An explicit threshold for the appearance of lift on the deck of a bridge

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Abstract

We set up the analytical framework for studying the threshold for the appearance of a *lift force* exerted by a viscous steady fluid (the wind) on the deck of a bridge. We model this interaction as in a wind tunnel experiment, where at the inlet and outlet sections the velocity field of the fluid has a *Poiseuille flow* profile. Since in a symmetric configuration the appearance of lift forces is a consequence of non-uniqueness of solutions, we compute an explicit threshold on the incoming flow ensuring uniqueness. This requires building an explicit solenoidal extension of the prescribed Poiseuille flow and bounding some embedding and cutoff constants.

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Keywords: Poiseuille flow, lift force, threshold of stability.

1 Introduction

The *lift force* is the component of the total force exerted by the fluid over an obstacle which is perpendicular to the stream, see (e.g.) the Introduction in [1] and an updated state-of-the art in [17]. Since the airplane flight is based on lift, improving the lift characteristics of aircrafts is highly desirable. Instead, if the fluid interacts with an obstacle representing a structure in civil engineering (e.g., a bridge or a skyscraper), the lift is an unpleasant factor of instability which needs to be avoided. In order to evaluate the lift force exerted on a designed structure, engineers usually exploit wind tunnel tests. We set up our model in the same context; we intend to compute an explicit threshold for the appearance of lift on the deck of a scaled bridge. The experiment is illustrated in Figure 1. In the left picture we sketch the wind tunnel with a bridge within it while the right picture is taken during a wind tunnel experiment at Politecnico di Milano: the appearance of vortices around the plate (deck) generates a lift due to the asymmetry of the vortex shedding.

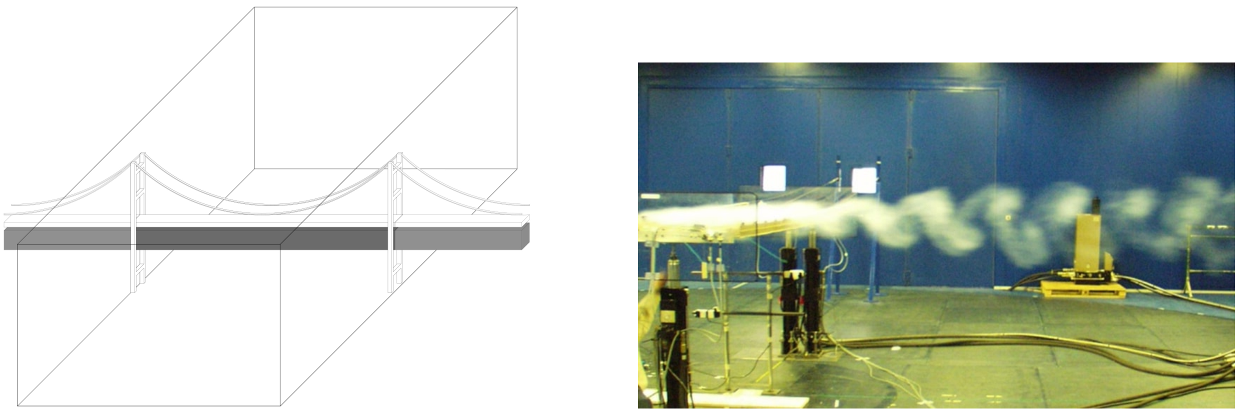

Figure 1: Left: sketch of a bridge within a wind tunnel. Right: wind tunnel experiment at Politecnico di Milano.

Denoting by Ω the 3D (non simply connected) domain consisting of a right parallelepiped (the wind tunnel) crossed by the plate, the fluid flow is assumed to be governed by the steady Navier-Stokes equations

$$-\mu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = 0 \qquad \nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^3$ is the unknown velocity vector field, $p : \Omega \to \mathbb{R}$ is the scalar pressure, μ is the coefficient of kinematic viscosity. We emphasize that we do not consider the action of any external force, in agreement with the experimental set-up in a wind tunnel where the flow is driven only by the inflow conditions. These conditions usually reproduce a *Poiseuille flow* profile, which we indicate with q, at the inlet and outlet sections. This leads to the following non-homogeneous boundary conditions, that we associate to (1.1),

$$u = q \quad \text{on } \partial T \qquad u = 0 \quad \text{on } \partial K; \tag{1.2}$$

here ∂T represents the boundary of the parallelepiped (tube) while ∂K represents boundary of the crossing plate. The velocity profile for the Poiseuille flow through a rectangular section was first derived by Boussinesq [6] and it correctly reproduces the (imposed) inflow and outflow conditions in a wind tunnel, if the latter is sufficiently long with respect to the characteristic length of the bridge (see, e.g. [5, Figure 11]), which justifies assuming the reorganization of the flow past the obstacle for sufficiently low Reynolds numbers.

Analyzing the well-posedness of (1.1)-(1.2), in order to obtain *explicit* bounds for the uniqueness of its solutions, is the main purpose of the present work; see Section 2 that contains the main result of the paper, Theorem 2.1. It turns out that, under symmetry assumptions on both the domain Ω and the boundary conditions (1.2), the obstacle K may suffer the action of a lift force exerted by the fluid only in presence of multiplicity of solutions. In Section 2 we also provide quantitative bounds on the Poiseuille flow for the occurrence of lift on the deck of some bridge models which were tested in the wind tunnel at Politecnico di Milano.

The subsequent sections are devoted to the proof of Theorem 2.1, which is organized in several steps. Section 3 is devoted to computing the bounds for the quantities that appear in the estimates needed for uniqueness. We then construct a suitable solenoidal extension of the Poiseuille flow q, which overcomes the presence of non-homogeneous boundary conditions in the problem and of the obstacle K pulled out from the domain. This enables us to obtain the sought estimates in Section 3.2. Then, in Section 3.3 we derive bounds for the Sobolev constants involved in the problem. Finally, in Section 4 we conclude the proof of Theorem 2.1: we prove existence and uniqueness for solutions of (1.1)-(1.2) and give a bound for uniqueness, which is fully explicit in view of the information derived in the previous sections. Proposition 4.3 aggregates all considerations to give an explicit expression for the bound which induces the appearance of a lift force over the obstacle.

Nowadays, computers are extremely precise but the importance of having *explicit theoretical bounds* remains unchanged. In the case of a suspension bridge subject to the wind, several different thresholds need to be compared in order to understand which phenomenon first triggers the instability; besides the appearance of the lift force (as in the present paper), one is also interested in thresholds for hangers slackening [16], and in the appearance of the so-called aerodynamic flutter [2]. The exact (or, at least, the explicit) value of the thresholds in general problems from mathematical physics is well-explained in the celebrated monograph by Pólya-Szegö [28], in particular for problems related to the electrostatic capacity, to the torsional rigidity, and to the principal frequency of a body; several further geometric inequalities are contained in the monograph. The techniques vary from symmetrization methods to a priori bounds and functional inequalities. These tools are also used in shape optimization problems [19] and in equimeasurable rearrangements of real-valued functions [29], both in calculus of variations and in partial differential equations. And variational problems within PDE's, such as the ∞-Laplacian, turn out to be extremely powerful in bounding solenoidal extensions for non-homogeneous boundary value problems in Navier-Stokes equations [13]. In this paper we derive bounds for some Sobolev embedding constants in a non simply connected 3D domain, a topic that is already quite involved in 2D domains [18]. Moreover, we need a precise bound on the solenoidal extension which is used to get rid of the non-homogeneous boundary condition. Bounds for solenoidal extensions are also needed in different areas of mathematical physics: a whole bunch of inequalities arises both in fluid mechanics and elasticity [4, 7, 14, 21, 23], and they are all linked to each other. Our approach and bounds may also be fruitfully employed for these problems.

2 Main result: appearance of the lift

We consider a steady fluid filling a three-dimensional cylindrical domain T which contains an obstacle K

$$T = (-L, L) \times \omega, \quad \omega = (-1,1) \times (-d,d), \quad K = (-l,l) \times (-1,1) \times (-h,h), \qquad \Omega = T \setminus \bar{K}. \tag{2.1}$$

The cross section of the cylinder is $\omega = (-1,1) \times (-d,d)$, L is the length of the cylinder and the obstacle K represents the deck of a bridge ($l < L$, $h < d$). The region of the flow is the domain Ω, see Figure 2 for two lateral views of Ω.

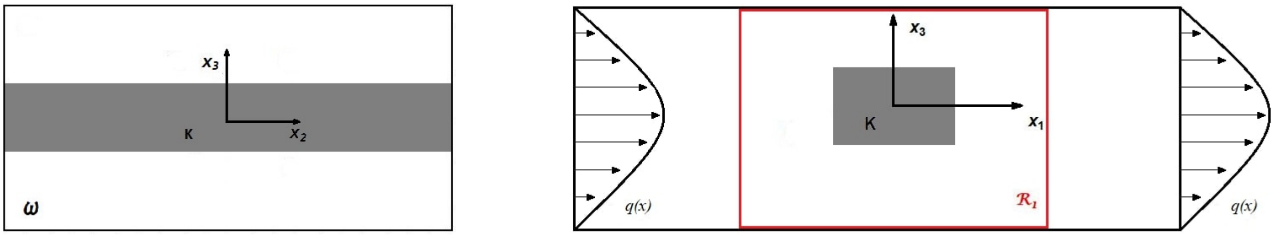

Figure 2: Left: rectangular cross-section ω of the cylinder T. Right: Poiseuille inflow-outflow.

The cylinder T (the space occupied by the wind tunnel) has a rectangular cross-section ω, see the left picture in Figure 2. This is the usual shape of a wind tunnel. The obstacle K has the section $(-l,l) \times (-h,h)$ on the plane x_1x_3, while the x_2-coordinate is confined in $(-1,1)$. Note that $\Omega = T \setminus \bar{K}$ *is not* simply connected. Around the cross-section of the obstacle we construct a "technical rectangle" $\mathcal{R}_1$ where the cut-off function will be supported.

At the inlet and outlet sections of the cylinder the flow is of Poiseuille-type, namely a unidirectional flow along the axis of the channel and defined on the rectangular cross-section ω; to this end, we define the function

$$g(x_2,x_3) = \Big[1 - \frac{x_3^2}{d^2} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^3} \frac{\cosh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})} \cos(\frac{\alpha_k x_3}{d})\Big] \qquad \forall (x_2,x_3) \in \omega\,, \tag{2.2}$$

where $\alpha_k = (2k-1)\frac{\pi}{2}$ ($k = 1, 2...$). In the boundary conditions (1.2), q is the profile of a Poiseuille flow, see the right picture in Figure 2. More precisely, we take

$$q(x) = \{v_1(x_2,x_3), 0, 0\} \quad \text{with} \quad v_1(x_2,x_3) = k_p \, \frac{g}{\|\nabla g\|_{L^2(\omega)}} \qquad \text{so that} \quad \|\nabla q\|_{L^2(\omega)} = k_p\,, \tag{2.3}$$

see Figure 3 for the plot; hence, the magnitude of the inflow is measured by the parameter $k_p = -\frac{1}{2\mu}\frac{\partial P}{\partial x_1} d^2 > 0$, the flow itself being driven by a (constant and negative) pressure drop $\frac{\partial P}{\partial x_1} < 0$.

We consider the space of vector fields vanishing only on the boundary of the obstacle

$$H^1_*(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \partial K\},$$

and two functional spaces of solenoidal vector fields

$$V_*(\Omega) = \{\phi \in H^1_*(\Omega) \,|\, \nabla \cdot \boldsymbol{\phi} = 0 \text{ in } \Omega\} \qquad V(\Omega) = \{\phi \in H^1_0(\Omega) \,|\, \nabla \cdot \boldsymbol{\phi} = 0 \text{ in } \Omega\}. \tag{2.4}$$

Note that if $u \in V_*(\Omega)$ satisfies (1.2), then its trace $u|_{\partial\Omega}$ is continuous. Then we introduce the standard trilinear form

$$\psi\,(u,v,w) = \int_\Omega (u \cdot \nabla) v \cdot w, \tag{2.5}$$

which is continuous in $H^1_*(\Omega) \times H^1_*(\Omega) \times H^1_*(\Omega)$, see e.g. [15, Lemma IX.1.1]). These tools enable us to define weak solutions of (1.1)-(1.2).

