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ORIGINAL PAPER

Rigid manifolds of general type with non-contractible universal cover

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Abstract
For each $n \geq 3$ we give examples of infinitesimally rigid projective manifolds of general type of dimension n with non-contractible universal cover. We provide examples with projective and examples with non-projective universal cover.

Keywords Rigid complex manifolds · Deformation theory · Fundamental group · Classifying space

Mathematics Subject Classification (2000) 32G05 · 14J10 · 14L30 · 32Q30 · 14J40 · 14B05

1 Introduction

In [2] several notions of *rigidity* have been discussed, the relations among them have been studied and many questions and conjectures have been proposed. In particular the authors showed that a rigid compact complex surface has Kodaira dimension $-\infty$ or 2, and observed that all known examples of rigid surfaces of general type are $K(\pi, 1)$ spaces. Recall that a CW complex with fundamental group π is called $K(\pi, 1)$ *space* if its universal cover is contractible, and that these spaces have the property that their homotopy type is uniquely determined by their fundamental group (cf. [19, §1.B]). This implies that the topological invariants, such as homology and cohomology, are determined by π. In [2] the following natural question has been posed.

Question 1 *Do there exist infinitesimally rigid surfaces of general type with non-contractible universal cover?*

The aim of this paper is to give a positive answer for the analogous question in higher dimensions. More precisely, we construct for each $n \geq 3$ an infinitesimally rigid manifold of

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general type of dimension n with non-contractible universal cover. For surfaces the question remains open. We recall now the notions of rigidity that are relevant for our purposes.

Definition 1 Let X be a compact complex manifold of dimension n.

1. A *deformation of* X is a proper smooth holomorphic map of pairs
$$f\colon (\mathfrak{X}, X) \to (\mathcal{B}, b_0),$$
where $(\mathcal{B}, b_0)$ is a connected (possibly not reduced) germ of a complex space.
2. X is said to be *rigid* if for each deformation of X, $f\colon (\mathfrak{X}, X) \to (\mathcal{B}, b_0)$ there is an open neighbourhood $U \subset \mathcal{B}$ of b_0 such that $X_t := f^{-1}(t) \simeq X$ for all $t \in U$.
3. X is said to be *infinitesimally rigid* if $H^1(X, \Theta_X) = 0$, where Θ_X is the sheaf of holomorphic vector fields on X.
4. X is said to be (infinitesimally) étale rigid if all finite étale covers $f\colon Y \to X$ are (infinitesimally) rigid.

Remark 1 (i) By Kodaira-Spencer-Kuranishi theory every infinitesimally rigid manifold is rigid. The converse does not hold in general as it was shown in [6] and [8] (cf. also [24]).
(ii) Beauville surfaces are examples of rigid, but not étale rigid manifolds (see [12]).

Both the examples constructed in [6] and Beauville surfaces are product quotient varieties, i.e. (resolutions of singularities of) finite quotients of product of curves with respect to a holomorphic group action. In recent years, product quotients turned out to be a very fruitful source of examples of rigid complex manifolds with additional properties. Besides the examples above, we mention [4], where the authors construct the first examples of rigid complex manifolds with Kodaira dimension 1 in arbitrary dimension $n \geq 3$, and [5] where they constructed new rigid three- and four-folds with Kodaira dimension 0. We refer to [11, 16–18, 21, 22] for other interesting examples of product quotient varieties.

The manifolds we construct are also product quotients. More precisely, inspired by the construction in [6] in Sect. 2 we consider for each $n \geq 3$ and $d \geq 4$, even and not divisible by 3 the n-fold product C^n of the Fermat curve C of degree d together with a suitable action of $\mathbb{Z}_d^2$. The quotient $X_{n,d} := C^n/\mathbb{Z}_d^2$ is a normal projective variety with isolated cyclic quotient singularities of type $\frac{1}{2}(1, \dots, 1)$, Kodaira dimension n and
$$H^1(X_{n,d}, \Theta_{X_{n,d}}) = H^1(C^n, \Theta_{C^n})^{\mathbb{Z}_d^2} = 0.$$
Blowing up the singular points, we obtain a resolution $\widehat{X_{n,d}} \to X_{n,d}$ such that $H^1(X_{n,d}, \Theta_{X_{n,d}}) = H^1(\widehat{X_{n,d}}, \Theta_{\widehat{X_{n,d}}})$. Therefore, $\widehat{X_{n,d}}$ is an infinitesimally rigid projective manifold of general type.

