Properties of the Capacity-Achieving Input of Non-Coherent Rayleigh Fading Channels

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***Abstract*—This work studies non-coherent Rayleigh fading channels subject to average- and peak-power constraints. Several properties of the optimal input distribution are derived based on the Karush-Kuhn-Tucker conditions. In particular, the capacity-achieving distribution is characterized in the small peak and average power regimes, upper and lower bounds on the optimal input probabilities are presented, insights about the locations of the support points are provided, and bounds on the channel capacity are established.**

I. Introduction

Rayleigh fading models help to characterize the impact of multipath propagation in wireless communication. Wireless channel signals encounter multiple paths due to reflections, diffractions, and scattering, resulting in time-varying signal distortions at the receiver. In this work, we consider non-coherent models where the channel state information (CSI) is unavailable at the receiver and transmitter. Non-coherent Rayleigh fading is often observed for low-power or low-cost wireless devices, short-range wireless links, or cases where maintaining coherent detection is impractical or undesirable.

Although significant progress has been made in understanding non-coherent communication, characterizing capacity remains a formidable challenge. The objective of this study is to investigate properties of capacity-achieving input distributions when subjected to average-power and peak-power constraints. We thereby gain insight into the limits of communication. Besides the theoretical interest, properties of the optimal distributions guide the design of practical coding schemes.

A. Channel Model

The input-output relationship of a Rayleigh fading channel with additive white Gaussian noise (AWGN) is

$$V = HU + W \tag{1}$$

where $H \sim \mathcal{CN}(0, \sigma_H^2)$, $W \sim \mathcal{CN}(0, \sigma_W^2)$, and the input U are mutually statistically independent. Neither the transmitter nor receiver knows the value of H, but both know the statistics of H. The input is subject to both average-and peak-power constraints given by

$$\mathbb{E}\left[|U|^2\right] \le \tilde{\mathsf{P}}, \tag{2}$$

$$\mathbb{P}\left[|U| \le \tilde{\mathsf{A}}\right] = 1 \left(\text{ or equiv. } |U| \le \tilde{\mathsf{A}} \text{ a.s.}\right) \tag{3}$$

for some $0 \le \tilde{\mathsf{P}}, \tilde{\mathsf{A}} \le \infty$. The capacity of the channel is

$$C_R(\tilde{\mathsf{P}}, \tilde{\mathsf{A}}) = \max_{P_U :\, \mathbb{E}[|U|^2] \le \tilde{\mathsf{P}},\, |U| \le \tilde{\mathsf{A}}} I(U; V). \tag{4}$$

B. Contributions and Paper Outline

This paper is organized as follows. Sec. II surveys the literature and Sec. III derives equivalent channel models and reviews the Karush-Kuhn-Tucker (KKT) conditions. Sec. IV presents our main results: a characterization of the capacity-achieving distribution in the small (but non-vanishing) peak and average power regime, upper bounds on the values of probability masses of the optimal input distribution, lower bounds on the probabilities of the maximum point and zero point of the capacity achieving distribution, and results on the location of the support points. Most proofs are relegated to the extended version of the paper [1] due to space constraints. Sec. V concludes the paper.

C. Notation

All logarithms are to the base e. Deterministic scalar quantities are denoted by lower-case letters and random variables by uppercase letters. For a random variable X and any measurable set $\mathcal{A} \subseteq \mathbb{R}$ we write the probability distribution of X by $P_X(\mathcal{A}) = \mathbb{P}[X \in \mathcal{A}]$. The support set of P_X is

$$\mathsf{supp}(P_X) = \{x : \text{for every open set } \mathcal{D} \ni x \text{ we have that } P_X(\mathcal{D}) > 0\}. \tag{5}$$

When X is a discrete random variable, we write $P_X(x)$ for $P_X(\{x\})$, i.e., P_X is understood as a probability mass function (pmf). The relative entropy between distributions P and Q is denoted by $D\left(P \,\|\, Q\right)$.

