A NEW APPROACH TO MGT-THERMOVISCOELASTICITY

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Abstract. In this paper we discuss some thermoelastic and thermoviscoelastic models obtained from the Gurtin theory, based on the invariance of the entropy under time reversal. We derive differential systems where the temperature and the velocity are ruled by generalized versions of the Moore-Gibson-Thompson equation. In the one-dimensional case, we provide a complete analysis of the evolution, establishing an existence and uniqueness result valid for any choice of the constitutive parameters. This result turns out to be new also for the MGT equation itself. Then, under suitable assumptions on the parameters, corresponding to the subcritical regime of the system, we prove the exponential stability of the related semigroup.

1. Introduction

A well-known drawback of the classical Fourier heat conduction law lies in the paradox of the instantaneous propagation of thermal waves, which conflicts with the basic principles of physics. Several authors tried in the years to overcome this difficulty, and a number of alternative theories of thermoelasticity and thermoviscoelasticity have been proposed. In this direction, we may recall the theory of Lord and Shulman [22], based on the Maxwell-Cattaneo law, or the one of Green and Lindsay [10], where the authors propose a second order in time hyperbolic equation for the energy (see also [23]). Both theories are strongly hyperbolic, and predict (as desired) that thermoelastic waves propagate with finite speed.

More recently, Green and Naghdi [11, 12, 13] devised three new theories, based on the axioms of thermomechanics, referred to as type I, type II and type III, respectively. The main difference among them is determined by the considered family of independent variables. The linear version of type I agrees with classical thermoelasticity. Type II is also called thermoelasticity without energy dissipation, as the energy is conserved. This theory takes as a new independent variable the gradient of the thermal displacement, that is, the integrated temperature gradient. The most general theory is the type III one, enclosing the former two as limiting cases. Here the independent variables are the gradient of the displacement, the gradient of the thermal displacement, the gradient of the temperature and the temperature. These theories became very popular, and nowadays the number of contributions about them in the literature is huge.

At the beginning of the Seventies of the last century, Gurtin [14] proposed a thermoelastic theory based on the so-called invariance under temporal inversion of the production of the entropy. In contrast with the abovementioned theories, this one has received a limited attention. Still, we can recall the works [1, 17]. In spite of that, as we will see later,

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the linear version of the type II theory can be seen as a particular case of the Gurtin's proposal. And the same can be said about the Moore-Gibson-Thompson (MGT) theory, which is lately encountering a great success (see [5, 4, 27, 31]).

Indeed, one of our goals is to emphasize the power of the Gurtin theory, which allows to obtain many interesting differential systems, currently under deep study. Accordingly, we present some thermoelastic and thermoviscoelastic models showing that, within the framework of Gurtin, equations of MGT type arise in a natural way. Then, for the sake of simplicity, we restrict our attention to one-dimensional thermoviscoelasticity, and we prove a general well-posedness result, without any restriction on the parameters of the differential system. In the physical situation where the parameters are strictly positive, and fulfill suitable assumptions (borrowed from the MGT theory), we show that the system generates a contraction semigroup of bounded linear operators, which turns out to be exponentially stable as well. For a particular choice of the structural parameters, we rediscover thermoelasticity.

Notation. Along this work, we will denote a vector $\boldsymbol{v} = (v_1, \dots v_N) \in \mathbb{R}^N$ by its generic i^{th}-component v_i. Given any function $u = u(\boldsymbol{x}, t)$, we will write $\partial_i u$ to mean its derivative with respect to the space variable x_i, and $\dot{u}$ to mean the derivative in time. We will also employ the Einstein notation, where $\partial_i v_i = \text{div } \boldsymbol{v}$.

2. Models of MGT Type

2.1. **The Moore-Gibson-Thompson equation.** We begin by briefly discussing the Moore-Gibson-Thompson (MGT) equation, that will arise in the thermoelastic and thermoviscoelastic models treated in this paper. Written in an abstract form, it reads

$$\dddot{u} + \alpha\ddot{u} + \beta A\dot{u} + \gamma A u = 0, \tag{2.1}$$

where A is a strictly positive operator on some Hilbert space H, and $\alpha, \beta, \gamma > 0$ are given constants. The MGT equation (2.1) has been originally introduced in connection with fluids mechanics [33], ruling the evolution of the acoustic velocity potential in thermally relaxing fluids (see also [19]). Nonetheless, it serves also as a model to describe the behavior of the displacement in certain viscoelastic materials (see [3, 7, 9, 23, 28]), as well as of the temperature displacement in type III heat conduction with a relaxation parameter (see [5, 31]). The feature of (2.1) is very peculiar, and strongly depends on the choice of the three constitutive parameters. A main role is played by the so-called stability number

