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On an uncountable family of graphs whose spectrum is a Cantor set

Matteo Cavaleri, Daniele D'Angeli, Alfredo Donno, and Emanuele Rodaro

Abstract. For each $p \geq 1$, the star automaton group $\mathcal{G}_{S_p}$ is an automaton group which can be defined starting from a star graph on $p+1$ vertices. We study Schreier graphs associated with the action of the group $\mathcal{G}_{S_p}$ on the regular rooted tree T_{p+1} of degree $p+1$ and on its boundary ∂T_{p+1}. With the transitive action on the n-th level of T_{p+1} is associated a finite Schreier graph Γ_n^p, whereas there exist uncountably many orbits of the action on the boundary, represented by infinite Schreier graphs which are obtained as the limits of the sequence $\{\Gamma_n^p\}_{n\geq 1}$ in the Gromov–Hausdorff topology. We obtain an explicit description of the spectrum of the graphs $\{\Gamma_n^p\}_{n\geq 1}$. Then, by using amenability of $\mathcal{G}_{S_p}$, we prove that the spectrum of each infinite Schreier graph is the union of a Cantor set of zero Lebesgue measure, which is the Julia set of the quadratic map $f_p(z) = z^2 - 2(p-1)z - 2p$, and a countable collection of isolated points supporting the Kesten–Neumann–Serre spectral measure. We also give a complete classification of the infinite Schreier graphs up to isomorphism of unrooted graphs, showing that they may have 1, 2 or $2p$ ends, and that the case of 1 end is generic with respect to the uniform measure on ∂T_{p+1}.

Dedicated to Rostislav Grigorchuk on the occasion of his 70th birthday

1. Introduction

Schreier graphs are very popular in automaton group theory. In fact, they describe in a very natural way the action of an invertible automaton on words over an alphabet or, equivalently, on a regular rooted tree. This relates algebraic properties of the automaton group with combinatorial properties of the corresponding Schreier graphs. This paper can be framed into the exciting research field involving groups acting by automorphisms on rooted trees. Many papers have been devoted to these topics in the last decades: the interested reader can refer to the following list of works (and bibliography therein) for more details [11, 14, 16, 17, 22].

Every automaton group acts by automorphisms on the rooted tree T. The action on finite levels is described by finite Schreier graphs. Going deeper and deeper in the tree,

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Added "the" before "limit" throughout the paper. Please check

one is led to the study of the dynamical system $(G, \partial T, \nu)$ carrying the measure ν invariant under the action of the group G on the boundary ∂T. One orbit of this action (i.e., a Schreier graph) can be seen as an infinite rooted graph obtained as the limit of a sequence of finite rooted Schreier graphs in the Gromov–Hausdorff topology. Finite and infinite Schreier graphs have been investigated from a combinatorial point of view in various contexts (e.g., [10,13]). Classifications of infinite Schreier graphs have been studied in several papers (see [4,5,8,9,21] for further discussions about this topic).

In this setting, another problem is of considerable interest: the study of the spectral properties of Schreier graphs associated with an automaton group. The determination of the spectrum of the Markov operator associated with a graph is, in general, a very difficult task, and only a few examples are known for families of graphs. This analysis is very important in the theory of random walks on groups and in geometric group theory. It is remarkable that the first examples of graphs whose spectrum is a Cantor set of Lebesgue measure zero, or the union of a Cantor set with a countable set of isolated points, have been obtained in the frame of Schreier graphs generated by automaton groups [1]. In this context, the self-similar form of the generators reflects into the block structure of the adjacency matrix and in some special cases, an appropriate manipulation allows to find recursive formulae for the determination of the spectrum [15, 18–20]. This method produces the sequence of spectra corresponding to finite levels, and this sequence approximates the spectrum corresponding to the boundary action. It is worth mentioning here that such approximation approach might also fail. In the case of the so-called Basilica group, the situation seems to be more complicated and the renormalization of the infinite graph instead of the finite approximation is used (see [6] for more details).

In the present work, we want to study the two problems introduced above for the Schreier graphs associated with an infinite family of automaton groups. More precisely, this paper can be seen as a natural continuation of the paper [7], where we defined a particular class of automaton groups, called *graph automaton groups*: starting from a graph $G = (V, E)$, we defined an invertible automaton $\mathcal{A}_G$ and then considered the associated group $\mathcal{G}_G$, whose generators are in a one-to-one correspondence with E, and which acts by automorphisms on the regular rooted tree of degree $|V|$. The automaton $\mathcal{A}_G$ is bounded, so that the group $\mathcal{G}_G$ is amenable. Under the hypothesis $|E| \geq 2$, we showed that $\mathcal{G}_G$ is a self-replicating group which is weakly regular branch over its commutator subgroup $\mathcal{G}'_G$; moreover, it contains elements of finite order and has a number of torsion relators coming from directed cycles in G. It turns out that right-angled Artin groups project onto the corresponding group obtained from the graph by this construction, which shows by the way that right-angled Artin groups have amenable self-replicating weakly branch quotients. We also studied in [7] some properties of finite Schreier graphs associated with $\mathcal{G}_G$ when G is a path graph or a cycle.

In the present paper, we consider a special class of graph automaton groups, obtained from a graph G which is a star. We call such groups *star automaton groups*. The star graph on $p+1$ vertices, consisting of a central vertex of degree p and p leaves, is denoted by S_p.

Schreier graphs associated with the action of $\mathcal{G}_{S_p}$ on the regular rooted tree T_{p+1} and its boundary ∂T_{p+1} are the main object of research of this paper.

In Section 2, we recall the construction of graph automaton groups, together with the notion of finite and infinite Schreier graphs. We also recall some basic facts about the Ihara zeta function, both for a finite regular graph and for an infinite graph obtained as the limit of a sequence of finite regular graphs; in particular, we focus on its integral representation by means of the Kesten–Neumann–Serre (KNS) spectral measure.

Section 3 is devoted to spectral computations for finite and infinite Schreier graphs associated with the group $\mathcal{G}_{S_p}$. In Section 3.1, all the details for the case $p = 3$ are given. We construct the adjacency matrices of the finite Schreier graphs: by using the Schur complement technique, we find a recursive description of their characteristic polynomials in terms of a quadratic map in Theorem 3.2. In Theorem 3.4, the spectra of these matrices are explicitly described. Then, using amenability of the group $\mathcal{G}_{S_3}$, we prove in Theorem 3.7 that the spectrum of any infinite Schreier graph associated with $\mathcal{G}_{S_3}$ is the union of a Cantor set of zero Lebesgue measure, which is the Julia set of the quadratic map, and a countable collection of isolated points supporting the KNS spectral measure. The knowledge of the KNS spectral measure is then used to obtain an integral representation of the Ihara zeta function. The results obtained for the case $p = 3$ are extended to the general case of any star graph S_p, and they are presented in Theorems 3.8, 3.9, and 3.10 of Section 3.2.

Section 4 is devoted to the investigation of topological and isomorphism properties of Schreier graphs associated with $\mathcal{G}_{S_p}$. The topological investigation developed for the finite case in Section 4.1 is preliminary to the results obtained in Section 4.2 in the infinite case, where we are able to classify, up to isomorphism of unrooted graphs, all infinite orbital Schreier graphs. We show that the limit graphs may have 1, 2 or $2p$ ends. In Theorem 4.16, we give an explicit classification of infinite Schreier graphs of $\mathcal{G}_{S_p}$ in terms of infinite words in $\{0, 1, \dots, p\}$, by characterizing the elements of the boundary of the tree T_{p+1} belonging to a graph with 1, 2, or $2p$ ends, showing that there exist uncountably many 1-ended and 2-ended orbits, but exactly one $2p$-ended orbit. Moreover, the case of 1 end is generic with respect to the uniform measure on ∂T_{p+1}. In Theorem 4.23, we provide necessary and sufficient conditions for two elements of ∂T_{p+1} to belong to isomorphic infinite Schreier graphs. In particular, we prove that there exists one isomorphism class of $2p$-ended graphs, consisting of one orbit; there exist uncountably many isomorphism classes of 2-ended graphs, each consisting of $2p$ graphs; there exist uncountably many isomorphism classes of 1-ended graphs, each consisting of uncountably many graphs. Finally, each isomorphism class is proven to have zero measure in Corollary 4.24.

2. Preliminaries

In this preliminary section, we recall some basic definitions and properties of automaton groups and their Schreier graphs, focusing on the special class of automaton groups, called graph automaton groups, which has been introduced by the authors in [7]. We also recall the notion of KNS spectral measure and Ihara zeta function, which will be investigated in Section 3 in the case of star automaton groups.

2.1. Graph automaton groups and Schreier graphs

Let us start by recalling the basic definition of automaton.

Definition 2.1. An *automaton* is a quadruple $\mathcal{A} = (S, X, \lambda, \eta)$, where

(1) S is the set of states;

(2) $X = \{1, 2, \dots, k\}$ is an alphabet;

(3) $\lambda\colon S \times X \to S$ is the transition map;

(4) $\eta\colon S \times X \to X$ is the output map.

The automaton $\mathcal{A}$ is *finite* if S is finite, and it is *invertible* if, for all $s \in S$, the transformation $\eta(s, \cdot)\colon X \to X$ is a permutation of X. An automaton $\mathcal{A}$ can be visually represented by its *Moore diagram*: this is a directed labeled graph whose vertices are identified with the states of $\mathcal{A}$. For every state $s \in S$ and every letter $x \in X$, the diagram has an arrow from s to $\lambda(s, x)$ labeled by $x|\eta(s, x)$. A sink id in $\mathcal{A}$ is a state with the property that $\lambda(\mathrm{id}, x) = \mathrm{id}$ and $\eta(\mathrm{id}, x) = x$ for any $x \in X$.

An important class of automata is given by bounded automata [23]. An automaton is said to be *bounded* if the sequence of numbers of paths of length n avoiding the sink state (along the directed edges of the Moore diagram) is bounded.

For each $n \geq 1$, let X^n denote the set of words of length n over the alphabet X and put $X^0 = \{\emptyset\}$, where $\emptyset$ is the empty word. Then the action of $\mathcal{A}$ can be naturally extended to the infinite set $X^* = \bigcup_{n=0}^{\infty} X^n$ and to the set $X^\infty = \{x_1x_2x_3 \dots : x_i \in X\}$ of infinite words over X.

For a state $s \in S$, we denote by $\mathcal{A}_s$ the transformation $\eta(s, \cdot)$ of $X^* \cup X^\infty$. Given the invertible automaton $\mathcal{A}$, the *automaton group* generated by $\mathcal{A}$ is by definition the group generated by the transformations $\mathcal{A}_s$ for $s \in S$, and it is denoted by $G(\mathcal{A})$. In the rest of the paper, we will often use the notation s instead of $\mathcal{A}_s$. Notice that the action of $G(\mathcal{A})$ on X^* preserves the sets X^n for each n.

It is a well-known fact that an automaton group can be regarded in a very natural way as a group of automorphisms of the regular rooted tree T_k in which each vertex has $|X| = k$ children, via the identification of the k^n vertices of the n-th level of T_k with the set X^n. Similarly, the action on X^∞ can be regarded as an action on the boundary ∂T_k of the tree, whose elements are infinite geodesic rays starting at the root of T_k. Notice that the set X^∞ can be equipped with the direct product topology; it is totally disconnected and homeomorphic to the Cantor set. We will denote by ν the uniform measure on X^∞ or, equivalently, on ∂T_k.

The group $G(\mathcal{A})$ is said to be *spherically transitive* if its action is transitive on X^n for any n. Let $g \in G(\mathcal{A})$. The action of g on X^* can be factorized by considering the action on X and $|X|$ restrictions as follows. Let $\mathrm{Sym}(k)$ be the symmetric group on k elements.

Then an element $g \in G(\mathcal{A})$ can be represented as

$$g = (g_1, \dots, g_k)\sigma, \tag{2.1}$$

where $g_i := \lambda(g, i) \in G(\mathcal{A})$ and $\sigma \in \mathrm{Sym}(k)$ describes the action of g on X. We say that equation (2.1) is the *self-similar representation* of g. In the tree interpretation of equation (2.1), the permutation σ corresponds to the action of g on the first level of T_k, and the automorphism g_i is the restriction of the action of g to the subtree (isomorphic to the whole T_k) rooted at the i-th vertex of the first level. Finally, it is known that if the automaton $\mathcal{A}$ is bounded, then the group $G(\mathcal{A})$ is amenable (see, e.g., [2]).

In [7], we introduced the following construction associating an invertible automaton with a given finite graph.

Let $G = (V, E)$ be a finite graph, where $V = \{x_1, \dots, x_k\}$ is its vertex set and E is its edge set. Let E' be the set of edges, where an orientation of each edge has been chosen. Notice that elements in E are unordered pairs of type $\{x_i, x_j\}$, whereas elements in E' are ordered pairs of type (x_i, x_j), meaning that the edge has been oriented from the vertex x_i to the vertex x_j.

We then define an automaton $\mathcal{A}_G = (E' \cup \{\mathrm{id}\}, V, \lambda, \eta)$ such that

- $E' \cup \{\mathrm{id}\}$ is the set of states;
- V is the alphabet;
- $\lambda\colon E' \times V \to E'$ is the transition map such that, for each $e = (x, y) \in E'$, one has

$$\lambda(e, z) = \begin{cases} e & \text{if } z = x, \\ \mathrm{id} & \text{if } z \neq x; \end{cases}$$

- $\eta\colon E' \times V \to V$ is the output map such that, for each $e = (x, y) \in E'$, one has

$$\eta(e, z) = \begin{cases} y & \text{if } z = x, \\ x & \text{if } z = y, \\ z & \text{if } z \neq x, y. \end{cases}$$

In other words, any directed edge $e = (x, y)$ is a state of the automaton $\mathcal{A}_G$, and it has just one transition to itself (given by $\lambda(e, x)$) and all other transitions to the sink id. Its action is nontrivial only on the letters x and y, which are switched since $\eta(e, x) = y$ and $\eta(e, y) = x$. It is easy to check that $\mathcal{A}_G$ is invertible for any G and any choice of the orientation of the edges. The *graph automaton group* $\mathcal{G}_G$ is defined as the automaton group generated by $\mathcal{A}_G$. In [7, Theorem 3.7], it is shown that, whenever $|E| \geq 2$, the automaton $\mathcal{A}_G$ is bounded, so that the group $\mathcal{G}_G$ is amenable; moreover, $\mathcal{G}_G$ is a self-replicating group, and it is weakly regular branch over its commutator subgroup $\mathcal{G}'_G$.