II. Background

The capacity-achieving input distribution of an AWGN channel with an average-power constraint is Gaussian [2]. Interestingly, if the average-power constraint is replaced by a peak-power constraint, then the capacity-achieving distribution is discrete. This result was shown by Smith in [3] who introduced a new approach connecting the support of the capacity-achieving distribution to the zeros of analytic functions. For the Rayleigh-fading channel, the capacity-achieving distribution with an average-power constraint was conjectured to be discrete by Richter in [4] by also appealing to properties of analytic functions. This conjecture was proved by Abou-Faycal *et al.* in [5] who also showed that there is positive-probability mass point at zero input.

The capacity-achieving distribution of other fading channels is also discrete. Gursoy *et al.* [6] considered a Rician channel with an additional fourth moment constraint and showed that the capacity-achieving distribution is discrete with a finite number of points. In [7], Katz and Shamai studied noncoherent AWGN channels with an average-power constraint and showed that the optimal input amplitude is discrete with an infinite number of mass points. Other channels for which either an input distribution is discrete or some component of the optimal input distribution (e.g., amplitude) is discrete include symmetric coherent vector additive Gaussian channels [8]–[11]; noncoherent block-independent AWGN channels [12]; and Poisson channels [13]. Attempts to generalize these results to additive channels are described in [14]–[17]. Generalizations to multi-user channels such as multiple access and wiretap channels can be found in [18] and [19]–[21], respectively. We refer the reader to [22] for a summary of related results.

Recently, alternative techniques were applied to refine our knowledge about the structure of the capacity-achieving distribution. For instance, [23] introduced two new techniques to upper and lower bound the probabilities of the support points; one of the methods relies on the strong data-processing inequality. Another method to bound the probabilities can be found in [24].

Characterizing the capacity-achieving input is often easier in the low-power regime. For example, several papers show that the optimal input distribution (or its magnitude for vector AWGN channel) in the low peak-power regime is supported on only two points [3], [10], [25]–[27]. We remark that all of these results hold in the small but non-vanishing peak-power regime. In contrast, for the non-coherent Rayleigh fading channel, to the best of our knowledge it has not been shown that two-point distributions are optimal at low power. Existing results show only that, as the power vanishes, two-point distributions attain the same first and second-order terms of the Taylor expansion of the capacity around zero power [28]. These binary distributions are called *flash-signals* in [28] since the zero signal has large probability and one extremely large signal value has a vanishing probability as the average power approaches infinity. In this work, we show that binary inputs are indeed optimal for small but non-vanishing power.

Bounds on the capacity of non-coherent fading channels have also been considered in [29]–[31]; the interested reader is referred to [32], [33] for a literature review. It is known that the capacity in low-power regime scales as P and in the large power regime scales as $\log\log \mathsf{P}$ where P denotes power.

III. Channel Models and KKT Conditions

This section presents two equivalent channel models that we use in our analysis. We also present auxiliary results and tools such as the KKT conditions, and we briefly argue why including an additional peak-power constraint is reasonable.

A. Equivalent Channel Models

We present a normalization of the Rayleigh channel model which lets us focus on the distribution of $X \triangleq |U|\frac{\sigma_H}{\sigma_W}$.

Proposition 1. *Let*

$$Y = |\tilde{H}X + \tilde{W}|^2 \tag{6}$$

where $\tilde{H} \sim \mathcal{CN}(0,1)$ and $\tilde{W} \sim \mathcal{CN}(0,1)$. Then, the capacity of the Rayleigh channel model in (1) *with average-power and peak-power constraints as in* (2) *and* (3) *can be equivalently expressed as the capacity of the channel in* (6)*:*

$$C(\tilde{\mathsf{P}},\tilde{\mathsf{A}}) = \max_{P_X:\, \mathbb{E}[X^2]\le \mathsf{P},\, X\le \mathsf{A}} I(X;Y) \triangleq C(\mathsf{P},\mathsf{A}), \tag{7}$$

where $\mathsf{P} \triangleq \frac{\sigma_H^2}{\sigma_W^2}\tilde{\mathsf{P}}$ and $\mathsf{A} \triangleq \frac{\sigma_H^2}{\sigma_W^2}\tilde{\mathsf{A}}$. Moreover, the capacity-achieving inputs and outputs of both channels can be related as follows:

$$|Y^\star| = \frac{|V^\star|^2}{\sigma_W^2}, \text{ and } |X^\star| = |U^\star|\frac{\sigma_H}{\sigma_W}. \tag{8}$$

We now propose a channel model based on Prop. 1. This model will be useful for studying capacity indirectly.