$$\varkappa = \alpha\beta - \gamma.$$

Indeed, although the equation is well posed for all $\alpha, \beta, \gamma > 0$, the asymptotic behavior of the solutions (or of the related energy) is dramatically different when $\varkappa$ is positive or negative. Indeed, in the more physical case where $\varkappa > 0$, referred to as the subcritical regime, the solutions exhibit an exponential decay as time goes to infinity. On the contrary, in the supercritical regime $\varkappa < 0$, there are solutions whose energy blows up exponentially fast. The critical case $\varkappa = 0$ sits in between: the equation generates a strongly continuous semigroup which can be proved to be bounded, but it is not stable, for there is a conserved pseudoenergy. We address the reader to the papers [2, 3, 6, 7, 8, 19, 20, 24, 28, 29] for more details on this topic.

2.2. **MGT-thermoelasticity and thermoviscoelasticity.** Our focus is the study of the dynamics in an elastic or a viscoelastic heat conductor of mass density $\rho > 0$ occupying a volume $\Omega \subset \mathbb{R}^N$ at rest, within the linear theory of Gurtin [14]. We present here three models, whose physical derivations will be discussed in detail in the final Appendix. The two variables in play are the (relative) temperature θ and the velocity vector u_i. The *star* and *hat* quantities appearing in the following systems are time-independent tensors, some of which required to be positive definite, and whose physical meanings are addressed in the Appendix, whereas $\tau > 0$ is a small relaxation time.

• The first model is the system of thermoelasticity without energy dissipation, otherwise called of type II (see Subsection A.1):

$$\begin{cases} \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r - l^*_{ij} \dot{\theta} \big], \\ a^* \ddot{\theta} = \partial_i \big[k^*_{ij} \partial_j \theta \big] - l^*_{ij} \partial_j \dot{u}_i. \end{cases} \tag{2.2}$$

• The second one is the general model of MGT-thermoelasticity, where the temperature obeys to an equation of MGT type (see Subsection A.2):

$$\begin{cases} \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r - l^*_{ij} \dot{\theta} \big], \\ a^* \tau \dddot{\theta} + a^* \ddot{\theta} = \partial_i \big[k^*_{ij} \partial_j \theta + \hat{k}_{ij} \partial_j \dot{\theta} \big] - l^*_{ij} (\partial_j \dot{u}_i + \tau \partial_j \ddot{u}_i). \end{cases} \tag{2.3}$$

In fact, introducing the variable $z_i = u_i + \tau \dot{u}_i$, and adding the first equation to its time-derivative multiplied by τ, the latter system can be more conveniently written in the form

$$\begin{cases} \rho \ddot{z}_i = \partial_j \big[c^*_{ijrs} \partial_s z_r - l^*_{ij} \dot{\theta} - l^*_{ij} \tau \ddot{\theta} \big], \\ a^* \tau \dddot{\theta} + a^* \ddot{\theta} = \partial_i \big[k^*_{ij} \partial_j \theta + \hat{k}_{ij} \partial_j \dot{\theta} \big] - l^*_{ij} \partial_j \dot{z}_i. \end{cases} \tag{2.4}$$

• The third model describes MGT-thermoviscoelasticity, combining two equations of MGT type, both in the velocity u_i and in the temperature θ (see Subsection A.3):

$$\begin{cases} \tau \rho \dddot{u}_i + \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r + \hat{c}_{ijrs} \partial_s \dot{u}_r \big] - \partial_j \big[l^*_{ij} \dot{\theta} + \hat{l}_{ij} \ddot{\theta} \big], \\ \hat{a} \dddot{\theta} + a^* \ddot{\theta} = \partial_i \big[k^*_{ij} \partial_j \theta + \hat{k}_{ij} \partial_j \dot{\theta} \big] - \partial_j \big[l^*_{ij} \dot{u}_i + \hat{l}_{ij} \ddot{u}_i \big]. \end{cases} \tag{2.5}$$

