; \frac{1}{n} \right)_{\overline{\mathbb{D}}}, \qquad n \in \mathbb{N},\ z \in \overline{\mathbb{D}}.$$

But by [38, pp. 55–56] $K_n(v)$ is a trigonometric polynomial of degree n, which by Theorem 2.2 implies that the degree of $L_n(f)(z)$ is $n + p - 1$, proving the corollary. □

Remark 2.5 The step $1/n$ inside the modulus of continuity in Corollary 2.4 can be put in accordance with the degree $n + p - 1$ of the p-analytic polynomials $L_n(f)(z)$, since there exists $C_p > 0$ (depending only on p) such that $\omega_1(f; 1/n)_{\overline{\mathbb{D}}} \leq C_p \omega_1(f; 1/(n + p - 1))_{\overline{\mathbb{D}}}$, for all $n \in \mathbb{N}$ and all $f \in H_p(\overline{\mathbb{D}})$. Indeed, it is good enough to choose $C_p > p$, which will imply that

$$\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}} \leq \omega_1\left(f; \frac{C_p}{n + p - 1}\right)_{\overline{\mathbb{D}}} \leq (C_p + 1)\omega_1\left(f; \frac{1}{n + p - 1}\right)_{\overline{\mathbb{D}}}.$$

Let us define higher moduli of smoothness of $f \in H_p(\overline{\mathbb{D}})$ by

$$\omega_q(f; \delta)_{\partial\mathbb{D}} = \sup_{r \in [0,1]} \sup\{|\Delta_h^q f(re^{ix})|; |x| \leq \pi, |h| \leq \delta\}, \tag{6}$$

where $q \in \mathbb{N}$, $q \geq 2$ and

$$\Delta_h^q f(re^{ix}) = \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} f(re^{i(x+jh)}).$$

The error estimate in the approximation of f by $L_n(f)(z)$ as in Corollary 2.4 can be expressed in terms of $\omega_2(f;\delta)_{\partial\mathbb{D}}$, as follows.

Theorem 2.6 *For $f \in H_p(\overline{\mathbb{D}})$, the p-analytic polynomials $L_n(f)(z)$ defined as in Corollary 2.4, give the estimate*

$$|f(z) - L_n(f)(z)| \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where $C > 0$ is an absolute constant.

Proof Indeed, we can write

$$\begin{aligned}
|f(z) - L_n(f)(z)| &\leq \frac{1}{c_n}\int_0^{2\pi} |f(z) - f(ze^{iv})|K_n(v)dv \\
&= \frac{1}{c_n}\int_{-\pi}^{\pi} |f(z) - f(ze^{iv})|K_n(v)dv \\
&= \frac{1}{c_n}\int_{-\pi}^{0} |f(z) - f(ze^{iv})|K_n(v)dv \\
&\quad + \frac{1}{c_n}\int_0^{\pi} |f(z) - f(ze^{iv})|K_n(v)dv \\
&= \frac{1}{c_n}\int_0^{\pi} |2f(z) - f(ze^{iv}) - f(ze^{-iv})|K_n(v)dv.
\end{aligned}$$

Writing $z = re^{ix}$, we now easily get

$$\begin{aligned}
|f(z) - L_n(f)(z)| &\leq \frac{1}{c_n}\cdot\int_0^{\pi} \omega_2(f;v)_{\partial\mathbb{D}} K_n(v)dv \\
&\leq \omega_2\left(f;\frac{1}{n}\right)_{\partial\mathbb{D}} \cdot \frac{1}{c_n}\cdot\int_0^{\pi}(nv+1)^2 K_n(v)dv \leq C\omega_2\left(f;\frac{1}{n}\right)_{\partial\mathbb{D}},
\end{aligned}$$

where for the last inequality we have applied the relations in Lorentz [38, p. 56].

Here we also have applied the property $\omega_2(f;\lambda\cdot\delta)_{\partial\mathbb{D}} \leq (1+\lambda)^2\cdot\omega_2(f;\delta)_{\partial\mathbb{D}}$. Also, notice that for ω_2 we used here a definition equivalent to (6) (in fact it is obtained from (6) by the simple substitution $x + h := y$)

$$\omega_2(f;\delta)_{\partial\mathbb{D}} = \sup_{r\in[0,1]} \sup\{|f(re^{i(y+h)}) - 2f(re^{iy}) + f(re^{i(y-h)})|;\ |y| \leq \pi, |h| \leq \delta\}.$$

The theorem is proved. □

More generally, let us attach to $f \in H_p(\overline{\mathbb{D}})$ the Jackson-type convolution operator given by the formula

$$I_{n,q}(f)(z) = -\int_{-\pi}^{\pi} K_{n,r}(v) \sum_{k=1}^{q+1} (-1)^k \binom{q+1}{k} f(ze^{ikv})dv, \tag{7}$$

where r is the smallest integer for which $r \geq (q+3)/2$, $q \in \mathbb{N}$ and

$$K_{n,r}(v) = \frac{1}{\lambda_{n',r}} \left(\frac{\sin \frac{n'v}{2}}{\sin \frac{v}{2}} \right)^{2r}, \qquad n' = \left[\frac{n}{r} \right] + 1$$

with $\lambda_{n',r}$ determined by $\int_{-\pi}^{\pi} K_{n,r}(v)dv = 1$.

According to [38, p. 57] $K_{n,r}$ is a trigonometric polynomial of degree n.