Proposition 2. *The capacity of the channel model of* (6) *is equivalent to the capacity of the exponential model*

$$Y = \frac{1}{S}T \tag{9}$$

where $T \sim Exp(1)$ and S are independent random variables, with input constraints

$$\mathbb{E}\left[\frac{1}{S}\right] \le 1+\mathsf{P}, \quad \frac{1}{1+\mathsf{A}^2} \le S \le 1. \tag{10}$$

The capacity is

$$C(\mathsf{P},\mathsf{A}) = \max_{P_S:\, \mathbb{E}\left[\frac{1}{S}\right]\le 1+\mathsf{P},\, S\in\left[\frac{1}{1+\mathsf{A}^2},1\right]} I(S;Y). \tag{11}$$

Proof. The channel transition law $f_{Y|X}$ is an exponential probability density function (pdf) of variable y with parameter $s = \frac{1}{1+x^2}$, i.e.,

$$f_{Y|S}(y|s) = s\exp\left(-sy\right), \qquad y \ge 0. \tag{12}$$

Hence, conditioned on $S = s$, we can write $Y = \frac{1}{s}T$ where $T \sim \text{Exp}(1)$. The other results follow easily. □

B. KKT Conditions

We next introduce the Karush-Kuhn-Tucker (KKT) conditions for the optimality of the input distributions P_X and P_S of the Rayleigh and exponential models, respectively.

Lemma 1. *Let $P_{X^\star}$ be an input distribution of the Rayleigh channel model of Prop. 1, and let $f_{Y^\star}$ be the induced optimal output pdf. Then, $P_{X^\star}$ is capacity-achieving if and only if there exists $\lambda \ge 0$ such that*

$$D\left(f_{Y|X}(\cdot|x)\,\middle\|\, f_{Y^\star}\right) \le C(\mathsf{P},\mathsf{A}) + \lambda(x^2-\mathsf{P}),\ x \in [0,\mathsf{A}], \tag{13}$$

$$D\left(f_{Y|X}(\cdot|x)\,\middle\|\, f_{Y^\star}\right) = C(\mathsf{P},\mathsf{A}) + \lambda(x^2-\mathsf{P}),\ x \in \mathsf{supp}(P_{X^\star}) \tag{14}$$

Proof. See Proposition 2 and [5]. □

Lemma 2. *Let $P_{S^\star}$ be an input distribution of the exponential model of Prop. 2, and let $f_{Y^\star}$ be the induced optimal output pdf. Then, $P_{S^\star}$ is capacity-achieving if and only if there exists $\lambda \geq 0$ such that*

$$\begin{aligned} &D\left(f_{Y|S}(\cdot|s) \,\middle\|\, f_{Y^\star}\right) \\ &\quad \leq C(\mathsf{P},\mathsf{A}) + \lambda\left(\frac{1}{s} - 1 - \mathsf{P}\right),\ s \in \left[\frac{1}{1+\mathsf{A}^2}, 1\right], \quad (15) \\ &D\left(f_{Y|S}(\cdot|s) \,\middle\|\, f_{Y^\star}\right) \\ &\quad = C(\mathsf{P},\mathsf{A}) + \lambda\left(\frac{1}{s} - 1 - \mathsf{P}\right),\ s \in \mathsf{supp}(P_{S^\star}). \quad (16) \end{aligned}$$

Proof. See [5]. □

Note that the Lagrange multiplier λ that accounts for the average-power constraint is the same for the different formulations of the KKT conditions. Indeed, λ depends only on the capacity as

$$\lambda = \frac{\partial}{\partial \mathsf{P}} C(\mathsf{P},\mathsf{A}) \tag{17}$$

and it is a function of P and A.