A particular instance of (2.5) that we want to highlight is when

$$\tau a^* = \hat{a} \qquad \text{and} \qquad \tau l^*_{ij} = \hat{l}_{ij}.$$

In that case, the system turns into

$$\begin{cases} \tau \rho \dddot{u}_i + \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r + \hat{c}_{ijrs} \partial_s \dot{u}_r \big] - \partial_j \big[l^*_{ij} \dot{\theta} + \tau l^*_{ij} \ddot{\theta} \big], \\ \tau a^* \dddot{\theta} + a^* \ddot{\theta} = \partial_i \big[k^*_{ij} \partial_j \theta + \hat{k}_{ij} \partial_j \dot{\theta} \big] - \partial_j \big[l^*_{ij} \dot{u}_i + \tau l^*_{ij} \ddot{u}_i \big]. \end{cases}$$

Here, the two MGT equations possess the same speed of diffusion $1/\tau$. The physical condition in order for the two MGT equations to stand in the subcritical regime translates into requiring that both tensors

$$\hat{c}_{ijrs} - \tau c^*_{ijrs} \qquad \text{and} \qquad \hat{k}_{ij} - \tau k^*_{ij}$$

are positive definite. If instead we have the equality

$$\hat{c}_{ijrs} = \tau c^*_{ijrs},$$

we recover (2.4) by setting $z_i = u_i + \tau \dot{u}_i$.

Summarizing, MGT-thermoviscoelasticity reduces to MGT-thermoelasticity when the MGT equation for the velocity is in the critical regime. If the equation for the temperature is critical as well, then the dissipation is completely lost, and we fall into the case of thermoelasticity of type II.

2.3. **Goal of the paper.** In this work, we will perform a detailed analysis of the one-dimensional version of (2.5), assuming also that the material is homogeneous, hence the parameters are independent of the space variable. Accordingly, (2.5) becomes

$$\begin{cases} \rho\tau\dddot{u} + \rho\ddot{u} - \hat{c}\dot{u}_{xx} - c^* u_{xx} = -l^*\dot{\theta}_x - \hat{l}\ddot{\theta}_x, \\ \hat{a}\dddot{\theta} + a^*\ddot{\theta} - \hat{k}\dot{\theta}_{xx} - k^*\theta_{xx} = -l^*\dot{u}_x - \hat{l}\ddot{u}_x. \end{cases} \tag{2.6}$$

In fact, our analysis will also cover the one-dimensional version of (2.4), as well as of (2.2), both obtained as particular cases of (2.6), for an appropriate choice of the parameters (see also the forthcoming Remarks 6.3 and 6.5).

Our main results can be subsumed as follows:

- ⋄ We prove the well-posedness of system (2.6) under the sole positivity restriction

  $$\hat{a} > 0, \quad \hat{c} > 0, \quad \hat{k} > 0.$$

  In particular, in absence of coupling (i.e., $l^* = \hat{l} = 0$), this provides a result which is new also for the MGT equation itself.
- ⋄ Assuming instead the positivity of all the parameters, and within the subcritical condition for the MGT equations involved, we show that the solution semigroup generated by (2.6) is exponentially stable.
- ⋄ With a different technique, due to the fact that now one of the equations is critical, we obtain the exponential stability of the solution semigroup generated by the one-dimensional version of the thermoelastic system (2.4).

Remark 2.1. Although here for simplicity we restrict to the one-dimensional analysis, the multiplier and the semigroup techniques used in the forthcoming proofs extend with no essential changes to the N-dimensional case at least if the material is homogeneous. In fact, we could say that the great advantage of the dimension one is mostly notational, besides a more immediate treatment of the boundary conditions.

3. MGT-Thermoviscoelasticity: The 1-D Case

Without loss of generality, we will work on the space domain $\Omega = (0, \pi)$.

3.1. **Notation.** Let $H = L^2(0,\pi)$ be the Hilbert space of square summable functions on the interval $(0,\pi)$, and let $H^1 = H_0^1(0,\pi)$ be the Sobolev space of square summable functions u on $(0,\pi)$, along with their derivatives, with the boundary condition

$$u(0) = u(\pi) = 0.$$

We denote by $\|u\|$ and $\langle u, v\rangle$ the norm and the inner product in H. In view of the Poincaré inequality

$$\|u\| \le \|u_x\|, \quad \forall u \in H^1,$$

the norm and the inner product in H^1 read $\|u_x\|$ and $\langle u_x, v_x\rangle$, respectively. The phase space of our problem will be the product Hilbert space

$$\mathcal{H} = H^1 \times H^1 \times H \times H^1 \times H^1 \times H,$$

endowed with the standard Euclidean product norm

$$\|(u,v,w,\theta,\phi,\psi)\|_{\mathcal{H}}^2 = \|u_x\|^2 + \|v_x\|^2 + \|w\|^2 + \|\theta_x\|^2 + \|\phi_x\|^2 + \|\psi\|^2.$$