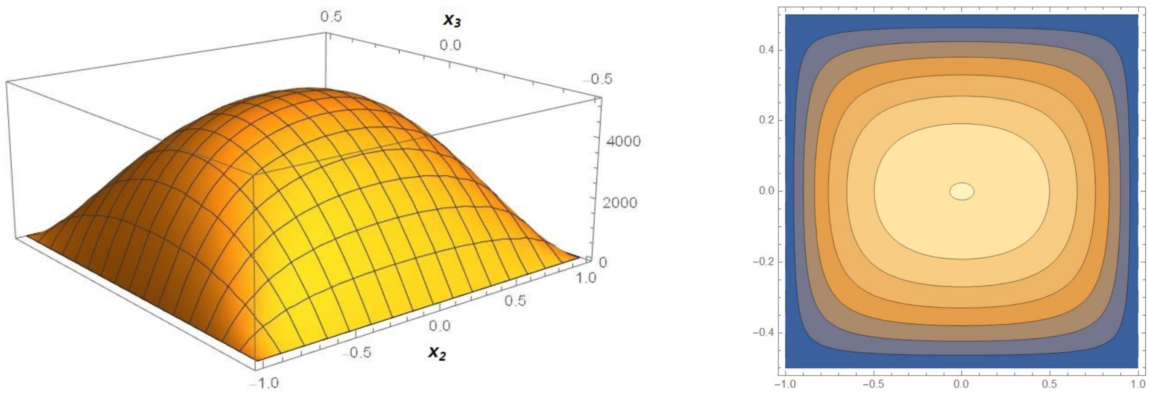

Figure 3: Profile of the Poiseuille flow through a rectangular parallelepiped, together with its velocity contours. The rectangular cross section is $(-1,1)\times(-0.5,0.5)$, the value of the parameter k_p is chosen to be $k_p \approx 0.84 \cdot 10^5$

Definition 2.1. Let Ω be as in (2.1). Given q as in (2.3), so that $q \in W^{1,\infty}(\partial T)$, a vector field $u : \Omega \to \mathbb{R}^3$ is called a weak solution to (1.1)-(1.2) if $u \in V_*(\Omega)$ satisfies (1.2) in the trace sense and

$$\mu(\nabla u, \nabla \phi)_{L^2(\Omega)} + \psi(u,u,\phi) = 0 \qquad \forall\, \phi \in V(\Omega). \tag{2.6}$$

Let us now define rigorously what is meant by *lift force* in this context. The stress tensor of an incompressible viscous fluid, whose velocity and pressure fields obey to the three-dimensional Navier-Stokes equations (1.1), is expressed through the following 3×3 matrix (see [26, Chapter 2])

$$\mathbf{T} = -p\mathbf{I} + \mu[\nabla u + (\nabla u)^T], \tag{2.7}$$

which combines the action of both the pressure p and the shear forces. In (2.7), $\mathbf{I}$ is the 3×3 - identity matrix. Hence, according to (2.1), the force exerted by the fluid over the obstacle K is

$$F_K = -\int_{\partial K} \mathbf{T}\cdot \hat{n}\, ds$$

where $\hat{n}$ is the outward unit normal to Ω, therefore directed towards the interior of K. But since we merely deal with weak solutions of (1.1)-(1.2), we need to weaken this definition and, as in [17, Definition 3.3], to redefine F_K by:

$$F_K = -\left\langle \mathbf{T}\cdot \hat{n}, 1\right\rangle_{\partial K}, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_{\partial K}$ is the duality between $W^{-\frac{2}{3},\frac{3}{2}}(\partial K)$ and $W^{\frac{2}{3},3}(\partial K)$. Accordingly, since the inflow velocity (2.3) only has the first component, if $\hat{k}$ denotes the unit vector along x_3, then the lift force exerted by the fluid on the obstacle K is

$$\mathcal{L}_K = F_K \cdot \hat{k}. \tag{2.9}$$

We can now state our main result :

Theorem 2.1. *Let Ω be as in* (2.1) *and q as in* (2.3)*, so that $q \in W^{1,\infty}(\partial T)$. For any $k_p > 0$, there exists a weak solution $(u,p) \in V_*(\Omega)\times L^2(\Omega)$ of* (1.1)-(1.2)*. Moreover, there exists $\bar{k}_p = \bar{k}_p(\mu, L, d, l, h)$ such that, if*

$$0 < k_p < \bar{k}_p(\mu, L, d, l, h) \tag{2.10}$$

then the weak solution is unique. Hence, in order to observe a lift force over the obstacle, it must be $k_p > \bar{k}_p$.

Theorem 2.1 deserves several **important comments** that explain its possible applications. The first statement does not come unexpected, existence and uniqueness for a non-homogeneous problem such as (1.1)-(1.2) usually hold under smallness assumptions on the data. The main novelty of Theorem 2.1 is the second statement since it allows to explicitly compute a threshold of stability for the obstacle K in terms of the flux at the inlet and outlet sections of the wind tunnel. This is possible because we are considering a symmetric framework, both for the domain and the boundary condition. We can give a quantitative form to $\bar{k}_p$, after determining its dependence on the physical parameters μ, L, d, l, h; the explicit form of $\bar{k}_p$ is given in (4.8), (see also Proposition 4.3). We emphasize that our purpose is not to determine the optimal (largest) value of $\bar{k}_p$; instead, we aim to provide an effective method to obtain an explicit expression for $\bar{k}_p$ yielding a *quantitative sufficient condition* for uniqueness of solutions of (1.1)-(1.2).

We now determine some numerical values of $\bar{k}_p$ computed through the final formula (4.8). We take real geometrical data from the online database [22] by referring to few experiments which took place in the wind tunnel (GVPM) at Politecnico di Milano. The coefficient of kinematic viscosity is chosen to be the one of air $\mu = 1.5 \cdot 10^{-5}$. The first data are taken from the model of the Izmit bay bridge in Turkey, a 1:30 sectional model. The second data come from another Turkish bridge: the model of the Third Bosphorus bridge. Finally, we took data from the model of the Talavera de la Reina Cable-stayed bridge near Toledo, in Spain. All parameters are made dimensionless with respect to the characteristic length of the problem, half of the channel's width, coinciding with half of the obstacle's length. The results are summarized in the following table.

bridge model	L	d	l	h	$\bar{k}_p \times 10^6$
Izmit bay	5	0.555	0.072	0.011	6.242
Third Bosphorus	5	0.555	0.084	0.008	6.245
Talavera de la Reina Cable-stayed	5	0.555	0.041	0.003	6.258

From a theoretical point of view, we consider the height d, the size of the bridge and the viscosity μ as fixed data for the problem, and we discuss the dependence of $\bar{k}_p$ on L (the length of the wind tunnel). We give here some qualitative properties on the behavior of $\bar{k}_p$, derived from its explicit form given in (4.8). As L diminishes, $\bar{k}_p$ grows, by making condition (2.10) less restrictive. This is as expected since a short channel does not let the velocity of the fluid deviate from the field prescribed at the inlet and outlet sections; the unique solution would tend to resemble the imposed Poiseuille flow $q(x)$, also fairly close to the obstacle. On the other hand, as L increases, $\bar{k}_p = \bar{k}_p(L)$ diminishes and tends to an horizontal asympote when $L \to \infty$. In other words, provided that we impose a sufficiently weak flux at the inlet and outlet sections, uniqueness is guaranteed even for an arbitrarily long channel (see also Remark 2).

Although L could tend to infinity, we do not consider an infinitely long channel. This would not be physically meaningful since our problem models an experimental test in a wind tunnel and we would also snag on a mathematical issue; existence of solutions for (1.1)-(1.2) would not be guaranteed for any value of the parameter k_p (as we have when $L < +\infty$), but only for sufficiently small values. Indeed, the problem would resemble the so-called Leray's problem (see [15, XII, Introduction]), for which the question around unconditional existence of solutions is still an open issue.

Finally, we briefly discuss the regularity of the solutions of (1.1)-(1.2).

Remark 1. *Weak solutions of* (1.1)-(1.2) *are smooth in the interior of the domain* Ω*, defined as in* (2.1)*, see for instance [15, Theorem IX.5.1]. Regularity up to the boundary is by far more difficult; although the obstacle* K *has a Lipschitz boundary, it generates non-convex corners within* Ω*. The* H^2*-regularity can be obtained for a convex polyhedron-like domain: see [8, 10, 9], where the authors precisely considers this type of domain and [24], where regularity of the Navier-Stokes system in three-dimensional domains with conic points is studied. However, arbitrary domains of polyhedral types which may possess reentrant corners, as in the case that we are considering, do not allow to consider solutions exhibiting better regularity than the minimal* $H^1(\Omega)$*, see for instance [27].*

3 Explicit expressions and bounds

3.1 Determination of the solenoidal extension

Given g as in (2.2), let b_2 and b_3 be the following functions, defined over ω:

$$b_2(x_2,x_3) = -\frac{k_p}{3\,\|\nabla g\|_{L^2(\omega)}} x_3\Big[2 - \frac{x_3^2}{d^2} + 6\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^3}\frac{\cosh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})}\cos(\frac{\alpha_k x_3}{d})\Big] \tag{3.1}$$

$$b_3(x_2,x_3) = \frac{k_p}{3\,\|\nabla g\|_{L^2(\omega)}}\Big\{x_2 + 6\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^4}\frac{\sinh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})}\Big[x_3\alpha_k\sin(\frac{\alpha_k x_3}{d}) + d\cos(\frac{\alpha_k x_3}{d})\Big]\Big\}. \tag{3.2}$$

Observe that

$$\frac{\partial}{\partial x_2}\, b_3(x_2,x_3) - \frac{\partial}{\partial x_3}\, b_2(x_2,x_3) = v_1(x_2,x_3),$$

where $v_1(x_2,x_3)$ describes the velocity profile of the Poiseuille flow in (2.3). Hence, if we define the vector field

$$b(x) = \{0, b_2(x_2,x_3), b_3(x_2,x_3)\}, \tag{3.3}$$

we obtain that $\nabla\times(b(x)) = q(x)$, where $q(x)$ is as in (2.3).

We aim to build a function $a(x)$ that plays the role of a "flux carrier", *i.e.* a smooth solenoidal extension of the prescribed velocity field at the inlet and outlet sections, vanishing on $\partial\Omega$. We seek a function $a(x)$ equal to $q(x)$ far away from the obstacle and equal to zero in a neighbourhood of the obstacle. Hence, following the classical procedure by Ladyzhenskaya [25], we take

$$a(x) = \nabla\times\big(b(x)\,\theta(x)\big)\,, \tag{3.4}$$

where b is as in (3.3) and $\theta(x)$ is a C^1 cut-off function equal to 1 at all points of Ω far away from ∂K and to 0 near ∂K (we shall later specify what we mean by "near"). Clearly $a \in H^1(\Omega)$, a is solenoidal and it vanishes close to the obstacle whereas it coincides with q far away from it. In order to give the explicit expression of the solenoidal extension $a(x)$, from (3.4), we need to determine both $\theta(x)$ and $b(x)$.