Next, we introduce the following functions:

$$\Xi_R(x; f_{Y^\star}) \triangleq D\left(f_{Y|X}(\cdot|x) \,\middle\|\, f_{Y^\star}\right) - C(\mathsf{P},\mathsf{A}) - \lambda(x^2 - \mathsf{P}), \tag{18}$$

$$\begin{aligned} \Xi_E(s; f_{Y^\star}) \triangleq{}& D\left(f_{Y|S}(\cdot|s) \,\middle\|\, f_{Y^\star}\right) - C(\mathsf{P},\mathsf{A}) \\ &- \lambda\left(\frac{1}{s} - 1 - \mathsf{P}\right), \end{aligned} \tag{19}$$

that let us rewrite the KKT conditions as follows:

$$\text{Rayleigh Model: } \begin{cases} \Xi_R(x; f_{Y^\star}) \leq 0 & x \in [0, \mathsf{A}] \\ \Xi_R(x; f_{Y^\star}) = 0 & x \in \mathsf{supp}(P_{X^\star}) \end{cases} \tag{20}$$

$$\text{Exponential Model: } \begin{cases} \Xi_E(s; f_{Y^\star}) \leq 0 & s \in \left[\frac{1}{1+\mathsf{A}^2}, 1\right] \\ \Xi_E(s; f_{Y^\star}) = 0 & s \in \mathsf{supp}(P_{S^\star}) \end{cases} \tag{21}$$

While the KKT conditions refer to channels with the same capacity, certain proofs are easier with one or another model. We will convert and report the results in the Rayleigh model. We also distinguish between two different types of zeros.

Definition 1. *Suppose that $s^\star$ is a zero of $\Xi_E(s; f_{Y^\star})$. Then, $s^\star$ is called a* non-nodal zero *if and only if both of the following two conditions hold:*

- *$s^\star \in (0,1)$ (i.e., $s^\star$ is not at the boundary of the support); and*
- *$\Xi_E'(s^\star; f_{Y^\star}) = 0$ (i.e., $s^\star$ is critical point) .*

Otherwise, $s^\star$ is referred to as nodal zero.

Remark 1. *$s^\star = 1$ is a nodal zero.*[1]

Remark 2. *All internal global maxima of Ξ_E are non-nodal zeros, and we have $\Xi_E''(s^\star; f_{Y^\star}) \leq 0$.*

[1]In [5], it was shown that $s^\star = 1$ is a support point.

Remark 3. *The point $s^\star = \frac{1}{1+\mathsf{A}^2}$ may or may not be a support point; when it is, it may or may not be a nodal zero. For example, if $\mathsf{P} = \infty$, then $s^\star = \frac{1}{1+\mathsf{A}^2}$ is a nodal zero. In addition, if A^2 is much larger than P, then $s^\star = \frac{1}{1+\mathsf{A}^2}$ might not be a support point.*

C. Why Peak Power is Needed

For non-coherent channels subject only to the average-power constraint, *flash signaling* achieves the capacity slope for $\mathsf{P} \to 0$ [28]. Flash signaling is not peak-power limited, i.e., the maximum support point $x^\star_{\max}$ converges to infinity as P goes to zero.

Proposition 3. *Let $P_{X^\star}$ be the capacity-achieving input distribution, and let*

$$x^\star_{\max}(\mathsf{P},\mathsf{A}) = \arg\max\{\mathsf{supp}(P_{X^\star})\} \tag{22}$$

be its largest amplitude. Then we have

$$\lim_{\mathsf{P}\to 0}\lim_{\mathsf{A}\to\infty} x^\star_{\max}(\mathsf{P},\mathsf{A}) = \infty, \quad \lim_{\mathsf{P}\to 0}\lim_{\mathsf{A}\to\infty} P_{X^\star}(x^\star_{\max}) = 0. \tag{23}$$

On the other hand, for $\mathsf{A} \to \infty$ and for P as small as 10^{-6}, we compute $x^\star_{\max} \approx 2.6746$. A similar argument can be made for those P's at which a new mass point appears in $P_{X^\star}$. Since at those P's the evolution of the capacity-achieving input distribution is abrupt, considering a peak power constraint has several advantages. First, the constraint is physically reasonable and, for sufficiently large $\mathsf{A} < \infty$, the capacity is almost identical to the capacity with only the average-power constraint while being mathematically more tractable. However, many of our results will hold for $\mathsf{A} = \infty$.

