Progress of Theoretical Physics, Vol. 126, No. 3, September 2011

Local Zeta Regularization and the Casimir Effect

Davide FERMI[1,*)] and Livio PIZZOCCHERO[2,3,**)]

[1] *Università di Milano*
[2] *Dipartimento di Matematica, Università di Milano, Via C. Saldini 50, I-20133 Milano, Italy*
[3] *Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy*

(Received June 14, 2011; Revised July 15, 2011)

The local zeta regularization allows to treat local divergences appearing in quantum field theory; these are renormalized by pure analytic continuation (in the parameter of the regulator), with no need to remove or subtract divergent terms. This approach can be applied to the stress-energy tensor of the Casimir effect, and works as well on curved space-times.

It is not useless to illustrate the power and elegance of this method in a simple case. In the present paper, our attention is devoted to the case of a neutral, massless scalar field in flat space-time, on a space domain with suitable (e.g., Dirichlet) boundary conditions. After a general outline of the local zeta method for the Casimir effect, we exemplify it in the typical case of a (Dirichlet) field between two parallel plates, or outside them. The results agree with the ones obtained by more popular methods, such as point splitting regularization. Connections with the existing literature on this subject are indicated.

Subject Index: 130, 132, 187

§1. Introduction

Zeta regularization is a method to give meaning to the divergent series appearing frequently in quantum field theory, reinterpreting them as analytic continuations. For example, the divergent series

$$\sum_{\ell=1}^{\infty} \ell^3 \qquad (1\cdot1)$$

is interpreted in this approach as the analytic continuation at $s = -3$ of the regularized series $\zeta(s) := \sum_{\ell=1}^{+\infty} 1/\ell^s$, that converges for $\Re s > 1$ and defines the familiar Riemann zeta function; in this sense, the sum (1·1) "equals" $\zeta(-3) = 1/120$. Tricks of this kind have been used for a long time in quantum field theory: for example, the above analysis of the series (1·1) appears in one of the most popular derivations of the total Casimir energy for a scalar or electromagnetic field between two parallel plates (see Refs. 1) and 2), or the issue "Casimir effect" in Wikipedia).

The computation of local quantities, such as (the vacuum expectation value of) the stress-energy tensor, can be done as well via a generalization of the above method; this procedure, called the *local zeta regularization*, is a bit less popular than its analogue for global quantities, such as the total energy.

The method of local zeta regularization arose from some ideas of Hawking[3)] and

*) E-mail: davide.fermi@studenti.unimi.it
**) Corresponding author. E-mail: livio.pizzocchero@unimi.it

Wald;[4] these were systematically developed by Moretti and co-authors in a long series of works, among which we quote the papers[5)–7)] and the subsequent book.[8]

These authors typically work on curved space-times, in a Euclidean framework (i.e., with a space-time metric of signature $(+, +, +, +)$);*) in this setting, the local zeta regularization is applied to divergent sums arising from path integrals.

Since the local zeta method is not so popular, in our opinion it is not useless to illustrate it in a much simpler framework (which, however, includes nontrivial boundary conditions); this is the aim of the present work.

In this paper we consider a (neutral, massless) scalar field $\widehat{\phi}$ in Minkowski space-time (with the usual metric of signature $(-, +, +, +)$), as viewed in a given inertial frame; the field is canonically quantized on a three-dimensional space domain Ω, with suitable (e.g., Dirichlet) boundary conditions on the frontier $\partial\Omega$. We are interested in the stress-energy tensor $\widehat{T}_{\mu\nu}$ or, more precisely, in the vacuum expectation value (VEV) of each stress-energy component: this is the so-called *local problem* in the theory of the Casimir effect.

To deal with the divergences related to this problem, we introduce a "zeta regularized field" $\widehat{\phi}^u$, depending on a complex parameter u and coinciding with $\widehat{\phi}$ for $u = 0$; this is used to define a "zeta regularized stress-energy tensor" $\widehat{T}^u_{\mu\nu}$, with finite VEV. The final step in this construction is the analytic continuation at $u = 0$ of such VEV, which give the Casimir stress-energy for the case under consideration. A very pleasant feature of this approach, typical of the zeta regularization method, is that one gets directly a finite expression for the Casimir stress-energy, with no need to remove or subtract divergent terms. This is a major difference with respect to other renormalization schemes, employed more frequently for the Casimir effect; among these alternative approaches, let us mention the point splitting method, occasionally considered in this paper for a comparison.

In the present work, the local zeta approach is mainly illustrated in the case of a field between two parallel plates, with Dirichlet boundary conditions (so, in this example $\Omega = (-\infty, +\infty)^2 \times (0, a)$, with a the distance between the plates); by simple variations of this setting, we then pass to the case of a field in the region outside one or two plates. The results obtained in these cases by local zeta regularization are compared with the ones derived by Deutsch and Candelas,[9] Fulling,[10] Milton[11] and by Esposito et al.[12] by point splitting, and they are found to coincide (incidentally, we take the occasion to show that the expressions for the Casimir stress-energy given in 10), 12) and 11), even though seemingly different, are in fact equal).

Let us also mention that the case of a scalar field between parallel plates is considered in a Euclidean framework in 6) and 8); however, the version of the local zeta approach applied to this case in the cited references produces the renormalized VEVs only for the square $\widehat{\phi}^{\,2}$ and for the trace $\widehat{T}^{\mu}_{\ \mu}$, while in the present work we renormalize the VEV for each component $\widehat{T}_{\mu\nu}$.

To conclude this Introduction, let us briefly describe the organization of the paper. In §2 we sketch the basic framework for the Casimir effect, in the situa-

*) This is obtained from a Lorentzian metric via a Wick rotation on the time coordinate (when this can be carried over nonambiguously, which typically happens for static space-times).

tion outlined before: a canonically quantized scalar field on a space domain Ω in Minkowski space-time, with given boundary conditions, its stress-energy tensor and the related VEV. In §3 we introduce the local zeta regularization scheme. In §4, we apply this scheme to the case of a Dirichlet field between parallel plates, giving all details about the necessary analytic continuations; comparison is made with the results of 9)–12). In §5, by simple geometric variations on the same theme, we derive the Casimir stress-energy outside one plate, or two parallel plates (again, comparing with the existing literature). In §6, the outcomes of §§4 and 5 are combined to derive the Casimir pressure on two parallel plates.

For completeness, in Appendices A and B we give some mathematical background on the analytic continuation of the polylogarithm, the function mainly involved in local zeta regularization for the case under investigation.

§2. Background for the scalar Casimir effect

Throughout this note we work in Minkowski space-time, which is identified with $\boldsymbol{R}^4$ using a set of inertial coordinates

$$x = (x^\mu)_{\mu=0,1,2,3} \equiv (t, x^1, x^2, x^3) \equiv (t, \boldsymbol{x}) \; . \tag{2·1}$$

We work in units where $c = 1, \hbar = 1$; the Minkowski metric is $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ (and is used to raise and lower indices).