Since $f \in H_p(\overline{\mathbb{D}})$, by using formula (1), we immediately obtain

$$\begin{aligned} I_{n,q}(f)(z) &= -\int_{-\pi}^{\pi} K_{n,r}(v) \sum_{k=1}^{q+1} (-1)^k \binom{q+1}{k} f(ze^{ikv})dv \\ &= \sum_{j=0}^{p-1} \overline{z}^j f_j(z) \left[-\sum_{k=1}^{q+1} (1)^k \binom{q+1}{k} \int_{-\pi}^{\pi} K_{n,r}(v) e^{ijkv} dv \right]. \end{aligned}$$

Reasoning as in the proof of Theorem 2.2, we have that each $I_{n,q}(f)(z)$ is a p-analytic polynomial.

Theorem 2.7 *If $f \in H_p(\overline{\mathbb{D}})$, then the p-analytic polynomials $I_{n,q}(f)(z)$ are of degree $n+p-1$ and give the error estimate*

$$|f(z) - I_{n,q}(f)(z)| \leq M \cdot \omega_{q+1}(f; 1/n)_{\partial\mathbb{D}}, \quad n \in \mathbb{N}, z \in \overline{\mathbb{D}}.$$

Proof As in [38, pp. 57–58] by taking into account the formula (7) and denoting $z = re^{ix}$, we get

$$\begin{aligned} |f(z) - I_{n,q}(f)(z)| &\leq \int_{-\pi}^{\pi} |\Delta_v^{q+1} f(re^{ix}) \cdot K_{n,r}(v)dv \\ &\leq \omega_{q+1}(f; 1/n)_{\partial\mathbb{D}} \cdot \int_{-\pi}^{\pi} (n|v|+1)^{q+1} K_{n,r}(v)dv \\ &\leq M\omega_{q+1}(f; 1/n)_{\partial\mathbb{D}}. \end{aligned}$$

The theorem is proved. □

Remark 2.8 Reasoning as in Remark 2.5, the estimate in Theorem 2.7 can be replaced by one of the form

$$|f(z) - I_{n,q}(f)(z)| \leq C_p \cdot \omega_{q+1}\left(f; \frac{1}{n+p-1}\right)_{\partial\mathbb{D}}.$$

Remark 2.9 For fixed order p and degree m, let us denote by $\mathcal{P}_{p,m}$ the class of all p-analytic polynomials of degree $\leq m$ and for $f \in H_p(\overline{\mathbb{D}})$ let us denote by

$$E_{p,m}(f) = \inf\{\|f - P_{p,m}\|_{\overline{\mathbb{D}}};\ P_{p,m} \in \mathcal{P}_{p,m}\},$$

the best approximation of f by p-analytic polynomials of degree $\leq m$, where $\|\cdot\|_{\overline{\mathbb{D}}}$ denotes the uniform norm. Concerning this quantity $E_{p,m}(f)$, there exist two interesting open problems: one is to find the degree of $E_{p,m}(f)$ for p-analytic functions in various subclasses of $H_p(\overline{\mathbb{D}})$, and the second one is, for given f, p and m, to prove the existence of $P^*_{p,m} \in \mathcal{P}_{p,m}$ with $E_{p,m}(f) = \|f - P^*_{p,m}\|_{\overline{\mathbb{D}}}$ and even to construct polynomials $Q_{m,p} \in \mathcal{P}_{p,m}$, $m \in \mathbb{N}$, for which $\|f - Q_{p,m}\|_{\overline{\mathbb{D}}} \leq C E_{p,m}(f)$, $m \in \mathbb{N}$, with $C > 1$ a constant independent of m (and possibly also independent of f).

In the first case, it is known for example that for Gevrey polyanalytic classes of functions f of order p, the degree of $E_{p,m}(f)$ was obtained in [51].

3 Polyanalytic Gauss–Weierstrass Complex Operators

In this section we deal with the approximation properties of the convolution based on the classical Gauss–Weierstrass kernel given by $K_t(u) = e^{-u^2/(2t)}$, $u \in \mathbb{R}$, $t > 0$, by introducing the polyanalytic Gauss–Weierstrass complex operator and showing that the family of these operators has all the properties of a semigroup on the space of polyanalytic functions of a given order. More exactly, if $f \in H_p(\overline{\mathbb{D}})$, then the p-analytic Gauss–Weierstrass complex operator is defined by

$$\begin{aligned} W_t(f)(z) &= \frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/(2t)}du \\ &= \frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^{+\infty} f(ze^{iu})e^{-u^2/(2t)}du, \qquad z \in \mathbb{D},\ t > 0. \end{aligned} \tag{8}$$

We have:

Theorem 3.1 *Let $f \in H_p(\overline{\mathbb{D}})$ be given by (1) with all f_j analytic in $\mathbb{D}$ given by $f_j(z) = \sum_{l=0}^{\infty} c_l^{(j)} z^l$, $j = 0, \ldots, p-1$. Then*

(i) *$W_t \in H_p(\overline{\mathbb{D}})$ and we have*

$$W_t(f)(z) = \sum_{j=0}^{p-1} \overline{z}^j \cdot \sum_{l=0}^{\infty} z^l d_{l,j}(t),$$

where

$$d_{l,j}(t) = c_l^{(j)} \cdot \frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^{+\infty} \cos[u(l-j)]e^{-u^2/(2t)}du = c_l^{(j)} e^{-(l-j)^2 t/2}.$$

(ii) *The following estimate holds:*

$$|W_t(f)(z) - f(z)| \leq C\omega_1(f; \sqrt{t}), \quad \text{for all } z \in \overline{\mathbb{D}},\ t > 0,$$

where $C > 0$ *is a constant independent of* t, z *and* f.

