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Some explicit solutions for nonlinear elastic depressed masonry arches loaded to collapse

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SUMMARY**.** In this paper the equilibrium problem for a masonry arch subject to vertical load is addressed. The arch is modelled as a curved, one-dimensional nonlinear elastic beam. The equilibrium problem is formulated in terms of nonlinear ordinary differential equations. In the case of a depressed arch subject to a uniformly distributed load, we show that it is possible to arrive at an explicit, albeit approximate, solution in terms of the displacements and rotations, by making a small number of simple hypotheses. The solution method is here concisely described for both statically determinate and indeterminate cases. Further details on this solution method along with the complete set of the analytical expressions here presented will be provided in forthcoming papers.

1 INTRODUCTION

Nowadays, nonlinear elastic models are widely used analysis tools to describe the mechanical response of masonry elements and structures. Following the pioneering work of Signorini [1, 2], many scholars mostly since the ’80 of last century, have contributed to this line of research. Within the context of nonlinear elasticity, the solution to equilibrium problems for masonry structures is usually sought via some numerical methods, such as FEM, BEM, DEM and so on. Here, on the contrary, we aim at illustrating how some explicit analytical solutions for depressed masonry arches may be found. The arch is schematised as a one-dimensional, nonlinear curved elastic beam, made of a material that is by hypothesis incapable of withstanding significant tensile stresses and with a limited compressive strength. By this way, describing the arch’s mechanical behaviour becomes a matter of solving nonlinear ordinary differential equations [3, 4, 5, 6, 7, 8]. We show that it is possible to find the explicit, albeit approximate, expressions for the displacements and rotations of the cross-sections of a depressed masonry arch subject to a uniformly distributed load, under a small number of simple hypotheses. The extension of the proposed method to general constraint conditions is briefly illustrated.

2 A ONE-DIMENSIONAL, NONLINEAR ELASTIC MODEL FOR MASONRY ARCHES

The masonry arch is here modelled as a deformable curved beam. Starting from a Signorini’s idea [1, 2], for each longitudinal fiber we adopt the following piecewise-linear constitutive relation between the longitudinal strain, ε_θ, and stress, σ_θ:

$$\sigma_\theta = \begin{cases} \sigma_c & \varepsilon_\theta \le \varepsilon_c, \\ E\varepsilon_\theta & \varepsilon_c < \varepsilon_\theta < \varepsilon_t, \\ \sigma_t & \varepsilon_\theta \ge \varepsilon_t, \end{cases} \tag{1}$$

where E, σ_t and σ_c are the masonry Young's modulus and tensile and compressive strength. For the notation here adopted reference is made to [8].

Under this hypothesis, and by making the usual assumptions in the theory of bending of beams, it is possible to subdivide the elastic domain drawn in the plane of the internal forces (n, m), where $n = N / \sigma_c h$ and $m = -M / \sigma_c h^2$ are the dimensionless axial force and bending moment, h is the thickness of the cross-section, into seven regions each characterised by a different nonlinear relation (Fig. 1a) between (n, m) and the axial strain and curvature (ε, χ).

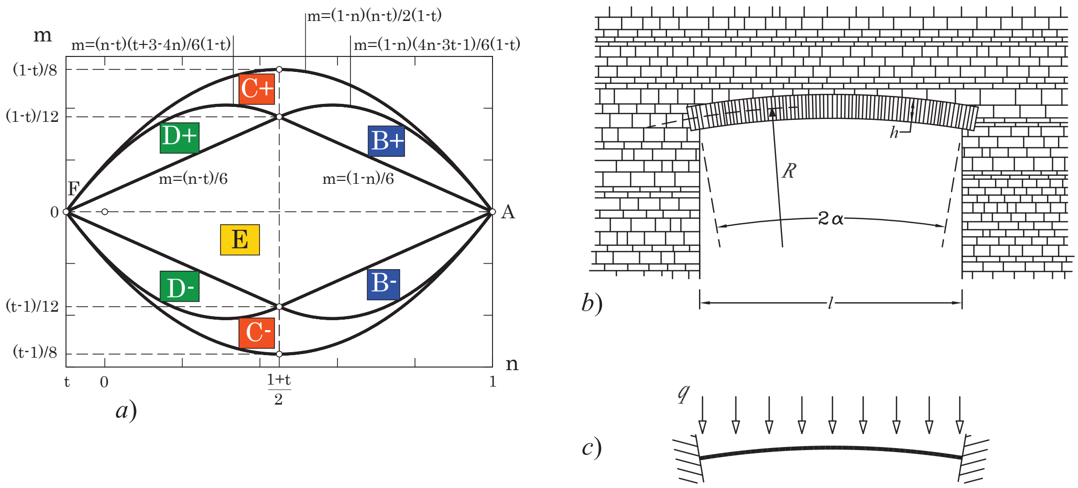


Figure 1. The elastic domain for the arch cross-section (*a*); the depressed circular arch (*b*, *c*).

It can be shown that, by using the one-dimensional nonlinear elastic model, the elastic equilibrium problem for the arch can be written in terms of simple ordinary differential equations [4]. An explicit solution can be found for simple cases, such as uniformly loaded flat or depressed arches [6, 8].