Let us fix a space domain $\Omega \subset \boldsymbol{R}^3$ where we consider a neutral, massless scalar field $\widehat{\phi}$; so, we have

$$\widehat{\phi} : \boldsymbol{R} \times \Omega \mapsto \mathcal{L}_{sa}(\mathcal{H}) \; ; \qquad 0 = \Box\widehat{\phi} = (-\partial_{tt} + \Delta)\widehat{\phi} \; . \tag{2·2}$$

Here we are considering the space $\mathcal{L}(\mathcal{H})$ of linear operators on the Fock space $\mathcal{H}$, and the subset $\mathcal{L}_{sa}(\mathcal{H})$ of the selfadjoint operators; $\Box := \partial^\mu \partial_\mu$ is the d'Alembertian and $\Delta := \sum_{i=1}^3 \partial_{ii}$ is the 3-Laplacian. We assume appropriate boundary conditions (e.g., the Dirichlet conditions $\widehat{\phi}(t, \boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial\Omega$).

To expand the field in normal modes, we consider a complete orthonormal set $(F_k)_{k \in K}$ of eigenfunctions for the Laplacian in $L^2(\Omega, \boldsymbol{C})$, with the given boundary conditions; K is a space of labels, for the moment unspecified, and we write the eigenvalues in the form $-\omega_k^2$. So,

$$F_k : \Omega \to \boldsymbol{C}; \quad \Delta F_k = -\omega_k^2 F_k \; (\omega_k > 0);$$
$$\int_\Omega d^3\boldsymbol{x} \overline{F_k}(\boldsymbol{x}) F_h(\boldsymbol{x}) = \delta(k, h). \quad (k, h \in K) \tag{2·3}$$

Any eigenvector label k can include different parameters, both discrete and continuous. We generically write $\int_K dk$ to indicate summation over all labels, (i.e., literal summation over the discrete parameters and integration over the continuous parameters, with a suitable measure); $\delta(h, k) = \delta(k, h)$ is the Dirac delta function for the labels space K (this reduces to the Kronecker symbol in the case of discrete parameters). The functions

$$f_k : \boldsymbol{R} \times \Omega \to \boldsymbol{C} \; , \qquad f_k(x) := F_k(\boldsymbol{x}) e^{-i\omega_k t} \tag{2·4}$$

fulfill $\Box f_k = 0$, and allow a unique expansion

$$\widehat{\phi}(x) = \int_K \frac{dk}{\sqrt{2\omega_k}} \left[\widehat{a}_k f_k(x) + \widehat{a}_k^\dagger \overline{f_k}(x)\right] \tag{2·5}$$

(with † indicating the adjoint operator, and $^-$ the complex conjugate). The destruction and creation operators $\widehat{a}_k, \widehat{a}_k^\dagger \in \mathcal{L}(\mathcal{H})$ fulfill the relations

$$[\widehat{a}_k, \widehat{a}_h] = 0 \ , \quad [\widehat{a}_k, \widehat{a}_h^\dagger] = \delta(h,k) \ , \qquad \widehat{a}_k|0\rangle = 0 \ , \tag{2·6}$$

where $|0\rangle \in \mathcal{H}$ is the vacuum state (of norm 1).

Let us pass to the stress-energy tensor. This depends on a parameter $\xi \in \boldsymbol{R}$, and its components $\widehat{T}_{\mu\nu} : \boldsymbol{R} \times \Omega \to \mathcal{L}_{sa}(\mathcal{H})$ are given by

$$\widehat{T}_{\mu\nu} := (1-2\xi)\, \partial_\mu \widehat{\phi} \circ \partial_\nu \widehat{\phi} - \left(\frac{1}{2} - 2\xi\right) \eta_{\mu\nu} \partial^\lambda \widehat{\phi} \partial_\lambda \widehat{\phi} - 2\xi\, \widehat{\phi} \circ \partial_{\mu\nu} \widehat{\phi} \ ; \tag{2·7}$$

in the above, we use the symmetrized operator product $\widehat{A} \circ \widehat{B} := (1/2)(\widehat{A}\widehat{B} + \widehat{B}\widehat{A})$.

To be precise, a theory involving merely a scalar field in flat space-time has a stress-energy tensor as above, with $\xi = 0$; the general form (2·7), with an arbitrary ξ, can be interpreted as the Minkowskian limit of the theory of a massless scalar field coupled with gravity, with ξ as the coupling constant.[*)]

Other authors have considered the general form (2·7) independently of the previous interpretation in terms of a gravitational coupling; these authors invoke the principle[13)] that one can add to the stress-energy tensor the divergence of an order three tensor (with suitable symmetries), and regard the terms proportional to ξ in (2·7) as additions of this kind.[10),11)]

To conclude these comments about ξ, we mention that the choice $\xi = 1/6$ gives a conformally invariant theory,[13),14)] where the tensor (2·7) has vanishing trace. For the above reasons, the term *conformal coupling* is usually employed to describe the case $\xi = 1/6$; one also speaks of a *minimal coupling* to indicate the case $\xi = 0$.

After this digression, we proceed towards the Casimir effect considering the vacuum expectation value (VEV) of $\widehat{T}_{\mu\nu}$. We use the expansion (2·5) for the field, with the relations $\langle 0|\widehat{a}_k \widehat{a}_h|0\rangle = 0$, $\langle 0|\widehat{a}_k^\dagger \widehat{a}_h^\dagger|0\rangle = 0$, $\langle 0|\widehat{a}_k^\dagger \widehat{a}_h|0\rangle = 0$ and $\langle 0|\widehat{a}_k \widehat{a}_h^\dagger|0\rangle =$

*) In the theory of a classical, massless scalar field coupled with gravity, the dynamical variables are the field ϕ and the space-time metric $g_{\mu\nu}$. The action functional is $S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^4x \sqrt{-g}\Big(\partial^\mu\phi\partial_\mu\phi - R(\frac{1}{8\pi} - \xi\phi^2)\Big)$ in units where the gravitational constant is 1, where $g := \det(g_{\mu\nu})$ and R is the scalar curvature of the metric: for more details see, e.g., Ref. 14) p. 43. One imposes the stationarity of the action with respect to variations of ϕ and $g_{\mu\nu}$: in this way one gets, respectively, the scalar field equation and Einstein's equations with the field stress-energy tensor $T_{\mu\nu}$. One can analyze the almost Minkowskian case where ϕ is small and $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ a small perturbation of the second order in ϕ. In this case the scalar field equation has the form $\Box\phi = O_2[\phi]$ with $\Box$ the Minkowski d'Alembertian $\partial^\mu\partial_\mu$, and the stress-energy tensor is $T_{\mu\nu} = (1-2\xi)\,\partial_\mu\phi\partial_\nu\phi - \left(\frac{1}{2} - 2\xi\right)\eta_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi - 2\xi\,\phi\partial_{\mu\nu}\phi + O_3[\phi]$; here, O_2 and O_3 indicate terms of orders 2 and 3. Of course, Einstein's equations relate $h_{\mu\nu}$ to ϕ. Neglecting the higher order terms, and quantizing the field, we obtain Eqs. (2·2) and (2·7).