(iii) *The following estimate holds:*

$$|W_t(f)(z) - W_s(f)(z)| \leq C_s|\sqrt{t} - \sqrt{s}|, \quad \text{for all } z \in \overline{\mathbb{D}},\ t \in V_s \subset (0, +\infty),$$

where $C_s > 0$ *is a constant depending on* f, *independent of* z *and* t *and* V_s *is any neighborhood of* s.

(iv) *The operator* W_t *is a contraction, that is,*

$$\|W_t(f)\| \leq \|f\|, \quad \text{for all } t > 0,\ f \in H_p(\overline{\mathbb{D}}).$$

(v) $(W_t, t \geq 0)$ *is a* (C_0)*-contraction semigroup of linear operators on the space* $H_p(\overline{\mathbb{D}})$ *and the unique solution* $v(t, z) \in H_p(\overline{\mathbb{D}})$, *for each fixed* t, *of the Cauchy problem*

$$\frac{\partial v}{\partial t}(t, z) = \frac{1}{2}\frac{\partial^2 v}{\partial \varphi^2}(t, z),\ (t, z) \in (0, +\infty) \times \mathbb{D}, \quad z = re^{i\varphi},\ z \neq 0, \tag{9}$$

$$v(0, z) = f(z), \quad z \in \overline{\mathbb{D}},\ f \in H_p(\overline{\mathbb{D}}), \tag{10}$$

is given by the formula

$$v(t, z) = W_t(f)(z) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/(2t)}\, du. \tag{11}$$

Proof (i) We obtain

$$\begin{aligned}
W_t(f)(z) &= \sum_{j=0}^{p-1} \overline{z}^j \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\left[\sum_{l=0}^{\infty} c_l^{(j)} z^l e^{iu(l-j)} e^{-u^2/(2t)}\right] du \\
&= \sum_{j=0}^{p-1} \overline{z}^j \sum_{l=0}^{\infty} z^l c_l^{(j)} \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\left[e^{iu(l-j)} e^{-u^2/(2t)}\right] du \\
&= \sum_{j=0}^{p-1} \overline{z}^j \sum_{l=0}^{\infty} z^l c_l^{(j)} \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} \cos[(u(l-j)] e^{-u^2/(2t)} du \\
&= \sum_{j=0}^{p-1} \overline{z}^j \sum_{l=0}^{\infty} z^l d_{l,j}(t),
\end{aligned}$$

where

$$d_{l,j}(t) = c_l^{(j)} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \cos[(u(l-j)]e^{-u^2/(2t)} du = c_l^{(j)} e^{-(l-j)^2 t/2},$$

where for the last equality we used, for example, the formula in Theorem 2.2.1 (i) in the book [25, p. 27] (see also [26, Thm. 2.1 (i)]).

It is easy to see that $|d_{l,j}(t)| \le |c_l^{(j)}|$, for all $j = 0, \ldots, p-1$ and $l \in \mathbb{N} \cup \{0\}$, which implies that $W_t(f)(z)$, $z \in \mathbb{D}$, is of the form (1).

It remains to prove that $W_t(f)(z)$ is continuous on all of $\overline{\mathbb{D}}$. In this sense, let $z_0 \in \overline{\mathbb{D}}$ and consider a sequence $z_n \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$, with $z_n \to z_0$ as $n \to \infty$.

We get

$$\begin{aligned}
|W_t(f)(z_n) - W_t(f)(z_0)| &\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(z_n e^{iu}) - f(z_0 e^{iu})| e^{-u^2/(2t)} du \\
&\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; |z_n e^{iu} - z_0 e^{iu}|) e^{-u^2/(2t)} du \\
&\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; |z_n - z_0|) e^{-u^2/(2t)} du \\
&\le \omega_1(f; |z_n - z_0|).
\end{aligned}$$

Therefore, passing to the limit with $n \to \infty$, since f is continuous on $\overline{\mathbb{D}}$ it follows from the continuity of $W_t(f)(z)$ for $z \in \overline{\mathbb{D}}$.

(ii) We obtain

$$\begin{aligned}
|W_t(f)(z) - f(z)| &\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(ze^{-iu}) - f(z)| e^{-u^2/(2t)}\, du \\
&\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \omega_1(f; |1 - e^{-iu}|) e^{-u^2/(2t)}\, du \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1\left(f; 2\left|\sin\frac{u}{2}\right|\right) e^{-u^2/(2t)}\, du \\
&\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; |u|) e^{-u^2/(2t)}\, du \\
&\le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; \sqrt{t}) \left(\frac{|u|}{\sqrt{t}} + 1\right) e^{-u^2/(2t)}\, du \\
&= \omega_1(f; \sqrt{t}) + \frac{\omega_1(f; \sqrt{t})}{\sqrt{t} \cdot \sqrt{2\pi t}} \int_0^{\infty} 2u e^{-u^2/(2t)}\, du.
\end{aligned}$$

Since

$$\int_0^{\infty} 2u e^{-u^2/(2t)} du = 2t \int_0^{\infty} e^{-v}\, dv = 2t,$$

we infer

$$|W_t(f)(z) - f(z)| \le \omega_1(f; \sqrt{t}) + \left[\omega_1(f; \sqrt{t})\right] \frac{2t}{t\sqrt{2\pi}} \le C\omega_1(f; \sqrt{t}).$$