IV. Main Results

The next theorem presents our main results. Let $x^\star_{\max} = \max \mathsf{supp}(P_{X^\star})$ and $\zeta = \frac{\mathbb{E}\left[(1+(X^\star)^2)^{-1}\right]}{\mathbb{E}\left[(1+(X^\star)^2)^{-2}\right]}$.

Theorem 1. *Suppose $\mathsf{P} \in (0,\infty]$ and $\mathsf{A} \in (0,\infty]$, and let $P_{X^\star}$ be the capacity-achieving input distribution. Then:*

1) (Small Average- and Peak-Power Regime)
   - *If $\mathsf{A}^2 \leq \mathsf{P}$, then $\lambda = 0$ (i.e., the average-power constraint is not active) and $x^\star_{\max} = \mathsf{A}$;*
   - *If $\mathsf{A}^2 < \sqrt{2}$, then $\lambda = 0$ or $\mathsf{supp}(P_{X^\star}) = \{0, \mathsf{A}\}$;*
   - *If $\lambda = 0$, then $\mathsf{supp}(P_{X^\star}) = \{0, \mathsf{A}\}$ or $\mathsf{A}^2 \geq \frac{\mathrm{e}}{3-\mathrm{e}}$.*
2) (On the Location of Support Points)
   - *$0 \in \mathsf{supp}(P_{X^\star})$;*
   - *If the average-power constraint is active, then*

$$\begin{aligned} \mathsf{P} \leq \frac{\mathsf{P}}{1 - P_{X^\star}(0)} &\leq (x^\star_{\max})^2 \\ &\leq \min\left\{\mathsf{A}^2, \frac{1+\mathsf{P}}{P_{X^\star}(x^\star_{\max})} - 1\right\}, \end{aligned} \tag{24}$$

$$\zeta - 1 \leq (x^\star)^2, \qquad x^\star \in \mathsf{supp}(P_{X^\star}) \setminus \{0\}. \tag{25}$$

Also, if a non-nodal support point $x^\star \neq 0$ exists, then

$$\max\left\{\sqrt{2}, \frac{3}{2}\zeta - 1\right\} \leq (x^\star_{\max})^2, \tag{26}$$

$$\frac{\sqrt{2}}{1-\frac{1}{1+(x^\star_{\max})^2}}-1\le (x^\star)^2,\quad x^\star\in\mathsf{supp}(P_{X^\star})\setminus\{0\}. \tag{27}$$

- *If the average-power constraint is not active, then*

$$\zeta-1\le (x^\star)^2\le \mathsf{A}^2,\qquad x^\star\in\mathsf{supp}(P_{X^\star})\setminus\{0\}. \tag{28}$$

Also, if a non-nodal support point $x^\star\neq 0$ exists, then

$$(x^\star)^2\le\frac{\mathsf{A}^2-1}{2},\qquad x^\star\in\mathsf{supp}(P_{X^\star})\setminus\{0\}. \tag{29}$$

3) (Upper Bounds on the Probabilities) *Let* $\tilde{\eta}=\frac{\mathsf{A}^2}{(1+\mathsf{A}^2)^{1+\frac{1}{\mathsf{A}^2}}}$. *Then:*

- *If the average-power constraint is not active:*

$$P_{X^\star}(x^\star)\le \mathrm{e}^{-\frac{C(\mathsf{P},\mathsf{A})}{\tilde{\eta}}},\qquad x^\star\in\mathsf{supp}(P_{X^\star}); \tag{30}$$