$\delta(k,h)$; in this way, we readily obtain the formal expression

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle = \int_K \frac{dk}{\omega_k}\left[\Big(\frac{1}{4}-\frac{\xi}{2}\Big)\Big(\partial_\mu f_k \partial_\nu \overline{f_k} + \partial_\nu f_k \partial_\mu \overline{f_k}\Big) - \Big(\frac{1}{4}-\xi\Big)\eta_{\mu\nu}\partial^\lambda f_k \partial_\lambda \overline{f_k} - \frac{\xi}{2}\Big(f_k \partial_{\mu\nu}\overline{f_k} + \overline{f_k}\partial_{\mu\nu} f_k\Big)\right]; \qquad (2{\cdot}8)$$

however, the above integral is divergent and some renormalization procedure is needed. A standard approach relies on the so-called point splitting regularization (besides the already cited 10), 12), see 15), 16); Ref. 11) essentially uses the same method). In this approach, in place of $\widehat{T}_{\mu\nu}(x)$ one considers

$$\widehat{T}_{\mu\nu}(x,x') := (1-2\xi)\,\partial_\mu\widehat{\phi}(x)\circ\partial_\nu\widehat{\phi}(x') - \left(\frac{1}{2}-2\xi\right)\eta_{\mu\nu}\partial^\lambda\widehat{\phi}(x)\circ\partial_\lambda\widehat{\phi}(x') - 2\xi\,\widehat{\phi}(x)\circ\partial_{\mu\nu}\widehat{\phi}(x'), \qquad (x,x'\in \boldsymbol{R}\times\Omega) \qquad (2{\cdot}9)$$

giving formally $\widehat{T}_{\mu\nu}(x)$ in the limit $x'\to x$. One then defines the renormalized VEV of $\widehat{T}_{\mu\nu}(x)$ as

$$\langle 0|\widehat{T}_{\mu\nu}(x)|0\rangle_{ren} := FP\Big|_{x'\to x}\langle 0|\widehat{T}_{\mu\nu}(x,x')|0\rangle\ , \qquad (2{\cdot}10)$$

where we have written FP to indicate the "finite part" in the limit $x'\to x$; this means that one writes down the VEV of $\widehat{T}_{\mu\nu}(x,x')$ and then removes the terms diverging for $x'\to x$.*)

Hereafter we will describe the alternative approach considered in this paper, i.e., the local zeta method.

§3. Local zeta regularization

In the sequel we keep all the notations of the previous section. Let us denote with u a complex parameter and consider the powers $(-\varDelta)^{-u/4}$, built from the 3-dimensional Laplacian $\varDelta$. From $\varDelta F_k = -\omega_k^2 F_k$ it follows $(-\varDelta)^{-u/4}F_k = \omega_k^{-u/2}F_k$, whence

$$(-\varDelta)^{-u/4} f_k = \omega_k^{-u/2} f_k\ ; \qquad (3{\cdot}1)$$

there are similar relations for the conjugate functions, starting from $\varDelta\overline{F_k} = -\omega_k^2\overline{F_k}$.

We now introduce the *smeared*, or *zeta-regularized* field operators and stress-energy tensor

$$\widehat{\phi}^u := (-\varDelta)^{-u/4}\widehat{\phi}\ , \qquad (3{\cdot}2)$$

*) Of course, the concept of "finite part" contains a basic ambiguity, that must be removed by a precise prescription. In the case of an electromagnetic field in Minkowski space-time, a prescription of this type has been given in 15); this approach could be adapted to the scalar case. An alternative strategy is to define the finite part in (2·10) as the $x'\to x$ limit of what remains after subtracting from $\langle 0|\widehat{T}_{\mu\nu}(x,x')|0\rangle$ the analogous VEV for a field without boundary conditions.[9),10)] For a critical analysis about these and other problematic aspects of point splitting, see 7).

$$\widehat{T}^u_{\mu\nu} := (1-2\xi)\,\partial_\mu\widehat{\phi}^u \circ \partial_\nu\widehat{\phi}^u - \left(\frac{1}{2}-2\xi\right)\eta_{\mu\nu}\partial^\lambda\widehat{\phi}^u\partial_\lambda\widehat{\phi}^u - 2\xi\,\widehat{\phi}^u \circ \partial_{\mu\nu}\widehat{\phi}^u \ , \qquad (3{\cdot}3)$$

which formally give $\widehat{\phi}$ and $\widehat{T}_{\mu\nu}$ in the limit $u \to 0$. Equation (2·5) implies

$$\widehat{\phi}^u(x) = \int_K \frac{dk}{\sqrt{2}\,\omega_k^{u/2+1/2}}\left[\widehat{a}_k f_k(x) + \widehat{a}_k^\dagger \overline{f_k}(x)\right] \ ; \qquad (3{\cdot}4)$$

now, a computation very similar to the one giving Eq. (2·8) produces the result

$$\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle = \int_K \frac{dk}{\omega_k^{u+1}}\left[\left(\frac{1}{4}-\frac{\xi}{2}\right)\left(\partial_\mu f_k\partial_\nu\overline{f_k} + \partial_\nu f_k\partial_\mu\overline{f_k}\right)\right.$$
$$\left. -\left(\frac{1}{4}-\xi\right)\eta_{\mu\nu}\partial^\lambda f_k\partial_\lambda\overline{f_k} - \frac{\xi}{2}\left(f_k\partial_{\mu\nu}\overline{f_k} + \overline{f_k}\partial_{\mu\nu}f_k\right)\right] . \qquad (3{\cdot}5)$$

The above integral typically converges for $\Re u$ sufficiently large and is an analytic function of u, a situation that will be exemplified hereafter. Equation (3·5), with $\Re u$ sufficiently large, is our regularization of the VEV for $\widehat{T}_{\mu\nu}$; we now define the renormalized VEV as

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren} := AC\Big|_{u=0}\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle \ , \qquad (3{\cdot}6)$$

where $AC|_{u=0}$ indicates that one should consider the analytic continuation of the function $u \mapsto \langle 0|\widehat{T}^u_{\mu\nu}|0\rangle$, and evaluate it at $u=0$. In the next section, the whole procedure will be exemplified in the classical case where Ω is the region between two parallel plates, with Dirichlet boundary conditions; in the subsequent section we will treat the region outside one or two plates.