We first proceed in building the cut-off function $\theta(x)$, merely depending on x_1 and x_3, whose profile is "specular" to a function supported in the rectangular region

$$\mathcal{R}_1 = \{(x_1,x_3) \in (-l-\alpha, l+\alpha)\times(-d,d)\} \tag{3.5}$$

that fully invades the domain Ω in the x_3-direction but not in the x_1-direction. In fact, the parameter α satisfies $0<\alpha<L-l$, is independent of L and is chosen so as to optimize the estimates for unique solvability of (1.1)-(1.2) while the same trick in the x_3-direction does not help because we numerically saw that this would not lead to better estimates. The rectangle $\mathcal{R}_1$, which contains the cross-section of the obstacle K (see the right picture in Figure 2), enables us to partition the domain Ω in (2.1) as follows

$$\begin{gathered}\Omega = \bigcup_{i=0}^{2}\Omega_i, \quad \Omega_0 = \mathcal{R}_1\times(-1,1)\setminus\bar{K}\\ \Omega_1 = \{x\in\mathbb{R}^3 : x_1 < -l-\alpha,\ (x_2,x_3)\in\omega\}, \qquad \Omega_2 = \{x\in\mathbb{R}^3 : x_1 > l+\alpha,\ (x_2,x_3)\in\omega\}\end{gathered} \tag{3.6}$$

We consider the functions

$$\theta_1(x_1) = \begin{cases} 1 & \text{if } x_1 < l \\ 0 & \text{if } x_1 > l+\alpha \\ \phi_1(x_1) & \text{otherwise}\end{cases}, \qquad \theta_2(x_3) = \begin{cases} 1 & \text{if } x_3 < h \\ 0 & \text{if } x_3 > d \\ \phi_2(x_3) & \text{otherwise}\end{cases},$$

with

$$\phi_1(x_1) = \frac{2\,x_1^3}{\alpha^3} - \frac{3x_1^2\,(\alpha+2l)}{\alpha^3} + \frac{6x_1\left(l^2+\alpha l\right)}{\alpha^3} - \frac{-\alpha^3+2l^3+3\alpha\, l^2}{\alpha^3}, \quad \phi_2(x_3) = -\frac{{x_3}^2}{(d-h)^2} + \frac{2h\,x_3}{(d-h)^2} + \frac{d(d-2h)}{(d-h)^2}.$$

Then we take

$$\theta(x_1,x_3) = 1 - \theta_1(x_1)\theta_1(-x_1)\theta_2(x_3)\theta_2(-x_3). \tag{3.7}$$

This function is represented in Figure 4.

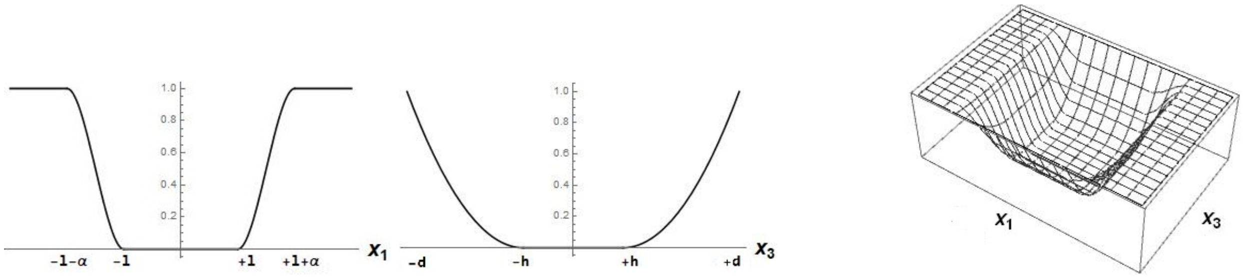

Figure 4: Left: restriction of θ to the x_1-axis and to the x_3-axis. Right: three-dimensional representation of θ.

The above construction enables us to state:

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be as in (2.1) and $q(x)$ as in (2.3). Let $a(x) = \nabla \times \big(b(x)\theta(x)\big)$, where $b(x)$ is as in (3.3) and $\theta(x)$ as in (3.7). Then, the vector field $a(x) \in H^1(\Omega)$ is such that*

$$\nabla \cdot a(x) = 0 \qquad a(x) = q(x) \quad \text{in } \Omega_i \qquad\quad a(x) = 0 \quad \text{on } \partial K. \tag{3.8}$$

with Ω_i, $i = 1,2$ defined in (3.6).

3.2 Bounds for the solenoidal extension

The aim of this subsection is to provide quantitative estimates for suitable norms of the solenoidal extension a, defined in Proposition 3.1, which will play a role in the uniqueness bound for solutions of (1.1)-(1.2). This bound will involve the L^4-norm of a and the L^2-norm of its gradient in the region Ω_0 defined by the partition (3.6); these are the quantities that we intend to estimate here.

To this end, we first prove some technical inequalities that involve the so-called Apéry constant [3]:

$$\zeta(3) = \frac{8}{7}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} \approx 1.202. \tag{3.9}$$

The next lemmas provide some estimates for the functions that we have introduced so far.

Lemma 3.1. *Let b_2 and b_3 be as in (3.1), (3.2). Then,*

$$|b_2(x_2,x_3)|^2 \leq \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}}\left(|2x_3 - \frac{x_3^3}{d^2}| + |x_3|\frac{42}{\pi^3}\,\zeta(3)\right)^2, \qquad |b_3(x_2,x_3)|^2 \leq \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}}\left(|x_2| + |x_3|\frac{42}{\pi^3}\,\zeta(3) + d\right)^2,$$

$$\left|\frac{\partial b_2(x_2,x_3)}{\partial x_2}\right|^2 \leq \frac{k_p^2}{d^2\,\|\nabla g\|^2_{L^2(\omega)}}\,x_3^2, \qquad \left|\frac{\partial b_2(x_2,x_3)}{\partial x_3}\right|^2 \leq \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}}\left(2 + 3\frac{x_3^2}{d^2} + \frac{42}{\pi^3}\,\zeta(3) + \frac{3|x_3|}{d}\right)^2,$$

$$\left|\frac{\partial b_3(x_2,x_3)}{\partial x_2}\right|^2 \leq \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}}\left(1 + \frac{42}{\pi^3}\,\zeta(3) + \frac{3|x_3|}{d}\right)^2, \qquad \left|\frac{\partial b_3(x_2,x_3)}{\partial x_3}\right|^2 \leq \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}}\left(\frac{3|x_3|}{d}\right)^2.$$

The bounds in Lemma 3.1 are obtained with some computations, by using (3.9) and the convergence of the series

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8},\qquad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}.$$

Then, we provide some bounds for the cut-off function.

Lemma 3.2. *Let θ be as in* (3.7)*. Then, given the partition* (3.6)*, in the region Ω_0 it holds that*

$$|\theta(x_1,x_3)|\le 1,$$

$$\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\le\frac{36(x_1-l)^2(d-x_3)^2(d-2h+x_3)^2(l-x_1+\alpha)^2}{\alpha^6(d-h)^4},\quad \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\le\frac{4(x_3-h)^2(l-x_1+\alpha)^4(2x_1-2l+\alpha)^2}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\le\frac{36(d-x_3)^2(d-2h+x_3)^2(2x_1-2l-\alpha)^2}{\alpha^6(d-h)^4},\quad \left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\le\frac{4(2x_1-2l+\alpha)^2(l-x_1+\alpha)^4}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3}\right|^2\le\frac{144(x_1-l)^2(x_3-h)^2(l-x_1+\alpha)^2}{\alpha^6(d-h)^4}.$$

Now we are ready to proceed. To begin with, we seek an upper bound for the L^4-norm of the solenoidal extension. We remark that all the integrals that we will encounter are well-defined, as we are considering smooth bounded functions over bounded domains.

Proposition 3.2. *Let $a=a(x)$ be the function defined in Proposition 3.1. Let Ω_0 be defined as in* (3.6)*. Let the constants δ_i, $i=1,..,24$ be defined as in the Appendix. Then*

$$\|a\|_{L^4(\Omega_0)}\le\Lambda_1 k_p,\qquad \|\nabla a\|_{L^2(\Omega_0)}\le\Lambda_2 k_p,$$

where Λ_1 and Λ_2 are defined by

$$\Lambda_1=\frac{\sqrt[4]{8}}{\|\nabla g\|_{L^2(\omega)}}\left\{\left[\delta_1+\delta_2+\delta_3+\left(\sqrt[4]{\delta_4}+\sqrt[4]{\delta_5}\right)^4+\left(\sqrt{2\delta_6}+\sqrt{2\delta_7}\right)^2+\left(\sqrt{2\delta_8}+\sqrt{2\delta_9}\right)^2\right]^{1/4}\right\},$$

$$\Lambda_2=\frac{\sqrt{8}}{\|\nabla g\|_{L^2(\omega)}}\left\{\left[\delta_{10}+\delta_{11}+(\sqrt{\delta_{12}}+\sqrt{\delta_{13}})^2+\delta_{14}+\delta_{15}+(\sqrt{\delta_{16}}+\sqrt{\delta_{17}})^2+(\sqrt{\delta_{18}}+\sqrt{\delta_{17}})^2+\right.\right.$$
$$\left.\left.(\sqrt{\delta_{19}}+\sqrt{\delta_{20}})^2+(\sqrt{\delta_{21}}+\sqrt{\delta_{22}}+\sqrt{\delta_{23}}+\sqrt{\delta_{24}})^2\right]^{1/2}\right\}.$$

Proof. The curl of the vector field $b(x)\theta(x)=\{0,b_2(x_2,x_3)\theta(x_1,x_3),b_3(x_2,x_3)\theta(x_1,x_3)\}$ reads

$$\nabla\times(b(x_2,x_3)\theta(x_1,x_3))=\left\{\theta(x_1,x_3)v_1(x_2,x_3)-b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3},-b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1},b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right\}.$$

The L^4-norm of this quantity involves both the square of each of the three components and the corresponding double products, as follows:

$$\|a\|^4_{L^4(\Omega_0)}=\|\nabla\times(b(x)\,\theta(x,\delta))\|^4_{L^4(\Omega_0)}=\int_{\Omega_0}\left|\nabla\times(b(x_2,x_3)\,\theta(x_1,x_3))\right|^4dx$$

$$=\int_{\Omega_0}\left(\left|\theta(x_1,x_3)v_1(x_2,x_3)-b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2+\left|b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2+\left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\right)^2dx.$$

The trinomial expansion gives six terms, that we estimate separately. The simplest terms can be estimated using Lemmas 3.1 and 3.2 (their use is hidden in the computation of the constants δ_i, but it does not appear explicitly here). Notice also that in the computation of the integrals we exploited the evenness of the function and the symmetries of the domain of integration:

$$\int_{\Omega_0} \left| b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^4 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_2, \qquad \int_{\Omega_0} \left| b_3(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^4 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_1,$$

$$\int_{\Omega_0} 2 \left| b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 \left| b_3(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_3,$$

while the remaining terms are estimated after an intermediate step which exploits the Minkowski inequality.

$$\begin{aligned}\int_{\Omega_0} \left| \theta(x_1,x_3) v_1(x_2,x_3) - b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_3} \right|^4 dx \le & \left[\left(\int_{\Omega_0} \left| \theta(x_1,x_3) v_1(x_2,x_3) \right|^4 dx \right)^{1/4} + \right. \\ & \left. \left(\int_{\Omega_0} b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_3} \right|^4 dx \right)^{1/4} \right]^4 \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}} \left(\sqrt[4]{\delta_4} + \sqrt[4]{\delta_5} \right)^4\end{aligned}$$

$$\begin{aligned}&\int_{\Omega_0} 2 \left| \theta(x_1,x_3) v_1(x_2,x_3) - b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_3} \right|^2 \left| b_3(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx \le \\ &2 \left[\left(\int_{\Omega_0} \left| \theta(x_1,x_3) v_1(x_2,x_3) b_3(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx \right)^{1/2} + \left(\int_{\Omega_0} \left| b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_3} b_3(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx \right)^{1/2} \right]^2 \le \\ &2 \cdot \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}} \left(\sqrt{\delta_6} + \sqrt{\delta_7} \right)^2\end{aligned}$$

$$\int_{\Omega_0} 2 \left| \theta(x_1,x_3) v_1(x_2,x_3) - b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_3} \right|^2 \left| b_2(x_2,x_3) \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx \le 2 \cdot \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}} \left(\sqrt{\delta_8} + \sqrt{\delta_9} \right)^2.$$

By combining these estimates, we obtain the L^4-bound.