For any integer $p \geq 1$, let $S_p = (V_p, E_p)$ denote the *star graph* on $p + 1$ vertices. Let us identify its vertex set V_p with the set $\{0, 1, 2, \dots, p\}$, where 0 corresponds to the central vertex, which is the only vertex of degree p, and the p leaves are identified with the vertex subset $\{1, 2, \dots, p\}$ (see Figure 1 for the case $p = 6$).

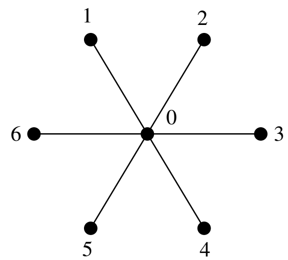

Figure 1. The star graph S_6.

In this paper, we will deal with *star automaton groups*, which are automaton groups obtained from S_p following the construction described above. The star automaton group defined starting from the graph S_p will be denoted by $\mathcal{G}_{S_p}$.

We conclude this subsection by recalling the definition of finite and infinite Schreier graphs associated with an automaton group $G(\mathcal{A})$.

Definition 2.2. The n-th Schreier graph $\Gamma_n = (V_{\Gamma_n}, E_{\Gamma_n})$ of the action of $G(\mathcal{A})$ on T_k, with respect to a symmetric generating set S, is the graph whose vertex set is X^n, where two vertices u and v are adjacent if and only if there exists $s \in S$ such that $s(u) = v$. If this is the case, the edge from u to v is labeled by s.

Notice that the Schreier graph Γ_n is a regular graph of degree $|S|$ on k^n vertices and it is connected for each n under the hypothesis of spherical transitivity. For each $n \geq 1$, let $\pi_{n+1}\colon \Gamma_{n+1} \to \Gamma_n$ be the map defined on $V_{\Gamma_{n+1}}$ as

$$\pi_{n+1}(x_1 \dots x_n x_{n+1}) = x_1 \dots x_n.$$

This map induces a surjective morphism from Γ_{n+1} onto Γ_n, which is a graph covering of degree k. In the rest of the paper, we will denote by A_n the adjacency matrix of the Schreier graph Γ_n: by definition, this is a symmetric square matrix of size k^n whose rows (and columns) sum to $|S|$. Since A_n is symmetric, all its eigenvalues are real: they constitute the adjacency spectrum (or spectrum) of Γ_n. Notice that the normalized adjacency matrix of Γ_n, which is given by $\frac{1}{|S|}A_n$, can be regarded as the transition matrix of the Markov operator M_n associated with the simple random walk on Γ_n.

For each $n \geq 1$, the Schreier graph Γ_n is nothing but the orbital graph of the action of $G(\mathcal{A})$ on the n-th level of the tree T_k or, equivalently, on the set X^n. On the other hand, it also makes sense to consider orbital graphs associated with the action of $G(\mathcal{A})$ on ∂T_k or, equivalently, on the set X^∞. Since the action of $G(\mathcal{A})$ on X^∞ has uncountably many orbits, there exist uncountably many distinct infinite Schreier graphs which are possibly nonisomorphic. It is therefore interesting to investigate isomorphism properties of infinite orbital Schreier graphs, regarded as unlabeled, unoriented, and unrooted graphs. We stress the fact that the isomorphism problem for labeled oriented Schreier graphs is much easier since rooted graphs are isomorphic if and only if the stabilizers of their roots coincide, so that unrooted graphs are isomorphic if and only if the corresponding stabilizers are conjugate.

Now take an infinite word $w = x_1x_2x_3\ldots \in X^\infty$ and denote by

$$w_n = x_1\ldots x_n \in X^n$$

its prefix of length n. It is known that the infinite Schreier graph Γ_w describing the orbit of w is approximated, as a rooted graph (Γ_w, w), by the sequence of finite Schreier graphs (Γ_n, w_n), in the space of rooted graphs of uniformly bounded degree endowed with the Gromov–Hausdorff convergence, provided, for example, by the following metric: given two rooted graphs (Γ_1, v_1) and (Γ_2, v_2), one puts

$$\mathrm{dist}((\Gamma_1, v_1), (\Gamma_2, v_2)) = \inf\left\{\frac{1}{r+1} : B_{\Gamma_1}(v_1, r) \text{ is isomorphic to } B_{\Gamma_2}(v_2, r)\right\},$$

where $B_{\Gamma_i}(v_i, r)$ is the ball of radius r in Γ_i centered in v_i (see [19, Theorem 3]).

According to the theory developed, for instance, in [1], under the hypothesis of amenability of the group $G(\mathcal{A})$, the spectrum of any infinite orbital Schreier graph Γ is obtained as

$$\mathrm{spectrum}(\Gamma) = \overline{\bigcup_{n=0}^{\infty} \mathrm{spectrum}(\Gamma_n)}.$$

2.2. Ihara zeta function

In this section, we recall the definition of Ihara zeta function for a finite regular graph Γ, which is an analog of the Riemann zeta function. For more details, the reader is referred to [20].

Definition 2.3. The Ihara zeta function $\zeta_\Gamma(t)$ for a finite regular graph Γ is the function

$$\zeta_\Gamma(t) = \exp\left(\sum_{r=1}^{\infty} \frac{c_r t^r}{r}\right),$$

where c_r is the number of closed, oriented loops of length r in the graph Γ.

It is also known that the Ihara zeta function of a finite regular graph $\Gamma = (V_\Gamma, E_\Gamma)$ of degree k satisfies the equation

$$\zeta_\Gamma(t) = (1-t^2)^{-\frac{k-2}{2}|V_\Gamma|} \det(1 - tkM + (k-1)t^2)^{-1},$$

where M is the Markov operator on Γ.

A notion of Ihara zeta function for an infinite rooted graph which is the limit of a sequence of finite regular rooted graphs can be given. Let (Γ_n, v_n) be a sequence of finite rooted graphs regular of degree k converging to the limit graph (Γ, v), and let M_n be the Markov operator on Γ_n whose transition matrix is the normalized adjacency matrix of Γ_n. The eigenvalues $\lambda_{i,n}$ of the operator M_n are said to be equidistributed with respect to a measure μ which has support in $[-1,1]$ if the sequence of counting measures

$$\mu_n = \sum_{i=1}^{|V_{\Gamma_n}|} \frac{\delta_{\lambda_{i,n}}}{|V_{\Gamma_n}|} \tag{2.2}$$

weakly converges to the measure μ. Moreover, it is known that given a covering sequence (Γ_n, v_n) of finite k-regular graphs, with associated Markov operators M_n, the eigenvalues of M_n are equidistributed with respect to some measure μ, which is called the Kesten–Neumann–Serre (KNS) spectral measure of the limit graph Γ. In particular,

$$\begin{aligned}\frac{1}{|V_{\Gamma_n}|}\ln \zeta_{\Gamma_n}(t) &= \sum_{r=1}^{\infty} \frac{c_r(\Gamma_n)t^r}{|V_{\Gamma_n}|r} \\ &= -\frac{k-2}{2}\ln(1-t^2) - \frac{1}{|V_{\Gamma_n}|}\ln\det(1-tkM_n+(k-1)t^2).\end{aligned}$$

When n goes to ∞, one gets

$$\ln \zeta_\Gamma(t) = \lim_{n\to\infty} \frac{1}{|V_{\Gamma_n}|}\ln \zeta_{\Gamma_n}(t) = \sum_{r=1}^{\infty} \frac{\widetilde{c_r}t^r}{r},$$

where $\widetilde{c_r}$ is the limit of the sequence $\frac{c_r(\Gamma_n)}{|V_{\Gamma_n}|}$. Moreover, the KNS spectral measure is uniquely determined by the Ihara zeta function $\zeta_\Gamma(t)$ according to the equation

$$\ln \zeta_\Gamma(t) = -\frac{k-2}{2}\ln(1-t^2) - \int_{-1}^{1}\ln(1-tk\lambda+(k-1)t^2)\,d\mu(\lambda) \quad \forall t : |t| < \frac{1}{k-1}.$$

We will apply this machinery in the setting of infinite orbital Schreier graphs, obtained as the limits of sequences of finite Schreier graphs, for the star automaton group $\mathcal{G}_{S_p}$.

3. Spectrum of Schreier graphs of the star automaton group $\mathcal{G}_{S_p}$

This section is devoted to the computation of the spectrum of both finite and infinite Schreier graphs associated with the action of the star automaton group $\mathcal{G}_{S_p}$ on the set $X^* \cup X^\infty$, where $X = \{0, 1, \dots, p\}$, or equivalently, on the regular rooted tree T_{p+1} and on its boundary. Since the same argument holds for every p, we prefer to present the explicit computation for the case $p = 3$ for the convenience of the reader; then we will extend the claim to the general case.

3.1. The case $p = 3$

Consider the oriented star graph S_3 on the four vertices $\{0, 1, 2, 3\}$ depicted in Figure 2.

The automaton associated with such orientation of S_3 is given in Figure 3.

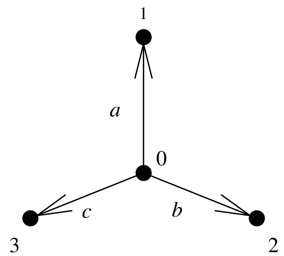

Figure 2. The oriented star graph S_3.

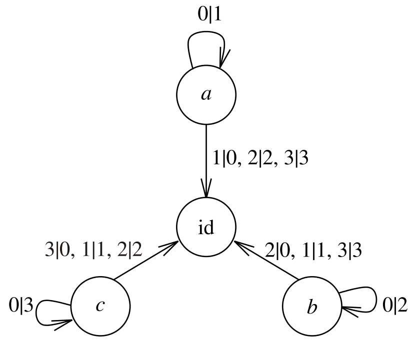

Figure 3. The automaton associated with the graph S_3 of Figure 2.

In particular, the star automaton group $\mathcal{G}_{S_3}$ is the group generated by the three automorphisms having the following self-similar representation (see [7]):

$$a = (a, \mathrm{id}, \mathrm{id}, \mathrm{id})(01), \quad b = (b, \mathrm{id}, \mathrm{id}, \mathrm{id})(02), \quad c = (c, \mathrm{id}, \mathrm{id}, \mathrm{id})(03). \tag{3.1}$$

Moreover, one has

$$a^{-1} = (\mathrm{id}, a^{-1}, \mathrm{id}, \mathrm{id})(01), \quad b^{-1} = (\mathrm{id}, \mathrm{id}, b^{-1}, \mathrm{id})(02), \quad c^{-1} = (\mathrm{id}, \mathrm{id}, \mathrm{id}, c^{-1})(03).$$

Let us denote by a_n, b_n, c_n the permutation matrices of size 4^n describing the action of the automorphisms a, b, c, respectively, on the set $\{0, 1, 2, 3\}^n$, so that the adjacency matrix A_n of the n-th Schreier graph Γ_n is given by

$$A_n = a_n + a_n^{-1} + b_n + b_n^{-1} + c_n + c_n^{-1}.$$

From equation (3.1), we get

$$a_{n+1} = \begin{pmatrix} 0 & a_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}, \quad b_{n+1} = \begin{pmatrix} 0 & 0 & b_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}, \quad c_{n+1} = \begin{pmatrix} 0 & 0 & 0 & c_n \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix},$$

where I_n is the identity matrix of size 4^n and 0 is the zero matrix of size 4^n. Similarly,

$$a_{n+1}^{-1} = \begin{pmatrix} 0 & I_n & 0 & 0 \\ a_n^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}, \quad b_{n+1}^{-1} = \begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ b_n^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}, \quad c_{n+1}^{-1} = \begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ c_n^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

Notice that

$$a_1 = a_1^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b_1 = b_1^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c_1 = c_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the adjacency matrix of the Schreier graph Γ_{n+1} is

$$A_{n+1} = \left(\begin{array}{c|c|c|c} 0 & a_n + I_n & b_n + I_n & c_n + I_n \\ \hline a_n^{-1} + I_n & 4I_n & 0 & 0 \\ \hline b_n^{-1} + I_n & 0 & 4I_n & 0 \\ \hline c_n^{-1} + I_n & 0 & 0 & 4I_n \end{array}\right) \quad \text{with } A_1 = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}.$$

We will make use of the following well-known result about determinant computation via the Schur complement formula (see, for instance, [24]).

Lemma 3.1. *Let M be a block matrix,*

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where A has size $k \times k$, B has size $k \times (n-k)$, C has size $(n-k) \times k$, and D has size $(n-k) \times (n-k)$. If D is nonsingular, one has

$$\det M = \det D \cdot \det(A - BD^{-1}C),$$

where the matrix $M/D := A - BD^{-1}C$ is called the Schur complement of D.

Theorem 3.2. *Let $P_n(\lambda)$ be the characteristic polynomial of the adjacency matrix A_n of the Schreier graph Γ_n for each $n \geq 1$. Then*

$$P_{n+1}(\lambda) = (\lambda - 4)^{2 \cdot 4^n} P_n(f(\lambda)) \tag{3.2}$$

with $f(\lambda) = \lambda^2 - 4\lambda - 6$ and $P_1(\lambda) = (\lambda - 6)(\lambda + 2)(\lambda - 4)^2$.

Proof. A direct computation gives $P_1(\lambda) = (\lambda - 6)(\lambda + 2)(\lambda - 4)^2$.