§4. Casimir effect between two parallel plates

Setting up the problem; the zeta-regularized stress-energy tensor Let the plates occupy the planes $x^3 = 0$ and $x^3 = a$ ($a > 0$); the region between the plates is

$$\Omega := \{(x^1, x^2, x^3) \mid x^1, x^2 \in \boldsymbol{R}\ , 0 < x^3 < a\} \ . \qquad (4{\cdot}1)$$

We assume the Dirichlet boundary conditions

$$\widehat{\phi}(t, x^1, x^3, x^3) = 0 \qquad \text{for } x^3 = 0, a \ . \qquad (4{\cdot}2)$$

Let us produce a complete orthonormal set $(F_k)_{k\in K}$ of Dirichlet eigenfunctions for Δ on Ω, and the corresponding eigenvalues $-\omega_k^2$. We can take

$$K := \{k = (k_1, k_2, k_3) \mid k_1, k_2 \in \boldsymbol{R}\ , k_3 \in \{\pi/a, 2\pi/a, 3\pi/a, \ldots\}\} \ , \qquad (4{\cdot}3)$$

$$\int_K dk := \int_{\boldsymbol{R}} dk_1 \int_{\boldsymbol{R}} dk_2 \sum_{k_3 \in\ \{\pi/a, 2\pi/a, 3\pi/a, \ldots\}} \ ;$$

$$F_k(\boldsymbol{x}) := \frac{1}{\pi\sqrt{2a}} e^{i(k_1 x^1 + k_2 x^2)} \sin(k_3 x^3) \ ; \qquad \omega_k := \sqrt{k_1^2 + k_2^2 + k_3^2} \ . \qquad (4{\cdot}4)$$

The above functions fulfill $\int_\Omega d^3\boldsymbol{x}\overline{F_k}F_h = \delta(k_1-h_1)\delta(k_2-h_2)\delta_{k_3,h_3}$; we can use them to build $f_k(t,\boldsymbol{x}) := F_k(\boldsymbol{x})e^{-i\omega_k t}$. Let us pass to the computation of the components $\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle$, and to their analytic continuation at $u=0$; we will start from the case $\mu=0$, $\nu=0$. From Eqs. (3·5), (4·3) and (4·4), we obtain

$$\begin{aligned}
&\langle 0|\widehat{T}^u_{00}|0\rangle \\
&= \frac{1}{8\pi^2 a}\sum_{k_3\in\ \{\pi/a,2\pi/a,3\pi/a,\dots\}}\int_{\boldsymbol{R}} dk_1\int_{\boldsymbol{R}} dk_2 \frac{k_1^2+k_2^2+k_3^2-(k_1^2+k_2^2+4\xi k_3^2)\cos(2k_3x^3)}{(k_1^2+k_2^2+k_3^2)^{u/2+1/2}} \\
&= \frac{1}{8\pi^{u-1}a^{4-u}}\sum_{\ell=1}^{+\infty}\int_{\boldsymbol{R}} dq_1\int_{\boldsymbol{R}} dq_2 \frac{q_1^2+q_2^2+\ell^2-(q_1^2+q_2^2+4\xi\ell^2)\cos(2\pi\ell x^3/a)}{(q_1^2+q_2^2+\ell^2)^{u/2+1/2}}, \qquad (4\cdot5)
\end{aligned}$$

where, in the last passage, we have performed a change of variables $k_1=(\pi/a)q_1$, $k_2=(\pi/a)q_2$, $k_3=(\pi/a)\ell$. We now pass to polar coordinates in the (q_1,q_2) plane, setting $q_1=\rho\cos\theta$, $q_2=\rho\sin\theta$, and then compute the integrals in θ,ρ; in this way we obtain

$$\begin{aligned}
\langle 0|\widehat{T}^u_{00}|0\rangle &= \frac{1}{8\pi^{u-1}a^{4-u}}\sum_{\ell=1}^{+\infty}\int_0^{2\pi} d\theta\int_0^{+\infty} d\rho\rho\frac{\rho^2+\ell^2-(\rho^2+4\xi\ell^2)\cos(2\pi\ell x^3/a)}{(\rho^2+\ell^2)^{u/2+1/2}} \\
&= \frac{1}{4\pi^{u-2}(u-3)a^{4-u}}\sum_{\ell=1}^{+\infty}\left[\frac{1}{\ell^{u-3}}-\frac{2+4(u-3)\xi}{(u-1)}\,\frac{\cos(2\pi\ell x^3/a)}{\ell^{u-3}}\right]. \qquad (4\cdot6)
\end{aligned}$$

The last series is clearly convergent if

$$\Re u > 4\ ; \qquad (4\cdot7)$$

under the same condition, all the expressions given previously for $\langle 0|\widehat{T}^u_{00}|0\rangle$ are meaningful and finite. To go on let us recall that the polylogarithm $(z,s)\mapsto Li_s(z)$ is defined by

$$Li_s(z) := \sum_{\ell=1}^{+\infty}\frac{z^\ell}{\ell^s} \qquad \text{for } z\in\boldsymbol{C},\ |z|\leqslant 1 \text{ and } s\in S_z, \qquad (4\cdot8)$$

where $S_z\subset\boldsymbol{C}$ is the set of values of s for which the above series converges: one finds

$$S_z = \begin{cases} \boldsymbol{C} & \text{if } |z|<1, \\ \{\Re s>0\} & \text{if } |z|=1,\ z\neq 1, \\ \{\Re s>1\} & \text{if } z=1. \end{cases} \qquad (4\cdot9)$$

(We note that, for $|z|>1$, there is no $s\in\boldsymbol{C}$ such that the series converges.) Let us also recall that the Riemann zeta function $s\mapsto\zeta(s)$ is defined setting

$$\zeta(s) := Li_s(1) = \sum_{\ell=1}^{+\infty}\frac{1}{\ell^s} \qquad \text{for } s\in\boldsymbol{C},\ \Re s>1\ . \qquad (4\cdot10)$$