3 THE DEPRESSED CIRCULAR ARCH

Let us consider the depressed circular arch shown in Figure 1b. We indicate with h, l, R and 2α the thickness of the cross-section, the clear span, the radius and the centre angle of the arch's line of axis, respectively, and we assume by hypothesis that the *ratio l/R is small* with respect to unity.

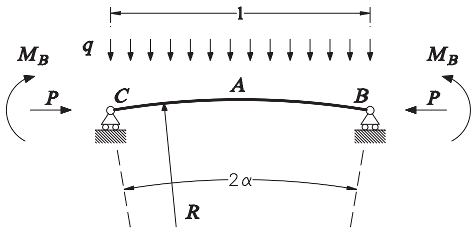
We model the masonry arch as a nonlinear elastic curved beam with clamped ends. For the sake of simplicity, we assume that the overlying wall applies a uniformly distributed vertical load, q, on the arch (Figure 1c). The stated equilibrium problem is statically indeterminate. Before solving it, we will illustrate the analytical method being proposed by considering a preliminary case, namely a simpler, statically determinate problem derived from the original. This problem has been examined in [8]. Here we will show that the same method, suitably enhanced with some small modifications, will also be used to solve the original statically indeterminate problem.

4 DISPLACEMENTS AND ROTATIONS FOR STATICALLY DETERMINED DEPRESSED MASONRY ARCHES

Before addressing the statically indeterminate problem, in this section we will take into consideration a statically determinate masonry arch.

Let us consider the depressed circular arch showed in Figure 2a. A uniformly distributed vertical load is exerted on the arch (applied, let's say, by some overlying wall). At both ends the horizontal thrust, P, and the couple of moment, M_B, are assigned, while the vertical component of the displacement is fully restrained (Figure 2a).

a)



b)

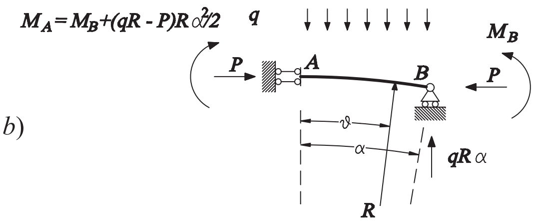


Figure 2: The depressed circular arch: a symmetrical, statically determinate problem (*a*); the corresponding mechanical scheme (*b*).

Because of symmetry, the analysis is addressed to the right half of the arch (Figure 2b). Let us indicate with θ ($0 \le \theta \le \alpha$) the angle formed between any given cross-section and the keystone section. In order to obtain explicit expressions for the displacements and rotations of the cross-sections, we make some simplifications by considering that α is sufficiently small with respect to unity. Thus, we may approximate the axial force as a constant and the bending moment according to a Taylor series up to the second order.

Since the arch is a statically determinate structure, the internal forces are known, and it is possible to establish the regions of the arch characterized by a different mechanical response (linear elastic, nonlinear in tension, etc.). Therefore, at each cross-section we can choose the suitable constitutive relation according to the elastic domain (Figure 1a), and obtain the explicit expressions for the cross-section's rotation and displacement by integration. To this end, we formally express the rotations φ and displacements, u and v, as

$$\varphi(\theta) = \varphi_0 + F(\theta), \qquad v(\theta) = (R\varphi_0 - u_0)\theta + v_0\left(1 - \frac{\theta^2}{2}\right) + G(\theta) - \theta H(\theta), \tag{2}$$

$$u(\theta)=u_0\left(1-\frac{\theta^2}{2}\right)+v_0\theta+\frac{R\varphi_0\theta^2}{2}-\frac{\theta^2}{2}H(\theta)+\theta G(\theta)+L(\theta),$$

in which

$$F(\theta)=-\int_0^\theta R\chi dt \text{ and } G(\theta)=\int_0^\theta t(R^2\chi+R\varepsilon)dt. \tag{3}$$

The expressions for $H(\theta)$ and $L(\theta)$, analogous to those for F and G, are omitted here for the sake of brevity.

The relations (2)-(3) can be integrated, thus yielding the explicit expressions for $F(\theta)$, $G(\theta)$, $H(\theta)$ e $L(\theta)$. Lastly, constants u_0, v_0 and φ_0 are determined by imposing the boundary conditions at the arch ends.