3.2. **The system.** Aiming for a notation closer to the one of the vast MGT literature, and in analogy with Subsection 2.1, we divide the first equation of (2.6) by $\rho\tau$, and the second one by $\hat{a}$. Then, setting

$$\alpha = \frac{1}{\tau}, \ \ \beta = \frac{\hat{c}}{\rho\tau}, \ \ \gamma = \frac{c^*}{\rho\tau}, \ \ p = \frac{l^*}{\rho\tau}, \ \ q = \frac{\hat{l}}{\rho\tau}, \ \ \hat{\alpha} = \frac{a^*}{\hat{a}}, \ \ \hat{\beta} = \frac{\hat{k}}{\hat{a}}, \ \ \hat{\gamma} = \frac{k^*}{\hat{a}}, \ \ \eta = \frac{\rho\tau}{\hat{a}},$$

we rewrite (2.6) as

$$\begin{cases} \dddot{u} + \alpha\ddot{u} - \beta\dot{u}_{xx} - \gamma u_{xx} = -p\dot{\theta}_x - q\ddot{\theta}_x, \\ \dddot{\theta} + \hat{\alpha}\ddot{\theta} - \hat{\beta}\dot{\theta}_{xx} - \hat{\gamma}\theta_{xx} = -\eta p\dot{u}_x - \eta q\ddot{u}_x. \end{cases} \tag{3.1}$$

System (3.1) is equipped with the Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0, \tag{3.2}$$

and fulfills the initial conditions assigned at the initial time $t = 0$

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \\ w(x,0) = w_0(x), \\ \theta(x,0) = \theta_0(x), \\ \phi(x,0) = \phi_0(x), \\ \psi(x,0) = \psi_0(x), \end{cases} \tag{3.3}$$

where $u_0, v_0, w_0, \theta_0, \phi_0, \psi_0 : [0,\pi] \to \mathbb{R}$ are prescribed data. As far as the parameters in play are concerned, we assume

$$\beta, \hat{\beta}, \eta > 0 \qquad \text{and} \qquad \alpha, \hat{\alpha}, \gamma, \hat{\gamma}, p, q \in \mathbb{R}.$$

Remark 3.1. It is worth noting that the axioms of thermomechanics do not imply the positivity of γ and $\hat{\gamma}$. It is therefore relevant also from the physical viewpoint to clarify the qualitative properties of (3.1) for arbitrary values of those parameters.

Remark 3.2. With this choice of the parameters, the two MGT equations have equal wave-speed when $\alpha = \hat{\alpha}$. In which case, the one-dimensional version of the thermoelastic system (2.4) is recovered if one further assumes $p = \alpha q$ and $\alpha\beta = \gamma$, being the latter the critical condition for the first equation.

The following well-posedness theorem holds.

Theorem 3.3. *For any fixed time $T > 0$ and any vector of initial data*

$$\boldsymbol{u}_0 = (u_0, v_0, w_0, \theta_0, \phi_0, \psi_0) \in \mathcal{H},$$

the Cauchy problem (3.1)-(3.3) *admits a unique solution*

$$\boldsymbol{u}(t) = (u(t), \dot{u}(t), \ddot{u}(t), \theta(t), \dot{\theta}(t), \ddot{\theta}(t)) \in \mathcal{C}([0, T], \mathcal{H}),$$

continuously depending on $\boldsymbol{u}_0$.

Accordingly, the boundary value problem (3.1)-(3.2) generates a strongly continuous semigroup of bounded linear operators

$$S(t) : \mathcal{H} \to \mathcal{H}$$

acting by the rule

$$S(t)\boldsymbol{u}_0 = \boldsymbol{u}(t).$$

Such a semigroup, without any further assumption on the parameters, is in general unbounded.

Remark 3.4. As we said, the well-posedness is obtained without any restriction on the structural parameters, with the exception of $\beta, \hat{\beta}, \eta$, which have to be strictly positive. The condition $\eta > 0$ is necessary, otherwise the coupling is destroyed. We now see that $\beta > 0$ is necessary as well. Indeed, if we set $v = \dot{u}$, we can write the first equation of (3.1) in the form of a wave equation

$$\ddot{v} - \beta v_{xx} = f,$$

which is well-posed only if $\beta > 0$. The same argument applies for the second equation, yielding $\hat{\beta} > 0$. In particular, by choosing $p = q = 0$ in (3.1), we establish a well-posedness result for the MGT equation (2.1) within the sole assumption $\beta > 0$.