Now put $A'_{n+1} = A_{n+1} - \lambda I_{n+1}$, so that

$$P_{n+1}(\lambda) = \det A'_{n+1} = \det \left(\begin{array}{c|c|c|c} -\lambda I_n & a_n + I_n & b_n + I_n & c_n + I_n \\ \hline a_n^{-1} + I_n & (4-\lambda)I_n & 0 & 0 \\ \hline b_n^{-1} + I_n & 0 & (4-\lambda)I_n & 0 \\ \hline c_n^{-1} + I_n & 0 & 0 & (4-\lambda)I_n \end{array}\right).$$

In order to compute $\det A'_{n+1}$, we use the Schur complement technique, where

$$A = -\lambda I_n, \quad B = \left(a_n + I_n \,\middle|\, b_n + I_n \,\middle|\, c_n + I_n\right), \quad C = \begin{pmatrix} a_n^{-1} + I_n \\ b_n^{-1} + I_n \\ c_n^{-1} + I_n \end{pmatrix}, \quad D = (4-\lambda)I_{3n}.$$

The Schur complement of the block $D = (4-\lambda)I_{3n}$ is given by

$$A'_{n+1}/D = -\lambda I_n - \begin{pmatrix} a_n + I_n & b_n + I_n & c_n + I_n \end{pmatrix} \cdot \frac{1}{4-\lambda} I_{3n} \cdot \begin{pmatrix} a_n^{-1} + I_n \\ b_n^{-1} + I_n \\ c_n^{-1} + I_n \end{pmatrix}$$
$$= -\lambda I_n - \frac{1}{4-\lambda}(a_n + a_n^{-1} + 2I_n + b_n + b_n^{-1} + 2I_n + c_n + c_n^{-1} + 2I_n)$$
$$= \Big(-\lambda - \frac{6}{4-\lambda}\Big) I_n - \frac{1}{4-\lambda} A_n = \frac{\lambda^2 - 4\lambda - 6}{4-\lambda} I_n - \frac{1}{4-\lambda} A_n.$$

Therefore, we have

$$\begin{aligned} \det A'_{n+1} &= \det D \cdot \det(A'_{n+1}/D) \\ &= (4-\lambda)^{3\cdot 4^n} \cdot \det\Big(\frac{\lambda^2 - 4\lambda - 6}{4-\lambda} I_n - \frac{1}{4-\lambda} A_n\Big) \\ &= (4-\lambda)^{2\cdot 4^n} \cdot \det(A_n - (\lambda^2 - 4\lambda - 6) I_n). \end{aligned}$$

This completes the proof. ■

Remark 3.3. Observe that, if we define Γ_0 to be the graph consisting of a single vertex endowed with three loops, so that it is a regular graph of degree 6 as the graph Γ_n is for each $n \geq 1$, then we have $P_0(\lambda) = \lambda - 6$ and equation (3.2) still holds with $n = 0$, in fact,

$$P_1(\lambda) = (\lambda - 4)^2 \cdot P_0(f(\lambda))$$

since $P_0(f(\lambda)) = \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$.

In Figures 4 and 5, the Schreier graphs Γ_n, for $n = 1, 2, 3$, associated with the group $\mathcal{G}_{S_3}$ are depicted. Vertices are labeled by words in $\{0, 1, 2, 3\}^n$.

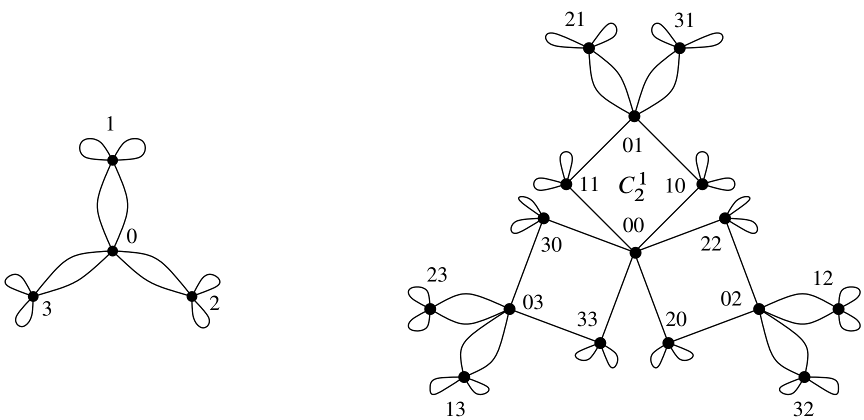

Figure 4. The Schreier graphs Γ_1 and Γ_2 associated with $\mathcal{G}_{S_3}$.

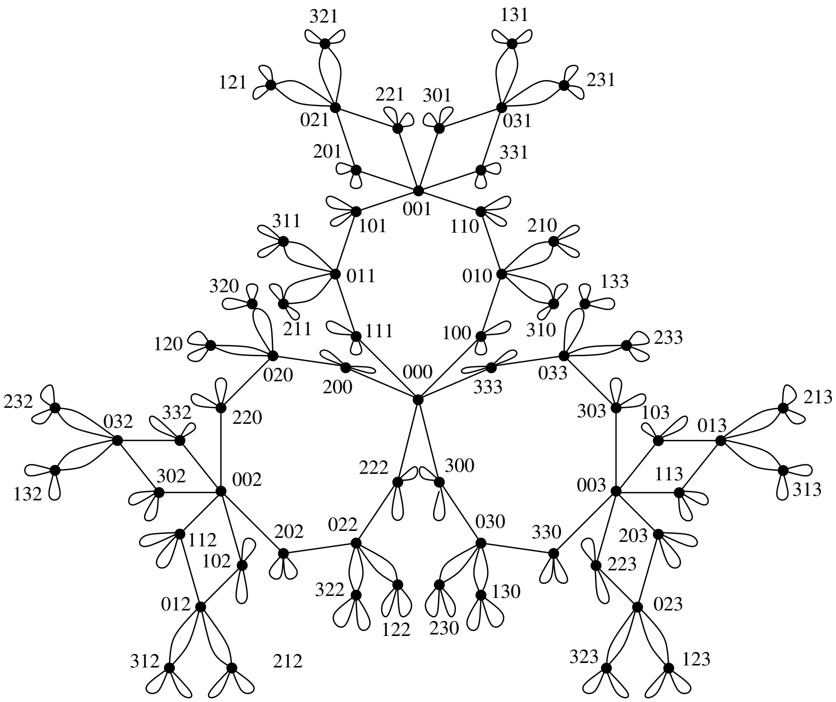

Figure 5. The Schreier graph Γ_3 associated with $\mathcal{G}_{S_3}$.

Theorem 3.4. *For each $n \geq 1$, the following factorization of the characteristic polynomial $P_n(\lambda)$ holds:*

$$P_n(\lambda) = (\lambda - 6) \cdot \prod_{i=0}^{n-1} (f^{\circ i}(\lambda) + 2) \cdot \prod_{i=0}^{n-1} (f^{\circ i}(\lambda) - 4)^{2 \cdot 4^{n-i-1}}, \tag{3.3}$$

where $f^{\circ i}(\lambda) = \underbrace{f(f(\dots f(\lambda)))}_{i \text{ times}}$. In particular, the adjacency spectrum of the graph Γ_n is

$$\{6\} \sqcup \left(\bigcup_{i=0}^{n-1} f^{-i}(-2) \right) \sqcup \left(\bigcup_{i=0}^{n-1} (f^{-i}(4))^{2 \cdot 4^{n-i-1}} \right).$$

Proof. Let $f(\lambda) = \lambda^2 - 4\lambda - 6$ as in Theorem 3.2. The factorization given in equation (3.3) can be proved by induction on n, using the recurrence

$$\begin{cases} P_{n+1}(\lambda) = (\lambda - 4)^{2 \cdot 4^n} P_n(f(\lambda)), \\ \quad P_1(\lambda) = (\lambda - 6)(\lambda + 2)(\lambda - 4)^2 \end{cases}$$

obtained in Theorem 3.2, and using the fact that

$$f(\lambda) - 6 = \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2),$$

which also implies that the iterated backward orbit of 6 under f can be written as

$$f^{-n}(6) = \{6\} \sqcup \left(\bigcup_{i=0}^{n-1} f^{-i}(-2) \right) \quad \forall n \geq 1.$$

The claim about the adjacency spectrum follows. ■

Remark 3.5. The eigenvalues of A_n given in Theorem 3.4 can be described more explicitly. In particular, a direct computation gives

$$f^{-1}(-2) = \{2 \pm 2\sqrt{2}\}, \quad f^{-2}(-2) = \left\{2 \pm \sqrt{12 \pm 2\sqrt{2}}\right\},$$

and, in general, it can be shown by induction that

$$f^{-n}(-2) = \left\{2 \pm \sqrt{12 \pm \sqrt{12 \pm \sqrt{\cdots \pm 2\sqrt{2}}}}\right\}, \quad n \geq 1,$$

where the double sign $\pm$ occurs n times. Similarly, one has

$$f^{-1}(4) = \{2 \pm \sqrt{14}\}, \quad f^{-2}(4) = \left\{2 \pm \sqrt{12 \pm \sqrt{14}}\right\}$$

and in general

$$f^{-n}(4) = \left\{2 \pm \sqrt{12 \pm \sqrt{12 \pm \sqrt{\cdots \pm \sqrt{14}}}}\right\}, \quad n \geq 1,$$

where also in this case the double sign $\pm$ occurs n times. In Figure 6, the histogram of the spectrum of the Schreier graph Γ_6 of the group $\mathcal{G}_{S_3}$ (in logarithmic scale) is represented.

Lemma 3.6. *Let $a, b > 0$, and let $\{a_n\}_{n\geq 1}$ be the sequence defined by recursion as*

$$\begin{cases} a_1 = \sqrt{b}, \\ a_{n+1} = \sqrt{a + a_n}, \quad n \geq 1. \end{cases}$$

If $\sqrt{b} < \frac{1}{2}(1 + \sqrt{1 + 4a})$, then the sequence $\{a_n\}_{n\geq 1}$ is increasing and

$$\lim_{n\to\infty} a_n = \frac{1}{2}(1 + \sqrt{1 + 4a}).$$

Proof. It is easy to show by induction that the sequence $\{a_n\}_{n\geq 1}$ is increasing and bounded, so that it admits a finite limit ℓ. By squaring, one can see that such a limit ℓ must satisfy the equation $\ell^2 - \ell - a = 0$, whose solutions are $\ell = \frac{1\pm\sqrt{1+4a}}{2}$. The solution corresponding to the sign $-$ cannot be accepted, since it must be $\ell > 0$, and we get the claim. ■

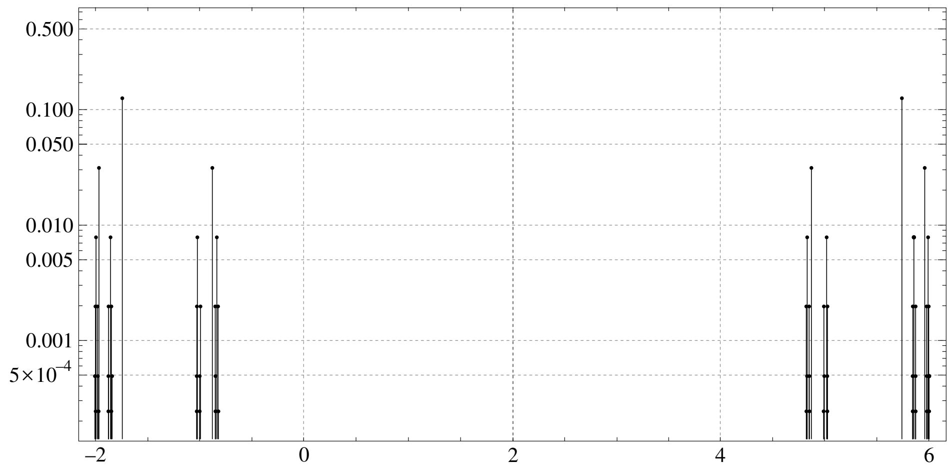

Figure 6. The histogram of the spectrum of the Schreier graph Γ_6 of the group $\mathcal{G}_{S_3}$.

Theorem 3.7. *The spectrum of each infinite Schreier graph* Γ *of* $\mathcal{G}_{S_3}$ *is the closure of the set of points*

$$\{4\} \cup \left\{ 2 \pm \underbrace{\sqrt{12 \pm \sqrt{12 \pm \sqrt{\cdots \pm \sqrt{14}}}}}_{n \text{ times}},\ n \geq 1 \right\}.$$

This set is the union of a Cantor set of zero Lebesgue measure which is symmetric about 2 *and a countable collection of isolated points supporting the KNS spectral measure* μ, *which is discrete and which has value* $\frac{1}{2\cdot 4^n}$ *at the points whose definition involves n radicals, for* $n \geq 1$, *and value* $\frac{1}{2}$ *at the point* 4.

Proof. Since the group $\mathcal{G}_{S_3}$ is amenable, the spectrum of each infinite Schreier graph Γ of $\mathcal{G}_{S_3}$ is given by

$$\overline{\{6\} \sqcup \left(\bigcup_{i=0}^{\infty} f^{-i}(-2) \right) \sqcup \left(\bigcup_{i=0}^{\infty} f^{-i}(4) \right)}.$$

Let us investigate the dynamics of the quadratic map $f(\lambda) = \lambda^2 - 4\lambda - 6$. As $f'(\lambda) = 2\lambda - 4$, the unique critical point of f is $\lambda_0 = 2$. Therefore, the critical value $f(2) = -10$ is the unique value of x such that the equation $f(\lambda) = x$ has a double root.

Now observe that Lemma 3.6 returns the limit value 4 for $a = 12$. It follows that, for each n, the spectrum of Γ_n is contained in the interval $[-2, 6]$. Now, it is easy to check that

$$f^{-1}[-2, 6] = [-2, 2 - 2\sqrt{2}] \cup [2 + 2\sqrt{2}, 6] \subseteq [-2, 6].$$

Since the critical value $-10 \notin f^{-1}[-2,6] \subseteq [-2,6]$, it follows that, for any value of x in $[-2,6]$, the entire backward orbit $f^{-i}(x)$ is still contained in $[-2,6]$ and the sets $f^{-i}(x)$, for each $i \geq 0$, consist of 2^i distinct real numbers. Moreover, it is known that, for such x, the sets $f^{-i}(x)$ are mutually disjoint for $i \geq 0$, provided x is not a periodic point (a point x is periodic if $f^k(x) = x$ for some positive integer k).