(iii) We have

$$|W_t(f)(z) - W_s(f)(z)| \le \frac{\|f\|}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| \frac{e^{-u^2/t}}{\sqrt{t}} - \frac{e^{-u^2/s}}{\sqrt{s}} \right| du.$$

Let us set $\sqrt{t} = a$, $\sqrt{s} = b$. By the mean value theorem, there is a value $c \in (a, b)$, such that

$$\left| \frac{e^{-u^2/a^2}}{a} - \frac{e^{-u^2/b^2}}{b} \right| = |a - b| e^{-u^2/c^2} \left[\frac{2u^2}{c^4} - \frac{1}{c^2} \right],$$

which combined with the fact that

$$\int_{-\infty}^{+\infty} e^{-u^2/(2c)} < \infty, \ \int_{-\infty}^{+\infty} u^2 e^{-u^2/(2c)} < \infty,$$

immediately implies the desired inequality for W_t.

(iv) Since

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-u^2/(2t)} du = 1,$$

we deduce

$$|W_t(f)(z)| \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(ze^{-iu})| e^{-u^2/(2t)} du \le \|f\|, \ \ z \in \overline{\mathbb{D}},$$

which yields $\|W_t(f)\| \le \|f\|$.

(v) Let $f \in H_p(\overline{\mathbb{D}})$, that is,

$$f(z) = \sum_{j=0}^{p-1} \overline{z}^j f_j(z) = \sum_{j=0}^{p-1} \overline{z}^j \sum_{l=0}^{\infty} c_l^{(j)} z^l, \ z \in \mathbb{D}.$$

If $z \in \mathbb{D}$, $z = re^{i\varphi}$, $0 < r < 1$, then by (i), we can write

$$W_t(f)(z) = \sum_{j=0}^{p-1} \sum_{l=0}^{\infty} c_l^{(j)} r^{l+j} e^{i\varphi(l-j)} e^{-(l-j)^2 t/2}.$$

It is easy to see that $W_{t+s}(f)(z) = W_s[W_t(f)](z)$, for all $t, s > 0$. If z is on the boundary of $\mathbb{D}$, then we may take a sequence $(z_n)_{n \in \mathbb{N}}$ of points in $\mathbb{D}$ with $\lim_{n \to \infty} z_n = z$ and we apply the continuity property from the above point (i). Also, denoting $W_t(f)(z)$ by $T(t)(f)$, it is easy to see that the property $\lim_{t\searrow 0} T(t)(f) = f$, the continuity of $T(\cdot)$ and its contraction property follow from (ii), (iii) and (iv), respectively. Therefore, all these facts show that $(W_t, t \geq 0)$ is a (C_0)-contraction semigroup of linear operators on the space $H_p(\overline{\mathbb{D}})$.

Furthermore, since from (i) the above series representation for $W_t(f)(z)$ is uniformly convergent in any compact disk included in $\mathbb{D}$, it can be differentiated term by term, with respect to t and φ. Then, we easily get that

$$\frac{\partial W_t(f)(z)}{\partial t} = \frac{1}{2}\frac{\partial^2 W_t(f)(z)}{\partial \varphi^2}.$$

Also, from the same series representation, it is easy to see that

$$W_0(f)(z) = f(z),\ z \in \overline{\mathbb{D}}.$$

Finally, we note that, in (9) we have to take $z \neq 0$ because $z = 0$ cannot be represented as function of φ. The theorem is proved. □

4 Approximation by Slice Quaternionic Polyanalytic Polynomials

The analogue of polyanalytic functions in the slice quaternionic setting have been introduced in [7–9] and subsequent papers.
To explain our results we need to introduce the necessary definitions and notation. The skew field of quaternions is defined to be

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k \quad ; \ x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the imaginary units satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

In $\mathbb{H}$ the conjugate and the norm of q are defined respectively by

$$\overline{q} = \mathrm{Re}(q) - \mathrm{Im}(q) \qquad \text{where } \mathrm{Re}(q) = x_0,\ \mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$$

and

$$|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

The set

$$\mathbb{S} = \left\{q = x_1 i + x_2 j + x_3 k;\ x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

contains all the imaginary units, namely all the elements q such that $q^2 = -1$. Any quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ can be written in a unique way as $q = x + Iy$ for some real numbers x and $y > 0$, and imaginary unit $I \in \mathbb{S}$, in fact

$$q = x_0 + \frac{x_1 i + x_2 j + x_3 k}{|x_1 i + x_2 j + x_3 k|} |x_1 i + x_2 j + x_3 k|.$$

For every $I \in \mathbb{S}$, we set $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ which is isomorphic to the complex plane $\mathbb{C}$. It is immediate that $\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$.
In this work, we are interested in the specific case of functions defined on the unit ball $\mathbb{B} = \{q \in \mathbb{H};\ |q| < 1\}$ and in this case slice p-polyanalytic functions are of the form

$$f(q) = f_0(q) + \overline{q} f_1(q) + \cdots + \overline{q}^{p-1} f_{p-1}(q), \qquad q \in \mathbb{B}, \tag{12}$$

where $f_j(q) = \sum_{l=0}^{+\infty} q^l c_l^{(j)}$, $c_l^{(j)} \in \mathbb{H}$, $j = 0, \ldots, p-1$, $l = 0, 1, \ldots,$ where the series is convergent in $\mathbb{B}$, i.e., $f_j(q)$ is a slice regular function. In particular, $f_j(q)$ can be a polynomial and if $f_j(q)$ is a polynomial for all $j = 0, \ldots, p-1$ we say that f is a slice p-polyanalytic polynomial whose degree $\deg(f)$ is defined as the maximum degree of the f_j's. We refer the reader to [16,32] for more information on this class of functions and to [30] for a summary of the approximation results in this framework.