In Sect. 3 we show that the universal cover $U_{n,d}$ of $\widehat{X_{n,d}}$ is non-contractible since it contains several $\mathbb{P}^{n-1}$ (see Propostion 3). We then discuss the finiteness of the fundamental group $\pi_1(X_{n,d}) = \pi_1(\widehat{X_{n,d}})$. The crucial ingredient here is Armstrong's description of the fundamental group of a quotient space [1] adapted to product quotients by [3]. The finiteness of $\pi_1(\widehat{X}_n)$ is equivalent to the finiteness of certain groups (Proposition 5: Finiteness criterion). This allows us to prove the following.

Theorem 1 *For each $n \geq 3$, $d \geq 4$, even and not divisible by 3 there exists an infinitesimally rigid projective n-dimensional manifold of general type $\widehat{X_{n,d}}$, whose universal cover $U_{n,d}$ is non-contractible. Moreover, the universal cover $U_{n,d}$ is projective if and only if $d = 4$.*

The construction actually works also for $n = 2$: the surface $\widehat{X_{2,4}}$ is not rigid, whereas the surface $\widehat{X_{2,d}}$ for $d \geq 8$ is rigid but not infinitesimally rigid (see [6]), and its universal cover is non-contractible.

Notation We work over the field of complex numbers, and we denote by $\mathbb{Z}_n$ the cyclic group of order n and by ζ_n a primitive n-th root of unity. The rest of the notation is standard in complex algebraic geometry.

2 The families

Let $C_d := \{x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{P}^2$ be the Fermat curve of degree d. Consider the group action

$$\phi_1 \colon \mathbb{Z}_d^2 \to \mathrm{Aut}(C_d), \qquad (a, b) \mapsto [(x_0 : x_1 : x_2) \mapsto (\zeta_d^a x_0 : \zeta_d^b x_1 : x_2)].$$

There are $3d$ points on C_d with non-trivial stabilizer. They form three orbits of length d. A representative of each orbit and a generator of the corresponding stabilizer is given in the table below:

Point	$(0 : 1 : \zeta_{2d})$	$(1 : 0 : \zeta_{2d})$	$(1 : \zeta_{2d} : 0)$
Generator	$(1, 0)$	$(0, 1)$	$(1, 1)$

Hence the quotient map

$$f \colon C_d \to \mathbb{P}^1, \qquad (x_0 : x_1 : x_2) \mapsto (x_0^d : x_1^d)$$

is branched in $(0 : 1)$, $(1 : 0)$ and $(1 : -1)$, each with branch index d.

2.1 The singular quotients $X_{n,d}$

From now on we fix $d \geq 4$, even and not divisible by 3, and denote C_d simply by C. Let A be the automorphism of $\mathbb{Z}_d^2$ given by the matrix

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}_d),$$

and let $\phi_2 := \phi_1 \circ A^{-1}$. For each $n \geq 2$ consider the $\mathbb{Z}_d^2$ diagonal action on C^n defined by

$$g(z_1, \ldots, z_n) := \big(\phi_1(g) \cdot z_1, \phi_2(g) \cdot z_2, \phi_2(g) \cdot z_3, \ldots, \phi_2(g) \cdot z_n\big)$$

and let $X_{n,d}$ be the quotient variety $X_{n,d} := C^n / \mathbb{Z}_d^2$.