In our case, the forward orbit of 4 under f goes to ∞, so that 4 is not a periodic point and the sets $f^{-i}(4)$ are mutually disjoint for $i \geq 0$. On the other hand, since $f(6) = 6$, so that 6 is a fixed point for f, the point -2 is not periodic and the sets $f^{-i}(-2)$ are mutually disjoint for $i \geq 0$. In particular, it follows that the number of distinct eigenvalues of the graph Γ_n is

$$1 + 2\sum_{i=0}^{n-1} 2^i = 2^{n+1} - 1 \quad \text{for each } n \geq 1.$$

Recall now that a periodic point x of f is repelling if $|f'(x)| > 1$. Since $f(6) = 6$ and $f'(6) = 8 > 0$, the point 6 is a repelling fixed point for the polynomial f. As the Julia set J of f is the closure of the set of repelling periodic points of f, then $6 \in J$. Now, by the total invariance of J, the backward orbit $\{6\} \sqcup (\bigcup_{i=0}^{\infty} f^{-i}(-2))$ of 6 is in J [12].

On the other hand, the value 4 is not in the Julia set, since its forward orbit goes to ∞, and therefore the set $\bigcup_{i=0}^{\infty}(f^{-i}(4))$ is a countable set of isolated points that accumulates to the Julia set J. It follows that the spectrum of Γ is given by

$$\overline{\bigcup_{i=0}^{\infty} f^{-i}(4)},$$

where, for each i, the set $f^{-i}(4)$ has been described in Remark 3.5. Notice that the Julia set J of f is a Cantor set since the map f is conjugate via the map $F(z) = z + 2$ to the quadratic map

$$z \mapsto z^2 - 12,$$

that is, $(F^{-1} \circ f \circ F) = z^2 - 12$, and $-12 < -2$ (see [12, Section 3.2]). Recall that the KNS spectral measure μ is the limit of the counting measures μ_n defined for Γ_n as in equation (2.2). We also know that in the spectrum of Γ_n, each eigenvalue in $\{6\} \sqcup (\bigcup_{i=0}^{n-1} f^{-i}(-2))$ has multiplicity 1, whereas each eigenvalue in $f^{-i}(4)$ has multiplicity $2 \cdot 4^{n-i-1}$ for each i. Now

$$\lim_{n\to\infty} \frac{2 \cdot 4^{n-i-1}}{4^n} = \frac{1}{2 \cdot 4^i} \quad \text{for each } i \geq 0.$$

Being $\sum_{i=0}^{\infty} \frac{2^i}{2 \cdot 4^i} = 1$, the KNS spectral measure is discrete and concentrated at these eigenvalues. ∎

The Ihara zeta function $\zeta_n(t)$ of the Schreier graph Γ_n of $\mathcal{G}_{S_3}$ satisfies the equation

$$\zeta_n(t) = (1 - t^2)^{-2 \cdot 4^n} \det(1 - tA_n + 5t^2)^{-1},$$

where A_n is the adjacency matrix of Γ_n. When passing to the limit, the following integral presentation holds:

$$\ln \zeta_\Gamma(t) = -2\ln(1-t^2) - \int_{-1}^{1} \ln(1-6t\lambda+5t^2)\, d\mu(\lambda) \quad \forall t : |t| < \frac{1}{5},$$

where μ is the KNS spectral measure and λ runs over the normalized spectrum of Γ. In our case, for each t such that $|t| < \frac{1}{5}$, we get

$$\begin{aligned}\ln \zeta_\Gamma(t) = & -2\ln(1-t^2) - \frac{1}{2}\ln(1-4t+5t^2) \\ & - \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{4^i} \ln\left(1 - t\left(\underbrace{2 \pm \sqrt{12 \pm \sqrt{12 \pm \sqrt{\cdots \pm \sqrt{14}}}}}_{\pm\ i \text{ times}}\right) + 5t^2\right).\end{aligned}$$

3.2. The general case

Let $p \geq 1$ be an integer number. The aim of this subsection is to generalize what we have seen in the previous subsection for the graph S_3 to the more general context of a star graph S_p on $p+1$ vertices. We will not give all the details presented in the case $p = 3$.

The star automaton group $\mathcal{G}_{S_p}$ is the group generated by p automorphisms e_i, $i = 1, \ldots, p$, having the following self-similar representation:

$$e_i = (e_i, \mathrm{id}, \ldots, \mathrm{id})(0i) \quad \text{for each } i = 1, \ldots, p. \tag{3.4}$$

Notice that

$$e_i^{-1} = (\mathrm{id}, \ldots, \mathrm{id}, \underbrace{e_i^{-1}}_{(i+1)\text{-th place}}, \mathrm{id}, \ldots, \mathrm{id})(0i) \quad \text{for each } i = 1, \ldots, p.$$

The group $\mathcal{G}_{S_p}$ acts on the rooted tree T_{p+1}. The n-th level of such a tree consists of $(p+1)^n$ vertices, identified with the set of words of length n over the alphabet $\{0, 1, \ldots, p\}$. As a consequence, the n-th Schreier graph is a regular graph of degree $2p$ on $(p+1)^n$ vertices, and its adjacency matrix A_n is a symmetric matrix of size $(p+1)^n$. We will adopt the notation Γ_n^p to denote the n-th Schreier graph associated with the action of $\mathcal{G}_{S_p}$. The following theorem holds.

Theorem 3.8. *Let $P_n(\lambda)$ be the characteristic polynomial of the adjacency matrix A_n of the Schreier graph Γ_n^p of the group $\mathcal{G}_{S_p}$ for each $n \geq 1$. Then*

$$P_{n+1}(\lambda) = (\lambda - 2(p-1))^{(p-1)(p+1)^n} P_n(f_p(\lambda))$$

with $f_p(\lambda) = \lambda^2 - 2(p-1)\lambda - 2p$ and $P_1(\lambda) = (\lambda - 2p)(\lambda + 2)(\lambda - 2(p-1))^{p-1}$.

Moreover, one can still define Γ_0^p to be the graph consisting of a single vertex endowed with p loops. In this way, one has $P_0(\lambda) = \lambda - 2p$, and the equation

$$P_1(\lambda) = (\lambda - 2(p-1))^{p-1} \cdot P_0(f_p(\lambda)),$$

is still satisfied.

Theorem 3.9. *For each $n \geq 1$, the following factorization of the characteristic polynomial $P_n(\lambda)$ holds:*

$$P_n(\lambda) = (\lambda - 2p) \cdot \prod_{i=0}^{n-1} (f_p^{\circ i}(\lambda) + 2) \cdot \prod_{i=0}^{n-1} (f_p^{\circ i}(\lambda) - 2(p-1))^{(p-1)(p+1)^{n-i-1}},$$

where $f_p^{\circ i}(\lambda) = \underbrace{f_p(f_p(\ldots f_p(\lambda)))}_{i \text{ times}}$.

In particular, the adjacency spectrum of the graph Γ_n^p is

$$\{2p\} \sqcup \left(\bigcup_{i=0}^{n-1} f_p^{-i}(-2) \right) \sqcup \left(\bigcup_{i=0}^{n-1} (f_p^{-i}(2(p-1)))^{(p-1)(p+1)^{n-i-1}} \right).$$

Proof. The proof proceeds as in Theorem 3.4 and uses the fact that

$$f_p^{-n}(2p) = \{2p\} \sqcup \left(\bigcup_{i=0}^{n-1} f_p^{-i}(-2) \right),$$

because $\lambda^2 - 2(p-1)\lambda - 4p = (\lambda - 2p)(\lambda + 2)$. ∎

In Figures 7 and 8, the Schreier graphs Γ_n^2, for $n = 1, 2, 3, 4$, associated with the group $\mathcal{G}_{S_2}$ are depicted. Vertices are labeled by words in $\{0, 1, 2\}^n$.

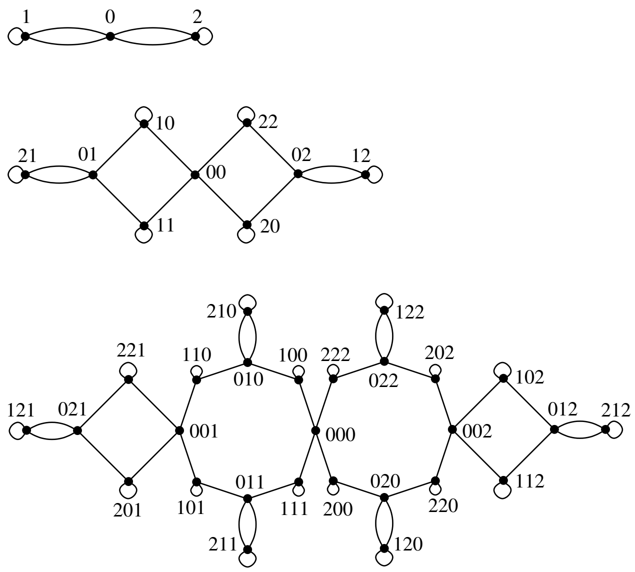

Figure 7. The Schreier graphs Γ_n^2 associated with $\mathcal{G}_{S_2}$, for $n = 1, 2, 3$.

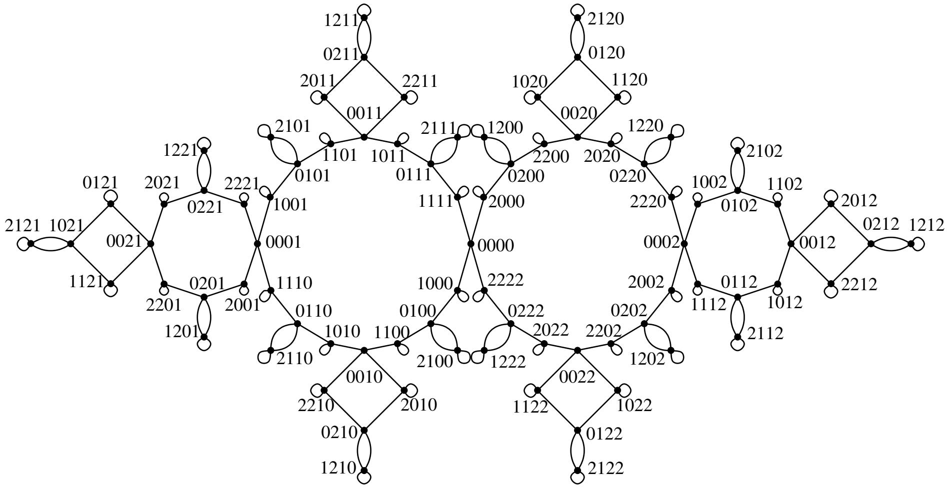

Figure 8. The Schreier graph Γ_4^2 associated with $\mathcal{G}_{S_2}$.

Theorem 3.10. *Let $p \geq 2$. The spectrum of each infinite Schreier graph Γ^p of $\mathcal{G}_{S_p}$ is the closure of the set of*

$$\{2(p-1)\} \cup \left\{ p-1 \pm \underbrace{\sqrt{p^2+p \pm \sqrt{p^2+p \pm \sqrt{\cdots \pm \sqrt{p^2+2p-1}}}}}_{n \text{ times}},\ n \geq 1 \right\}.$$

This set is the union of a Cantor set of zero Lebesgue measure which is symmetric about $p-1$ and a countable collection of isolated points supporting the KNS spectral measure μ, which is discrete and which has value $\frac{p-1}{(p+1)^{n+1}}$ at the points whose definition involves n radicals, for $n \geq 1$, and value $\frac{p-1}{p+1}$ at the point $2(p-1)$.

Proof. The proof proceeds as in the case $p = 3$. The spectrum of Γ^p is given by

$$\overline{\{2p\} \sqcup \left(\bigcup_{i=0}^{\infty} f_p^{-i}(-2) \right) \sqcup \left(\bigcup_{i=0}^{\infty} f_p^{-i}(2(p-1)) \right)},$$

with $f_p(\lambda) = \lambda^2 - 2(p-1)\lambda - 2p$. A direct computation gives, for $n \geq 1$,

$$f_p^{-n}(-2) = \left\{ p-1 \pm \sqrt{p^2+p \pm \sqrt{p^2+p \pm \sqrt{\cdots \pm \sqrt{p^2-1}}}} \right\},$$

$$f_p^{-n}(2(p-1)) = \left\{ p-1 \pm \sqrt{p^2+p \pm \sqrt{p^2+p \pm \sqrt{\cdots \pm \sqrt{p^2+2p-1}}}} \right\},$$

where the double sign $\pm$ occurs n times. By using Lemma 3.6, it is easy to check that, for each n, the spectrum of Γ_n^p is contained in the interval $[-2, 2p]$. Here, the countable set of isolated points that accumulates to the Julia set J of f_p is the set $\bigcup_{i=0}^{\infty}(f_p^{-i}(2(p-1)))$. Notice that the Julia set J of f_p has the structure of a Cantor set since the map f_p is conjugate via the map $F_p(z) = z + (p-1)$ to the quadratic map

$$z \mapsto z^2 - p(p+1),$$

and $-p(p+1) < -2$ for every $p \geq 2$. ■

The Ihara zeta function $\zeta_n(t)$ of the Schreier graph Γ_n^p satisfies the equation

$$\zeta_n(t) = (1-t^2)^{-(p-1)(p+1)^n} \det(1 - tA_n + (2p-1)t^2)^{-1},$$

where A_n is the adjacency matrix of Γ_n^p. When passing to the limit, the following integral presentation holds:

$$\ln \zeta_{\Gamma^p}(t) = -(p-1)\ln(1-t^2) - \int_{-1}^{1} \ln(1 - 2pt\lambda + (2p-1)t^2) d\mu(\lambda) \ \ \forall t : |t| < \frac{1}{2p-1},$$

where μ is the KNS spectral measure and λ runs over the normalized spectrum of Γ^p. In particular, we obtain for each t such that $|t| < \frac{1}{2p-1}$,

$$\begin{aligned}\ln \zeta_{\Gamma^p}(t) = & -(p-1)\ln(1-t^2) - \frac{p-1}{p+1}\ln(1 - 2(p-1)t + (2p-1)t^2) \\ & - \frac{p-1}{p+1}\sum_{i=1}^{\infty}\frac{1}{(p+1)^i} \\ & \times \ln\left(1 - t\left(p - 1 \pm \sqrt{p^2 + p \pm \sqrt{\cdots \pm \sqrt{p^2+2p-1}}}\right) + (2p-1)t^2\right).\end{aligned}$$

Remark 3.11. In Theorem 3.10, we have supposed $p \geq 2$. In fact, for $p = 1$, the star S_1 is the path graph P_2 on 2 vertices. The associated star automaton group $\mathcal{G}_{S_1}$ is the group acting on the binary rooted tree T_2 generated by the automorphism a having the self-similar representation

$$a = (a, \mathrm{id})(01),$$

which is isomorphic to the group $\mathbb{Z}$ and which is classically known as *adding machine*. For each $n \geq 1$, the n-th Schreier graph Γ_n^1 is a cycle on 2^n vertices. Moreover, one has $f_1 = \lambda^2 - 2$, and the Julia set of this quadratic map is the whole interval $[-2, 2]$.