To introduce the corresponding convolution operators of a quaternion variable, we need a suitable exponential function of a quaternion variable. For any $I \in \mathbb{S}$, we choose the following well-known definition for the exponential: $e^{It} = \cos(t) + I\sin(t),\ t \in \mathbb{R}$, see [33]. The Euler's formula holds:

$$(\cos(t) + I\sin(t))^k = \cos(kt) + I\sin(kt),$$

and therefore we can write $[e^{It}]^k = e^{Ikt}$.

For any $q \in \mathbb{H} \setminus \mathbb{R}$, let $r := \|q\|$; then, see [33], there exists a unique $a \in (0, \pi)$ such that $\cos(a) := x_1/r$ and a unique $I_q \in \mathbb{S}$, such that

$$q = re^{I_q a}, \text{ with } I_q = iy + jv + ks,\ y = \frac{x_2}{r\sin(a)},\ v = \frac{x_3}{r\sin(a)},\ s = \frac{x_4}{r\sin(a)}.$$

Now, if $q \in \mathbb{R}$, then we choose $a = 0$, if $q > 0$ and $a = \pi$ if $q < 0$, and as I_q we choose an arbitrary fixed $I \in \mathbb{S}$. So that if $q \in \mathbb{R} \setminus \{0\}$, then again we can write $q = \|q\|(\cos(a) + I\sin(a))$ (but with a non unique I). The above is called the trigonometric form of the quaternion number $q \neq 0$. For $q = 0$ we do not have a trigonometric form for q (exactly as in the complex case). Analogously to the case of a complex variable, we can introduce the following convolution operator of a quaternionic variable.

For $p \in \mathbb{N}$ and $\mathbb{B}$ the open unit ball in $\mathbb{H}$, let us denote by $SP_p(\overline{\mathbb{B}})$ the space of all slice p-polyanalytic functions in $\mathbb{B}$ which are continuous in $\overline{\mathbb{B}}$, endowed with the uniform norm $\|\cdot\|$.

Also, let $K_{n,r}(v)$ be an even, classical, positive-valued, trigonometric polynomial of degree $d_{n,r} \in \mathbb{N}$, with $K_{n,r}(v) \geq 0$, for all $v \in [0, 2\pi]$ and $n, r \in \mathbb{N}$.

For $f \in SP_p(\overline{\mathbb{B}})$, $q \in \mathbb{H} \setminus \mathbb{R}$ and $n \in \mathbb{N}$, let us define the convolution operator

$$\begin{aligned} L_{n,r}(f)(q) &= \frac{1}{c_{n,r}} \cdot \int_0^{2\pi} f(qe^{I_q v}) K_{n,r}(v) dv \\ &= \frac{1}{c_{n,r}} \cdot \int_{-\pi}^{\pi} f(qe^{I_q v}) K_{n,r}(v) dv, \end{aligned} \tag{13}$$

where

$$c_{n,r} = \int_0^{2\pi} K_{n,r}(v) dv. \tag{14}$$

In this section, we will use the trigonometric kernels

$$K_{n,r}(v) = \left(\frac{\sin \frac{nv}{2}}{\sin \frac{v}{2}} \right)^{2r}.$$

According to Lorentz [38, p. 55] they are even and positive trigonometric polynomials of degree $r(n-1)$, which can be written in the form

$$K_{n,r}(v) = \sum_{s=0}^{r(n-1)} A_{r,s} \cdot \cos(s \cdot v),$$

with $A_{r,s} \in \mathbb{R}$, for all $r \in \mathbb{N}$, $r \geq 2$ and $s = 1, \ldots, r(n-1)$.

Firstly, we prove the following:

Lemma 4.1 *The functions $L_{n,r}(f)(q)$ are slice p-polyanalytic polynomials.*

Proof Since $q, \overline{q}, e^{I_q v}$ and $e^{-I_q v}$ are on the slice $\mathbb{C}_{I_q}$ determined by I_q, they commute. Therefore it is immediate that $L_{n,r}(f)(q)$ can be written in the form

$$L_{n,r}(f)(q) = \sum_{j=0}^{p-1} \overline{q}^j \cdot \frac{1}{c_{n,r}} \cdot \int_0^{2\pi} e^{-I_q j v} \cdot f_j(qe^{I_q v}) \cdot K_{n,r}(v) dv, n \in \mathbb{N}, \qquad q \in \overline{\mathbb{B}}. \tag{15}$$