- *If the average-power constraint is active:*

$$P_{X^\star}(x^\star_{\max})\le \mathrm{e}^{-\frac{C(\mathsf{P},\mathsf{A})}{\tilde{\eta}}}. \tag{31}$$

Moreover, if $\mathsf{A}=\infty$, *then:*

$$P_{X^\star}(x^\star_{\max})\le \mathrm{e}^{-\frac{C(\mathsf{P},\mathsf{A})-\frac{(1+\mathsf{P})}{1+(x^\star_{\max})^2}+1}{\tilde{\eta}}}\le \mathrm{e}^{-\frac{C(\mathsf{P},\mathsf{A})}{\tilde{\eta}}}. \tag{32}$$

- *Location dependent bound:*

$$P_{X^\star}(x^\star)\le\frac{\min\{\mathsf{P},\mathsf{A}^2\}}{(x^\star)^2},\qquad x^\star\in\mathsf{supp}(P_{X^\star}). \tag{33}$$

Moreover, if the average-power constraint is active, then

$$P_{X^\star}(0)\le 1-\frac{\mathsf{P}}{(x^\star_{\max})^2}\le 1-\frac{\mathsf{P}}{\mathsf{A}^2}. \tag{34}$$

4) (Lower Bounds on the Probabilities)

- *If the average-power constraint is active and there exists a non-nodal support point, then*

$$P_{X^\star}(0)\ge 1-\frac{\min\{\mathsf{P},\mathsf{A}^2\}}{\sqrt{2}-1}; \tag{35}$$

- *If the average-power constraint is not active and there exists a non-nodal support point, then*

$$P_{X^\star}(x^\star_{\max})\ge\frac{0.3}{(\mathsf{A}^2+1)\mathsf{A}^2\mathrm{e}^{\mathsf{A}^2+1}(1-\mathrm{e}^{-1})}. \tag{36}$$

The results in Theorem 1 depend on the value $C(\mathsf{P},\mathsf{A})$ and on the Lagrange multiplier λ which are unknown. Therefore, it is useful to have upper and lower bounds.

Proposition 4. *Fix some* $\mathsf{P}\in(0,\infty]$ *and* $\mathsf{A}\in(0,\infty]$. *Then,*

$$\underline{C}(\mathsf{P}_\mathsf{A})\le C(\mathsf{P},\mathsf{A})\le\overline{C}(\mathsf{P}_\mathsf{A}), \tag{37}$$

where $\mathsf{P}_\mathsf{A}:=\min(\mathsf{P},\mathsf{A}^2)$ *and*

$$\overline{C}(\mathsf{P}_\mathsf{A})=\Big\{\log(1+\mathsf{P}_\mathsf{A}), \tag{38}$$
$$\log(1+\log(1+\mathsf{P}_\mathsf{A}))+1+\Gamma\left(1-\frac{1}{1+\log(1+\mathsf{P}_\mathsf{A})}\right)\Big\},$$
$$\underline{C}(\mathsf{P}_\mathsf{A})=\log\left(\sqrt{(\mathrm{e}^{-\gamma-1}\log(\mathsf{P}_\mathsf{A}+1))^2+1}\right)$$
$$\ge\max\{0,\log(\log(1+\mathsf{P}_\mathsf{A})))-\gamma-1\}, \tag{39}$$

where Γ is the Gamma function and $\gamma\approx 0.577$ is the Euler-Mascheroni constant. Moreover, the gap between the bounds is at most 4 nats.

We emphasize that Proposition 4 does not give the tightest possible bounds but shows how capacity scales in the large power regime.