Remark 2 The diagonal action is not free, indeed

$$\mathrm{Fix}(\phi_1(g)) \neq \emptyset \text{ and } \mathrm{Fix}(\phi_2(g)) \neq \emptyset \Longleftrightarrow g \in H := \left\langle \left(\frac{d}{2}, 0\right), \left(0, \frac{d}{2}\right) \right\rangle.$$

Noting that $\phi_{1|H} = \phi_{2|H}$, we see that a point $(z_1, \ldots, z_n) \in C^n$ has a non-trivial stabilizer if and only if all its coordinates z_i belong to one and only one of the three $\mathbb{Z}_d^2$-orbits displayed in the table above.

Proposition 1 *For $n \geq 3$ the projective variety $X_{n,d}$ is infinitesimally rigid and of general type. The singular locus of $X_{n,d}$ consists of $6 \cdot d^{n-2}$ cyclic quotient singularities of type $\frac{1}{2}(1, \ldots, 1)$.*

Proof By Remark 2 there are $3 \cdot d^n$ points on C^n with non-trivial stabilizer, each generated by one of the order 2 elements in $\mathbb{Z}_d^2$. Thus, $X_{n,d}$ has $(3 \cdot d^n)/(d^2/2) = 6 \cdot d^{n-2}$ singularities of type $\frac{1}{2}(1, \ldots, 1)$.

These singularities are terminal if $n \geq 3$, see [26, p. 376 Theorem]. Since the quotient map $C^n \to X_{n,d}$ is quasi-étale, $g(C) = (d-1)(d-2)/2 \geq 3$ and $X_{n,d}$ is terminal, its Kodaira dimension is $\kappa(X_{n,d}) = \kappa(C^n) = n$ (cf. [13, p. 51]).

According to Schlessinger [27], isolated quotient singularities in dimension at least three are rigid, i.e. $\mathcal{E}xt^1(\Omega^1_{X_{n,d}}, \mathcal{O}_{X_{n,d}}) = 0$. Thus the local-to-global Ext spectral sequence yields

$$H^1(X_{n,d}, \Theta_{X_{n,d}}) \simeq \mathrm{Ext}^1(\Omega^1_{X_{n,d}}, \mathcal{O}_{X_{n,d}}).$$

Hence it suffices to verify that $X_{n,d}$ has no equisingular deformations. Since $g(C) \geq 3$ we have $H^0(C, \Theta_C) = 0$, hence by Künneth formula we get

$$H^1(C^n, \Theta_{C^n}) = \bigoplus_{i=1}^{n} H^1(C, \Theta_C).$$

Using the fact that the quotient map $C^n \to X_{n,d}$ is quasi-étale and the action is diagonal, we obtain

$$H^1(X_{n,d}, \Theta_{X_{n,d}}) = H^1(C^n, \Theta_{C^n})^{\mathbb{Z}_d^2} = \bigoplus_{i=1}^{n} H^1(C, \Theta_C)^{\mathbb{Z}_d^2}.$$

The branch locus B of $f : C \to C/\mathbb{Z}_d^2 \simeq \mathbb{P}^1$ consists of 3 points p_i with branch indices $m_{p_i} = d$, thus by [7, Ex.VI.12] we have

$$\begin{aligned} \dim H^1(C, \Theta_C)^{\mathbb{Z}_d^2} &= \dim H^0(C, 2K_C)^{\mathbb{Z}_d^2} = h^0(\mathbb{P}^1, 2K_{\mathbb{P}^1} + \sum_{p_i \in B} p_i \cdot \lfloor 2(1 - \frac{1}{m_{p_i}}) \rfloor) \\ &= h^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0. \end{aligned}$$

□

2.2 Resolution of singularities of type $\frac{1}{2}(1, \ldots, 1)$

Proposition 2 *A singularity $U := \mathbb{C}^n/\mathbb{Z}_2$ of type $\frac{1}{2}(1, \ldots, 1)$ admits a resolution $\rho : \widehat{U} \to U$ by a single blow-up, with exceptional prime divisor $\mathbb{P}^{n-1}$. If $n \geq 3$,*

$$\rho_* \Theta_{\widehat{U}} = \Theta_U \quad \textit{and} \quad R^1 \rho_* \Theta_{\widehat{U}} = 0.$$

For a proof we refer to [27, proof of Theorem 4], see also [4, Corollary 5.9, Proposition 5.10].