From formula (15), we need to calculate

$$\int_0^{2\pi} e^{-I_q j v} \cdot f_j(ze^{I_q v}) \cdot K_n(v) dv.$$

Again from the fact that I_q, $e^{-I_q jv}$, q^l and $e^{I_q lv}$ commute, we get

$$\begin{aligned} e^{-I_q jv} \cdot f_j(qe^{I_q v}) \cdot K_{n,r}(v) &= e^{-I_q jv} \left(\sum_{l=0}^{\infty} q^l e^{I_q lv} c_l^{(j)} \right) \left(\sum_{s=0}^{r(n-1)} A_{r,s} \cdot \cos(s \cdot v) \right) \\ &= \left(\sum_{l=0}^{\infty} \sum_{s=0}^{r(n-1)} q^l e^{I_q (l-j)v} c_l^{(j)} A_{r,s} \cos(s \cdot v) \right) \\ &= \sum_{l=0}^{\infty} \sum_{s=0}^{r(n-1)} q^l c_l^{(j)} A_{r,s} \cos((l-j)v) \cos(s \cdot v) \\ &\quad + I_q \sum_{l=0}^{\infty} \sum_{s=0}^{r(n-1)} q^l c_l^{(j)} A_{r,s} \sin((l-j)v) \cos(s \cdot v) \\ &:= S_1 + I_q S_2. \end{aligned}$$

Now, by integrating the sum S_1 with respect to v from 0 to 2π, it easily follows that the only terms which are different from zero are the terms for which $l = j - 1$, with the maximum value $l = p - 1 + nr - r$, while integrating S_2 we get that all its terms are equal to zero.

Consequently, formula (15) shows that $L_{n,r}(f)(q)$ is a slice $(p-1)$-polyanalytic polynomial of degree $p - 1 + nr - 1$. □

Denoting

$$\omega_1(f; \delta)_{\overline{\mathbb{B}}} = \sup\{|f(q_1) - f(q_2)|; |q_1 - q_2| \le \delta, q_1, q_2 \in \overline{\mathbb{B}}\},$$

we are now in position to prove the first main result of this section.

Theorem 4.2 *For $f \in SP_p(\overline{\mathbb{B}})$, $q \in \mathbb{H} \setminus \mathbb{R}$ and $n \in \mathbb{N}$, let us define the convolution operator*

$$L_{n,2}(f)(q) = \frac{1}{c_{n,2}} \cdot \int_0^{2\pi} f(qe^{I_q v}) K_{n,2}(v) dv = \frac{1}{c_{n,2}} \cdot \int_{-\pi}^{\pi} f(qe^{I_q v}) K_{n,2}(v) dv, \tag{16}$$

where

$$K_{n,2}(v) = \left(\frac{\sin \frac{n'v}{2}}{\sin \frac{v}{2}} \right)^4$$

with $n' = [n/2] + 1$, $n \in \mathbb{N}$ and

$$c_{n,2} = \int_0^{2\pi} K_{n,2}(v) dv. \tag{17}$$

Then $L_{n,2}(f)(q)$, $n \in \mathbb{N}$ are slice p-polyanalytic polynomials of degree $n + p - 1$, which satisfy the quantitative estimate

$$|f(q) - L_{n,2}(f)(q)| \leq 2(1 + M)\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{B}}}, n \in \mathbb{N}, q \in \overline{\mathbb{B}},$$

where $M > 0$ is a constant independent of q, f and n.

Proof Taking $q = re^{I_q x} \in \overline{\mathbb{B}}$, we can write

$$\begin{aligned} L_{n,2}(f)(q) &= \frac{1}{c_{n,2}} \int_0^{2\pi} f(re^{I_q(v+x)})K_{n,2}(v)dv \\ &= \frac{1}{c_{n,2}} \int_0^{2\pi+x} f(re^{I_q t})K_{n,2}(t-x)dt \\ &= \frac{1}{c_{n,2}} \int_0^{2\pi} f(re^{I_q t})K_n(t-x)dt. \end{aligned}$$

It follows that

$$|f(q) - L_{n,2}(f)(q)| \leq \frac{1}{c_{n,2}} \int_0^{2\pi} |f(q) - f(qe^{I_q v})|K_{n,2}(v)dv$$

and using calculations similar to those in the proof of the second part of Theorem 2.2 (by replacing α_n, i, z by n, I_q, q respectively) we obtain the required estimate. □

Remark 4.3 Reasoning as in Remark 2.5, the step $1/n$ inside the modulus of continuity in Theorem 4.2 can be put in accordance with the degree $n + p - 1$ of the slice p-polyanalytic polynomials $L_{n,2}(f)(q)$, since there exists $C_p > 0$ (depending only on p) such that $\omega_1(f; 1/n)_{\overline{\mathbb{B}}} \leq C_p\omega_1(f; 1/(n + p - 1))_{\overline{\mathbb{B}}}$, for all $n \in \mathbb{N}$ and all $f \in SP_p(\overline{\mathbb{B}})$. Indeed, it is good enough to choose $C_p > p$, which will imply that

$$\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{B}}} \leq \omega_1\left(f; \frac{C_p}{n+p-1}\right)_{\overline{\mathbb{B}}} \leq (C_p + 1)\omega_1\left(f; \frac{1}{n+p-1}\right)_{\overline{\mathbb{B}}}.$$

Now, if we define higher moduli of smoothness of $f \in SP_p(\overline{\mathbb{B}})$ by

$$\omega_m(f; \delta)_{\partial\mathbb{B}} = \sup_{I_q \in \mathbb{S}} \sup_{r \in [0,1]} \sup\{|\Delta_h^m f(re^{I_q x})|; |x| \leq \pi, |h| \leq \delta\}, \tag{18}$$

where $m \in \mathbb{N}$, $m \geq 2$ and

$$\Delta_h^m f(re^{I_q x}) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(re^{I_q(x+jh)}),$$

then the error estimate in the approximation of f by $L_{n,2}(f)(q)$ as in Theorem 4.2, can be expressed in terms of $\omega_2(f; \delta)_{\partial\mathbb{B}}$, as follows.