As an example, we consider the case in which the arch's line of axis may be subdivided into two parts: starting from the keystone, the first part presents a nonlinear under tension response (region D+, Figure 1a), the second one presents a linear elastic response (region E). Starting from relations (2)-(3), the following expressions can be obtained:

$$\varphi(\theta)=\begin{cases}F^{D+}(\theta), & 0\le\theta\le\theta_1,\\ F^E(\theta)+F_1, & \theta_1\le\theta\le\alpha,\end{cases} \tag{4}$$

$$v(\theta)=v_0\left(1-\frac{\theta^2}{2}\right)+\begin{cases}G^{D+}(\theta)-\theta H^{D+}(\theta), & 0\le\theta\le\theta_1,\\ G^E(\theta)+G_1-\theta[H^{D+}(\theta)+H_1], & \theta_1\le\theta\le\alpha,\end{cases} \tag{5}$$

$$u(\theta)=v_0\theta+\begin{cases}L^{D+}(\theta)-\dfrac{\theta^2H^{D+}(\theta)}{2}+\theta G^{D+}(\theta), & 0\le\theta\le\theta_1,\\ L^E(\theta)+L_1-\dfrac{\theta^2[H^E(\theta)+H_1]}{2}+\theta[G^E(\theta)+G_1], & \theta_1\le\theta\le\alpha,\end{cases} \tag{6}$$

in which, for example, $F_1=F^{D+}(\theta_1)-F^E(\theta_1)$, while G_1, H_1 and L_1 have analogous expressions omitted here for brevity.

Some of the analytical expressions appearing in (4)-(6), all functions of angle θ, have been given in [8]; the complete set will be provided in a forthcoming paper. By way of example, in region D+, we have

$$F^{D+}(\theta)=k_1\left(\frac{\arctan\theta\sqrt{a}}{\sqrt{a}}+\frac{\theta}{1+a\theta^2}\right);\ G^{D+}(\theta)=k_2\left(\frac{b}{2a^2}\ln(1+a\theta^2)+\frac{(a-b)\theta^2}{2a(1+a\theta^2)}\right)+Rt\varepsilon_c\frac{\theta^2}{2},$$

where a and b are two dimensionless parameters, while $H^{D+}(\theta)$ and $L^{D+}(\theta)$ have analogous expressions here omitted. Lastly, by imposing the proper constraint conditions, it is possible to determine v_0.

As an example, we consider a 10 m span, 100 cm thick depressed arch (Figure 2a) of constant radius R = 50 m, central angle 2α = 0.2 rad and unit width (1 m) in the transverse direction. Both the arch's end sections are subjected to a horizontal thrust, P = 2000 kN, and a bending moment, M_B = -100 kNm. We suppose that a vertical load per unit length of the horizontal projection of the line of axis, q = 120 kN/m, is uniformly distributed throughout the arch. Lastly, we assume that E = 7 GPa is the masonry Young's modulus and that the masonry tensile and compressive strengths are equal to σ_t = 0.3 MPa and σ_c = - 20 MPa, respectively. The diagram of transverse displacement of the line of axis, v, is plotted in Figure 3. In the case under examination it is an easy matter to verify that the nonlinear response predicted by this model and the one that would be obtained by assuming linear elastic behavior largely differ.

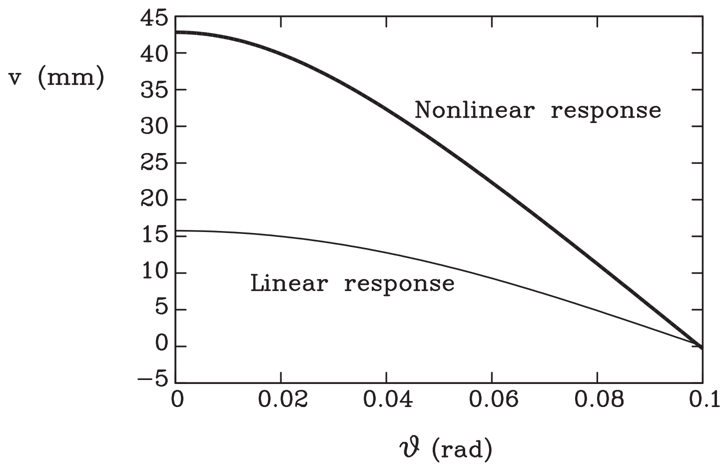


Figure 3: Transverse displacements of the right side of the line of axis.

5 THE SOLUTION METHOD FOR STATICALLY INDETERMINATE ARCHES

In the case of a statically indeterminate arch, the determination of the arch's line of axis displacements and cross-section rotations by means of the integral expressions (2)-(3) requires the preliminary assessment of the unknown redundant reaction components. For example, the solution to the problem shown in Figure 1c requires determining two components of the unknown redundant reaction. We choose the thrust, P, and bending moment, M_B, at the end section as unknowns. The (P, M_B) values are evaluated by imposing that the end sections rotation and horizontal displacement are zero.

The nonlinear system of equations corresponding to the boundary conditions is solved via a semi-analytical method. The expressly developed incremental solution procedure enables determining the redundant reaction components and internal forces for each load step. The choice of the trial values to attribute to the unknowns, (P, M_B), is an important issue. An initial value very different from the actual one may lengthen the calculation time and render the solution process inefficient. In this regard, an effective strategy from the operational point of view is to follow an incremental loading process starting at zero.

6 SOME FURTHER REMARKS

Via the solution method here presented, the evolution of the reactive forces and the distribution of the arch regions that behave non-linearly can be accurately followed for increasing loads up to collapse. In two forthcoming papers we will show some more results regarding statically determined and indeterminate problems. More details on the semi-analytical solution procedure will be given, by providing the complete set of the analytical expressions for the response functions. Furthermore, we will show that a conventional value of the limit load can be assessed.

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