The functions Li, ζ can be extended to larger domains by analytic continuation, as reviewed in Appendix A. Comparing Eqs. (4·6), (4·8), (4·10), and noting that $\cos(2\pi\ell x^3/a) = (1/2)(e^{2i\pi x^3/a})^\ell + (1/2)(e^{-2i\pi x^3/a})^\ell$, we see that

$$\langle 0|\widehat{T}^u_{00}|0\rangle$$
$$= \frac{1}{4\pi^{u-2}(u-3)a^{4-u}}\left\{\zeta(u-3) - \frac{1+2(u-3)\xi}{(u-1)}\left[Li_{u-3}(e^{2i\pi x^3/a}) + Li_{u-3}(e^{-2i\pi x^3/a})\right]\right\} \tag{4·11}$$

for $\Re u > 4$. The other components $\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle$ are treated similarly. More precisely, we find

$$\langle 0|\widehat{T}^u_{ii}|0\rangle$$
$$= \frac{1}{8\pi^2 a}\sum_{k_3\in\{\pi/a,2\pi/a,3\pi/a,\ldots\}}\int_{\boldsymbol{R}} dk_1 \int_{\boldsymbol{R}} dk_2 \frac{k_i^2 - (k_i^2 + (1-4\xi)k_3^2)\cos(2k_3 x^3)}{(k_1^2+k_2^2+k_3^2)^{u/2+1/2}} \quad (i=1,2); \tag{4·12}$$

$$\langle 0|\widehat{T}^u_{33}|0\rangle$$
$$= \frac{1}{8\pi^2 a}\sum_{k_3\in\;\{\pi/a,2\pi/a,3\pi/a,\ldots\}}\int_{\boldsymbol{R}} dk_1 \int_{\boldsymbol{R}} dk_2 \frac{k_3^2}{(k_1^2+k_2^2+k_3^2)^{u/2+1/2}}\;; \tag{4·13}$$

$$\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle = 0 \qquad \text{for } \mu\neq\nu\;; \tag{4·14}$$

indeed, one checks that $\langle 0|\widehat{T}^u_{22}|0\rangle = \langle 0|\widehat{T}^u_{11}|0\rangle$ with a change of variables $k_2 \leftrightarrow k_1$. The expressions (4·12) and (4·13) can now be treated with the same method employed for $\langle 0|\widehat{T}^u_{00}|0\rangle$: one makes a change of variables $k_1 = (\pi/a)q_1$, $k_2 = (\pi/a)q_2$, $k_3 = (\pi/a)\ell$, passes to polar coordinates (ρ,θ) in the (q_1,q_2) plane, integrates in these coordinates and then expresses the remaining sum over ℓ in terms of the zeta function and of the polylogarithm. The results of such computations can be summarized in the formula

$$\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle\Big|_{\mu,\nu=0,1,2,3} = A^u\begin{pmatrix} u-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & u-3\end{pmatrix}$$
$$+B^u(x^3)\begin{pmatrix} -1-2(u-3)\xi & 0 & 0 & 0\\ 0 & 1-\frac{u}{2}+2(u-3)\xi & 0 & 0\\ 0 & 0 & 1-\frac{u}{2}+2(u-3)\xi & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \text{for } \Re u > 4,$$
$$A^u := \frac{\zeta(u-3)}{4\pi^{u-2}(u-3)(u-1)a^{4-u}}, \quad B^u(x^3) := \frac{Li_{u-3}(e^{2i\pi x^3/a}) + Li_{u-3}(e^{-2i\pi x^3/a})}{4\pi^{u-2}(u-3)(u-1)a^{4-u}}\,. \tag{4·15}$$

Renormalization by analytic continuation Due to (4·15), the problem of the analytic continuation of $\langle 0|\widehat{T}^u_{\mu\nu}|0\rangle$ at $u=0$ is reduced to the problem of continuing the functions $s \mapsto \zeta(s), Li_s(z)$ (for fixed z) up to the point $s = -3$. As reviewed in Appendix B, such continuations are given by

$$Li_{-3}(z) = \frac{z(z^2+4z+1)}{(z-1)^4} \quad \text{for } z \neq 1; \qquad \zeta(-3) = Li_{-3}(1) = \frac{1}{120}\,. \tag{4·16}$$

(Note a discontinuity with respect to z presented by the continuations: $\lim_{z\to 1} Li_{-3}(z) = \infty \neq Li_{-3}(1)$. For an interpretation of this fact, we refer again to Appendix B).

Keeping in mind these facts we return to Eq. (4·15), from which we infer that $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren} := AC\Big|_{u=0}\langle 0|\widehat{T}^{u}_{\mu\nu}|0\rangle$ is as follows:

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren}\Big|_{\mu,\nu=0,1,2,3} = A\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} + (1-6\xi)B(x^3)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4·17}$$

Here $A := A^0 = \frac{\pi^2}{12a^4}\zeta(-3)$ and $B(x^3) := B^0(x^3) = \frac{\pi^2}{12a^4}\Big[Li_{-3}(e^{2i\pi x^3/a}) + Li_{-3}(e^{-2i\pi x^3/a})\Big]$, i.e., using the expressions (4·16),

$$A = \frac{\pi^2}{1440a^4}\,,$$

$$B(x^3) = \frac{\pi^2}{12a^4}\,\frac{2+\cos(2\pi x^3/a)}{[1-\cos(2\pi x^3/a)]^2} = \frac{\pi^2}{48a^4}\frac{3-2\sin^2(\pi x^3/a)}{\sin^4(\pi x^3/a)}. \qquad (0 < x^3 < a) \tag{4·18}$$

(The first expression above for $B(x^3)$ follows using (4·16) with $z = e^{\pm 2i\pi x^3/a}$; the second expression follows from the duplication formula for the cosine.)
Let us remark the following:
i) For $\mu = 0, 1, 2$ the components $\langle 0|\widehat{T}_{\mu\mu}|0\rangle_{ren}$ depend on x^3 through the function B, except in the conformal case $\xi = 1/6$ where they are constant. The component $\langle 0|\widehat{T}_{33}|0\rangle_{ren}$ is constant in any case.
ii) The function $B(x^3)$ diverges like $1/(x^3)^4$ in the limit $x^3 \to 0$, and like $1/(x^3-a)^4$ in the limit $x^3 \to a$; the same can be said of $\langle 0|\widehat{T}_{\mu\mu}|0\rangle_{ren}$ for $\mu = 0, 1, 2$ and $\xi \neq 1/6$. The appearing of divergences at the boundaries is a fairly general feature, that was deeply investigated in 9) for boundaries of arbitrary shape (not only for the scalar, but also for the electromagnetic field).

Equations (4·17)–(4·18) are our final result for the renormalized stress-energy VEV. We have now checked the following claim of the Introduction: the local zeta method, based on analytic continuation, gives directly a finite stress-energy tensor, *with no need to remove divergent terms*. We already indicated this fact as a relevant difference between this approach and the point splitting method; however the renormalized tensors derived by these two approaches coincide, as illustrated hereafter.

Comparison with the results obtained by point splitting Let us compare our equations (4·17) and (4·18) with the results obtained via point splitting by Deutsch and Candelas,[9] Fulling,[10] and by Esposito et al.[12] (As already mentioned, Ref. 9) considers boundaries of arbitrary shape. In the cited reference the case of plane boundaries is treated as the simplest example, omitting the computational details; these can be found in 10) and 12). In addition to the formula written hereafter, Ref. 12) gives as well the corrections of order one due to an external, weak gravitational field.)