Theorem 4.4 *For $f \in SP_p(\overline{\mathbb{B}})$, the slice p-polyanalytic polynomials $L_{n,2}(f)(q)$ defined in* (16) *give the estimate*

$$|f(q) - L_{n,2}(f)(q)| \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{B}}, \quad q \in \overline{\mathbb{B}},$$

where $C > 0$ is an absolute constant.

Proof Indeed, we can write

$$|f(q) - L_{n,2}(f)(q)| \leq \frac{1}{c_{n,2}} \int_0^{2\pi} |f(q) - f(qe^{I_q v})| K_{n,2}(v) dv.$$

Then, writing $q = re^{I_q x}$, and reasoning as in the proof of Theorem 2.6 (where i, z must be replaced by I_q, q respectively) we now easily get the assertion. □

More generally, for $f \in SP_p(\overline{\mathbb{B}})$, let us attach the generalized Jackson-type convolution operator given by the formula

$$I_{n,m}(f)(q) = -\int_{-\pi}^{\pi} K_{n,r}(v) \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} f(qe^{I_q kv}) dv, \tag{19}$$

where r is the smallest integer for which $r \geq (m+3)/2$, $m \in \mathbb{N}$, and

$$K_{n,r}(v) = \frac{1}{\lambda_{n',r}} \left(\frac{\sin \frac{n'v}{2}}{\sin \frac{v}{2}}\right)^{2r}, \qquad n' = \left[\frac{n}{r}\right] + 1,$$

with $\lambda_{n',r}$ determined by $\int_{-\pi}^{\pi} K_{n,r}(v) dv = 1$. According to [38, p. 57] $K_{n,r}$ is an even trigonometric polynomial of degree n.

We now set $K_{n,r}(v) = \sum_{s=0}^{n} A_{r,s} \cos(s \cdot v)$ and we consider $f \in SP_p(\overline{\mathbb{B}})$. Using reasonings and calculations similar to those ones in the proof of Lemma 4.1, by formula (12) we immediately obtain

$$\begin{aligned} I_{n,m}(f)(q) &= -\int_{-\pi}^{\pi} K_{n,r}(v) \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} f(qe^{I_q kv}) dv \\ &= -\sum_{j=0}^{p-1} \overline{q}^j \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} \sum_{l=0}^{\infty} \sum_{s=0}^{n} \\ &\quad \times \left\{ A_{r,s} \int_0^{2\pi} \cos[v(l-kj)] \cos(s \cdot v) dv \right\} q^l c_l^{(j)}. \end{aligned}$$

Again reasoning as in the proof of Lemma 4.1, we get that each $I_{n,m}(f)(z)$ is a slice p-polyanalytic polynomial of degree $n + (p-1)(m+1)$.

Theorem 4.5 *If $f \in SP_p(\overline{\mathbb{B}})$, then the slice p-polyanalytic polynomials $I_{n,m}(f)(q)$ are of degree $n + (p-1)(m+1)$ and give the error estimate*

$$|f(q) - I_{n,m}(f)(q)| \leq M \cdot \omega_{m+1}\left(f; \frac{1}{n}\right)_{\partial\mathbb{B}}, \qquad n \in \mathbb{N},\ q \in \overline{\mathbb{B}},$$

where $M > 0$ is independent of f, q and n.

Proof As in [38, pp. 57–58] by taking into account the formula (19) and denoting $q = re^{I_q x}$, we get

$$\begin{aligned}|f(q) - I_{n,m}(f)(q)| &\leq \int_{-\pi}^{\pi} |\Delta_v^{m+1} f(re^{I_q x}) \cdot K_{n,r}(v)dv \\ &\leq \omega_{m+1}(f; 1/n)_{\partial\mathbb{B}} \cdot \int_{-\pi}^{\pi} (n|v|+1)^{m+1} K_{n,r}(v)dv \\ &\leq M\omega_{q+1}\left(f; \frac{1}{n}\right)_{\partial\mathbb{B}}, \qquad q \in \overline{\mathbb{B}},\end{aligned}$$

and the theorem is proved. □

Remark 4.6 Reasoning as in Remark 4.3, the estimate in Theorem 4.5, can be replaced by

$$|f(q) - I_{n,m}(f)(q)| \leq C_{m,p} \cdot \omega_{m+1}\left(f; \frac{1}{n+(p-1)(m+1)}\right)_{\partial\mathbb{B}}.$$

5 Polyanalytic Gauss–Weierstrass Quaternionic Operators

Keeping the notation in Sect. 3, for $SP_p(\overline{\mathbb{B}})$ let us introduce now the polyanalytic Gauss–Weierstrass quaternionic operators given by the formula

$$\begin{aligned}&W_t(f)(q) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(qe^{I_q u})e^{-u^2/(2t)}du, \qquad q \in \mathbb{H}\setminus\mathbb{R}, q = re^{I_q a} \in \mathbb{B},\\ &W_t(f)(q) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(qe^{I u})e^{-u^2/(2t)}du, \qquad q \in \mathbb{R}\setminus\{0\}, q = re^{I a} \in \mathbb{B},\ a = 0 \text{ or } \pi,\\ &W_t(f)(0) = f(0),\end{aligned} \tag{20}$$

where $I \in \mathbb{S}$ is fixed (but arbitrary).