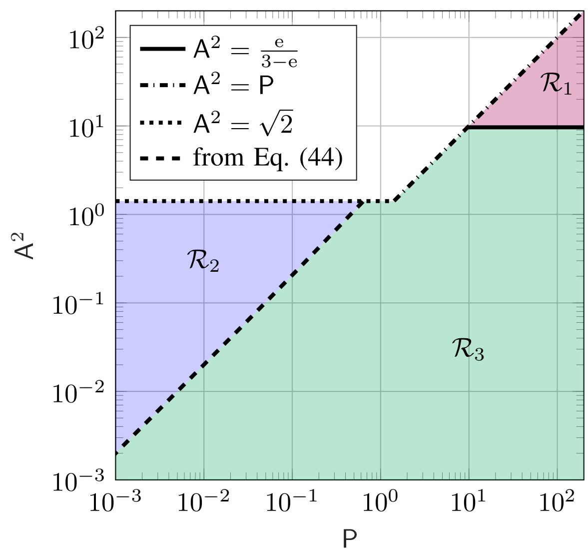

Fig. 1. Small average- and peak-power regime regions.

Proposition 5. *If the average-power constraint is active, then:*

- $\mathsf{P}\mapsto\lambda$ *is a decreasing function of* P*;*
- *If* $\mathsf{A}<\infty$, *then*

$$0\le\lambda\le\frac{C(\mathsf{P},\mathsf{A})}{\mathsf{P}}\le 1; \tag{40}$$

Also, if there exists a non-nodal support point $x^\star$, then

$$\lambda\ge\frac{3\mathrm{e}^{-1}}{1+(x^\star_{\max})^2}-\zeta(3\mathrm{e}^{-1}-1). \tag{41}$$

- *If* $\mathsf{A}=\infty$, *then*

$$0\le\frac{1}{1+(x^\star_{\max})^2}\le\lambda\le\frac{C(\mathsf{P},\mathsf{A})}{\mathsf{P}}\le 1. \tag{42}$$

A. Discussion

Fig. 1 subdivides the space spanned by P and A^2 into regions, each characterized by specific properties. The region $\mathcal{R}_0$ includes the pairs $(\mathsf{P},\mathsf{A}^2)$ outside of the Small Average- and Peak-Power Regime defined in Theorem 1. The union $\bigcup_{i=1}^3\mathcal{R}_i$ constitutes the Small Average- and Peak-Power Regime region. To better understand when the average-power constraint is active, consider the following result.

Lemma 3. *Suppose* $\mathsf{supp}(P_{X^\star})=\{0,\mathsf{A}\}$ *and define*

$$\rho:=\mathbb{E}\left[\log f_Y(Y)\mid X=0\right]-\mathbb{E}\left[\log f_Y(Y)\mid X=\mathsf{A}\right]$$
$$-\log(1+\mathsf{A}^2) \tag{43}$$

$$\int_0^\infty \left(\frac{1}{1+\mathsf{A}^2} \mathrm{e}^{-\frac{y}{1+\mathsf{A}^2}} - \mathrm{e}^{-y} \right) \log \left((1 - P_{X^\star}(\mathsf{A}))\mathrm{e}^{-y} + \frac{P_{X^\star}(\mathsf{A})}{(1+\mathsf{A}^2)} \mathrm{e}^{-\frac{y}{1+\mathsf{A}^2}} \right) \mathrm{d}y + \log(1+\mathsf{A}^2) = 0. \tag{44}$$

TABLE I
PROPERTIES OF THE SMALL-POWER REGIONS.

Region	Average-power constraint	$P_{X^\star}$ properties
$\mathcal{R}_1$	OFF	$\max \mathsf{supp}(P_{X^\star}) = \mathsf{A}$
$\mathcal{R}_2$	ON	$\mathsf{supp}(P_{X^\star}) = \{0, \mathsf{A}\}$
$\mathcal{R}_3$	OFF	$\mathsf{supp}(P_{X^\star}) = \{0, \mathsf{A}\}$

where

$$f_Y(y) = \left(1 - \frac{\mathsf{P}}{\mathsf{A}^2}\right) \mathrm{e}^{-y} + \frac{\mathsf{P}}{\mathsf{A}^2} \frac{1}{1+\mathsf{A}^2} \mathrm{e}^{-\frac{y}{1+\mathsf{A}^2}}, \qquad y \geq 0. \tag{45}$$

If $\rho > 0$, then the average-power constraint is active with $\lambda = \frac{\rho}{\mathsf{A}^2}$ and $P_{X^\star}(\mathsf{A}) = \frac{\mathsf{P}}{\mathsf{A}^2}$. Otherwise, if $\rho \leq 0$, the average-power constraint is not active ($\lambda = 0$) and $P_{X^\star}(\mathsf{A})$ is a solution to the equation in (44)*.*

By combining Theorem 1 and Lemma 3, the regions $\mathcal{R}_1$ to $\mathcal{R}_3$ are characterized by properties summarized in Table I.