Remark 3 (see [4, Remark 5.4]) Both properties are not obvious and in general even false. For any resolution $\rho : Z' \to Z$ of a normal variety Z, the direct image $\rho_* \Theta_{Z'}$ is a subsheaf of the reflexive sheaf Θ_Z, and this inclusion is in general strict: e.g. take the blow-up of the origin of $\mathbb{C}^2$.

The vanishing of $R^1\rho_*\Theta_{Z'}$ is also not automatic: take the resolution of an A_1 surface singularity (i.e. $\frac{1}{2}(1,1)$) by a -2 curve, then $R^1\rho_*\Theta_{Z'}$ is a skyscraper sheaf at the singular point with value $H^1(\mathbb{P}^1,\mathcal{O}(-2))\cong\mathbb{C}$. More generally, for canonical ADE surface singularities $R^1\rho_*\Theta_{Z'}$ is never zero, cf. [10, 25, 27].

Corollary 1 *Let Z_n be a projective variety of dimension $n\geq 3$ with only singularities of type $\frac{1}{2}(1,\dots,1)$. Then there exists a resolution $\rho\colon\widehat{Z}_n\to Z_n$, such that*

$$H^1(Z_n,\Theta_{Z_n})\simeq H^1(\widehat{Z}_n,\Theta_{\widehat{Z}_n}).$$

In particular, if Z_n is infinitesimally rigid, so is $\widehat{Z}_n$.

Proof Since the singularities of Z_n are isolated, we resolve them simultaneously using Proposition 2 and we get a resolution $\rho\colon\widehat{Z}_n\to Z_n$ having the same properties:

$$\rho_*\Theta_{\widehat{Z}_n}=\Theta_{Z_n}\quad\text{and}\quad R^1\rho_*\Theta_{\widehat{Z}_n}=0.$$

Leray's spectral sequence implies $H^1(\widehat{Z}_n,\Theta_{\widehat{Z}_n})\simeq H^1(Z_n,\Theta_{Z_n})$. □

By the corollary, for $n\geq 3$ there exists a resolution $\widehat{X_{n,d}}\to X_{n,d}$ of the singularities of $X_{n,d}$, which is infinitesimally rigid. By Remark 3, for $n=2$ the minimal resolution $\widehat{X_{2,d}}$ of $X_{2,d}$ is not infinitesimally rigid, nevertheless the main theorem of [6] shows that $\widehat{X_{2,d}}$ is rigid for $d\geq 8$, whereas $\widehat{X_{2,4}}$ is a numerical Campedelli surface, whose Kuranishi family has dimension 6.

2.3 Non-étale infinitesimally rigidity

We conclude this section constructing an étale cover of $\widehat{X_{n,d}}$ which is not infinitesimally rigid, thus $\widehat{X_{n,d}}$ is not étale infinitesimally rigid.

Let $H:=\left\langle\left(\frac{d}{2},0\right),\left(0,\frac{d}{2}\right)\right\rangle$ be as in Remark 2.

Lemma 1 *Let $Y_{n,d}:=C^n/H$ be the quotient with respect to the restricted diagonal action, then:*

1. *The natural morphism $\psi\colon Y_{n,d}\to X_{n,d}$ is an unramified Galois cover with group $\mathbb{Z}_{d/2}^2$.*
2. $h^1(Y_{n,d},\Theta_{Y_{n,d}})=3n\cdot\left(\frac{d^2-2d}{8}\right)$.

Proof (1) Since H is a normal subgroup of $\mathbb{Z}_d^2$ the map ψ is a Galois cover with group $\mathbb{Z}_d^2/H\cong\mathbb{Z}_{d/2}^2$. By Remark 2 the stabilizer of a point $z\in C^n$ with respect to the $\mathbb{Z}_d^2$-action is contained in H, whence the map ψ is unramified.