3.3. **Proof of Theorem 3.3.** The proof is quite technical, so let us first give a road map addressing the main points.

- The first step consists in rendering all the parameters positive, so to fall into the classical MGT scheme. To this end, a "pumping" technique is needed, with the price of inheriting some extra lower order terms with the bad sign in the equations.

- It is then crucial to construct a new norm in $\mathcal{H}$, equivalent to the natural one, along the line of what is commonly done to treat the case of a single MGT equation.

- Finally, we introduce the new variables z and ζ, allowing to rephrase the system as two coupled wave-type equations, plus lower order terms. At this point, we can exploit the classical multipliers, that is, $\dot{z}$ and $\dot{\zeta}$.

Remark 3.5. Differently from what happens in most models in the literature, the present coupling is generally not fully helpful. In particular, it might not be able to transfer damping in a satisfactory way between the two equations. Indeed, what typically occurs is that the coupling terms cancel each other when performing the basic energy estimate. On the contrary, here some residual terms remain, which play against dissipation. This feature renders the identification of the good equivalent norm quite challenging.

We are now ready to start the proof. First, we choose $r \geq 0$ large enough that

$$\gamma_r = \gamma + r > 0 \qquad \text{and} \qquad \hat{\gamma}_r = \hat{\gamma} + r > 0.$$

Next, we arbitrarily select $\varepsilon > 0$, and we set

$$\alpha_m = \alpha + m.$$

The value $m \geq 0$ is taken large enough that $\alpha_m > 0$,

$$\varkappa = \alpha_m \beta - \gamma_r > r\sqrt{\alpha_m}, \tag{3.4}$$

$$\kappa = \alpha_m \hat{\beta} - \hat{\gamma}_r > r\sqrt{\alpha_m}, \tag{3.5}$$

and the following inequalities hold:

$$\sqrt{\alpha_m} > \frac{r}{\gamma_r}\frac{(1+\varepsilon)}{\varepsilon}\left[1 + \frac{2(1+\varepsilon)\gamma_r}{\varepsilon(\varkappa - r\sqrt{\alpha_m}\,)}\right], \tag{3.6}$$

$$\sqrt{\alpha_m} > \frac{r}{\hat{\gamma}_r}\left[1 + \frac{2\hat{\gamma}_r}{\kappa - r\sqrt{\alpha_m}}\right]. \tag{3.7}$$

Finally, we choose $n \geq 0$ large enough that

$$\hat{\alpha}_n = \hat{\alpha} + n \geq \alpha_m,$$

and

$$\omega = \alpha_m(\hat{\alpha}_n - \alpha_m) \geq \frac{1+\varepsilon}{\varkappa}\,\eta\alpha_m(p - \alpha_m q)^2. \tag{3.8}$$

The key ingredient is a suitable equivalent norm in $\mathcal{H}$.

Lemma 3.6. *For $\boldsymbol{u} = (u, v, w, \theta, \phi, \psi) \in \mathcal{H}$, the function*

$$\begin{aligned}
|\boldsymbol{u}|^2_{\mathcal{H}} = \eta\alpha_m\|w + \alpha_m v\|^2 + \eta\gamma_r\|v_x + \alpha_m u_x\|^2 + \eta\varkappa\|v_x\|^2 \\
+ \alpha_m\|\psi + \alpha_m\phi\|^2 + \hat{\gamma}_r\|\phi_x + \alpha_m\theta_x\|^2 + \kappa\|\phi_x\|^2 + \omega\alpha_m\|\phi\|^2 \\
- 2\eta r\alpha_m\langle u_x, v_x\rangle - 2r\alpha_m\langle\theta_x, \phi_x\rangle - 2\eta\alpha_m(p - \alpha_m q)\langle v_x, \phi\rangle
\end{aligned}$$

defines a norm in $\mathcal{H}$ which is equivalent to the original one.