The upper bound for the Dirichlet norm of a is obtained through the very same procedure, even though it turns out to be slightly more elaborate, since the gradient of the vector field $\nabla \times (b\,\theta)$ returns the following 3×3 matrix

$$\nabla(\nabla \times (b(x)\,\theta(x))) = \left\{ \nabla \left[\theta(x_1,x_3) v_1(x_2,x_3) - b_2(x_2,x_3) \tfrac{\partial \theta(x_1,x_3)}{\partial x_3} \right], \nabla \left[-b_3(x_2,x_3) \tfrac{\partial \theta(x_1,x_3)}{\partial x_1} \right], \nabla \left[b_2(x_2,x_3) \tfrac{\partial \theta(x_1,x_3)}{\partial x_1} \right] \right\}^T.$$

In order not to burden the writing of equations, we rewrite this term as

$$\nabla(\nabla \times (b(x)\,\theta(x))) = \left\{ \nabla A, \nabla B, \nabla C \right\}^T.$$

Thus we obtain

$$\|\nabla a\|^2_{L^2(\Omega_0)} = \|\nabla(\nabla \times (b(x)\theta(x)))\|^2_{L^2(\Omega_0)} = \|\nabla A\|^2_{L^2(\Omega_0)} + \|\nabla B\|^2_{L^2(\Omega_0)} + \|\nabla C\|^2_{L^2(\Omega_0)}.$$

We start estimating the second and third integral, which are slightly simpler. Analogously to what has been done before, we obtain, after having exploiting the Minkowski inequality and Lemmas 3.1 and 3.2,

$$\begin{aligned}\|\nabla B\|^2_{L^2(\Omega_0)} = & \int_{\Omega_0} \left| b_3(x_2,x_3) \frac{\partial^2 \theta(x_1,x_3)}{\partial x_1^2} \right|^2 dx + \int_{\Omega_0} \left| \frac{\partial b_3(x_2,x_3)}{\partial x_2} \frac{\partial \theta(x_1,x_3)}{\partial x_1} \right|^2 dx + \\ & \int_{\Omega_0} \left| -\frac{\partial b_3(x_2,x_3)}{\partial x_3} \frac{\partial \theta(x_1,x_3)}{\partial x_1} - b_3(x_2,x_3) \frac{\partial^2 \theta(x_1,x_3)}{\partial x_1 \partial x_3} \right|^2 dx \le \frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}} \left[\delta_{10} + \delta_{11} + (\sqrt{\delta_{12}} + \sqrt{\delta_{13}})^2 \right]\end{aligned}$$

$$\|\nabla C\|^2_{L^2(\Omega_0)} = \int_{\Omega_0} \left| b_2(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2} \right|^2 dx + \int_{\Omega_0} \left| \frac{\partial b_2(x_2,x_3)}{\partial x_2}\frac{\partial\theta(x_1,x_3)}{\partial x_1} \right|^2 dx +$$
$$\int_{\Omega_0} \left| \frac{\partial b_2(x_2,x_3)}{\partial x_3}\frac{\partial\theta(x_1,x_3)}{\partial x_1} + b_2(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3} \right|^2 dx \le \frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}} \left[\delta_{14} + \delta_{15} + (\sqrt{\delta_{16}} + \sqrt{\delta_{17}})^2\right].$$

For what concerns the third integral, we obtain

$$\|\nabla A\|^2_{L^2(\Omega_0)} = \int_{\Omega_0} \left| v_1(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_1} - b_2(x_2,x_3)\tfrac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3} \right|^2 dx + \int_{\Omega_0} \left| \theta(x_1,x_3)\tfrac{\partial v_1(x_2,x_3)}{\partial x_2} - \tfrac{\partial b_2(x_2,x_3)}{\partial x_2}\tfrac{\partial\theta(x_1,x_3)}{\partial x_3} \right|^2 dx +$$
$$\int_{\Omega_0} \left| \theta(x_1,x_3)\tfrac{\partial v_1(x_2,x_3)}{\partial x_3} + v_1(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_3} - \tfrac{\partial b_2(x_2,x_3)}{\partial x_3}\tfrac{\partial\theta(x_1,x_3)}{\partial x_3} - b_2(x_2,x_3)\tfrac{\partial^2\theta(x_1,x_3)}{\partial x_3^2} \right|^2 dx \le$$
$$\frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}} \left[(\sqrt{\delta_{18}} + \sqrt{\delta_{17}})^2 + (\sqrt{\delta_{19}} + \sqrt{\delta_{20}})^2 + (\sqrt{\delta_{21}} + \sqrt{\delta_{22}} + \sqrt{\delta_{23}} + \sqrt{\delta_{24}})^2\right].$$

Thus, if we blend these bounds, we obtain the claimed estimate for the Dirichlet norm of a. □

3.3 Bounds for the Sobolev embedding constants

We preliminarily remark that a slight modification of the procedure developed in [17, Theorem 2.2] yields

$$\sigma_* \|u\|^2_{L^4(\omega)} \le \|\nabla u\|^2_{L^2(\omega)} \qquad \text{with} \ \ \sigma_* = \sqrt{3}\left(\frac{\pi}{2}\right)^{3/2} \frac{\sqrt{1+d^2}}{d} \qquad \forall\, u \in H^1_0(\omega). \tag{3.10}$$

Note that σ_* provides a lower bound for the Sobolev constant σ_0 of the embedding $H^1_0(\omega) \subset L^4(\omega)$ in the 2D-rectangle ω, see (2.1), defined by

$$\sigma_0 = \min_{v\in H^1_0(\omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\omega)}}{\|v\|^2_{L^4(\omega)}}. \tag{3.11}$$

This section is devoted to computing an explicit lower bound $\mathcal{S}_*$ for the Sobolev constant

$$\mathcal{S}_0 = \min_{v\in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^4(\Omega)}} \tag{3.12}$$

for the (compact) embedding $H^1_0(\Omega) \subset L^4(\Omega)$, appearing in the estimate ensuring uniqueness for (1.1)-(1.2). A more significant modification of [17, Theorem 2.3] allows to find a constant Γ^*, as small as possible, satisfying

$$\|v\|^2_{L^4(\Omega)} \le \Gamma^* \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall\, v \in H^1_0(\Omega),$$

so that $\mathcal{S}_0 \ge \mathcal{S}_* = 1/\Gamma^*$. We emphasize that Γ^* is sought as small as possible, in order to obtain less restrictive conditions ensuring uniqueness of the solutions of (1.1). In order to obtain an explicit form for $\mathcal{S}_*$, we need the following bound for the Poincaré constant.

Lemma 3.3. *Let Ω be the domain in (2.1). For any scalar function $w \in H^1_0(\Omega)$, one has*

$$\|w\|_{L^2(\Omega)} \le \min\left\{ \frac{2}{\pi}\frac{Ld}{(d^2+L^2+L^2d^2)^{1/2}}\, ,\ \sqrt[3]{\frac{6\,(Ld-lh)}{\pi^4}} \right\} \|\nabla w\|_{L^2(\Omega)}. \tag{3.13}$$

Proof. We start by proving the first bound in (3.13). The least eigenvalue $\lambda_1 > 0$ of $-\Delta$ in T under homogeneous Dirichlet boundary conditions is given by

$$\lambda_1 = \frac{\pi^2}{4\,L^2} + \frac{\pi^2}{4\,d^2} + \frac{\pi^2}{4} = \frac{\pi^2}{4}\frac{d^2 + L^2 + L^2 d^2}{L^2 d^2},$$

as it is associated with the eigenfunction $\cos(\frac{\pi x_1}{2L})\cos(\frac{\pi x_2}{2})\cos(\frac{\pi x_3}{2d})$. Hence, the Poincaré inequality returns

$$\|w\|^2_{L^2(T)} \le \frac{4}{\pi^2}\frac{L^2 d^2}{d^2 + L^2 + L^2 d^2}\,\|\nabla w\|^2_{L^2(T)} \qquad \forall\, w \in H^1_0(T).$$

Since any function in $H^1_0(\Omega)$ can be extended by 0 in K, thereby becoming a function in $H^1_0(T)$, this inequality proves the first bound in (3.13).

For the second bound in (3.13) we invoke the Faber-Krahn inequality (see [28]) which states that

$$\min_{w \in H^1_0(\Omega)} \frac{\|\nabla w\|_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}} \ge \min_{w \in H^1_0(\Omega^*)} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}},$$

where Ω^* is a ball having the same volume as Ω. In order to compute the right-hand side in this inequality, we recall that the Poincaré constant in the unit sphere is given by π, which corresponds to the first zero of the spherical Bessel function of order 0, $\frac{\sin x}{x}$. Then, in the ball Ω^*, it holds that

$$\min_{w \in H^1_0(\Omega^*)} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}} = \frac{\pi}{R},$$

where R is the radius of this ball

$$R = \sqrt[3]{\frac{6\,(Ld - lh)}{\pi}},$$

and we used the fact that $|\Omega| = |T| - |K| = 8\,(Ld - lh)$. Therefore we obtain the second bound in (3.13). □

Note that equality between the two upper bounds in (3.13) occurs whenever

$$\frac{4\pi}{3} = (Ld - lh)\left(1 + \frac{1}{L^2} + \frac{1}{d^2}\right)^{3/2};$$

therefore which bound is better depends on the relative size of the obstacle K within T. Now, we are ready to prove:

Proposition 3.3. *For any $v \in H^1_0(\Omega)$, one has*

$$\|v\|^2_{L^4(\Omega)} \le \min\left\{\frac{Ld}{\pi^3(d^2 + L^2 + L^2 d^2)^{1/2}}\,,\ \sqrt[3]{\frac{3\,(Ld - lh)}{4\,\pi^{10}}}\right\}^{1/2} \|\nabla v\|^2_{L^2(\Omega)} \tag{3.14}$$

This inequality holds both for scalar functions and vector-valued functions.

Proof. We first prove (3.14) for scalar functions w for which del Pino-Dolbeault [11, Theorem 1] obtained the following optimal Gagliardo-Nirenberg inequality in $\mathbb{R}^3$:

$$\|w\|^2_{L^4(T)} \le \frac{1}{2^{1/3}\,\pi^{2/3}}\,\|\nabla w\|_{L^2(T)}\|w\|_{L^3(T)} \quad \forall w \in H^1_0(T). \tag{3.15}$$

Here we exploit the fact that functions in $H^1_0(T)$ can be extended by zero outside T becoming functions defined on the whole space $\mathbb{R}^3$. To get rid of the L^3-norm, we use the Hölder inequality

$$\|w\|^3_{L^3(T)} = \int_T |w||w|^2\,dx \le \|w\|_{L^2(T)}\|w\|^2_{L^4(T)}$$

which, combined with (3.15), gives

$$\|w\|^2_{L^4(T)} \le \frac{1}{\sqrt{2}\,\pi}\ \|\nabla w\|^{3/2}_{L^2(T)}\,\|w\|^{1/2}_{L^2(T)}. \tag{3.16}$$

Next, we estimate the term $\|w\|^{1/2}_{L^2(T)}$ through Lemma 3.3. Using (3.13) within (3.16) leads to (3.14) (for scalar functions) since $H^1_0(\Omega) \subset H^1_0(T)$.