(2) Since $C\to C/H$ is branched in $\frac{3d}{2}$ points, we have

$$\dim\left(H^1(C^n,\Theta_{C^n})^H\right)=n\cdot\dim\left(H^1(C,\Theta_C)^H\right)=3n\cdot\left(\frac{d^2-2d}{8}\right)$$

arguing as in Proposition 1. □

3 The universal cover of $\widehat{X_{n,d}}$

In this section we prove that the universal cover $U_{n,d}$ of $\widehat{X_{n,d}}$ is non-contractible, and then we discuss whether it is projective or not.

Proposition 3 *Let X be a compact Kähler manifold, containing a $\mathbb{P}^m$. Then the universal cover U of X is non-contractible.*

Proof Since $\mathbb{P}^m$ is simply connected, the inclusion map $i\colon \mathbb{P}^m \hookrightarrow X$ lifts to a map $f\colon \mathbb{P}^m \to U$. Looking for a contradiction, assume that U is contractible, then f is homotopic to a constant map, therefore the inclusion i is also homotopic to a constant map. In particular we see that the induced linear map $i^*: H^2(X, \mathbb{C}) \to H^2(\mathbb{P}^m, \mathbb{C})$ is the zero map. Now let $[\omega]$ be a Kähler class of X. Its restriction $i^*([\omega])$ is a Kähler class of $\mathbb{P}^m$, whence non zero, contradiction. □

Corollary 2 *The universal cover $U_{n,d}$ of $\widehat{X_{n,d}}$ is non-contractible.*

Proof By Proposition 2 the manifold $\widehat{X_{n,d}}$ contains several $\mathbb{P}^{n-1}$. □

3.1 The Fundamental Group

In this section we discuss the finiteness of the fundamental group $\pi_1(\widehat{X_{n,d}})$. In order to do this we use the main theorem of [1] in the case of product quotient varieties following [3, 15]. We briefly recall their strategy and we refer to them for further details.

Let G be a finite group acting diagonally on a product $Z := C_1 \times \ldots \times C_n$ of curves of genus at least 2, and consider the group $\mathbb{G}$ of all possible lifts of automorphisms induced by the action of G on Z to the universal cover $u : \mathbb{H}^n \to Z$. The group $\mathbb{G}$ acts properly discontinuously on $\mathbb{H}^n$ and u is equivariant with respect to the natural map $\mathbb{G} \to G$, hence we have an isomorphism $\mathbb{H}^n/\mathbb{G} \cong Z/G$. Since $\mathbb{H}^n$ is simply connected we can apply Armstrong's results (see [1]) and get the following.

Proposition 4 *Let $\mathrm{Fix}(\mathbb{G})$ be the normal subgroup of $\mathbb{G}$ generated by the elements having non-empty fixed locus. Then*

$$\pi_1(Z/G) = \mathbb{G}/\,\mathrm{Fix}(\mathbb{G}).$$

Assume that the G-action on Z restricts to a faithful action ϕ_i on each factor C_i. Let $\mathbb{T}_i$ be the group of all possible lifts of automorphisms induced by the action of G on C_i to the universal cover $\mathbb{H}$ of C_i, and let $\varphi_i : \mathbb{T}_i \to G$ be the natural map. In this setting, the above group $\mathbb{G}$ is the preimage of the diagonal subgroup $\Delta_G \subset G^n$ under $\varphi_1 \times \ldots \times \varphi_n$:

$$\mathbb{G} = \{(x_1, \ldots, x_n) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \mid \varphi_1(x_1) = \ldots = \varphi_n(x_n)\}.$$

There is also a similar description of $\mathbb{G}$ in the non-faithful case, see [15, Proposition 3.3].

Remark 4 (i) The group $\mathbb{T}_i$ has a simple presentation (see also [14, Example 29]): let g' be the genus of C_i/G and $m_1, \ldots, m_r$ be the ramification indices of the branch points of the covering map $C_i \to C_i/G$, then

$$\mathbb{T}_i = \mathbb{T}(g'; m_1, \dots, m_r) := \Big\langle a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r \mid c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdots c_r \Big\rangle.$$

(ii) The group $\mathbb{T}(g'; m_1, \dots, m_r)$ is called the *orbifold surface group* of type $[g'; m_1, \dots, m_r]$.