B. Numerical Results for the Optimal Input Distribution

We provide further insights on $P_{X^\star}$ and its support for the regions $\mathcal{R}_0$–$\mathcal{R}_3$ of Fig. 1. For some values of A^2, we evaluated an estimate $\hat{P}_{X^\star}$ of the capacity-achieving distribution by adapting the numerical algorithm in [21] to the Rayleigh fading channel. The new algorithm iteratively updates a tentative pmf $\hat{P}_{X^\star}$ and stops when the KKT conditions in (20) are satisfied within a small tolerance ε, see [21, Sec. 5.1].

Fig. 2 shows the support of the pmf estimate $\hat{P}_{X^\star}$ vs. P, for $\mathsf{A}^2 = 1, 6.25, 100$. The size of the blue dots qualitatively shows the probability associated with each mass point. Comparing Fig. 2 to Fig. 1 shows that our definition of the small power regime regions $\mathcal{R}_i$'s is conservative, especially for $\mathcal{R}_1$, and that one could improve the characterization of $\mathcal{R}_1$ and $\mathcal{R}_3$. Moreover, as expected, for each A^2 the pmf estimates do not change in the regions with inactive average power constraint, i.e., in $\mathcal{R}_1$ and $\mathcal{R}_3$. Finally, for the wide range of points given by $\mathsf{P}, \mathsf{A}^2 \leq 200$, we have $|\mathsf{supp}(P_{X^\star})| = 2$.

V. CONCLUSION

We studied non-coherent Rayleigh channels and developed properties of the capacity-achieving input distribution based on the KKT conditions. Both average- and peak-power constraints were considered. Our main focus was on small, but non-vanishing, average- and peak powers. We showed that the capacity-achieving distribution is binary for sufficiently small values of the power constraints. We also derived upper and lower bounds on the optimal input probability values, provided insight into the mass point positions, and derived bounds on the channel capacity.

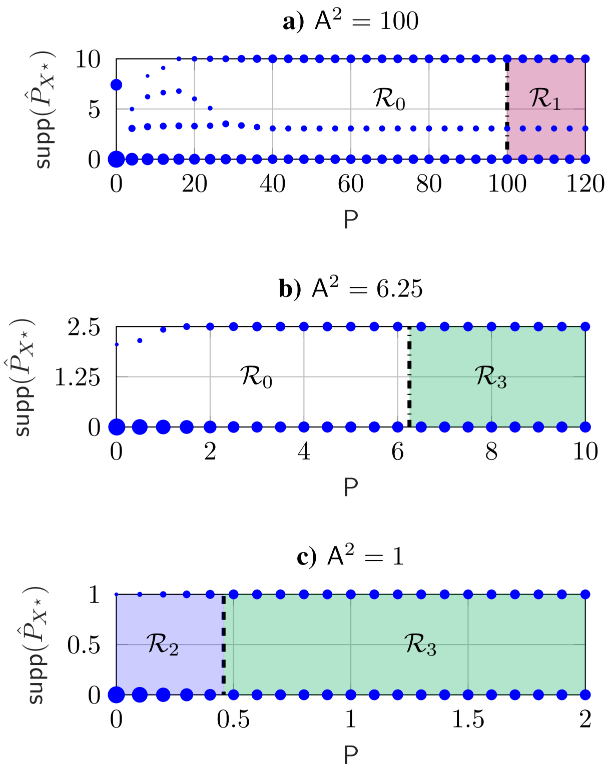

Fig. 2. Approximate optimal pmf support vs. P for $\mathsf{A}^2 = 100, 6.25, 1$.

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