Once we have obtained (3.14) for scalar functions, we claim that it also holds for vector-valued functions. Indeed, let $v = (v_1, v_2, v_3) \in H^1_0(\Omega)$; then, applying the Minkowski inequality, we can consider the L^4-norm of each component of this function individually and we use (3.14) as follows

$$\begin{aligned}\|v\|^4_{L^4(\Omega)} &\le \left(\|v_1\|^2_{L^4(\Omega)} + \|v_2\|^2_{L^4(\Omega)} + \|v_3\|^2_{L^4(\Omega)}\right)^2 \\ &\le \min\left\{\frac{Ld}{\pi^3(d^2+L^2+L^2d^2)^{1/2}}\,,\ \sqrt[3]{\frac{3\,(Ld-lh)}{4\,\pi^{10}}}\right\}\left(\|\nabla v_1\|^2_{L^2(\Omega)} + \|\nabla v_2\|^2_{L^2(\Omega)} + \|\nabla v_3\|^2_{L^2(\Omega)}\right)^2 \\ &= \min\left\{\frac{Ld}{\pi^3(d^2+L^2+L^2d^2)^{1/2}}\,,\ \sqrt[3]{\frac{3\,(Ld-lh)}{4\,\pi^{10}}}\right\}\|\nabla v\|^4_{L^2(\Omega)}.\end{aligned}$$

This proves (3.14) also for vector fields in $H^1_0(\Omega)$. □

We point out that (3.16) significantly improves the usual interpolation inequalities in fluid mechanics [15, (II.3.10)]. Proposition 3.3 yields the lower bound for the Sobolev constant $\mathcal{S}_0$, defined in (3.12), which reads as

$$\mathcal{S}_0 \ge \mathcal{S}_* = \max\left\{\pi^3\left(1+\frac{1}{L^2}+\frac{1}{d^2}\right)^{1/2},\ \sqrt[3]{\frac{4\,\pi^{10}}{3\,(Ld-lh)}}\right\}^{1/2}. \tag{3.17}$$

Remark 2. *Once we have fixed the height d of the wind tunnel and the size of the obstacle K by choosing l and h, there exists a critical threshold L^* where the two bounds within the maximum in* (3.17) *coincide. If $L < L^*$ then $\mathcal{S}_*(L)$ equals the second bound in* (3.17) *while if $L > L^*$ then $\mathcal{S}_*(L)$ equals the first bound in* (3.17) *and, therefore,*

$$\lim_{L\to\infty} \mathcal{S}_*(L) = \pi^{3/2}\left(1+\frac{1}{d^2}\right)^{1/4}.$$

The existence of an horizontal asympote for the function $\mathcal{S}_*(L)$ is **particularly significant** *in terms of uniqueness of solutions for the problem* (1.1)-(1.2)*. Indeed, $\mathcal{S}_*$ is part of the expression of $\overline{k}_p$ which determines the condition for uniqueness of solutions (see Theorem 2.1) and is the only parameter in this expression depending on L; however, since $\mathcal{S}_*$ loses this dependence by virtue of the presence of such an asymptote, we infer that L does not play a direct role in terms of uniqueness of the solution. Figure 5 shows the behaviour of the map $L \mapsto \mathcal{S}_*(L)$ with the parameters from the model of the Izmit bay bridge (*$d = 0.555$, $h = 0.011$, $l = 0.072$*), see Section 2; in this case, $L^* \approx 0.0014$.*

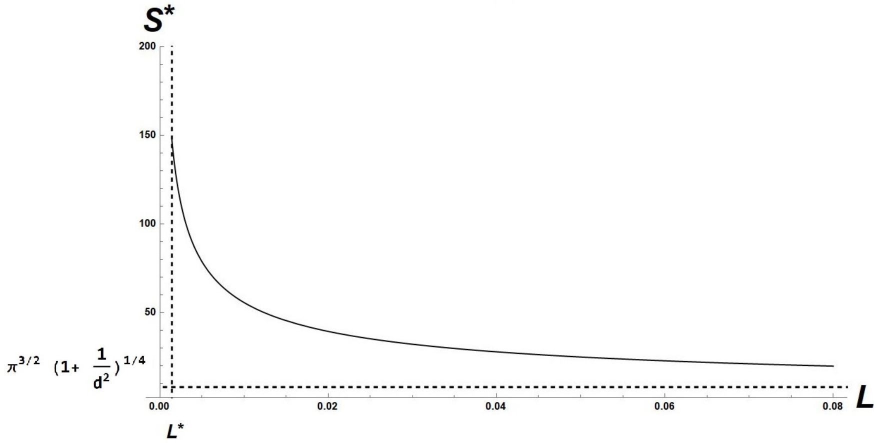

Figure 5: Graph of the function $L \mapsto \mathcal{S}_*(L)$ with the parameters from the model of the Izmit bay bridge.

4 Proof of Theorem 2.1

4.1 Existence and uniqueness

The idea of the proof is quite standard but, for our purposes, it is mandatory to fully report it since we need to emphasize the role played by each of the constants appearing in the a priori estimates.

In order to prove existence of a weak (or *generalized*) solution of (1.1)-(1.2) we look for velocity fields of the form

$$u = \hat{u} + s \tag{4.1}$$

where s is a sufficiently smooth *general solenoidal extension* of the prescribed velocity at the boundary q, which reproduces a Poiseuille-flow, while $\hat{u}$ (weakly, see Definition 2.1) solves the following problem:

$$-\mu\,\Delta\hat{u} + (\hat{u}\cdot\nabla)\,\hat{u} + (\hat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\hat{u} + \nabla p = f := \mu\,\Delta s - (s\cdot\nabla)\,s, \quad \nabla\cdot\hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u} = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

It is well-known (see for instance [15, Theorem IX.4.1]) that the existence of a solution follows once we find a uniform bound on $\|\nabla\hat{u}\|_{L^2(\Omega)}$ depending only on the data. On the other hand, uniqueness of solutions relies on some *a priori* bound from the data, this is why we need the following statement.

Lemma 4.1. *Let Ω be as in* (2.1) *and $q \in W^{1,\infty}(\partial T)$ as in* (2.3)*. Let $\hat{u}$ be a weak solution of* (4.2) *defined as in* (4.1)*. If $\mathcal{S}_*$ is as in* (3.17)*, σ_* as in* (3.10)*, the constants Λ_i $(i = 1, 2)$ as in Proposition 3.2, and if*

$$k_p < \frac{\mu\,\sigma_*}{\mathcal{S}_* + \Lambda_2\sigma_*}, \tag{4.3}$$

then the following a priori estimate holds:

$$\|\nabla\hat{u}\|_{L^2(\Omega)} \leq \frac{\mu\,\Lambda_2 k_p + \Lambda_1 \frac{\Lambda_2}{\sqrt{\mathcal{S}_*}} k_p^2}{\mu - \frac{k_p}{\sigma_*} - \frac{k_p\Lambda_2}{\mathcal{S}_*}}. \tag{4.4}$$

Proof. Consider (4.2) where we substitute s with the *specific* solenoidal extension a from Proposition 3.1. Multiply (4.2) by $\hat{u}$ and integrate by parts over Ω and, recalling that $\hat{u} = 0$ on $\partial\Omega$, obtain

$$\mu\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \psi\,(\hat{u},\hat{u},\hat{u}) + \psi\,(\hat{u},a,\hat{u}) + \psi\,(a,\hat{u},\hat{u}) = \langle\, f,\hat{u}\rangle = -\mu(\nabla a,\nabla\hat{u})_{L^2(\Omega)} - \psi\,(a,a,\hat{u})$$

with ψ as in (2.5). The properties of ψ guarantee that the second and fourth terms on the left-hand side vanish:

$$\mu\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \psi\,(\hat{u}, a, \hat{u}) = \langle\, f, \hat{u}\rangle = -\mu(\nabla a, \nabla\hat{u})_{L^2(\Omega)} - \psi\,(a, a, \hat{u}). \tag{4.5}$$

For the right-hand side of this equation we first exploit the partition (3.6)

$$(\nabla a, \nabla\hat{u})_{L^2(\Omega)} = \int_{\Omega_0} \nabla a : \nabla\hat{u}\, dx + \int_{\Omega_1} \nabla a : \nabla\hat{u}\, dx + \int_{\Omega_2} \nabla a : \nabla\hat{u}\, dx$$

and we remark that

$$\int_{\Omega_1} \nabla q : \nabla\hat{u}\, dx = \int_{\Omega_1} \Delta q \cdot \hat{u}\, dx = -2\frac{k_p}{d^2\,\|\nabla g\|_{L^2(\omega)}} \int_{-L}^{-l-\alpha} \left[\int_\omega \hat{u}\cdot\hat{n}\, dx_2\, dx_3\right] dx_1 = 0$$

as $\hat{u}$ carries no flux being divergence-free on Ω. For the same reason, also the integral over Ω_2 vanishes and we obtain

$$\left|\mu(\nabla a, \nabla\hat{u})_{L^2(\Omega)}\right| = \mu\left|\int_{\Omega_0} \nabla a : \nabla\hat{u}\, dx\right| \le \mu\|\nabla a\|_{L^2(\Omega_0)}\|\nabla\hat{u}\|_{L^2(\Omega_0)} \le \mu\,\Lambda_2 k_p\|\nabla\hat{u}\|_{L^2(\Omega)}$$

where we used the bound in Ω_0 given in Proposition 3.2. Since $a = q$ in $\Omega_1 \cup \Omega_2$ by Proposition 3.1 and since $(q\cdot\nabla)q \equiv 0$, we have that

$$\left|\psi\,(a, a, \hat{u})\right| = \left|\int_{\Omega_0} (a\cdot\nabla)a\cdot\hat{u}\, dx\right| \le \|a\|_{L^4(\Omega_0)}\|\nabla a\|_{L^2(\Omega_0)}\|\hat{u}\|_{L^4(\Omega_0)}\,.$$

Using again Proposition 3.2 and collecting terms we may then bound the right hand side of (4.5) as

$$|\langle f, \hat{u}\rangle| \le \left(\mu\,\|\nabla a\|_{L^2(\Omega_0)} + \|a\|_{L^4(\Omega_0)}\frac{\|\nabla a\|_{L^2(\Omega_0)}}{\sqrt{\mathcal{S}_0}}\right)\|\nabla\hat{u}\|_{L^2(\Omega)} \le \left(\mu\,\Lambda_2 k_p + \frac{\Lambda_1\Lambda_2}{\sqrt{\mathcal{S}_*}}k_p^2\right)\|\nabla\hat{u}\|_{L^2(\Omega)}$$

where we also used (3.17).

On the other hand, since a obeys (3.8),

$$|\psi\,(\hat{u}, a, \hat{u})| \le \left|\int_{\Omega_1} (\hat{u}\cdot\nabla)q\cdot\,\hat{u}\, dx + \int_{\Omega_2} (\hat{u}\cdot\nabla)q\cdot\,\hat{u}\, dx + \int_{\Omega_0} (\hat{u}\cdot\nabla)a\cdot\hat{u}\, dx\right| \le \left(\frac{k_p}{\sigma_0} + \frac{\|\nabla a\|_{L^2(\Omega_0)}}{\mathcal{S}_0}\right)\|\nabla\hat{u}\|^2_{L^2(\Omega)}, \tag{4.6}$$

where we used the Hölder inequality together with the Sobolev inequalities in (3.11) and (3.12) as follows:

$$\left|\int_{\Omega_1} (\hat{u}\cdot\nabla)q\cdot\,\hat{u}\, dx\right| \le \int_{-L}^{-l-\alpha} \|\hat{u}\|_{L^4(\omega)}\|\nabla q\|_{L^2(\omega)}\|\hat{u}\|_{L^4(\omega)}\, dx_1 \le k_p\int_{-L}^{-l-\alpha} \frac{\|\nabla_{x'}\hat{u}\|^2_{L^2(\omega)}}{\sigma_0}\, dx_1 \le \frac{k_p}{\sigma_0}\|\nabla\hat{u}\|^2_{L^2(\Omega_1)},$$

where $x' = (x_2, x_3)$, while ∇ indicates the gradient with respect to the three variables (x_1, x_2, x_3). The integral over Ω_2 can be treated analogously. Finally, by exploiting the inequality $\|\nabla\hat{u}\|^2_{L^2(\Omega_1)} + \|\nabla\hat{u}\|^2_{L^2(\Omega_2)} \le \|\nabla\hat{u}\|^2_{L^2(\Omega)}$, we see that the left-hand side of (4.5) can be lower bounded by

$$-\frac{k_p}{\sigma_0}\|\nabla\hat{u}\|^2_{L^2(\Omega)} - \frac{\Lambda_2\, k_p}{\mathcal{S}_0}\|\nabla\hat{u}\|^2_{L^2(\Omega)} \le -\frac{k_p}{\sigma_0}\|\nabla\hat{u}\|^2_{L^2(\Omega)} - \frac{\|\nabla a\|_{L^2(\Omega_0)}}{\mathcal{S}_0}\|\nabla\hat{u}\|^2_{L^2(\Omega)} \le \psi\,(\hat{u}, a, \hat{u}),$$

where we used again the bounds in Proposition 3.2. At last, we exploit the inequalities (3.10) and (3.17) to obtain

$$-\frac{k_p}{\sigma_*}\|\nabla\hat{u}\|^2_{L^2(\Omega)} - \frac{\Lambda_2\, k_p}{\mathcal{S}_*}\|\nabla\hat{u}\|^2_{L^2(\Omega)} \le \psi\,(\hat{u}, a, \hat{u}). \tag{4.7}$$

By plugging (4.7) into (4.5) and dividing by $\|\nabla\hat{u}\|_{L^2(\Omega)}$, we obtain the bound (4.4), provided that (4.3) holds. $\square$

We are now in position to prove the following statement, whose major result is the form of the quantitative bound for uniqueness of solutions of (1.1)-(1.2); this bound will be used for the overall conclusion in Proposition 4.3 below.