The non-trivial stabilizers of the $\mathbb{T}_i$-action on $\mathbb{H}$ are cyclic and generated by the conjugates of the elements c_k. The restriction of φ_i to each one of these subgroups is an isomorphism onto its image, which is the stabilizer of a point in C_i. Conversely, all non-trivial stabilizers of the G-action on C_i are of this form (see [3]).

Definition 2 Let $L_i \subset \mathbb{T}_i$ be set of the elements $c_j^{l_j} \in \mathbb{T}_i$ such that $\varphi_i(c_j^{l_j}) \in G$ has non-empty fixed locus on $Z = C_1 \times \ldots \times C_n$, where $j \in \{1, \dots, r\}$ and $l_j \in \{1, \dots, m_j - 1\}$.

We denote by $\langle\langle L_i \rangle\rangle_{\mathbb{T}_i}$ the normal subgroup of $\mathbb{T}_i$ generated by L_i.

Proposition 5 *(Finiteness criterion) The group $\pi_1(Z/G) = \mathbb{G}/\operatorname{Fix}(\mathbb{G})$ is finite if and only if the groups $\mathbb{T}_i/\langle\langle L_i \rangle\rangle_{\mathbb{T}_i}$ are finite.*

Proof According to [3, pag.1018-1019] the group $\mathbb{G}/\operatorname{Fix}(\mathbb{G})$ fits in an exact sequence

$$1 \to E \to \mathbb{G}/\operatorname{Fix}(\mathbb{G}) \to \mathbf{H} \to 1,$$

where E is a finite group and $\mathbf{H}$ is a subgroup of finite index of the product

$$\mathbb{T}_1/\langle\langle L_1 \rangle\rangle_{\mathbb{T}_1} \times \cdots \times \mathbb{T}_n/\langle\langle L_n \rangle\rangle_{\mathbb{T}_n}.$$

□

Remark 5 Let X be a normal variety with only quotient singularities, and let $\rho\colon \widehat{X} \to X$ be a resolution of singularities. Then $\rho_*\colon \pi_1(\widehat{X}) \to \pi_1(X)$ is an isomorphism, by [20, Theorem 7.8]. In particular, $\pi_1(\widehat{X_{n,d}}) \simeq \pi_1(X_{n,d})$.

According to the description of $X_{n,d}$ given in the previous section its associated orbifold surface groups $\mathbb{T}_i$ are all of type $[0; d, d, d]$, and applying this discussion to our situation we get the following.

Theorem 2 *The universal cover $U_{n,d}$ of $\widehat{X_{n,d}}$ is projective if and only if $d = 4$.*

Proof The universal cover $U_{n,d}$ of $\widehat{X_{n,d}}$ is projective if and only if the fundamental group $\pi_1(\widehat{X_{n,d}})$ is finite. Therefore, by Proposition 5 the universal cover $U_{n,d}$ is projective if and only if the groups $\mathbb{T}_i/\langle\langle L_i \rangle\rangle_{\mathbb{T}_i}$ are finite. Let $k := \frac{d}{2}$. Since the elements in $\mathbb{Z}_d^2$ fixing points on C^n are exactly the elements in $H = \langle (k, 0), (0, k) \rangle$, by Remark 4 (ii) we see that $L_i = \{c_1^k, c_2^k, c_3^k\}$, whence

$$\mathbb{T}_i/\langle\langle L_i \rangle\rangle_{\mathbb{T}_i} \cong \mathbb{T}(0; d, d, d)/\langle\langle c_1^k, c_2^k, c_3^k \rangle\rangle = \langle c_1, c_2, c_3 | c_1^k, c_2^k, c_3^k, c_1c_2c_3 \rangle \cong \mathbb{T}(0; k, k, k).$$

The statement follows since the group $\mathbb{T}(0; 2, 2, 2) \cong \mathbb{Z}_2^2$ is finite, whereas $\mathbb{T}(0; k, k, k)$ is infinite for $k > 2$. □

Remark 6 By Lemma 1 the universal cover $U_{n,4}$ of $\widehat{X_{n,4}}$ is not infinitesimally rigid.