The case $p = 2$ corresponds to the path graph P_3 on 3 vertices. The associated star automaton group $\mathcal{G}_{S_2}$ is known as *tangled odometer*, and it is the group acting on the rooted ternary tree T_3 generated by the automorphisms a and b having the following self-similar representation:

$$a = (a, \mathrm{id}, \mathrm{id})(01), \quad b = (b, \mathrm{id}, \mathrm{id})(02).$$

It is worth mentioning that $\mathcal{G}_{S_2}$ can be also obtained as the iterated monodromy group of the polynomial $\frac{-z^3+3z}{2}$. This case has been investigated in [7], where the family of groups associated with the path graph P_k, for every $k \geq 2$, has been treated in detail. Notice that, for $p = 2$, our spectral results recover the ones given for this group in [17, Theorem 6.3].

4. Schreier graphs of star automaton groups

In this section, we give a complete classification of the infinite Schreier graphs associated with the star automaton group $\mathcal{G}_{S_p}$. By complete classification we mean the following: we have already remarked that, given an infinite sequence $w = x_1x_2x_3\ldots \in X^\infty$, one can define the rooted graph (Γ_w, w) as the limit of the sequence of rooted graphs $(\Gamma_n, x_1\ldots x_n)$. Now we can forget the root and consider the corresponding (unrooted) infinite Schreier graph. We want to describe the isomorphism classes of such infinite graphs arising from the action of the star automaton group on X^∞. In what follows, given a subgraph Θ of Γ_n^p, we denote by Θw the set of vertices obtained by appending the word $w \in X^* \cup X^\infty$ to the vertices of Θ. When it is clear from the context, with abuse of notation, we identify a set of vertices of a graph with its induced subgraph. The geodesic distance (or distance for short) between the vertices u, v is denoted by $d(u, v)$.

Definition 4.1. (1) Two infinite sequences $\xi = x_1x_2x_3\ldots$ and $\eta = y_1y_2y_3\ldots$ in X^∞ are cofinal if there exists $k \in \mathbb{N}$ such that $x_n = y_n$ for any $n \geq k$.

(2) Two sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ of integers are compatible if there exist $l, h \in \mathbb{N}$ such that $x_{l+n} = y_{h+n}$ for any $n \in \mathbb{N}$.

In other words, two infinite words over X are cofinal if they differ only for prefixes of equal length. In this case, we write $\xi \sim \eta$. The cofinality is an equivalence relation, and we denote by $\mathrm{Cof}(\xi)$ the equivalence class of words cofinal to ξ. Two sequences are compatible if they coincide after removing from them some initial terms (possibly a different number of them). Notice that also being compatible is an equivalence relation. We want to stress the fact that two compatible words differ for prefixes which do not have to be of the same length.

4.1. Finite Schreier graphs

From now on, we fix a star S_p and use the same representation from Figure 2. In this case, $X = \{0, 1, \ldots, p\}$, where 0 is the vertex of degree p. We denote by e_i the (directed) edge connecting 0 to $i \in \{1, \ldots, p\}$, so that $e_i = (e_i, \mathrm{id}, \ldots, \mathrm{id})(0i)$ (see equation (3.4)). We will denote by Γ_n^p the n-th Schreier graph of the group $\mathcal{G}_{S_p}$.

Observe that the generator e_i of $\mathcal{G}_{S_p}$ acts like an adding machine on the set $\{0, i\}^*$. More precisely, when we let it act on a vertex of type $0^t jw$, with $j \neq 0, i$ and $|w| = n - t - 1$, we obtain a cycle of length 2^t whose vertex set is the whole set $\{0, i\}^t jw$.

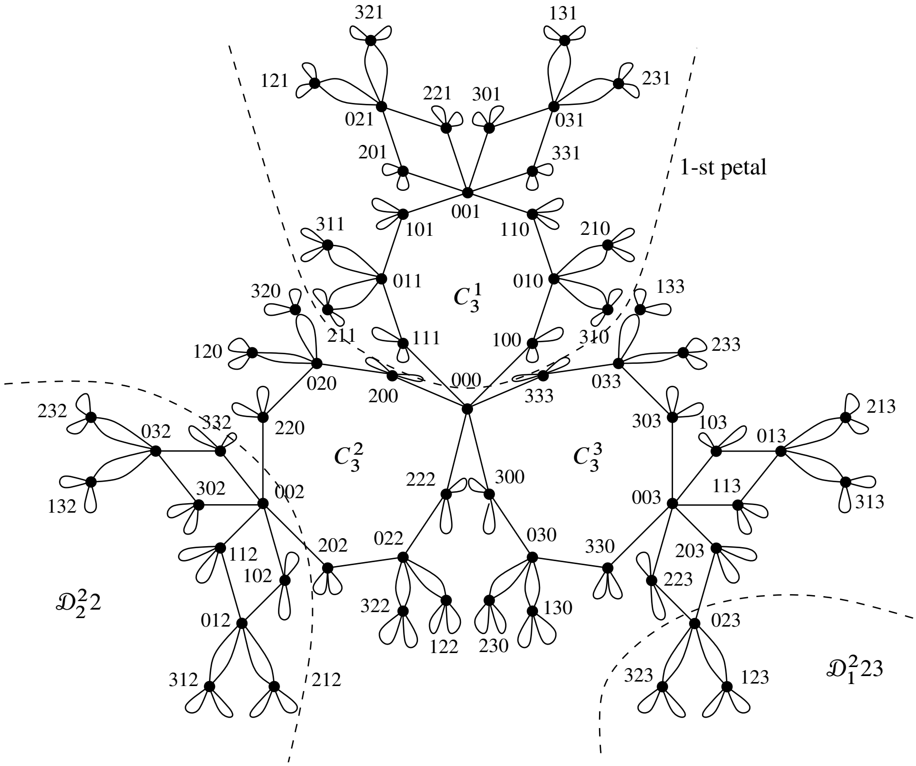

Figure 9. The Schreier graph Γ_3^3.

Let us denote by C_n^i the (maximal) cycle of length 2^n labeled by e_i for $i = 1, \ldots, p$. Notice that the maximal cycles in Γ_n^p are exactly those generated by the e_i's and containing 0^n.

Example 4.2. In Figure 9, which represents the Schreier graph Γ_3^3, the three maximal cycles have length 8. With respect to (3.1), one has $e_1 = a$, $e_2 = b$, $e_3 = c$. In particular,

- the cycle C_3^1, containing the adjacent vertices 000 and 111, is obtained by letting a act on the vertex 000;
- the cycle C_3^2, containing the adjacent vertices 000 and 222, is obtained by letting b act on the vertex 000;
- the cycle C_3^3, containing the adjacent vertices 000 and 333, is obtained by letting c act on the vertex 000.

Lemma 4.3. *If u, v are adjacent vertices in Γ_n^p, then the vertices uw and vw are adjacent in $\Gamma_{n+|w|}^p$ for any $w \in X^*$ with the only exception, for $i = 1, \ldots, p$, given by $\{u, v\} = \{0^n, i^n\}$ and w starting with 0 or i.*

Proof. It is enough to notice that if u, v are adjacent vertices in Γ_n^p, then there exists i such that $e_i(u) = v$, i.e., a directed path in the generating automaton labeled by u and v and starting from the state e_i. Such a path must either end up in the sink (when $\{u, v\} \neq \{0^n, i^n\}$) or end up in e_i (when $\{u, v\} = \{0^n, i^n\}$). In the first case, we can append to u any $w \in X^*$ in such a way that $e_i(uw) = vw$. In the second case, if w starts with a letter $j \neq 0, i$, the path labeled by $0^n j$ and $e_i(0^n j) = i^n j$ ends up in the trivial state. Hence also in the case $\{u, v\} = \{0^n, i^n\}$ and w not starting by 0, i, one has that uw and vw are adjacent in $\Gamma_{n+|w|}^p$. ■

Remark 4.4. Lemma 4.3 implies that any cycle C in Γ_n^p labeled by e_i and different from C_n^i appears $(p+1)^k$ times in the Schreier graph Γ_{n+k}^p, with vertices Cw for any $w \in X^k$. The same can be said for cycles of the form $C_n^i v$ where v does not start with 0 or i.

From Lemma 4.3, we deduce that, passing from Γ_n^p to Γ_{n+1}^p, each cycle in Γ_n^p is preserved just by adding to all its vertices the same letter $k \in X$ except for some of the p maximal cycles C_n^i. In fact, $C_n^i j$ with $j \neq 0, i$ also corresponds to a subgraph in Γ_{n+1}^p that is a copy of C_n^i, whereas $C_n^i 0$ and $C_n^i i$ correspond to the two halves of the new maximal cycle C_{n+1}^i of Γ_{n+1}^p.

Example 4.5. Look at Figures 4 and 5, where $p = 3$. We have that the maximal cycle C_2^1 in Γ_2^3 produces the cycles $C_2^1 2$ and $C_2^1 3$ of length 4 in Γ_3^3, which are attached to the vertices 002 and 003, respectively. On the other hand, the cycles $C_2^1 0$ and $C_2^1 1$ do not appear in Γ_3^3, but they constitute the two halves of the maximal cycle C_3^1 (the edge connecting 001 and 111 and the edge connecting 000 and 110 do not appear, whereas two new edges connecting the vertices 001 and 110, and the vertices 000 and 111, appear).

Recall that a *cut-vertex* of a graph is a vertex whose deletion increases the number of connected components of the graph (see, for instance, [3]). Following [7, Proposition 4.7], we have that $0u$ is a cut-vertex in Γ_n^p for any $u \in X^{n-1}$. In particular, 0^n is a cut-vertex. Notice that, by removing 0^n from Γ_n^p, we obtain p connected components that we call *petals*. More precisely, the vertex 0^n belongs to the maximal cycle C_n^i, for each $i = 1, \dots, p$, and the connected component containing this maximal cycle generated by e_i is called the i-th petal. One can show that the i-th petal consists of the set of vertices ending with a suffix $i0^k$, for $k = 0, 1, \dots, n-1$. See, for instance, Figure 9, representing the Schreier graph Γ_3^3, where the 1-st petal is highlighted in the upper part of the graph.

All other vertices of Γ_n^p, those beginning with $i \neq 0$, have $p-1$ loops corresponding to the actions of the generators e_j, with $j \neq i$ (we consider loops as cycles of length 1). In the remaining part of the paper, we will consider also such vertices as cut-vertices. In particular, it follows that Γ_n^p has a cactus structure. In particular, the following lemma holds.

Lemma 4.6. *The vertex 0^n is a cut-vertex belonging to C_n^i for any $i = 1, \dots, p$. Any vertex $v \in \{0, i\}^n \setminus \{0^n\}$ is a cut-vertex belonging to C_n^i and to other $p-1$ cycles labeled by e_j, with $j \neq i$, whose size is 2^k if $v = 0^k i v'$, with $0 \leq k \leq n-1$.*

Definition 4.7. Let Γ_n^p be n-th Schreier graph of the group $\mathcal{G}_{S_p}$. Let $i \in \{1, \dots, p\}$. The n-decoration $\mathcal{D}_n^i$ is the subgraph of Γ_n^p obtained by removing from Γ_n^p the i-th petal.

Notice that $\mathcal{D}_n^i$ contains 0^n and is connected. Basically, it is the union of the petals different from the i-th one together with the vertex 0^n. Moreover, $\mathcal{D}_n^i$ and $\mathcal{D}_n^j$ are isomorphic graphs for any i, j. When we are not interested in the specific decoration, but just in its structure, we only write $\mathcal{D}_n$.

From Lemmas 4.3 and 4.6, it follows that $\mathcal{D}_n^i iw$ induces a subgraph in $\Gamma_{n+1+|w|}^p$ which is a copy of $\mathcal{D}_n^i$ via the map $viw \mapsto v$. In particular, $\mathcal{D}_n^i i$ is attached to the vertex $0^n i$ of the maximal cycle C_{n+1}^i generated by e_i in Γ_{n+1}^p. From this, it follows that $\mathcal{D}_n^i i$ is a subgraph of $\mathcal{D}_{n+1}^j$ for every $j \neq i$. When we want to highlight the fact that its structure comes from the n-th level, we say that such subgraph of Γ_{n+1}^p is an n-decoration of C_{n+1}^i. By using an analogous argument, we deduce that the subgraphs $\mathcal{D}_n^i ii$ and $\mathcal{D}_n^i i0$ inside Γ_{n+2}^p are n-decorations attached to C_{n+2}^i at the vertices $0^n ii$ and $0^n i0$. By iterating this argument, we can conclude that, for every $u \in \{0, i\}^m$, the subgraph $\mathcal{D}_n^i iu$ is an n-decoration in Γ_{n+m+1}^p attached to C_{n+m+1}^i at the vertex $0^n iu$. Hence in Γ_n^p, we have attached to C_n^i:

- the decoration $\mathcal{D}_n^i$ at the vertex 0^n;
- one $(n-1)$-decoration given by $\mathcal{D}_{n-1}^i i$ at the vertex $0^{n-1} i$;
- 2^k copies of an $(n-k-1)$-decoration given by $\mathcal{D}_{n-k-1}^i iu$ at the vertex $0^{n-k-1} iu$, with $k = 1, \dots, n-1$ for every $u \in \{0, i\}^k$.