Proposition 4.1. *Let Ω be as in* (2.1) *and $q \in W^{1,\infty}(\partial T)$ as in* (2.3)*. Then, there exists at least one weak solution u to problem* (1.1)-(1.2) *with corresponding $p \in L^2(\Omega)$.*

Moreover, if $\mathcal{S}_*$ is as in (3.17)*, σ_* as in* (3.10)*, the constants Λ_i $(i = 1, 2)$ as in Proposition 3.2, and*

$$k_p < \bar{k}_p := \mu\sigma_* \frac{2\mathcal{S}_* + \sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_* - \sigma_*\sqrt{(\sqrt{\mathcal{S}_*}\Lambda_1 + 2\Lambda_2)^2 + \frac{4\mathcal{S}_*\Lambda_2}{\sigma_*}}}{2\mathcal{S}_* + 2\sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_*}. \tag{4.8}$$

then the weak solution is unique.

Proof. Existence of u satisfying (2.6) follows from [15, Theorem IX.4.1], provided that we have an a priori bound on $\|\nabla\hat{u}\|_{L^2(\Omega)}$, where $\hat{u}$ solves (4.1)-(4.2) in a weak sense. Multiply (4.2) by $\hat{u}$ and integrate by parts over Ω: the two terms $\psi(\cdot, \hat{u}, \hat{u})$ vanish and we bound the right-hand side through the Hölder inequality and (3.12):

$$\mu\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \psi(\hat{u}, s, \hat{u}) \leq \left(\mu\|\nabla s\|_{L^2(\Omega)} + \|s\|_{L^4(\Omega)} \frac{\|\nabla s\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\right) \|\nabla\hat{u}\|_{L^2(\Omega)}. \tag{4.9}$$

We draw attention to the fact that s is here a sufficiently smooth *general* solenoidal extension of q; that is why its norms in (4.9) live on the whole domain Ω, rather than on a component of the partition (3.6).

The flux of q is null across the two connected components of the boundary $\partial\Omega = \partial K \cup \partial T$, i.e.

$$\int_{\partial T} q \cdot \hat{n} = \int_{\partial K} q \cdot \hat{n} = 0.$$

Hence, in view of [15, Lemma IX.4.2], there exists a Hopf extension [20], namely for any $\eta > 0$ there exists a solenoidal extension s satisfying

$$|\psi(v, s, v)| \leq \eta \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall v \in V(\Omega).$$

By choosing $\eta < \mu$ and plugging this bound into (4.9), we obtain the desired a priori bound on $\hat{u}$ ensuring existence of u satisfying (2.6) for any given value of $\mu > 0$. The existence of a pressure field $p \in L^2(\Omega)$ corresponding to the weak solution u follows, for instance, from [15, Lemma IX.1.2].

We now turn to uniqueness. Let us suppose that u_0 and u_1 are two weak solutions of (1.1)-(1.2). Define $w = u_0 - u_1$; it satisfies the following identity

$$\mu(\nabla w, \nabla\phi)_{L^2(\Omega)} + \psi(u_0, w, \phi) + \psi(w, u_1, \phi) = 0 \qquad \forall\, \phi \in V(\Omega)$$

where $V(\Omega)$ is defined in (2.4). Since $w \in V(\Omega)$, we may substitute ϕ with it and obtain

$$\mu\|\nabla w\|^2_{L^2(\Omega)} = -\psi(w, u_1, w).$$

Then, we obtain an upper bound for the right-hand side. If we define $u_1 = \hat{u}_1 + a$, where a is the *specific* solenoidal extension built in Proposition 3.1 (not the above Hopf extension), we can divide this member in two terms:

$$-\psi(w, u_1, w) = \psi(w, w, u_1) = \psi(w, w, \hat{u}_1 + a) = \psi(w, w, \hat{u}_1) + \psi(w, w, a). \tag{4.10}$$

For the first term, by applying the Hölder inequality, the Sobolev inequality in Ω and finally the lower bound for $\mathcal{S}_0$ in (3.12), labelled as $\mathcal{S}_*$, we deduce that

$$\begin{aligned} |\psi(w, w, \hat{u}_1)| &\leq \|w\|_{L^4(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|\hat{u}_1\|_{L^4(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}} \|\hat{u}_1\|_{L^4(\Omega)} \\ &\leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\mathcal{S}_0} \|\nabla\hat{u}_1\|_{L^2(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\mathcal{S}_*} \|\nabla\hat{u}_1\|_{L^2(\Omega)}\ . \end{aligned}$$

The second term in (4.10) can be treated similarly to (4.6), after using the property of the trilinear form ψ and both the lower bounds (3.10) and (3.17):

$$|\psi\,(w,w,a)| = |\psi\,(w,a,w)| \leq \left(\frac{k_p}{\sigma_0} + \frac{\|a\|_{L^4(\Omega_0)}}{\sqrt{\mathcal{S}_0}}\right)\|\nabla w\|^2_{L^2(\Omega)} \leq \left(\frac{k_p}{\sigma_*} + \frac{\|a\|_{L^4(\Omega_0)}}{\sqrt{\mathcal{S}_*}}\right)\|\nabla w\|^2_{L^2(\Omega)}\,.$$

By combining these bounds and using the result of Proposition 3.2 we infer

$$\mu\|\nabla w\|^2_{L^2(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}}\Big(\frac{\|\nabla \hat{u}_1\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} + \frac{k_p}{\sigma_*}\sqrt{\mathcal{S}_*} + \|a\|_{L^4(\Omega_0)}\Big) \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}}\Big(\frac{\|\nabla \hat{u}_1\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} + \frac{k_p}{\sigma_*}\sqrt{\mathcal{S}_*} + \Lambda_1\, k_p\Big)\,.$$

Then, provided that (4.3) holds, we insert the a priori bound (4.4) for the gradient of $\hat{u}_1$ and we obtain

$$\mu\|\nabla w\|^2_{L^2(\Omega)} \leq \|\nabla w\|^2_{L^2(\Omega)}\,\frac{-k_p^2\left(\frac{\mathcal{S}_*}{\sigma_*^2} + \frac{\sqrt{\mathcal{S}_*}\Lambda_1}{\sigma_*} + \frac{\Lambda_2}{\sigma_*}\right) + k_p\left(\sqrt{\mathcal{S}_*}\Lambda_1\mu + \Lambda_2\mu + \frac{\mathcal{S}_*\mu}{\sigma_*}\right)}{\mathcal{S}_*\,\mu - \frac{k_p}{\sigma_*}\mathcal{S}_* - k_p\Lambda_2}\,,$$

which implies $w = 0$ if the following condition holds:

$$-k_p^2\left(\frac{\mathcal{S}_*}{\mu\sigma_*^2} + \frac{\sqrt{\mathcal{S}_*}\Lambda_1}{\mu\sigma_*} + \frac{\Lambda_2}{\mu\sigma_*}\right) + k_p\left(\sqrt{\mathcal{S}_*}\Lambda_1 + 2\Lambda_2 + \frac{2\mathcal{S}_*}{\sigma_*}\right) < \mu\,\mathcal{S}_*.$$

This is a condition of negativity on a concave parabola as a function of k_p, which crosses the vertical axis in $-\mu\mathcal{S}_0$. Hence, it is fulfilled if k_p is less than the smallest between the two roots of the second-order polynomial, which reads as (4.8). Some tedious computations show that the right-hand side of inequality (4.8) is smaller than the right-hand side of inequality (4.3), thus (4.8) implies (4.3): this proves uniqueness. □

Remark 3. *Notice that $\bar{k}_p > 0$ since the denominator is strictly positive as well as the same can be easily check to go for the numerator, and it can be rewritten as*

$$\bar{k}_p = \mu\sigma_*\left(1 - \frac{\sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + \sigma_*\sqrt{(\sqrt{\mathcal{S}_*}\Lambda_1 + 2\Lambda_2)^2 + \frac{4\mathcal{S}_*\Lambda_2}{\sigma_*}}}{2\mathcal{S}_* + 2\sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_*}\right).$$

4.2 Threshold for the appearance of the lift

Before stating the main result of this section, in the spirit of [17] we recall the following implications for solutions of (1.1)-(1.2)

$$\text{uniqueness} \implies \text{symmetry} \implies \text{no } \textit{lift} \text{ exerted over } K.$$

It is a well-know experimental fact that lift vanishes whenever the obstacle is symmetric with respect to the angle of attack of the fluid; see e.g. [12, Figure 7.21]. We state here a small variant of [17, Proposition 4.1]:

Proposition 4.2. *Let Ω be as in (2.1) and $q \in W^{1,\infty}(\partial T)$ be as in (2.3). Let $u = (u_1, u_2, u_3) \in V_*(\Omega)$ be a weak solution of problem (1.1)-(1.2). Let $\mathcal{S}_*$ be as in (3.17), σ_* as in (3.10), the constants Λ_i, $i = 1, 2$ as in Proposition 3.2. Then also $w = (w_1, w_2, w_3) \in V_*(\Omega)$ defined by*

$$w_1(x_1,x_2,x_3) = u_1(x_1,x_2,-x_3) \qquad w_2(x_1,x_2,x_3) = -u_2(x_1,x_2,-x_3) \qquad w_3(x_1,x_2,x_3) = u_3(x_1,x_2,-x_3)$$

for a.e. $(x_1,x_2,x_3) \in \Omega$, solves (1.1)-(1.2) in a weak sense. Moreover, if (4.8) is valid, the weak solution of (1.1)-(1.2) is unique and it satisfies the symmetry property

$$u_1(x_1,x_2,x_3) = u_1(x_1,x_2,-x_3) \qquad u_2(x_1,x_2,x_3) = -u_2(x_1,x_2,-x_3) \qquad u_3(x_1,x_2,x_3) = u_3(x_1,x_2,-x_3)$$

for a.e. $(x_1,x_2,x_3) \in \Omega$

Proposition 4.2 stands because Ω is symmetric with respect to all three axes x_1, x_2, x_3 and because the boundary datum q is x_3-even in its first component v_1 and null in its other components. Proposition 4.2 then shows that uniqueness implies symmetry. Then, [17, Theorem 3.7] shows that symmetry implies no lift exerted on the obstacle K. We recall that we adopt a generalized definition for the lift force, (2.9), since we are considering weak solutions.

In view of [17, Theorem 3.7], we can state the following proposition, which embodies the explicit realisation of the purpose of this work.

Proposition 4.3. *Let Ω be as in* (2.1) *and $q \in W^{1,\infty}(\partial T)$ as in* (2.3). *Let δ_i, $i = 1, 2, .., 24$ be reported in the Appendix, where we also emphatized the dependence on the parameter α, used to define the rectangle $\mathcal{R}_1$ in* (3.5). *Let F_K be the total force exerted by the fluid over the obstacle K, given in* (2.8). *For any $k_p \geq 0$, there exists a weak solution $(u, p) \in V_*(\Omega) \times L^2(\Omega)$ of* (1.1)-(1.2).

Moreover, given the constants Λ_1 and Λ_2 in Proposition 3.2, given σ_* as in (3.10) *and $\mathcal{S}_*$ as in* (3.17), *if the parameter k_p, regulating the inlet and outlet flow, is such that $k_p < \bar{k}_p$ (see* (4.8)*), then the weak solution is unique and the fluid exerts no lift on the obstacle K, that is*

$$\left\langle \{-p\mathbf{I} + \mu[\nabla u + (\nabla u)^T]\} \cdot \hat{n}, 1 \right\rangle_{\partial K} = 0$$

It remains to show how to compute the constants δ_i, $i = 1, 2, .., 24$, depending on α (defining the region $\mathcal{R}_1$ in (3.5)), which come from the explicit estimates for the norms of the solenoidal extension $a(x)$ given in Proposition 3.2.

We computed the δ_i's with the software *Mathematica*, that was also used to compute the value of α maximizing $\bar{k}_p$, once we know the structural parameters of the problem, in particular for the table in Section 2. Since the computations are unpleasant, we give their explicit value in the Appendix.