Proof. On account of the Cauchy-Schwarz and the Poincaré inequalities, it is apparent that

$$|\boldsymbol{u}|_{\mathcal{H}} \leq C\|\boldsymbol{u}\|_{\mathcal{H}},$$

for some $C > 0$. So, we are left to attain the converse. By the Young inequality, used several times hereafter,

$$-2\eta r\alpha_m\langle u_x, v_x\rangle \geq -\frac{\eta(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}\,\|u_x\|^2 - \frac{\eta\varepsilon r}{1+\varepsilon}\sqrt{\alpha_m}\,\|v_x\|^2,$$
$$-2r\alpha_m\langle \theta_x, \phi_x\rangle \geq -r\sqrt{\alpha_m^3}\,\|\theta_x\|^2 - r\sqrt{\alpha_m}\,\|\phi_x\|^2.$$

Besides, we estimate the last term of $|\boldsymbol{u}|^2_{\mathcal{H}}$ as

$$-2\eta\alpha_m(p-\alpha_m q)\langle v_x, \phi\rangle \geq -\frac{\eta^2\alpha_m(p-\alpha_m q)^2}{\omega}\|v_x\|^2 - \omega\alpha_m\|\phi\|^2.$$

Since from (3.8)

$$\eta\varkappa - \frac{\eta^2\alpha_m(p-\alpha_m q)^2}{\omega} \geq \frac{\eta\varepsilon\varkappa}{1+\varepsilon},$$

we obtain the inequality

$$\begin{aligned}|\boldsymbol{u}|^2_{\mathcal{H}} \geq \eta\Big[&\alpha_m\|w+\alpha_m v\|^2 + \gamma_r\|v_x+\alpha_m u_x\|^2 + \frac{\varepsilon\varkappa_*}{1+\varepsilon}\|v_x\|^2 - \frac{(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}\,\|u_x\|^2\Big] \\ &+ \alpha_m\|\psi+\alpha_m\phi\|^2 + \hat{\gamma}_r\|\phi_x+\alpha_m\theta_x\|^2 + \kappa_*\|\phi_x\|^2 - r\sqrt{\alpha_m^3}\,\|\theta_x\|^2,\end{aligned}$$

where, on account of (3.4)-(3.5), we defined

$$\varkappa_* = \varkappa - r\sqrt{\alpha_m} > 0 \qquad \text{and} \qquad \kappa_* = \kappa - r\sqrt{\alpha_m} > 0.$$

Let us first tackle the terms in the first line. For $\nu_1, \nu_2 \in (0,1)$ to be properly chosen later, we have

$$\alpha_m\|w+\alpha_m v\|^2 \geq \nu_1\alpha_m\|w\|^2 - \frac{\nu_1\alpha_m^3}{1-\nu_1}\|v_x\|^2,$$
$$\gamma_r\|v_x+\alpha_m u_x\|^2 \geq \nu_2\gamma_r\alpha_m^2\|u_x\|^2 - \frac{\nu_2\gamma_r}{1-\nu_2}\|v_x\|^2.$$

Thus,

$$\begin{aligned}&\alpha_m\|w+\alpha_m v\|^2 + \gamma_r\|v_x+\alpha_m u_x\|^2 + \frac{\varepsilon\varkappa_*}{1+\varepsilon}\|v_x\|^2 - \frac{(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}\,\|u_x\|^2 \\ &\geq \nu_1\alpha_m\|w\|^2 + \left[\frac{\varepsilon\varkappa_*}{1+\varepsilon} - \frac{\nu_2\gamma_r}{1-\nu_2} - \frac{\nu_1\alpha_m^3}{1-\nu_1}\right]\|v_x\|^2 + \left[\nu_2\gamma_r\alpha_m^2 - \frac{(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}\right]\|u_x\|^2.\end{aligned}$$

At this point, we set

$$\nu_2 = \frac{\varepsilon\varkappa_*}{\varepsilon\varkappa_* + 2(1+\varepsilon)\gamma_r},$$

so that

$$\frac{\nu_2\gamma_r}{1-\nu_2} = \frac{1}{2}\frac{\varepsilon\varkappa_*}{1+\varepsilon}.$$

It is then apparent that, up to fixing ν_1 suitably small, the coefficient of $\|v_x\|^2$ becomes strictly positive. Moreover, making use of (3.6),

$$\nu_2\gamma_r\alpha_m^2 > \frac{(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}.$$

In conclusion, we proved that

$$\alpha_m\|w+\alpha_m v\|^2+\gamma_r\|v_x+\alpha_m u_x\|^2+\frac{\varepsilon\varkappa_*}{1+\varepsilon}\|v_x\|^2-\frac{(1+\varepsilon)r}{\varepsilon}\sqrt{\alpha_m^3}\,\|u_x\|^2$$
$$\geq \delta\Big[\|w\|^2+\|v_x\|^2+\|u_x\|^2\Big],$$

for some $\delta>0$. Arguing exactly in the same manner, exploiting (3.7) in place of (3.6), we get

$$\alpha_m\|\psi+\alpha_m\phi\|^2+\hat\gamma_r\|\phi_x+\alpha_m\theta_x\|^2+\kappa_*\|\phi_x\|^2-r\sqrt{\alpha_m^3}\,\|\theta_x\|^2$$
$$\geq \delta\Big[\|\psi\|^2+\|\phi_x\|^2+\|\theta_x\|^2\Big],$$

for a possibly different $\delta>0$. Collecting the last two inequalities, the claim follows. $\square$