Here, by 0-decoration we mean a vertex with $p-1$ loops attached. In Figure 9, representing the Schreier graph Γ_3^3, the 2-decoration $\mathcal{D}_2^2 2$ and one 1-decoration given by $\mathcal{D}_1^2 23$ are depicted, attached to the vertices 002 and 023, respectively.

Remark 4.8. Notice that the vertex $0^k iv'$ of C_n^i has attached the k-decoration $\mathcal{D}_k^i iv'$.

Proposition 4.9. *Let ϕ_n^i be the nontrivial automorphism of C_n^i fixing 0^n. Then for any $v \in C_n^i$, the vertices v and $\phi_n^i(v)$ have attached decorations that are isomorphic. In particular, $\phi_n^i(v)$ is the only vertex of C_n^i satisfying $d(0^n, v) = d(0^n, \phi_n^i(v))$.*

Proof. The vertices of C_n^i can be identified with the numbers $0, \dots, 2^n - 1$ by using the binary expansion (from the left to the right) of such numbers by identifying i with 1. Notice that the automorphism ϕ_n^i is a reflection around the axis connecting 0^n and $0^{n-1} i$ and it acts in such a way that $v + \phi_n^i(v) \equiv 0 \mod 2^n$. In particular, if $u = 0^k iv$, with $k \in \{0, 1, \dots, n-1\}$, then $\phi_n^i(u) = 0^k iv'$, where v' is the word obtained from v by switching any 0 to i and vice versa. By Remark 4.8, such vertices have attached the same $(n - |v| - 1)$-decoration. The claim follows. ∎

Any vertex of Γ_n^p is a cut-vertex belonging to p different cycles. If the vertex u belongs to the i-th petal of Γ_n^p, then there is a unique path of cycles, connecting u to C_n^i. The first cycle in this path is the one containing u in the direction of C_n^i. Notice that the path of cycles is not defined for 0^n. From now on, we do not consider this vertex.

We denote by $\mathcal{P}_n^u = \{P_1^n(u), \ldots, P_{m_u}^n(u)\}$ the *path of cycles* associated with $u \in \Gamma_n^p$. Notice that $P_{m_u}^n(u)$ is C_n^i if u belongs to the i-th petal. Moreover, we denote by $\mathcal{L}_n^u = \{L_1^n(u), \ldots, L_{m_u}^n(u)\}$ the set of the lengths of the cycles in $\mathcal{P}_n^u$, i.e., $L_i^n(u)$ is the length of the cycle $P_i^n(u)$. In what follows, with a small abuse of notation, we identify the graph $P_i^n(u)$ with its vertex set.

Example 4.10. Looking at the graph Γ_3^3 in Figure 9, we have $\mathcal{L}_3^{121} = \{2, 4, 8\}$; $\mathcal{L}_3^{201} = \{4, 8\}$; $\mathcal{L}_3^{130} = \{2, 8\}$.

Given a word $u \in X^n \setminus \{0^n\}$, we can write $u = 0^k a_1 u_1 a_2 u_2 \ldots a_t u_t$, where $0 \leq k \leq n-1$, $a_i \in \{1, \ldots, p\}$, $a_i \neq a_{i+1}$ and $u_i \in \{0, a_i\}^*$. We call this writing the *decomposition* of u.

Lemma 4.11. *Let $u \in X^n \setminus \{0^n\}$, and let $u = 0^k a_1 u_1 a_2 u_2 \ldots a_t u_t$ be its decomposition. Then*

(1) $m_u = t$;

(2) $P_1^n(u) = C^{a_1}_{|u_1|+k+1} a_2 u_2 \ldots a_t u_t$, $P_2^n(u) = C^{a_2}_{|u_1|+|u_2|+k+2} a_3 u_3 \ldots a_t u_t, \ldots$, $P_i^n(u) = C^{a_i}_{\sum_{\ell=1}^{i} |u_\ell|+k+i} a_{i+1} u_{i+1} \ldots a_t u_t, \ldots$, $P_{m_u}^n(u) = C_n^{a_t}$;

(3) $\mathcal{L}_n^u = \{2^{|u_1|+k+1}, 2^{|u_1|+|u_2|+k+2}, \ldots, 2^{\sum_{\ell=1}^{i} |u_\ell|+k+i}, \ldots, 2^n\}$.

Proof. We proceed by induction on the value of t in the decomposition of u.

If $t = 1$, then $u = 0^k a_1 u_1$ and such vertex belongs to $C^{a_1}_{|u|}$ and the claim is true.

Let $t = \ell + 1$, so that $u = 0^k a_1 u_1 a_2 u_2 \ldots a_\ell u_\ell a_{\ell+1} u_{\ell+1}$. Notice that the vertices u and $v = 0^{k+1+|u_1|} a_2 u_2 \ldots a_\ell u_\ell a_{\ell+1} u_{\ell+1}$ belong to the same cycle $C^{a_1}_{k+1+|u_1|} a_2 u_2 \ldots a_\ell u_\ell a_{\ell+1} u_{\ell+1}$, whose length is $2^{k+1+|u_1|}$. Notice that the index t of the decomposition of v equals ℓ. By using the inductive hypothesis and the uniqueness of the path of cycles, one can show the asserts. ∎

Proposition 4.12. *Let $w = x_1 x_2 \ldots \in X^\infty \setminus \{0^\infty\}$, and consider the sequence of sets $\{\mathcal{P}_n^{x_1 \ldots x_n}\}_{n \geq 1}$. Then $|\mathcal{P}_n^{x_1 \ldots x_n}| \leq |\mathcal{P}_{n+1}^{x_1 \ldots x_n x_{n+1}}|$. Moreover, $\lim_n |\mathcal{P}_n^{x_1 \ldots x_n}| < \infty$ if and only if w is cofinal to a word in $\{0, i\}^\infty$ for some $i \in X$.*

Proof. Suppose that $w_n = x_1 \ldots x_n$ ends with a suffix $i0^k$ for some $k \geq 0$ and $i \in \{1, \ldots, p\}$, so that it belongs to the i-th petal. Then, by using Lemma 4.11, passing from Γ_n^p to Γ_{n+1}^p we have two possible situations:

(1) If $x_{n+1} \in \{0, i\}$, the index t of the decomposition of w_n and w_{n+1} is the same.

(2) If $x_{n+1} \neq 0, i$, the index t of the decomposition of w_{n+1} increases by one with respect to that of w_n.

The length of the path of cycles remains the same if and only if we add, after some prefix of w ending with a suffix $i0^k$, only letters from the alphabet $\{0, i\}$, for some i. In particular, it follows that, in the second case, we have a nested path of cycles associated with the prefixes w_n of w. ∎

Notice that the analogous statement clearly holds by substituting $\mathcal{P}$ by $\mathcal{L}$.

Remark 4.13. Lemma 4.11 and Proposition 4.12 imply that $\mathcal{P}_n^{x_1\dots x_n}$ and $\mathcal{P}_{n+1}^{x_1\dots x_n x_{n+1}}$ are such that either they have the same size (and in this case they differ just for the last cycle that has length 2^n in one case and 2^{n+1} in the other case) or the path $\mathcal{P}_{n+1}^{x_1\dots x_n x_{n+1}}$ contains one cycle more than $\mathcal{P}_n^{x_1\dots x_n}$ that is its subset. In particular, the length of the path of cycles associated with $u = 0^k a_1 u_1 a_2 u_2 \dots a_t u_t$ is t. Any time we read a new letter a_i, the sequence increases by one.

Remark 4.13 implies that one can define the path of cycles associated with $w = x_1 x_2 \dots \in X^\infty$ as the limit of $\mathcal{P}_n^{x_1\dots x_n}$. The same can be said for the sequence of the lengths. We denote them by $\mathcal{P}^w = \{P_1^w, P_2^w, \dots\}$ and $\mathcal{L}^w = \{L_1^w, L_2^w, \dots\}$, respectively. Moreover, we can also define the decomposition of an infinite word $u = 0^k a_1 u_1 \dots \in X^\infty$.

4.2. From finite to infinite Schreier graphs

We start this section with the following result that is standard in this setting.

Lemma 4.14. *Let $w \in X^\infty$. If $w \in \mathrm{Cof}(0^\infty) \cup \dots \cup \mathrm{Cof}(p^\infty)$, then the orbit of w under $\mathcal{G}_{S_p}$ coincides with* $\mathrm{Cof}(0^\infty) \cup \dots \cup \mathrm{Cof}(p^\infty)$. *Otherwise, the orbit of w under $\mathcal{G}_{S_p}$ coincides with* $\mathrm{Cof}(w)$.

Proof. Notice that the only infinite paths in the generating automaton, that do not fall into the sink, are those labeled by $0^\infty | i^\infty$ starting at e_i, with $i \in \{1, \dots, p\}$ (in particular, all the words i^∞'s are in the orbit of 0^∞). This implies that the action of $\mathcal{G}_{S_p}$ changes infinitely many letters only on words of type $w = i^\infty$, with $i \in \{0, 1, \dots, p\}$. Therefore, if $w \in \mathrm{Cof}(0^\infty) \cup \dots \cup \mathrm{Cof}(p^\infty)$, its orbit is contained in $\mathrm{Cof}(0^\infty) \cup \dots \cup \mathrm{Cof}(p^\infty)$; similarly, if $w \notin \mathrm{Cof}(0^\infty) \cup \dots \cup \mathrm{Cof}(p^\infty)$, one has that its orbit is contained in $\mathrm{Cof}(w)$. In order to show the opposite inclusions, we use that $\mathcal{G}_{S_p}$ is self-replicating and spherically transitive (see [7]). In particular, given u and w cofinal, there exist prefixes u_n, w_n of length n such that $u = u_n v$ and $w = w_n v$. By transitivity, there exists $g \in \mathcal{G}_{S_p}$ such that $g(w_n) = u_n$. Let $g(w) = u_n v'$. Since $\mathcal{G}_{S_p}$ is self-replicating, there exists $g' \in \mathcal{G}_{S_p}$ such that $g'(u_n v') = u_n v$. Then $g' g(w) = u$, so that u belongs to the orbit of w. ∎

The particular structure of the Schreier graphs allows to keep trace of the dynamic of an infinite word u.

Lemma 4.15. *Let $u \in X^\infty$ with decomposition $u = 0^k a_1 u_1 \dots \in X^\infty$, then $P_i^u \cap P_{i+1}^u = \{0^{k+i+\sum_{j=1}^{i} |u_j|} a_{i+1} u_{i+1} \dots\}$.*

Proof. Take n such that $n > k + i + 1 + \sum_{j=1}^{i+1} |u_j|$, and let u_n be the prefix of u of length n such that $u = u_n u'$. Notice that if $P_i^{u_n} \cap P_{i+1}^{u_n} = \{w\}$, then $P_i^u \cap P_{i+1}^u = wu'$. This means that we can study the intersection of cycles in Γ_n^p for n large enough. A new cycle appears whenever we read a letter $a_{i+1} \neq a_i$. In this case, u_n becomes an element of $\mathcal{D}_n^{a_i} a_{i+1}$. In particular, the last two cycles are connected in $0^{k+i+\sum_{j=1}^{i} |u_j|} a_{i+1}$. ∎

Using the previous results, we are ready to prove the following classification theorem. We recall that an *end* for an infinite graph Γ is an equivalence class of rays that remain in the same connected component whenever we remove a finite subgraph from Γ. An infinite graph Γ is said to be k*-ended* if it contains exactly k ends. Equivalently, Γ is k-ended if the supremum of the number of connected infinite components of Γ, when a finite subgraph is removed from Γ, equals k. For each $w \in X^\infty$, let us denote by Γ_w^p the infinite Schreier graph of the group $\mathcal{G}_{S_p}$ containing the vertex w, that is, the graph describing the orbit of $w \in \partial T_{p+1}$ under the action of $\mathcal{G}_{S_p}$. Put $E_k = \{w \in X^\infty : \Gamma_w^p \text{ is } k\text{-ended}\}$.

Notice that, using the spherical transitivity of $\mathcal{G}_{S_p}$, one can show that any invariant measurable subset of X^∞ must have measure 0 or 1 (see [16]).

Theorem 4.16. *Let $v \in X^\infty$. Then Γ_v^p is either $2p$-ended, or 2-ended, or 1-ended. In particular,*

(1) $E_{2p} = \mathrm{Cof}(0^\infty) \cup \mathrm{Cof}(1^\infty) \cup \cdots \cup \mathrm{Cof}(p^\infty)$ *and consists of one orbit.*

(2) $E_2 = (\bigcup_{i=1}^p \bigcup_{w \in \{0,i\}^\infty} \mathrm{Cof}(w)) \setminus E_{2p}$ *and consists of uncountably many orbits.*

(3) $E_1 = X^\infty \setminus (E_{2p} \cup E_2)$ *and consists of uncountably many orbits.*

Moreover, $\nu(E_1) = 1$.

Proof. (1) The vertex 0^n belongs to C_n^i for any $n \geq 1$ and for every $i \in \{1, \dots, p\}$. When n goes to infinity, the length of C_n^i goes to infinity giving rise to 2 rays that can be disconnected by removing the vertex 0^∞. The same can be said for the other cycles containing 0^n, and this implies that $\Gamma_{0^\infty}^p$ is at least $2p$-ended. Any other vertex of $\Gamma_{0^\infty}^p$ belongs to some decoration $\mathcal{D}_k$ for some $k \in \mathbb{N}$, that is, a finite graph attached to exactly one of the $2p$ rays described above. This implies that $\Gamma_{0^\infty}^p$ is $2p$-ended. Moreover, it follows from Lemma 4.14 that $\Gamma_{0^\infty}^p = \mathrm{Cof}(0^\infty) \cup \mathrm{Cof}(1^\infty) \cup \cdots \cup \mathrm{Cof}(p^\infty)$. This shows that $\mathrm{Cof}(0^\infty) \cup \mathrm{Cof}(1^\infty) \cup \cdots \cup \mathrm{Cof}(p^\infty) \subseteq E_{2p}$. The claim will follow from the remaining part of the proof.