The essence of point splitting has been reviewed in Eqs. (2·9) and (2·10) (which are implemented in 9), 10), 12) using a Green function method, fully equivalent to the eigenfunction expansion for the Laplacian). The cited works produce the formal result

$$\lim_{x'\to x} \langle 0|\widehat{T}_{\mu\nu}(x,x')|0\rangle = \left(A + \frac{1}{2\pi^2}\lim_{x^{3\prime}\to x^3}\frac{1}{(x^3-x^{3\prime})^4}\right)\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix}$$

$$+ (1-6\xi)\, B(x^3)\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad (4·19)$$

where A, B are as in our Eq. (4·18). As indicated in (2·10), in this approach renormalization amounts to subtract the divergent term proportional to $\lim_{x^{3\prime}\to x^3}(x^3 - x^{3\prime})^{-4}$; so, the renormalized stress-energy VEV agrees with ours.

A stress-energy VEV renormalization, based essentially on point splitting, appears as well in the book of Milton[11] who gives for A the expression in (4·18) but obtains, in place of B, the function

$$\mathcal{B}(x^3) = \frac{1}{16\pi^2 a^4}\left[\zeta(4, x^3/a) + \zeta(4, 1-x^3/a)\right] \; ; \qquad (4·20)$$

here $(z,s)\mapsto \zeta(s,z)$ is the Hurwitz zeta function defined by

$$\zeta(s,z) = \sum_{\ell=0}^{+\infty}\frac{1}{(\ell+z)^s}. \qquad (4·21)$$

Indeed, the Milton function $\mathcal{B}$ coincides with the function B in Eq. (4·18). To show this, we refer to the known identity (see 17), p. 608, Eq. (25.11.12))

$$\zeta(s+1,z) = \frac{(-1)^{s+1}}{s!}\psi^{(s)}(z) \qquad \text{for } s = 1,2,3,\ldots\,, \qquad (4·22)$$

where the right-hand side contains the polygamma function $\psi^{(s)}(z) := (d/dz)^{s+1}\ln\Gamma(z)$, for $s = 1,2,3,\ldots$; this implies

$$\mathcal{B}(x^3) = \frac{1}{96\pi^2 a^4}\left[\psi^{(3)}(x^3/a) + \psi^{(3)}(1-x^3/a)\right]\,. \qquad (4·23)$$

Another relation, known to hold for the polygamma function, is

$$\psi^{(s)}(1-z) + (-1)^{s+1}\psi^{(s)}(z) = (-1)^s\pi\frac{d^s}{dz^s}\cot(\pi z) \qquad \text{for } s = 1,2,3,\ldots\,; \qquad (4·24)$$

(see 17), p. 144, Eq. (5.15.6)); this entails

$$\mathcal{B}(x^3) = -\frac{1}{96\pi a^4}\left(\frac{d^3}{dz^3}\cot(\pi z)\right)\Big|_{z=x^3/a}$$
$$= \frac{\pi^2}{48a^4}\left[\frac{3-2\sin^2(\pi x^3/a)}{\sin^4(\pi x^3/a)}\right] = B(x^3) \text{ as in } (4{\cdot}18)\ . \qquad (4{\cdot}25)$$

As a final comment on this result, we mention that the equality $\mathcal{B} = B$ is a special case of a more general relation between the polylogarithm and the Hurwitz zeta function (see the classical paper of Jonquière[18] on these functions; the handbook[17] reports the same results in modern notations).

§5. The Casimir effect outside one plate, or two parallel plates

The case of a single plate Let the plate occupy the plane $x^3 = 0$; hereafter we determine the renormalized VEV of $\widehat{T}_{\mu\nu}$ in one of the half-spaces bounded by the plane, say, in

$$\Omega_\infty := \{(x^1,x^2,x^3) \mid x^1,x^2 \in \boldsymbol{R}\ , x^3 > 0\}\ . \qquad (5{\cdot}1)$$

As before, we assume for the (scalar, gravity coupled) field $\widehat{\phi}$ the Dirichlet boundary conditions

$$\widehat{\phi}(t,x^1,x^2,x^3) = 0 \qquad \text{for } x^3 = 0\ . \qquad (5{\cdot}2)$$

To treat this case, it is not even necessary to set up a framework as in the previous sections, starting from the Dirichlet eigenfunctions of Δ in Ω_∞. In fact, it suffices to view Ω_∞ as the $a \to +\infty$ limit of the domain

$$\Omega_a := \{(x^1,x^2,x^3) \mid x^1,x^2 \in \boldsymbol{R}\ , 0 < x^3 < a\} \qquad (5{\cdot}3)$$

and define the renormalized VEV of $\widehat{T}_{\mu\nu}$ in Ω_∞ as

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\infty,ren} := \lim_{a\to+\infty}\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{a,ren}\ , \qquad (5{\cdot}4)$$

where the right-hand side contains the renormalized VEV in Ω_a; the latter is known from the previous section, see Eqs. (4·17) and (4·18). So,

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\infty,ren} = \left(\lim_{a\to+\infty} A_a\right)\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{pmatrix}$$
$$+ (1-6\xi)\left(\lim_{a\to+\infty} B_a(x^3)\right)\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad A_a := A,\ B_a(x^3) := B(x^3)$$
$$\text{as in } (4{\cdot}18). \qquad (5{\cdot}5)$$

From Eq. (4·18), it is evident that $A_a \to 0$, $B_a(x^3) \to 1/(16\pi^2(x^3)^4)$ for $a \to +\infty$; so,

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\infty,ren} = \frac{1-6\xi}{16\pi^2(x^3)^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (0 < x^3 < +\infty) \qquad (5·6)$$

The above result appears in 9) and 10), and is derived by different means in 19). Of course, one obtains similar conclusions in the half space $\{-\infty < x^3 < 0\}$.

We observe that, if the coupling parameter takes the conformal value $\xi = 1/6$, the stress-energy tensor vanishes everywhere outside the plate.

The case outside two parallel plates We now consider, as in the previous section, two plates occupying the planes $x^3 = 0$ and $x^3 = a$; we are interested in the renormalized VEV of $\widehat{T}_{\mu\nu}$ in the region outside the plates, which is the disjoint union of the half spaces $\{x^3 < 0\}$ and $\{x^3 > a\}$. This can be obtained by obvious adaptations of the result (5·6) on the half space $\{x^3 > 0\}$; the conclusion is

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren} = \frac{1-6\xi}{16\pi^2(x^3)^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (-\infty < x^3 < 0) \qquad (5·7)$$

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren} = \frac{1-6\xi}{16\pi^2(x^3-a)^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (a < x^3 < +\infty) \qquad (5·8)$$

Note that, in the conformal case $\xi = 1/6$, $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{ren}$ is identically zero outside the plates. If $\xi \neq 1/6$, the components with $\mu = \nu = 0, 1, 2$ of this tensor diverge like $1/(x^3)^4$ and $1/(x^3-a)^4$ for $x^3 \to 0^-$ and $x^3 \to a^+$, respectively; we recall that similar divergences were found as well for the stress-energy tensor between the plates.