Remark 7 (i) The first Betti number b_1 of $Y_{n,4}$ is zero, because the quotient C/H is isomorphic to the projective line. Indeed by Künneth formula and [23, §1.2] we have

$$H^1(Y_{n,4},\mathbb{C}) = H^1(C^n,\mathbb{C})^H = \bigoplus H^1(C,\mathbb{C})^H = \bigoplus H^1(\mathbb{P}^1,\mathbb{C}) = 0.$$

Assuming $d=4$, we can actually prove that $g^2=1$ for all $g\in\pi_1(Y_{n,4}) = \mathbb{G}/\operatorname{Fix}(\mathbb{G})$. This tells us that $\pi_1(Y_{n,4}) = \pi_1(\widehat{Y_{n,4}}) \cong \mathbb{Z}_2^s$ for some $s\in\mathbb{N}$.

The element g is represented by an n-tuple

$$(w_1,\dots,w_n)\in\mathbb{G} = \mathbb{T}_1\times_H\cdots\times_H\mathbb{T}_n$$

where $\mathbb{T}_k = \mathbb{T}(0;2,2,2,2,2,2)$ and all the maps $\varphi_k:\mathbb{T}_k\to H$ are equal, as we consider the same action on each factor (see Remark 2). Since $\varphi_k(w_k^2) = (0,0)\in H=\mathbb{Z}_2^2$, the tuple

$$(1,\dots,1,w_k^2,1\dots,1)$$

belongs to $\mathbb{G}$, and to prove the claim it suffices to show that this tuple is contained in $\operatorname{Fix}(\mathbb{G})$.

Note that the number of occurrences n_i of the letter c_i in the word w_k^2 is even. Observe now, that in any group a product $a\cdot b$ can be written as $b\cdot(b^{-1}\cdot a\cdot b)$, hence we can write w_k^2 as

$$w_k^2 = \left(\prod_{i=1}^{n_1} g_i^{-1}c_1g_i\right)\cdot\ldots\cdot\left(\prod_{j=1}^{n_6} h_j^{-1}c_6h_j\right), \tag{1}$$

for certain $g_i,\dots,h_j\in\mathbb{T}_k$.

By Remark 4 (ii) and since H is abelian, we get $(c_1,\dots,c_1,g_i^{-1}c_1g_i,c_1,\dots,c_1)\in\operatorname{Fix}(\mathbb{G})$. We conclude that

$$(1,\dots,1,\prod_{i=1}^{n_1} g_i^{-1}c_1g_i,1\dots,1) = \prod_{i=1}^{n_1}(c_1,\dots,c_1,g_i^{-1}c_1g_i,c_1,\dots,c_1)\in\operatorname{Fix}(\mathbb{G})\,.$$

The same applies to each factor in the RHS of (1) and so $(1,\dots,1,w_k^2,1\dots,1)\in\operatorname{Fix}(\mathbb{G})$. This shows $g^2=1$, whence $\pi_1(Y_{n,4})$ is abelian, and it is finite since $\pi_1(Y_{n,4}) = \pi_1(Y_{n,4})^{ab} = H_1(Y_{n,4},\mathbb{Z})$ has rank 0.

(ii) We implemented Proposition 4 using the computer algebra system MAGMA [9], and we found $\pi_1(Y_{n,4}) = \mathbb{Z}_2^{n-1}$ and $\pi_1(X_{n,4}) = \mathbb{Z}_2^{n+1}$ for $n=2,3,4,5$. In particular, the universal cover of the varieties $X_{n,4}$ and $Y_{n,4}$ has $3\cdot 2^{3n-2}$ singularities of type $\frac{1}{2}(1,\dots,1)$. We expect the above to generalize to any dimension.

Remark 8 The surfaces $\widehat{X_{2,d}}$ with $d\geq 8$ are rigid but not infinitesimally rigid (see [6]), and their universal cover is non-contractible. This answer partially the question posed in the Introduction in the case of surfaces.

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