(2) Let $w = x_1x_2\dots$ be cofinal to $u \in \{0, i\}^\infty \setminus (\mathrm{Cof}(0^\infty) \cup \mathrm{Cof}(i^\infty))$ for some $i \in \{1, \dots, p\}$. By Proposition 4.12, the path of cycles associated with w is finite. Moreover, $d(w, u) < \infty$. This implies that there exists $N \in \mathbb{N}$ such that $m_u = N$, and so $P_{m_u}(x_1 \dots x_n) = P_N(x_1 \dots x_n) = C_n^i$ for every n large enough. The length of C_n^i is 2^n and goes to infinity. Hence w belongs to a decoration attached at u to an infinite double ray, and so Γ_w^p is 2-ended. Finally, Lemma 4.14 implies that each orbit coincides with a cofinality class.

(3) Any $w \in X^\infty \setminus (\bigcup_{i=1}^p \bigcup_{w \in \{0,i\}^\infty} \mathrm{Cof}(w))$ gives rise to an infinite path of cycles which is by construction unique. It follows that Γ_w^p is 1-ended. Also in this case, Lemma 4.14 implies that each orbit coincides with a cofinality class.

For the last claim, first observe that E_{2p} is countable and so $\nu(E_{2p}) = 0$. In order to prove that $\nu(E_2) = 0$, we notice that

$$E_2 \subset \bigcup_{i=1}^{p} \bigcup_{n \geq 0} X^n \{0, i\}^\infty.$$

Let $i \in \{1, \dots, p\}$. Let us show that $\nu(\{0,i\}^\infty) = 0$. A direct computation gives

$$\nu(\{0,i\}^\infty) = 1 - (p-1)\sum_{j=1}^{\infty} \frac{2^{j-1}}{(p+1)^j} = 0.$$

It follows that

$$\nu(E_2) \le p \sum_{n=0}^{\infty} (p+1)^n \nu(\{0,i\}^\infty) = 0.$$

Therefore,

$$1 = \nu(E_1) + \nu(E_2) + \nu(E_{2p}) = \nu(E_1). \qquad \blacksquare$$

In words, we can say that E_2 consists of infinite words containing, after any arbitrary finite prefix long enough, both the letters 0 and i, for one fixed $i \in \{1, \dots, p\}$, and only them. On the other hand, the set E_1 consists of infinite words containing, after any arbitrary finite prefix, at least two letters in $\{1, \dots, p\}$.

Remark 4.17. Theorem 4.16 can be directly proven by using the techniques developed in [5].

Now we pass to the study of isomorphism classes for the infinite Schreier graphs $\{\Gamma_w^p\}_{w \in X^\infty}$.

Let $w = x_1 x_2 \ldots \in X^\infty$. Recall that (Γ_w^p, w) is the rooted graph obtained as the limit of the finite rooted graphs $(\Gamma_n^p, x_1 \ldots x_n)$ in the Gromov–Hausdorff topology. Once we get (Γ_w^p, w), we forget the root and consider the infinite graph Γ_w^p. Given $u, v \in X^\infty$, we ask when Γ_u^p and Γ_v^p are isomorphic.

Observe that the vertices belonging to E_{2p} give rise to one isomorphism class since they belong to the same orbit (the one containing 0^∞). Moreover, it is clear that graphs with different number of ends cannot be isomorphic.

We start with the following result. Recall that, given $w = x_1 x_2 \ldots \in X^\infty$, we have denoted by $\mathcal{L}^w$ the limit of the sequence of the lengths of the cycles which constitute the path of cycles $\mathcal{P}_n^{x_1 \ldots x_n}$ for $n \to \infty$.

Lemma 4.18. (1) *Let $u, v \in E_1$ with $v \in \Gamma_u^p$. Then the sequences $\mathcal{L}^u$ and $\mathcal{L}^v$ are compatible.*

(2) *Let $u, v \in E_1$ such that Γ_u^p is isomorphic to Γ_v^p. Then the sequences $\mathcal{L}^u$ and $\mathcal{L}^v$ are compatible.*

Proof. Let us start by proving the first claim. The sequences of cycles associated with u and v must eventually coincide. This exactly means that after some possibly different initial paths, the sequences must join. This implies that the sequences of the lengths of these cycles are compatible.

For the second claim, notice that if Γ_u^p is isomorphic to Γ_v^p, then there exist $w \in \Gamma_v^p$ and an isomorphism $\phi\colon \Gamma_u^p \to \Gamma_v^p$ such that $\phi(u) = w$. This implies that $\mathcal{L}^u = \mathcal{L}^w$.

Since $w \in \Gamma_v^p$, claim (1) implies that $\mathcal{L}^v$ and $\mathcal{L}^w$ are compatible. This concludes the proof. ■

Whenever $w \in E_1$, there is also a sequence of vertices $\{w(n)\}_{n\in\mathbb{N}}$ defined by $w(0)=w$ and $\{w(i)\} = P_i^w \cap P_{i+1}^w$. If $w = 0^k a_1 u_1 \dots a_i u_i \dots$ with $a_i \in \{1,\dots,p\}$, $u_i \in \{0,a_i\}^*$ and $a_i \neq a_{i+1}$, then from Lemma 4.15 $w(i) = 0^{k+i+\sum_{j=1}^{i}|u_j|} a_{i+1}u_{i+1}\dots$ Moreover, we define the sequence of distances $\{d_n^w\}_{n\in\mathbb{N}}$ such that $d_i^w = d(w(i-1), w(i))$.

Proposition 4.19. *Let $u, v \in E_1$ such that $\{d_n^u\}_{n\in\mathbb{N}}$ and $\{d_n^v\}_{n\in\mathbb{N}}$ are compatible, then $\mathcal{L}^u$ and $\mathcal{L}^v$ are compatible.*

Proof. Let $u = 0^k a_1 u_1 \dots a_i u_i \dots$ with $a_i \in \{1,\dots,p\}$, $u_i \in \{0,a_i\}^*$, $a_i \neq a_{i+1}$ and $v = 0^m b_1 v_1 \dots b_i v_i \dots$ with $b_i \in \{1,\dots,p\}$, $v_i \in \{0,b_i\}^*$, $b_i \neq b_{i+1}$, and suppose that $\mathcal{L}^u$ and $\mathcal{L}^v$ are not compatible. Then, for every $h, l \geq 0$, there exist infinitely many $n \in \mathbb{N}$ such that $L_{h+n}^u \neq L_{l+n}^v$. By Lemma 4.11, this is equivalent to say

$$2^{\sum_{\ell=1}^{h+n}|u_\ell|+k+h+n} \neq 2^{\sum_{\ell=1}^{l+n}|v_\ell|+m+l+n} \tag{4.1}$$

for infinitely many $n \in \mathbb{N}$. Put $q = k+h+n+1+\sum_{j=1}^{h+n+1}|u_j|$. Notice that, by virtue of Lemma 4.15,

$$\begin{aligned} d_{h+n+1}^u &= d(u(h+n), u(h+n+1)) \\ &= d(0^{k+h+n+\sum_{j=1}^{h+n}|u_j|} a_{h+n+1}u_{h+n+1}\dots, 0^q a_{h+n+2}u_{h+n+2}\dots) \\ &= d(0^{k+h+n+\sum_{j=1}^{h+n}|u_j|} a_{h+n+1}u_{h+n+1}, 0^q). \end{aligned}$$

In other words, the distance d_{h+n+1}^u can be computed within the finite Schreier graph Γ_q^p. The last distance relies to vertices belonging to the same cycle (the maximal cycle $C_q^{a_{h+n+1}}$) and can be explicitly computed: suppose that $u_{h+n+1} = x_1 \dots x_{|u_{h+n+1}|}$, where $x_i \in \{0, a_{h+n+1}\}$. Put $t = k+h+n+\sum_{j=1}^{h+n}|u_j|$. Then, by using the adding machine structure, one has

$$d_{h+n+1}^u = \min\left\{2^t + \sum_{i:x_i\neq 0} 2^{t+i}, 2^t + \sum_{i:x_i=0} 2^{t+i}\right\}.$$

Analogously, if $s = m+l+n+\sum_{j=1}^{l+n}|v_j|$ and $v_{h+n+1} = y_1 \dots y_{|v_{l+n+1}|}$, where $y_i \in \{0, b_{l+n+1}\}$, one has

$$d_{l+n+1}^v = \min\left\{2^s + \sum_{i:y_i\neq 0} 2^{s+i}, 2^s + \sum_{i:y_i=0} 2^{s+i}\right\}.$$

In any case, 2^t is the smallest addend of d_{h+n+1}^u, and 2^s is the smallest addend of d_{l+n+1}^v. Since by equation (4.1) it must be $2^t \neq 2^s$, we get $d_{h+n+1}^u \neq d_{l+n+1}^v$ for infinitely many n. The claim follows. ■

Proposition 4.20. *Let $w, v \in E_1$. Then Γ_w^p is isomorphic to Γ_v^p if and only if the sequences $\{d_n^w\}_{n\in\mathbb{N}}$ and $\{d_n^v\}_{n\in\mathbb{N}}$ are compatible.*

Proof. Suppose that Γ_w^p and Γ_v^p are isomorphic. Then there exists $z \in \Gamma_v^p$ such that (Γ_w^p, w) and (Γ_v^p, z) are isomorphic as rooted graphs. Since the paths of cycles associated with w and z coincide, we have $d_n^w = d_n^z$ for every $n \geq 0$. Therefore, $\{d_n^z\}_{n\in\mathbb{N}}$ and $\{d_n^v\}_{n\in\mathbb{N}}$ are compatible, because $z \in \Gamma_v^p$ and so the paths of cycles of z and v must join. The claim follows.

Vice versa, first suppose that $d_n^w = d_n^v$ for every $n \in \mathbb{N}$. Then by adapting the proof of Proposition 4.19, we deduce that $\mathcal{L}^w = \mathcal{L}^v$. We want to define an isomorphism $\phi\colon \Gamma_w^p \to \Gamma_v^p$. First of all, put $\phi(w) = v$. Notice that $w(1)$ (resp. $v(1)$) is the only vertex of P_1^w (resp. P_1^v) that is attached to a cycle isomorphic to P_2^w (resp. P_2^v). Moreover, the cycles P_2^w and P_2^v are isomorphic by assumption. Since $d(w, w(1)) = d(v, v(1))$, we can put $\phi(w(1)) = v(1)$. By iterating the same argument for each n, we deduce that it must be $\phi(w(n)) = v(n)$ for any $n \geq 0$. It follows that $\phi(P_n^w) = P_n^v$ for every n. Since such cycles have the same size, they have attached subgraphs that are isomorphic. This implies that ϕ can be extended to an isomorphism between Γ_w^p and Γ_v^p.

If the sequences $\{d_n^w\}_{n\in\mathbb{N}}$ and $\{d_n^v\}_{n\in\mathbb{N}}$ are compatible, then there exist i, j such that $d_{i+n}^w = d_{j+n}^v$ for every $n \in \mathbb{N}$. Define $\phi(w(i+n)) = v(j+n)$ for each n. Notice that $\Gamma_w^p \setminus \{w(i)\}$ contains one infinite connected component which is isomorphic to the only infinite one of $\Gamma_v^p \setminus \{v(j)\}$. The remaining parts of the two graphs Γ_w^p and Γ_v^p are finite subgraphs attached to the isomorphic cycles P_{i+1}^w and P_{j+1}^v, and so they are isomorphic. This gives an isomorphism between Γ_w^p and Γ_v^p. ■

Pay attention to the fact that there are infinite sequences $u, v \in X^\infty$ such that $\mathcal{L}^u = \mathcal{L}^v$, but Γ_u^p and Γ_v^p are not isomorphic.

Example 4.21. Consider the vertices $u = (1002)^\infty$ and $v = (1012)^\infty$. The paths of cycles associated with $u = (1002)^\infty$ and $v = (1012)^\infty$ are the same. However, by using Proposition 4.20, one can check that there is no isomorphism between the graphs Γ_u^p and Γ_v^p, since $u(n)$ and $v(n)$ belong to two cycles of the same length, but

$$d(u(n), u(n+1)) \neq d(v(n), v(n+1))$$

for each n.

For every $i, j \in \{1, \ldots, p\}$, let us define the map $\phi_{i,j}\colon X \to X$ as

$$\phi_{i,j}(k) = \begin{cases} k & \text{if } k \neq i, \\ j & \text{if } k = i. \end{cases}$$

In particular, the map $\phi_{i,j}$ fixes 0 for any i, j; moreover, $\phi_{i,i}$ is the identity map. Given $w = x_1 x_2 \ldots \in X^* \cup X^\infty$, we define $\phi_{i,j}(w) = \phi_{i,j}(x_1)\phi_{i,j}(x_2)\ldots$

Given $u \in \{0, i\}^* \cup \{0, i\}^\infty$, we denote by u' the word obtained from u by switching 0 to i and vice versa. For a given element $u \in X^\infty$ cofinal to a word in $\{0, i\}^\infty$, having the form $u_1 i u_2$, where $i u_2$ is the maximal suffix of u in $\{0, i\}^\infty$, we put $\overline{u} = u_1 i u_2'$.