§6. Pressure on the plates

In this section we always use the spatial indices $i, j \in \{1, 2, 3\}$. Let us consider any one of the two plates at $x^3 = 0$ or $x^3 = a$, and evaluate the force per unit area acting on it; in principle, this computation should take into account the action of the field both inside and outside the plates. The force per unit area produced on the given plate by the field in the inner region is $p^i_{in} = \langle 0|\widehat{T}^i{}_j|0\rangle_{in} n^j_{out}$ where $\langle 0|\widehat{T}^i{}_j|0\rangle_{in}$ is the renormalized stress-energy tensor in the inner region and n^j_{out} the normal unit vector to the plate pointing towards the outer region. On the other hand, the force per unit area produced on the same plate by the field in the outer region is $p^i_{out} = \langle 0|\widehat{T}^i{}_j|0\rangle_{out} n^j_{in}$, where the subscripts $_{in}$, $_{out}$ have an obvious meaning. So, the total force per unit area on the plate is

$$p^i = \langle 0|\widehat{T}^i{}_j|0\rangle_{in} n^j_{out} + \langle 0|\widehat{T}^i{}_j|0\rangle_{out} n^j_{in} \, . \qquad (6·1)$$

For the plate located at $x^3 = 0$, we have $(n^j_{in}) = (0, 0, 1)$, $(n^j_{out}) = (0, 0, -1)$, so

$$p^i\Big|_{x^3=0} = \langle 0|\widehat{T}^i{}_3|0\rangle_{out} - \langle 0|\widehat{T}^i{}_3|0\rangle_{in}\Big|_{x^3=0}; \tag{6·2}$$

for the plate at $x^3 = a$ the inner and outer normals are reverted, so

$$p^i\Big|_{x^3=a} = -\langle 0|\widehat{T}^i{}_3|0\rangle_{out} + \langle 0|\widehat{T}^i{}_3|0\rangle_{in}\Big|_{x^3=a}. \tag{6·3}$$

Now, we take the expressions of $\langle 0|\widehat{T}^i{}_3|0\rangle_{in,out} = \langle 0|\widehat{T}_{i3}|0\rangle_{in,out}$ from Eqs. (4·17), (4·18) and (5·7), (5·8); for both plates $\langle 0|\widehat{T}_{i3}|0\rangle_{out}$ vanishes and $(\langle 0|\widehat{T}_{i3}|0\rangle_{in}) = (0, 0, -3A)$ with $A = \pi^2/1440a^4$, as usually; in conclusion

$$\left(p^i\Big|_{x^3=0}\right) = \left(0, 0, \frac{\pi^2}{480a^4}\right); \tag{6·4}$$

$$\left(p^i\Big|_{x^3=a}\right) = \left(0, 0, -\frac{\pi^2}{480a^4}\right). \tag{6·5}$$

Thus the plates are subject to a reciprocal attraction inversely proportional to the fourth power of their distance. We note that, once more, the result obtained agrees with the ones reported in Refs. 10)–12).

Acknowledgements

This work was partly supported by INdAM, INFN and by MIUR, PRIN 2008 Research Project "Geometrical methods in the theory of nonlinear waves and applications".

We are grateful to Giuseppe Molteni for useful indications about the polylogarithm, to Valter Moretti and to the anonymous referee for useful comments and bibliographical indications.

Appendix A

—— *Analytic Continuation of the Polylogarithm (and of the Zeta Function)* ——

Let us report Eqs. (4·8) and (4·10)

$$Li_s(z) := \sum_{\ell=1}^{+\infty} \frac{z^\ell}{\ell^s} \text{ for } z \in \boldsymbol{C},\ |z| \leqslant 1 \text{ and } s \in S_z;$$

$$\zeta(s) := Li_s(1) := \sum_{\ell=1}^{+\infty} \frac{1}{\ell^s} \text{ for } s \in \boldsymbol{C},\ \Re s > 1\ .$$

(In the above S_z is the subset of $\boldsymbol{C}$ such that the series for $Li_s(z)$ converges, see Eq. (4·9)). Our problem is continuing analytically (in s) the functions defined as

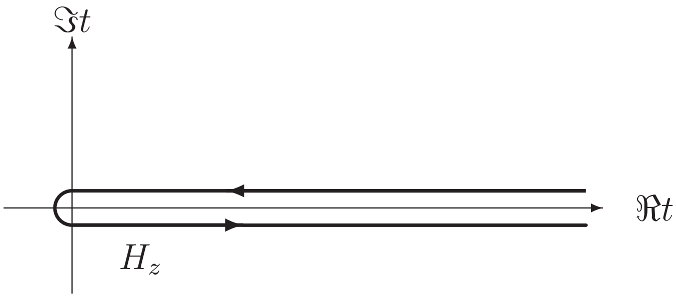

Fig. 1.

above; the solution is well known,[18] and reported here for completeness. Indeed, let us define

$$Li_s(z) := -\frac{\Gamma(1-s)\, z}{2\pi i} \int_{H_z} dt \frac{(-t)^{s-1}}{e^t - z} \qquad \text{for } z \in \boldsymbol{C},\ s \in \boldsymbol{C} \setminus \{1,2,3,..\}\ , \qquad (\text{A}\cdot 1)$$

$$Li_s(z) := \lim_{s' \to s} Li_{s'}(z) \text{ for } z \in \boldsymbol{C} \setminus \{1\},\ s \in \{1,2,3,...\} \text{ or } z = 1,\ s \in \{2,3,...\}; \quad (\text{A}\cdot 2)$$

$$\zeta(s) := Li_s(1) \qquad \text{for } s \in \boldsymbol{C} \setminus \{1\}\ . \qquad (\text{A}\cdot 3)$$

In Eq. (A·1), Γ is the usual Gamma function. Furthermore:
i) H_z is a Hankel contour in the complex t plane, starting from infinity in the direction of the positive real axis, turning counterclockwise around $t = 0$ and returning to infinity in the direction of the positive real axix (see Fig. 1); this contour is chosen so that *all* the solutions t of the equation $e^t = z$ are outside the region bounded by H_z, except the solution $t = 0$ appearing if $z = 1$.
ii) For each $t \in H_z$ we intend

$$(-t)^{s-1} := e^{-i(s-1)\pi} t^{s-1}\ , \qquad t^{s-1} := |t|^{s-1} e^{i(s-1)\arg t}, \qquad (\text{A}\cdot 4)$$

where $t \mapsto \arg t$ is the unique continuous function on H_z such that $\arg t \to 0$ when t tends to the beginning of the path.