Appendix: explicit values of the constants δ_i's

In the sequel, we report the values of the constants δ_i's used in Proposition 3.2. Notice that the domain of integration has been reduced exploiting the fact that the cut-off function $\theta(x)$ in (3.7) is equal to 0 in the region $I = \{(x_1, x_3) \in (-l, l) \times (-h, h)\}$.

$$\delta_1 = \int_l^{\alpha+l}\int_0^1\int_h^d \tfrac{1}{81}\left|\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 (d + x_2 + \tfrac{42\zeta(3)}{\pi^3} x_3)^4 dx_3 dx_2 dx_1 + \int_l^{\alpha+l}\int_0^1\int_0^h \tfrac{1}{81}\left|\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 (d + x_2 + \tfrac{42\zeta(3)}{\pi^3} x_3)^4 dx_3 dx_2 dx_1 =$$
$$\tfrac{8d^5}{70945875\pi^{12}\alpha^3(d-h)^8}(130d^6(466754400h^2\zeta(3)^4+1764\pi^6(4554h^2-2145h+128)\zeta(3)^2+1386\pi^9(282h^2-187h+21)\zeta(3)+$$
$$11\pi^{12}(1044h^2-837h+128)+222264\pi^3(418h-103)h\zeta(3)^3)-715d^5(74680704h^3\zeta(3)^4+148176\pi^3(104h-57)h^2\zeta(3)^3+$$
$$3528\pi^6(396h^2-414h+65)h\zeta(3)^2+21\pi^9(3456h^3-5076h^2+1496h-63)\zeta(3)+2\pi^{12}(1176h^3-2088h^2+837h-64))+143d^4(124467840h^4\zeta(3)^4+$$
$$2963520\pi^3(9h-13)h^3\zeta(3)^3+105840\pi^6(24h^2-66h+23)h^2\zeta(3)^2+210\pi^9(672h^3-2592h^2+1692h-187)h\zeta(3)+\pi^{12}(5040h^4-23520h^3+$$
$$20880h^2-4185h+128))+429\pi^3d^3h(4445280h^3\zeta(3)^3+70560\pi^3(12h-11)h^2\zeta(3)^2+$$
$$630\pi^6(112h^2-192h+47)h\zeta(3)+\pi^9(3360h^3-7840h^2+3480h-279))+3432\pi^6d^2h^2(35280h^2\zeta(3)^2+\pi^6(420h^2-490h+87)+420\pi^3(14h-9)h\zeta(3))-$$
$$65d^7(476089488h\zeta(3)^4+111132\pi^3(824h-77)\zeta(3)^3+10584\pi^6(715h-128)\zeta(3)^2+462\pi^9(748h-189)\zeta(3)+11\pi^{12}(837h-256))+$$
$$20d^8(189189\pi^9\zeta(3)+4402944\pi^6\zeta(3)^2+55621566\pi^3\zeta(3)^3+298722816\zeta(3)^4+4576\pi^{12})+48048\pi^9dh^3(105h\zeta(3)+\pi^3(15h-7))+144144\pi^{12}h^4)$$

$$\delta_2 = \int_l^{\alpha+l}\int_h^d \tfrac{1}{81}\left|\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 \left(2x_3 - \tfrac{x_3^3}{d^2} + x_3\tfrac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3 dx_1 + \int_l^{\alpha+l}\int_0^h \tfrac{1}{81}\left|\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 \left(2x_3 - \tfrac{x_3^3}{d^2} + x_3\tfrac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3 dx_1 =$$
$$\tfrac{4c^9}{4583103525\pi^{12}\alpha^3(d-h)^8}(36d^2h^2(58924852\pi^9\zeta(3)+2151459072\pi^6\zeta(3)^2+35193676896\pi^3\zeta(3)^3+217766858400\zeta(3)^4+609689\pi^{12})-$$
$$d^3h(1054256952\pi^9\zeta(3)+38811729096\pi^6\zeta(3)^2+640378146240\pi^3\zeta(3)^3+3998199520224\zeta(3)^4+10822633\pi^{12})+1792d^4(110664\pi^9\zeta(3)+$$
$$4108104\pi^6\zeta(3)^2+68372640\pi^3\zeta(3)^3+430747632\zeta(3)^4+1127\pi^{12})-152dh^3(12599244\pi^9\zeta(3)+456297408\pi^6\zeta(3)^2+7400798496\pi^3\zeta(3)^3+$$
$$45387197856\zeta(3)^4+131377\pi^{12})+152h^4(4306848\pi^9\zeta(3)+154738080\pi^6\zeta(3)^2+2488764096\pi^3\zeta(3)^3+15129065952\zeta(3)^4+45251\pi^{12}))$$

$$
\begin{aligned}
\delta_3 =& 2\int_l^{\alpha+l}\int_0^1\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4\left(x_2+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)\right)^2 dx_2dx_3dx_1\\
&+2\int_l^{\alpha+l}\int_0^1\int_0^h \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4\left(x_2+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)\right)^2 dx_2dx_3dx_1 =
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{2d^7}{723647925\pi^{12}\alpha^3(d-h)^8}(136d^4(36406843200h^2\zeta(3)^4+1764\pi^6(264006h^2-69591h+1664)\zeta(3)^2+42\pi^9(363726h^2-135987h+5888)\zeta(3)+\\
&\pi^{12}(193554h^2-89571h+5248)+666792\pi^3(9846h-1339)h\zeta(3)^3)-136d^3h(32038022016h^2\zeta(3)^4+1764\pi^6(244308h^2-144612h+9295)\zeta(3)^2+\\
&42\pi^9(345480h^2-288567h+33202)\zeta(3)+\pi^{12}(188292h^2-193554h+29857)+2889432\pi^3(2048h-627)h\zeta(3)^3)+816d^2h^2(1779890112h^2\zeta(3)^4+\\
&1764\pi^6(14304h^2-22542h+3289)\zeta(3)^2+84\pi^9(10387h^2-22975h+5928)\zeta(3)+\pi^{12}(11596h^2-31382h+10753)+1926288\pi^3(175h-143)h\zeta(3)^3)-\\
&51d^5(49513306752h\zeta(3)^4+444528\pi^3(19632h-1001)\zeta(3)^3+7056\pi^6(85551h-8476)\zeta(3)^2+147\pi^9(131328h-18545)\zeta(3)+8\pi^{12}(29857h-5248))+\\
&6d^6(29527827\pi^9\zeta(3)+946266048\pi^6\zeta(3)^2+14013152496\pi^3\zeta(3)^3+81252605952\zeta(3)^4+356864\pi^{12})+7072\pi^3dh^3(11002068h\zeta(3)^3+\\
&58212\pi^3(28h-11)\zeta(3)^2+63\pi^6(1283h-880)\zeta(3)+\pi^9(1338h-1207))+14144\pi^6h^4(10164\pi^3\zeta(3)+116424\zeta(3)^2+223\pi^6))
\end{aligned}
$$

$$
\delta_4 =(l+\alpha)\int_h^d\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\,\zeta(3)\right)^4 dx_3+\alpha\int_0^h\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\,\zeta(3)\right)^4 dx_3 =
$$

$$
\begin{aligned}
&\tfrac{(d-h)(2\alpha+l)}{315\pi^{12}c^8}(-\pi^3d^6h^2(4788\pi^6\zeta(3)+43218\pi^3\zeta(3)^2+144060\zeta(3)^3+187\pi^9)+\pi^6d^5h^3(4032\pi^3\zeta(3)+\\
&18522\zeta(3)^2+233\pi^6)+\pi^6d^4h^4(4032\pi^3\zeta(3)+18522\zeta(3)^2+233\pi^6)-5\pi^9d^3h^5(252\zeta(3)+29\pi^3)-5\pi^9d^2h^6(252\zeta(3)+29\pi^3)-\pi^3d^7h(4788\pi^6\zeta(3)+\\
&43218\pi^3\zeta(3)^2+144060\zeta(3)^3+187\pi^9)+d^8(4032\pi^9\zeta(3)+49392\pi^6\zeta(3)^2+288120\pi^3\zeta(3)^3+756315\zeta(3)^4+128\pi^{12})+35\pi^{12}dh^7+35\pi^{12}h^8)
\end{aligned}
$$

$$
\delta_5 =\int_l^{\alpha+l}\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)\right)^4 dx_3dx_1+\int_0^l\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)\right)^4 dx_3dx_1 =
$$

$$
\begin{aligned}
&\tfrac{4(\alpha+l)^9}{44349279975\pi^{12}}(191\alpha^4+2160\alpha^2l^2-2200\alpha l^3+880l^4-1004\alpha^3l)(-3d^{10}h^2(1770720\pi^9\zeta(3)-71971200\pi^6\zeta(3)^2-5239503360\pi^3\zeta(3)^3-\\
&75645329760\zeta(3)^4+37687\pi^{12})+d^9h^3(-11743872\pi^9\zeta(3)-300839616\pi^6\zeta(3)^2+654937920\pi^3\zeta(3)^3+75645329760\zeta(3)^4-124511\pi^{12})-\\
&d^8h^4(7174272\pi^9\zeta(3)+283566528\pi^6\zeta(3)^2+3012714432\pi^3\zeta(3)^3-15129065952\zeta(3)^4+52723\pi^{12})+6\pi^3d^7h^5(-148512\pi^6\zeta(3)-17393040\pi^3\zeta(3)^2-\\
&327468960\zeta(3)^3+2453\pi^9)+18\pi^3d^6h^6(118048\pi^6\zeta(3)+1079568\pi^3\zeta(3)^2-21831264\zeta(3)^3+1971\pi^9)+126\pi^6d^5h^7(9520\pi^3\zeta(3)+199920\zeta(3)^2+99\pi^6)-\\
&6\pi^6d^4h^8(5712\pi^3\zeta(3)-839664\zeta(3)^2+1021\pi^6)-15\pi^9d^3h^9(11424\zeta(3)+313\pi^3)-21\pi^9d^2h^{10}(1632\zeta(3)+7\pi^3)+d^{11}h(25715424\pi^9\zeta(3)+\\
&1793522304\pi^6\zeta(3)^2+51347132928\pi^3\zeta(3)^3+529517308320\zeta(3)^4+114463\pi^{12})+28d^{12}(3544296\pi^9\zeta(3)+182626920\pi^6\zeta(3)^2+4257096480\pi^3\zeta(3)^3+\\
&37822664880\zeta(3)^4+\\
&26291\pi^{12})+495\pi^{12}dh^{11}+99\pi^{12}h^{12})\alpha^{12}d^8(d-h)^3
\end{aligned}
$$

$$
\begin{aligned}
\delta_6 =& \int_l^{\alpha+l}\int_0^1\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(x_2+\frac{x_3^2}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\,\zeta(3)\right)^2 dx_2dx_3dx_1\\
&+\int_l^{\alpha+l}\int_0^1\int_0^h \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(x_2+\frac{x_3^2}{d^2}+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\,\zeta(3)\right)^2 dx_2dx_3dx_1 =
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{d^3}{155925\pi^{12}\alpha(d-h)^4}(11d^2(21781872h^2\zeta(3)^4+294\pi^6(3264h^2-1893h+56)\zeta(3)^2+126\pi^9(534h^2-495h+32)\zeta(3)+\pi^{12}(2088h^2-2511h+256)+\\
&37044\pi^3(190h-49)h\zeta(3)^3)-33d^3(10890936h\zeta(3)^4+6174\pi^3(544h-35)\zeta(3)^3+1029\pi^6(422h-61)\zeta(3)^2+42\pi^9(682h-159)\zeta(3)+\\
&pi^{12}(837h-256))+6d^4(51282\pi^9\zeta(3)+825699\pi^6\zeta(3)^2+6723486\pi^3\zeta(3)^3+22819104\zeta(3)^4+1408\pi^{12})+33\pi^3dh(432180h\zeta(3)^3+\\
&686\pi^3(204h-25)\zeta(3)^2+14\pi^6(1179h-308)\zeta(3)+3\pi^9(232h-93))+264\pi^6h^2(441\pi^3\zeta(3)+1715\zeta(3)^2+29\pi^6))
\end{aligned}
$$

$$\delta_7 = \int_l^{\alpha+l} \int_0^1 \int_h^d \frac{1}{81} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 \left(x_2 + x_3 \frac{42}{\pi^3}\ \zeta(3) + d\right)^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3)\right)^2 dx_2 dx_3 dx_1 =$$