Example 4.22. Let $w = 0^k a_1 u_1 a_2 u_2 a_3 u_3 \dots$ with $a_i \in \{1, 2, \dots, p\}$, $u_i \in \{0, a_i\}^*$ and $a_{i+1} \neq a_i$ as usual. We have

$$d_1^w = d(w, w(1)), \qquad d_2^w = d(w(1), w(2)),$$
$$d_3^w = d(w(2), w(3)), \quad d_4^w = d(w(3), w(4))$$

with

$$w(1) = 0^{k+1+|u_1|} a_2 u_2 \dots, \qquad w(2) = 0^{k+2+|u_1|+|u_2|} a_3 u_3 \dots,$$
$$w(3) = 0^{k+3+|u_1|+|u_2|+|u_3|} a_4 u_4 \dots, \quad w(4) = 0^{k+4+|u_1|+|u_2|+|u_3|+|u_4|} a_5 u_5 \dots$$

Now consider the vertex $v = 0^k a_1 u_1' a_2 u_2 a_3 u_3 \dots$ with u_1' obtained from u_1 by switching 0 to a_1 in u_1 and vice versa (see Figure 10). Notice that $w(n) \equiv v(n)$ for each $n \geq 1$, since w and v belong to the same cycle. It follows that $\mathcal{P}^w = \mathcal{P}^v$, so that $\mathcal{L}^w = \mathcal{L}^v$.

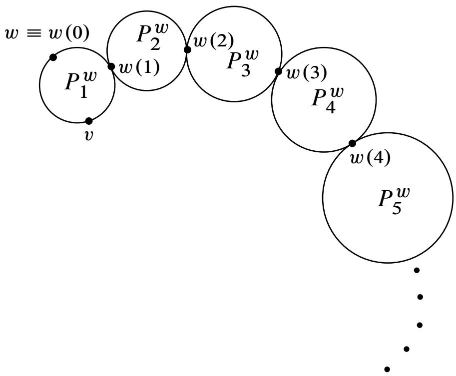

Figure 10. The path of cycles of the word w of Example 4.22.

Finally, a comparison between the decompositions of w and v ensures that $d_1^w = d(w, w(1)) = d_1^v = d(v, v(1))$, so that we also have $d_n^w = d_n^v$ for each $n \geq 1$.

Theorem 4.23. (1) *There is one isomorphism class of* $2p$*-ended graphs consisting of the graph* $\Gamma_{0^\infty}^p$.

(2) *Let* $u, v \in E_2$*, Then* Γ_u^p *is isomorphic to* Γ_v^p *if and only if either* $v \sim \phi_{i,j}(u)$ *or* $v \sim \overline{\phi_{i,j}(u)}$ *for some* $i, j \in \{1, \dots, p\}$*. In particular, there are uncountably many isomorphism classes of* 2*-ended graphs, each consisting of* $2p$ *graphs.*

(3) *Let* $u, v \in E_1$ *with* $u = 0^k a_1 u_1 a_2 u_2 \dots a_i u_i \dots$*,* $a_i \in \{1, \dots, p\}$*,* $a_i \neq a_{i+1}$ *and* $u_i \in \{0, a_i\}^*$*. Then the graph* Γ_u^p *is isomorphic to the graph* Γ_v^p *if and only if* $v \sim 0^k b_1 v_1 b_2 v_2 \dots b_i v_i \dots$ *where, for any* $i \in \mathbb{N}$*,* $b_i \in \{1, \dots, p\}$*,* $|v_i| = |u_i|$*,* $b_i \neq b_{i+1}$ *and either* $v_i = \phi_{a_i,b_i}(u_i)$ *or* $v_i = \phi_{a_i,b_i}(u_i)'$*. In particular, there are uncountably many isomorphism classes of* 1*-ended graphs, each consisting of uncountably many graphs.*

Proof. (1) The first statement is clear by Theorem 4.16.

(2) Let $u \in E_2$. Then by Theorem 4.16, there exists $i \in \{1, \dots, p\}$ such that $u = u_1 i u_2$, where iu_2 is the maximal suffix of u in $\{0, i\}^\infty$. This implies that u belongs to a decoration isomorphic to $\mathcal{D}_{|u_1|}$ and attached at $0^{|u_1|} i u_2$ to the corresponding infinite double ray. Notice that, in view of the decomposition of u, the graphs Γ_u^p and $\Gamma_{\phi_{i,j}(u)}^p$ are isomorphic. Such isomorphism maps the decoration attached at $0^{|u_1|} i u_2$ to the isomorphic one attached at $0^{|u_1|}\phi_{i,j}(iu_2)$. This isomorphism maps the vertices of the infinite double ray $e_i^t(0^{|u_1|} i u_2)$ to $e_j^t(0^{|u_1|}\phi_{i,j}(iu_2))$ for each $t \in \mathbb{Z}$. Similarly, the graphs Γ_u^p and $\Gamma_{\overline{\phi_{i,j}(u)}}^p$ are isomorphic, and the isomorphism maps $0^{|u_1|} i u_2$ to $0^{|u_1|}\overline{\phi_{i,j}(iu_2)}$. It follows that the vertices of the infinite double ray $e_i^t(0^{|u_1|} i u_2)$ are mapped to $e_j^{-t}(0^{|u_1|}\overline{\phi_{i,j}(iu_2)})$ for each $t \in \mathbb{Z}$, as one can deduce from Proposition 4.9. Finally, by using Lemma 4.14, it is easy to show that if v is either cofinal to $\phi_{i,j}(u)$ or cofinal to $\overline{\phi_{i,j}(u)}$, then Γ_u^p is isomorphic to Γ_v^p.

Vice versa, suppose $u = x_1 x_2 \dots = u_1 i u_2$ and $v = y_1 y_2 \dots = v_1 j v_2$, where iu_2 is the maximal suffix of u in $\{0, i\}^\infty$ and iv_2 is the maximal suffix of v in $\{0, j\}^\infty$. Assume that Γ_u^p and Γ_v^p are isomorphic through ϕ in such a way that $\phi(u) = v$. Then ϕ must induce an isomorphism of the finite rooted graphs $(\Gamma_n^p, x_1 \dots x_n)$ and $(\Gamma_n^p, y_1 \dots y_n)$ for each n. Take n large enough so that $n > |u_1| + 1$. By Proposition 4.9, the graphs $(\Gamma_n^p, x_1 \dots x_n)$ and $(\Gamma_n^p, y_1 \dots y_n)$ are isomorphic if and only if either $jy_{|u_1|+2} \dots y_n = \phi_{i,j}(ix_{|u_1|+2} \dots x_n)$ or $jy_{|u_1|+2} \dots y_n = j\phi_{i,j}(x_{|u_1|+2} \dots x_n)'$. The claim follows.

(3) First let $v = 0^k b_1 v_1 b_2 v_2 \dots b_i v_i \dots$, where for any $i \in \mathbb{N}$, $b_i \in \{1, \dots, p\}$, $|v_i| = |u_i|$, $b_i \neq b_{i+1}$ and either $v_i = \phi_{a_i, b_i}(u_i)$ or $v_i = \phi_{a_i, b_i}(u_i)'$. We claim that $d_n^v = d_n^u$ for any $n \in \mathbb{N}$. Recall that

$$\begin{aligned} d_n^v &= d(v(n-1), v(n)) \\ &= d(0^{k+\sum_{j=1}^{n-1}|v_j|} b_n v_n \dots, 0^{k+\sum_{j=1}^{n}|v_j|} b_{n+1} v_{n+1} \dots) \\ &= d(0^{k+\sum_{j=1}^{n-1}|v_j|} b_n v_n, 0^{k+\sum_{j=1}^{n}|v_j|}). \end{aligned}$$

From the proof of Proposition 4.19, it follows that the value of d_n^v only depends on the position of the 0's and b_n's in v_n, and so it is independent from the specific b_n. By assumption, $k + \sum_{j=1}^n |v_j| = k + \sum_{j=1}^n |u_j|$, and by Proposition 4.9, the vertex v satisfies $d_n^v = d_n^u$ if $v_i = \phi_{a_i, b_i}(u_i)$ or $v_i = \phi_{a_i, b_i}(u_i)'$. This implies that Γ_v^p and Γ_u^p are isomorphic (as rooted graphs). Finally, Lemma 4.14 implies that if v' is cofinal to v, then $\Gamma_v^p = \Gamma_{v'}^p$ and so $\Gamma_{v'}^p$ is isomorphic to Γ_u^p.

Vice versa, suppose $v = 0^m b_1 v_1 b_2 v_2 \dots b_i v_i \dots$ with $b_i \in \{1, \dots, p\}$, $b_i \neq b_{i+1}$ and $v_i \in \{0, b_i\}^*$. First notice that Lemma 4.18 implies that if $u, v \in E_1$ are such that Γ_u^p is isomorphic to Γ_v^p, then the sequences $\mathcal{L}^u$ and $\mathcal{L}^v$ must be compatible. By Lemma 4.11, this is equivalent to say that there exist l, h such that

$$|v_{l+i}| = |u_{h+i}| \ \forall\, i \geq 1 \quad \text{and} \quad m + l + \sum_{j=1}^{l} |v_j| = k + h + \sum_{j=1}^{h} |u_j|. \tag{4.2}$$

Moreover, Proposition 4.20 implies that also the sequences $\{d_n^u\}_{n\in\mathbb{N}}$ and $\{d_n^v\}_{n\in\mathbb{N}}$ must be compatible. As before, one has

$$\begin{aligned} d_{l+i}^v &= d(v(l+i-1), v(l+i)) \\ &= d(0^{k+l+i-1+\sum_{j=1}^{l+i-1}|v_j|} b_{l+i} v_{l+i} \ldots, 0^{k+l+i+\sum_{j=1}^{l+i}|v_j|} b_{l+i+1} v_{l+i+1} \ldots) \\ &= d(0^{k+l+i-1+\sum_{j=1}^{l+i-1}|v_j|} b_{l+i} v_{l+i}, 0^{k+l+i+\sum_{j=1}^{l+i}|v_j|}). \end{aligned}$$

Then, using equation (4.2), one can check that this quantity equals d_{h+i}^u if and only if, for any $i \geq 1$, there exists $b_i \in \{1, \ldots, p\}$ such that either $v_{l+i+1} = \phi_{a_i,b_i}(u_{h+i+1})$ or $v_{l+i+1} = \phi_{a_i,b_i}(u_{h+i+1})'$. The claim follows. ∎

For a given $w \in X^\infty$, put

$$I_w = \{z \in X^\infty : \Gamma_z^p \text{ and } \Gamma_w^p \text{ are isomorphic}\}.$$

Corollary 4.24. *For every $w \in X^\infty$, one has $\nu(I_w) = 0$.*

Proof. Notice that, by virtue of Theorem 4.16, if $w \in E_{2p}$ or $w \in E_2$, then $\nu(I_w) = 0$. Therefore, we can restrict our attention to the case $w \in E_1$. Let $w = 0^k a_1 u_1 \ldots$, and let us consider its decomposition. By Theorem 4.23, we have $I_w = \bigcup_{u \in E'_w} \mathrm{Cof}(u)$, where

$$E'_w = \{z \in X^\infty : (\Gamma_z^p, z) \text{ and } (\Gamma_w^p, w) \text{ are isomorphic as rooted graphs}\}.$$

In particular, the claim follows if we prove that $\nu(E'_w) = 0$. By claim (3) of Theorem 4.23, in order to measure E'_w, we first must remove from X^∞ all subsets of type $0^t m X^\infty$, with $t \leq k-1$ and $m \neq 0$. Then in $0^k X^\infty$ we remove the subset $0^{k+1}X^\infty$. After that consider the subsets $0^k a X^\infty$ with $a \in \{1, \ldots, p\}$. In each of these p subsets of X^∞, all subsets of type $0^k a v X^\infty$ must be removed, for any v of length u_1, except for the two v's giving rise to isomorphism. After that, for each of the remaining rays, we proceed as before, according to the decomposition of w: the subsets of type $0^k a v b X^\infty$ with $b \in \{0, a\}$ must be removed. By iterating this argument, a direct computation gives

$$\begin{aligned} \nu(E'_w) = 1 - p\sum_{j=1}^{k} \frac{1}{(p+1)^j} - \frac{1}{(p+1)^{k+1}} - p\frac{(p+1)^{|u_1|}-2}{(p+1)^{|u_1|+k+1}} - 2\frac{2p}{(p+1)^{|u_1|+k+2}} \\ - 2p(p-1)\frac{(p+1)^{|u_2|}-2}{(p+1)^{|u_1|+|u_2|+k+2}} - 2\frac{4p(p-1)}{(p+1)^{|u_1|+|u_2|+k+3}} \\ - 4p(p-1)^2\frac{(p+1)^{|u_3|}-2}{(p+1)^{|u_1|+|u_2|+|u_3|+k+3}} - \cdots. \end{aligned}$$

We can rearrange the sum as follows:

$$\nu(E'_w) = 1 - p\sum_{j=1}^{k} \frac{1}{(p+1)^j} - \frac{1}{(p+1)^{k+1}} - \frac{p}{(p+1)^{k+1}} +$$

$$
\begin{aligned}
&- \sum_{m=1}^{\infty} \Big[\frac{2^m p(p-1)^{m-1}}{(p+1)^{|u_1|+\cdots+|u_m|+k+m}} - \frac{2^{m+1} p(p-1)^{m-1}}{(p+1)^{|u_1|+\cdots+|u_m|+k+m+1}} \\
&\qquad - \frac{2^m p(p-1)^m}{(p+1)^{|u_1|+\cdots+|u_m|+k+m+1}} \Big] \\
&= 0 + \sum_{m=1}^{\infty} \frac{2^m p(p-1)^{m-1}}{(p+1)^{|u_1|+\cdots+|u_m|+k+m+1}} [(p+1) - 2 - p + 1] = 0. \qquad \blacksquare
\end{aligned}
$$

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Matteo Cavaleri
Università degli Studi Niccolò Cusano, Via Don Carlo Gnocchi, 3, 00166 Roma, Italy;
matteo.cavaleri@unicusano.it

Daniele D'Angeli
Università degli Studi Niccolò Cusano, Via Don Carlo Gnocchi, 3, 00166 Roma, Italy;
daniele.dangeli@unicusano.it

Alfredo Donno
Università degli Studi Niccolò Cusano, Via Don Carlo Gnocchi, 3, 00166 Roma, Italy;
alfredo.donno@unicusano.it

Emanuele Rodaro
Politecnico di Milano, Piazza Leonardo da Vinci, 32, 20133 Milano, Italy;
emanuele.rodaro@polimi.it