For each fixed z with $|z| \leqslant 1$, the function $s \mapsto Li_s(z)$ defined via (A·1) (A·2) is the (unique) analytic continuation of the function $s \in S_z \mapsto Li_s(z)$ previously defined via the power series (4·8); to prove this, it suffices to prove that the definition (A·1) for $Li_s(z)$ via a contour integral implies a series expansion as in (4·8), if $s \in S_z$. To this purpose, we reexpress the function of z and t in (A·1) in the following way:

$$z\frac{(-t)^{s-1}}{e^t - z} = (-t)^{s-1} \frac{z e^{-t}}{1 - z e^{-t}} = (-t)^{s-1} \sum_{\ell=1}^{+\infty} (z e^{-t})^{\ell}\ ; \qquad (\text{A}\cdot 5)$$

inserting this result into Eq. (A·1), we obtain

$$Li_s(z) = -\frac{\Gamma(1-s)}{2\pi i} \sum_{\ell=1}^{+\infty} z^{\ell} \int_{H_z} (-t)^{s-1} e^{-\ell t}\ . \qquad (\text{A}\cdot 6)$$

On the other hand, the known Hankel's integral representation for $1/\Gamma$[17)] implies

$$-\frac{1}{2\pi i}\int_{H_z}(-t)^{s-1}e^{-\ell t}=\frac{1}{\Gamma(1-s)\ell^s} \tag{A·7}$$

for $\ell = 1, 2, 3...$. The last two equations yield the wanted expansion $Li_s(z) = \sum_{\ell=1}^{+\infty} z^\ell/\ell^s$, of the form (4·8).

The above manipulations have hidden a problem: to grant convergence of the series expansion (A·5) and the exchange between the summation over ℓ and the integration over H_z, one should have $|ze^{-t}| < 1$ uniformly in $t \in H_z$: on the other hand, $|ze^{-t}|$ can be larger than 1 when t is on the arc in the half plane $\Re t < 0$, turning around the origin. Let us sketch how to overcome this difficulty; the basic idea is that $Li_s(z)$ defined in (A·1) does not change if we shrink the path H_z around the origin. If $|z| < 1$, we can shrink H_z so that $|ze^{-t}| < 1$ uniformly in H_z, including the arc that turns around the origin. The case $|z| = 1$ is a bit more technical: one isolates from the integral over H_z the contribution of the arc encircling the origin, makes a series expansion of the integrand in the remaining part of H_z, and finally proves that the contribution from the arc can be made arbitrarily small by shrinking.

The previous results on the analytic continuation of the function $s \mapsto Li_s(z)$ hold, in particular, for $z = 1$; so, the function $s \in \boldsymbol{C} \setminus \{1\} \mapsto \zeta(s)$ in (A·3) is the (unique) analytic continuation of the function defined previously by (4·10).

Another property of the function (A·1)(A·2) is that it is jointly analytic in (z, s), when these variables range in a suitable open subset of $\boldsymbol{C}^2$; outside this open set, some pathologies can appear. In particular, for a given s, this function can happen to be discontinuous in z at the specific point $z = 1$: see, e.g., the case $s = -3$ discussed hereafter.

Appendix B
—— *The Polylogarithm (and the Zeta Function) at $s = -3$* ——

Let us consider the analytic continuation of the polylogarithm described in Appendix A, and evaluate it at $s = -3$. For this choice of s, Eq. (A·1) takes the form

$$Li_{-3}(z) = -\frac{6\,z}{2\pi i}\int_{H_z}\frac{dt}{t^4(e^t-z)} \qquad \text{for } z \in \boldsymbol{C}\ ; \tag{B·1}$$

the integral therein is easily computed by the method of residues, as briefly sketched hereafter.*) First of all, the integral in (A·1) involves a meromorphic function of t, whose only singularity in the region bounded by H_z is a pole at $t = 0$. The order of the pole is 4 if $z \neq 1$, while it is 5 if $z = 1$, and one finds

$$\mathrm{Res}\left[\frac{1}{t^4(e^t-z)}\right]_{t=0} = -\frac{(z^2+4z+1)}{6(z-1)^4} \ \text{ if } z \neq 1;\ \ \mathrm{Res}\left[\frac{1}{t^4(e^t-1)}\right]_{t=0} = -\frac{1}{720}\ .$$

*) Of course, here we are viewing the path H_z as the limit of a closed loop, with one side going to infinity along the semiaxis $\Re t > 0$; the contribution to the integral of this part of the loop goes to zero in the above limit, due to the term $1/e^t$ in the integrand.

When these results are inserted into (B·1), the residue theorem gives

$$Li_{-3}(z) = \frac{z(z^2+4z+1)}{(z-1)^4} \text{ for } z \in \boldsymbol{C} \setminus \{1\}; \quad \zeta(-3) = Li_{-3}(1) = \frac{1}{120} \ ;$$

these are the statements (4·16), which are now justified.

To conclude, we note a discontinuity of the type mentioned at the end of Appendix A: $\lim_{z\to 1} Li_{-3}(z) = \infty \neq Li_{-3}(1)$. This is basically due to the jump in the order of the pole (from 4 to 5) when z goes to 1.

References

1) S. K. Blau, M. Visser and A. Wipf, Nucl. Phys. B **310** (1988), 163.
2) E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications* (World Scientific, Singapore, 1994).
3) S. W. Hawking, Commun. Math. Phys. **55** (1977), 133.
4) R. M. Wald, Commun. Math. Phys. **70** (1979), 221.
5) V. Moretti, Phys. Rev. D **56** (1997), 7797.
6) D. Iellici and V. Moretti, Phys. Lett. B **425** (1998), 33.
7) V. Moretti, Commun. Math. Phys. **201** (1999), 327.
8) A. A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, *Analytic aspects of quantum fields* (World Scientific, 2003).
9) D. Deutsch and P. Candelas, Phys. Rev. D **20** (1979), 3063.
10) S. A. Fulling, *Aspects of quantum field theory in curved space time* (Cambridge University Press, 1989).
11) K. A. Milton, *The Casimir effect - Physical manifestations of zero-point energy* (World Scientific, 2001).
12) G. Esposito, G. M. Napolitano and L. Rosa, Phys. Rev. D **77** (2008), 105011, arXiv:0803.0861v1.
13) J. Schwinger, *Particles, sources, and fields* (Perseus Books, Reading, Massachusets, 1998).
14) L. E. Parker and D. J. Toms, *Quantum field theory in curved spacetime* (Cambridge University Press, 2009).
15) L. S. Brown and G. J. Maclay, Phys. Rev. **184** (1969), 1272.
16) S. M. Christensen, Phys. Rev. D **14** (1976), 2490.
17) F. W. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of mathematical functions* (Cambridge University Press, 2010).
18) A. Jonquière, Bull. Soc. Math. France **17** (1889), 142, available at `http://www.numdam.org/`
19) A. Romeo and A. A. Saharian, J. of Phys. A **35** (2002), 1297.