The results in Sect. 3 can be generalized to this case and we have:

Theorem 5.1 *Let $f \in SP_p(\overline{\mathbb{B}})$ be given by (12) with all f_j slice regular functions in $\mathbb{B}$ given by $f_j(q) = \sum_{l=0}^{\infty} q^l c_l^{(j)}$, $c_l^{(j)} \in \mathbb{H}$, $j = 0, \ldots, p-1$, $l = 0, 1, 2, \ldots$. Then*

(i) $W_t \in SP_p(\overline{\mathbb{B}})$ *and we have*

$$W_t(f)(q) = \sum_{j=0}^{p-1} \overline{q}^j \cdot \sum_{l=0}^{\infty} q^l d_{l,j}(t),$$

where

$$d_{l,j}(t) = \left[\frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^{+\infty} \cos[u(l-j)] e^{-u^2/(2t)} du\right] c_l^{(j)} = e^{-(l-j)^2 t/2} \cdot c_l^{(j)}.$$

(ii) *The following estimate holds:*

$$|W_t(f)(q) - f(q)| \le C\omega_1(f; \sqrt{t}) \qquad \text{for all } q \in \overline{\mathbb{B}},\ t > 0,$$

where $C > 0$ *is a constant independent of* t, q *and* f.

(iii) *The following estimate holds:*

$$|W_t(f)(q) - W_s(f)(q)| \le C_s|\sqrt{t} - \sqrt{s}| \qquad \text{for all } z \in \overline{\mathbb{B}},\ t \in V_s \subset (0, +\infty),$$

where $C_s > 0$ *is a constant depending on* f, *independent of* q *and* t *and* V_s *is any neighborhood of* s.

(iv) *The operator* W_t *is a contraction, that is,*

$$\|W_t(f)\| \le \|f\| \qquad \text{for all } t > 0,\ f \in SP_p(\overline{\mathbb{B}}).$$

(v) $(W_t, t \ge 0)$ *is a* (C_0)*-contraction semigroup of linear operators on the space* $SP_p(\overline{\mathbb{B}})$ *and the unique solution* $u(t, q) \in SP_p(\overline{\mathbb{B}})$, *for each fixed* t, *of the Cauchy problem*

$$\frac{\partial v}{\partial t}(t, q) = \frac{1}{2}\frac{\partial^2 v}{\partial \varphi^2}(t, q), \qquad (t, q) \in (0, +\infty) \times \mathbb{B},\ q = re^{I_q\varphi},\ q \ne 0, \tag{21}$$

$$v(0, q) = f(q),\ q \in \overline{\mathbb{B}}, \qquad f \in SP_p(\overline{\mathbb{B}}), \tag{22}$$

is given by the formula

$$v(t, q) = W_t(f)(q). \tag{23}$$

Proof (i) Since $\overline{q}^j, q^l, e^{I_q lu}, e^{-I_q ju}$ are on the same slice, they commute and therefore we obtain

$$W_t(f)(q) = \sum_{j=0}^{p-1} \overline{q}^j \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \left[\sum_{l=0}^{\infty} q^l e^{I_q u(l-j)} c_l^{(j)} e^{-u^2/(2t)}\right] du$$

$$= \sum_{j=0}^{p-1} \overline{q}^j \sum_{l=0}^{\infty} q^l \left\{\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \left[e^{I_q u(l-j)} e^{-u^2/(2t)}\right] du\right\} c_l^{(j)}$$

$$= \sum_{j=0}^{p-1} \overline{q}^j \sum_{l=0}^{\infty} q^l \left\{ \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \cos[(u(l-j)]e^{-u^2/(2t)} du \right\} c_l^{(j)}$$

$$= \sum_{j=0}^{p-1} \overline{q}^j \sum_{l=0}^{\infty} q^l d_{l,j}(t),$$

where

$$d_{l,j}(t) = \left\{ \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \cos[(u(l-j)]e^{-u^2/(2t)} du \right\} c_l^{(j)} = e^{-(l-j)^2 t/2} c_l^{(j)},$$

where for the last equality we used, for example, the formula in Theorem 2.2.1, (i) in the book [25, p. 27] (see also [26, Thm. 2.1 (i)]).
It is immediate that $|d_{l,j}(t)| \leq |c_l^{(j)}|$, for all $j = 0, \ldots, p-1$ and $l \in \mathbb{N} \cup \{0\}$, which implies that $W_t(f)(q)$, $q \in \mathbb{B}$, is of the form (1).

The continuity of $W_t(f)(q)$ on $\overline{\mathbb{B}}$ is obtained exactly as that in the complex case in the proof Theorem 3.1,(i).

Since the proofs of (ii), (iii), (iv) follow exactly the lines in the proof of Theorem 3.1, (ii), (iii), (iv) and (v), we omit them here.

Also, for the proof of (v) it is enough to observe that denoting $q = re^{I_q \varphi}$, $0 < r < 1$, by using the point (i) we can write

$$W_t(f)(q) = \sum_{j=0}^{p-1} \sum_{l=0}^{\infty} r^{l+j} e^{I_q \varphi(l-j)} e^{-(l-j)^2 t/2} c_l^{(j)}$$

and from this point we follow exactly the lines in the proof of Theorem 3.1, (v). The theorem is proved. □

Remark 5.2 For $p = 1$ the results in Sects. 4, 5 give the corresponding results in the quaternionic slice regular case, see [30].

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