We rewrite (3.1) as

$$\begin{cases}\dddot u+\alpha_m\ddot u-\beta\dot u_{xx}-\gamma_r u_{xx}=-p\dot\theta_x-q\ddot\theta_x+m\ddot u-ru_{xx},\\ \dddot\theta+\hat\alpha_n\ddot\theta-\hat\beta\dot\theta_{xx}-\hat\gamma_r\theta_{xx}=-\eta p\dot u_x-\eta q\ddot u_x+n\ddot\theta-r\theta_{xx}.\end{cases}\tag{3.9}$$

For an arbitrarily fixed time $T>0$, let

$$\boldsymbol{u}(t)\in\mathcal{C}([0,T],\mathcal{H})\qquad\text{with}\qquad\boldsymbol{u}(0)=\boldsymbol{u}_0$$

be a regular solution to (3.9) on $[0,T]$. Introducing the new variables

$$z=\dot u+\alpha_m u\qquad\text{and}\qquad\zeta=\dot\theta+\alpha_m\theta,$$

system (3.9) becomes

$$\begin{cases}\alpha_m\ddot z-\gamma_r z_{xx}-\varkappa\dot u_{xx}=-p\dot\zeta_x+(p-\alpha_m q)\ddot\theta_x+m\alpha_m\ddot u-r\alpha_m u_{xx},\\ \alpha_m\ddot\zeta-\hat\gamma_r\zeta_{xx}+\omega\ddot\theta-\kappa\dot\theta_{xx}=-\eta p\dot z_x+\eta(p-\alpha_m q)\ddot u_x+n\alpha_m\ddot\theta-r\alpha_m\theta_{xx}.\end{cases}\tag{3.10}$$

We multiply the first equation of (3.10) by $\eta\dot z$ in H. From the equalities

$$\langle\dot u_{xx},\dot z\rangle=-\frac{1}{2}\frac{d}{dt}\|\dot u_x\|^2-\alpha_m\|\dot u_x\|^2,$$
$$\langle u_{xx},\dot z\rangle=-\frac{d}{dt}\langle u_x,\dot u_x\rangle+\|\dot u_x\|^2-\alpha_m\langle u_x,\dot u_x\rangle,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\Big[\eta\alpha_m\|\dot z\|^2+\eta\gamma_r\|z_x\|^2+\eta\varkappa\|\dot u_x\|^2-2\eta r\alpha_m\langle u_x,\dot u_x\rangle\Big]+\eta\alpha_m\varkappa\|\dot u_x\|^2$$
$$=-\eta p\langle\dot\zeta_x,\dot z\rangle+\eta(p-\alpha_m q)\langle\ddot\theta_x,\dot z\rangle+\eta m\alpha_m\langle\ddot u,\dot z\rangle-\eta r\alpha_m\|\dot u_x\|^2+\eta r\alpha_m^2\langle u_x,\dot u_x\rangle.$$

Then, we multiply the second equation of (3.10) by $\dot\zeta$ in H. This time, we exploit the equalities

$$
\begin{aligned}
\langle \dot\theta_{xx}, \dot\zeta\rangle &= -\frac{1}{2}\frac{d}{dt}\|\dot\theta_x\|^2 - \alpha_m\|\dot\theta_x\|^2,\\
\langle \theta_{xx}, \dot\zeta\rangle &= -\frac{d}{dt}\langle \theta_x, \dot\theta_x\rangle + \|\dot\theta_x\|^2 - \alpha_m\langle \theta_x, \dot\theta_x\rangle,\\
\langle \ddot\theta, \dot\zeta\rangle &= \frac{1}{2}\frac{d}{dt}\alpha_m\|\ddot\theta\|^2 + \|\ddot\theta\|^2,\\
\langle \ddot u_x, \dot\zeta\rangle &= \frac{d}{dt}\alpha_m\langle \dot u_x, \dot\theta\rangle + \langle \dot z_x, \ddot\theta\rangle - 2\alpha_m\langle \dot u_x, \ddot\theta\rangle,
\end{aligned}
$$