$$\frac{29}{842836995 d^4 (d-h)\alpha\pi^{12}} ((42dh^5\pi^6((35+41h)\pi^6+3h(1099+834h)\pi^3\zeta(3)+105840h^2\zeta(3)^2)+14h^6\pi^6(41\pi^6+3753h\pi^3\zeta(3)+117936h^2\zeta(3)^2)+$$
$$2d^3h^3\pi^3((-4193-3777h+2205h^2)\pi^9-42(2834+12081h+4284h^2)\pi^6\zeta(3)-26460h(403+714h)\pi^3\zeta(3)^2-326728080h^2\zeta(3)^3)+$$
$$2d^2h^4\pi^3((-1259+2205h+861h^2)\pi^9+42(-1222-2142h+3297h^2)\pi^6\zeta(3)-111132h(39+20h)\pi^3\zeta(3)^2-127579536h^2\zeta(3)^3)+24d^8(1388\pi^{12}+$$
$$146328\pi^9\zeta(3)+5976873\pi^6\zeta(3)^2+111928446\pi^3\zeta(3)^3+809040960\zeta(3)^4)+3d^7((11104+10973h)\pi^{12}+168(5252+8587h)\pi^9\zeta(3)+8820(2717+$$
$$7829h)\pi^6\zeta(3)^2+518616(429+2804h)\pi^3\zeta(3)^3+11488381632h\zeta(3)^4)+d^4h^2((511-25158h-7554h^2)\pi^12-21(-10712+44721h+81984h^2)\pi^9\zeta(3)-$$
$$1764(-3289-4836h+30816h^2)\pi^6\zeta(3)^2+111132h(3861+296h)\pi^3\zeta(3)^3+11407477536h^2\zeta(3)^4)+d^5h((10973+1533h-25158h^2)\pi^{12}-$$
$$63(-10712-20775h+18478h^2)\pi^9\zeta(3)+1764(5863+38103h+15588h^2)\pi^6\zeta(3)^2+333396h(2717+6198h)\pi^3\zeta(3)^3+26455639392h^2\zeta(3)^4)+$$
$$d^6((11104+32919h+1533h^2)\pi^12+42(14144+75624h+46257h^2)\pi^9\zeta(3)+1764(4576+56862h+84063h^2)\pi^6\zeta(3)^2+666792h(1573+5947h)\pi^3\zeta(3)^3+$$
$$36406843200h^2\zeta(3)^4)))$$

$$\delta_8 = \int_l^{\alpha+l} \int_h^d \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3)\right)^2 \left(1 - \frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3 dx_1$$
$$+ \int_l^{\alpha+l} \int_0^h \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3)\right)^2 \left(1 - \frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3 dx_1 =$$

$$\frac{d^5}{675675\pi^{12}\alpha(d-h)^4} (16d^2(42980\pi^9\zeta(3)+1007097\pi^6\zeta(3)^2+10005996\pi^3\zeta(3)^3+37081044\zeta(3)^4+656\pi^{12})-dh(1931202\pi^9\zeta(3)+44515471\pi^6\zeta(3)^2+$$
$$432612180\pi^3\zeta(3)^3+1557403848\zeta(3)^4+29857\pi^{12})+2h^2(687050\pi^9\zeta(3)+15590575\pi^6\zeta(3)^2+148324176\pi^3\zeta(3)^3+519134616\zeta(3)^4+10753\pi^{12}))$$

$$\delta_9 = \int_l^{\alpha+l} \int_h^d \frac{1}{81} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3 dx_1 =$$

$$\frac{29}{90745449795\pi^{12}\alpha d^8(d-h)} (26132023008d^8\zeta(3)^4\cdot(80d^4+142d^3h+150d^2h^2+109dh^3+47h^4)+95721696\pi^3d^6\zeta(3)^3(3040d^6+5095d^5h+4845d^4h^2+$$
$$2718d^3h^3+74d^2h^4-1470dh^5-574h^6)+13674528\pi^6d^4\zeta(3)^2(1128d^8+1775d^7h+1477d^6h^2+493d^5h^3-553d^4h^4-966d^3h^5-210d^2h^6+210dh^7+78h^8)+$$
$$1596\pi^9d^2\zeta(3)(231104d^{10}+339379d^9h+236757d^8h^2-788d^7h^3-219196d^6h^4-256410d^5h^5+3402d^4h^6+124740d^3h^7+$$
$$31788d^2h^8-17655dh^9-6369h^{10})+\pi^{12}(3354976d^{12}+4567786d^{11}h+2486970d^{10}h^2-1418721d^9h^3-4509979d^8h^4-$$
$$4150776d^7h^5+1393616d^6h^6+3466134d^5h^7+408114d^4h^8-1033230d^3h^9-280038d^2h^{10}+110253dh^{11}+39039h^{12}))$$

$$\delta_{10} = \int_l^{\alpha+l} \int_0^1 \int_h^d \frac{1}{9} \left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2 \left(x_2 + x_3 \frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2 dx_3 dx_1$$
$$+ \int_l^{\alpha+l} \int_0^1 \int_0^h \frac{1}{9} \left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2 \left(x_2 + x_3 \frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2 dx_3 dx_1 =$$

$$\frac{4d^3}{135\pi^6\alpha^3} (d^2(10584h^2\zeta(3)^2+\pi^6(60h^2-75h+8)+126\pi^3(10h-7)h\zeta(3))-3d^3(5292h\zeta(3)^2+21\pi^3(28h-5)\zeta(3)+\pi^6(25h-8))+6d^4(105\pi^3\zeta(3)+$$
$$1008\zeta(3)^2+4\pi^6)+5\pi^3dh(126h\zeta(3)+\pi^3(12h-5))+20\pi^6h^2)(d-h)^4$$

$$\delta_{11} = \int_l^{\alpha+l} \int_h^d \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(1 + \frac{42}{\pi^3}\ \zeta(3) + \frac{3\,x_3}{d}\right)^2 dx_3 dx_1 + \int_l^{\alpha+l} \int_0^h \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(1 + \frac{42}{\pi^3}\ \zeta(3) + \frac{3\,x_3}{d}\right)^2 dx_3 dx_1 =$$

$$\frac{2d^3}{1575\pi^6\alpha(d-h)^4} (d^2(9114\pi^3\zeta(3)+98784\zeta(3)^2+233\pi^6)-14dh(1932\pi^3\zeta(3)+22050\zeta(3)^2+47\pi^6)+28h^2(735\pi^3\zeta(3)+8820\zeta(3)^2+17\pi^6))$$

$$\delta_{12} = \int_l^{\alpha+l} \int_h^d \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(\frac{3\,x_3}{d}\right)^2 dx_3 dx_1 + \int_l^{\alpha+l} \int_0^h \frac{1}{9} \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(\frac{3\,x_3}{d}\right)^2 dx_3 dx_1 = \frac{2d^3(8d^2 - 21dh + 14h^2)}{175(d-h)^4\alpha}$$

$$\delta_{13} = \int_l^{\alpha+l} \int_0^1 \int_h^d \frac{1}{9} \left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1 x_3}\right|^2 \left(x_2 + x_3 \frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2 dx_3 dx_1 =$$

$$\frac{4}{675\pi^6\alpha(d-h)} \left[10(1+3d+3d^2)\pi^6 + 315(1+2d)(3d+h)\pi^3\zeta(3) + 5292(6d^2+3dh+h^2)\zeta(3)^2\right]$$

$$\delta_{14} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_l^{\alpha+l}\int_0^h \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{2d^5}{10395\pi^6\alpha^3(d-h)^4}(48d^2(1540\pi^3\zeta(3)+19404\zeta(3)^2+31\pi^6)-11dh(17892\pi^3\zeta(3)+222264\zeta(3)^2+365\pi^6)+22h^2(6048\pi^3\zeta(3)+74088\zeta(3)^2+125\pi^6))$$

$$\delta_{15} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\frac{x_3^2}{d^2}\, dx_3dx_1 + \int_l^{\alpha+l}\int_0^h \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\frac{x_3^2}{d^2}\, dx_3dx_1 = \frac{2d^3(8d^2-21dh+14h^2)}{175(d-h)^4\alpha}$$

$$\delta_{16} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_l^{\alpha+l}\int_0^h \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{d^3}{3150\pi^6\alpha(d-h)^4}(d^2(63336\pi^3\zeta(3)+395136\zeta(3)^2+2819\pi^6)-2dh(94080\pi^3\zeta(3)+617400\zeta(3)^2+3971\pi^6)+16h^2(8967\pi^3\zeta(3)+61740\zeta(3)^2+359\pi^6))$$

$$\delta_{17} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1x_3}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{2}{4725\pi^6\alpha d^4(d-h)}\Big[74088d^4\zeta(3)^2\big[6d^2+3dh+h^2\big)+1008\pi^3d^2\zeta(3)\big(27d^4+11d^3h+d^2h^2-3dh^3-h^4\big)+\pi^6(428d^6+129d^5h-45d^4h^2-94d^3h^3-$$
$$18d^2h^4+15dh^5+5h^6)\Big]$$

$$\delta_{18} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 + \int_l^{\alpha+l}\int_0^h \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 =$$
$$\frac{c^3}{525\pi^6\alpha(d-h)^4}(16d^2(252\pi^3\zeta(3)+1029\zeta(3)^2+16\pi^6)-3dh(4312\pi^3\zeta(3)+17150\zeta(3)^2+279\pi^6)+24h^2(441\pi^3\zeta(3)+1715\zeta(3)^2+29\pi^6))$$

$$\delta_{19} = (l+\alpha)\int_h^d \frac{1}{d^2}4\, dx_3 + \alpha\int_0^h \frac{1}{d^2}4 = 4(l+\alpha)\,\frac{d-h}{d^2} + 4\alpha\frac{h}{d^2}$$

$$\delta_{20} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\frac{x_3^2}{d^2}\, dx_3dx_1 + \int_0^l\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\frac{x_3^2}{d^2}\, dx_3dx_1 = \frac{26\alpha\left(6d^2+3dh+h^2\right)}{525d^2(d-h)}$$

$$\delta_{21} = (l+\alpha)\int_h^d \frac{1}{d^2}\left(2\frac{x_3}{d^2}+2\right)^2 dx_3 + \alpha\int_0^h \frac{1}{d^2}\left(2\frac{x_3}{d^2}+2\right)^2 dx_3 = \frac{4}{3c^6}(-3c^2h^2l-h^3l+c^3(l+\alpha)+3c^5(l+\alpha)+3c^4(l-hl+\alpha))$$

$$\delta_{22} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 + \int_0^l\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 =$$
$$\frac{4(\alpha+l)^5(13\alpha^2+20l^2-30\alpha l)}{3675\pi^6\alpha^6d^4(d-h)}(-\pi^3d^2h^2(49\zeta(3)+\pi^3)-\pi^3d^3h(147\zeta(3)+11\pi^3)+d^4(196\pi^3\zeta(3)+1715\zeta(3)^2+8\pi^6)+3\pi^6dh^3+\pi^6h^4)$$

$$\delta_{23} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_0^l\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{2(\alpha+l)^5}{33075\pi^6\alpha^6c^4(c-h)}(13\alpha^2+20l^2-30\alpha l)(12\pi^3c^2h^2(147\zeta(3)+37\pi^3)+3\pi^3c^3h(3234\zeta(3)+403\pi^3)+2c^4(17787\pi^3\zeta(3)+61740\zeta(3)^2+1346\pi^6)+$$
$$117\pi^6ch^3+18\pi^6h^4)$$

$$\delta_{24} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_0^l\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{4(\alpha+l)^5}{33075\pi^6\alpha^6d^4(d-h)^3}(13\alpha^2+20l^2-30\alpha l)(d^4h^2(4116\pi^3\zeta(3)+61740\zeta(3)^2+71\pi^6)-3\pi^3d^3h^3(588\zeta(3)+23\pi^3)-3\pi^3d^2h^4(588\zeta(3)+23\pi^3)+$$
$$d^5h(4116\pi^3\zeta(3)+61740\zeta(3)^2+71\pi^6)+d^6(4116\pi^3\zeta(3)+61740\zeta(3)^2+71\pi^6)+15\pi^6dh^5+15\pi^6h^6)$$

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