to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\alpha_m\|\dot\zeta\|^2 + \hat\gamma_r\|\zeta_x\|^2 + \omega\alpha_m\|\dot\theta\|^2 + \kappa\|\dot\theta_x\|^2 - 2r\alpha_m\langle\theta_x, \dot\theta_x\rangle - 2\eta\alpha_m(p-\alpha_m q)\langle \dot u_x, \dot\theta\rangle\Big]\\
&\quad + \alpha_m\kappa\|\dot\theta_x\|^2 + \omega\|\ddot\theta\|^2\\
&= -\eta p\langle \dot z_x, \dot\zeta\rangle + \eta(p-\alpha_m q)\langle \dot z_x, \ddot\theta\rangle - 2\eta\alpha_m(p-\alpha_m q)\langle \dot u_x, \ddot\theta\rangle\\
&\quad + n\alpha_m\langle \ddot\theta, \dot\zeta\rangle - r\alpha_m\|\dot\theta_x\|^2 + r\alpha_m^2\langle \theta_x, \dot\theta_x\rangle.
\end{aligned}
$$

Defining the energy $\mathsf{E} = \mathsf{E}(t)$ as

$$
\begin{aligned}
\mathsf{E} &= \frac{1}{2}|\boldsymbol{u}|^2_{\mathcal{H}}\\
&= \frac{1}{2}\Big[\eta\alpha_m\|\dot z\|^2 + \eta\gamma_r\|z_x\|^2 + \eta\varkappa\|\dot u_x\|^2 + \alpha_m\|\dot\zeta\|^2 + \hat\gamma_r\|\zeta_x\|^2 + \kappa\|\dot\theta_x\|^2 + \omega\alpha_m\|\dot\theta\|^2\\
&\quad - 2r\alpha_m\langle \theta_x, \dot\theta_x\rangle - 2\eta r\alpha_m\langle u_x, \dot u_x\rangle - 2\eta\alpha_m(p-\alpha_m q)\langle \dot u_x, \dot\theta\rangle\Big],
\end{aligned}
$$

and collecting the identities above we end up with

$$
\begin{aligned}
&\frac{d}{dt}\mathsf{E} + \eta\alpha_m\varkappa\|\dot u_x\|^2 + \alpha_m\kappa\|\dot\theta_x\|^2 + \omega\|\ddot\theta\|^2\\
&\quad = -2\eta\alpha_m(p-\alpha_m q)\langle \dot u_x, \ddot\theta\rangle + \eta m\alpha_m\langle \ddot u, \dot z\rangle - \eta r\alpha_m\|\dot u_x\|^2 + \eta r\alpha_m^2\langle u_x, \dot u_x\rangle\\
&\qquad + n\alpha_m\langle \ddot\theta, \dot\zeta\rangle - r\alpha_m\|\dot\theta_x\|^2 + r\alpha_m^2\langle \theta_x, \dot\theta_x\rangle.
\end{aligned}
\tag{3.11}
$$

The right-hand side of (3.11) is clearly controlled by the norm of $\boldsymbol{u}$. On the other hand, we know from Lemma 3.6 that

$$
\frac{1}{C}\|\boldsymbol{u}(t)\|^2_{\mathcal{H}} \le \mathsf{E}(t) \le C\|\boldsymbol{u}(t)\|^2_{\mathcal{H}},
$$

for some $C > 1$. Accordingly, we deduce from (3.11) that

$$
\frac{d}{dt}\mathsf{E} \le \mu\mathsf{E},
$$

for some $\mu > 0$. Then, the Gronwall lemma yields the estimate

$$
\mathsf{E}(t) \le \mathsf{E}(0)e^{\mu t}.
$$

For any fixed $\boldsymbol{u}_0 \in \mathcal{H}$, this provides the uniform bound in $L^\infty(0,T;\mathcal{H})$ of any sequence $\boldsymbol{u}^n$ of Galerkin approximations with initial data $\boldsymbol{u}_0^n \to \boldsymbol{u}_0$ in $\mathcal{H}$, implying the weak-$*$ convergence (up to a subsequence)

$$\boldsymbol{u}^n \rightharpoonup \boldsymbol{u} \quad \text{in } L^\infty(0,T;\mathcal{H}),$$

for some weak solution $\boldsymbol{u}$. By linearity, the same estimate holds for the energy of the difference $\boldsymbol{u}^n - \boldsymbol{u}^k$, yielding the convergence of the entire sequence $\boldsymbol{u}^n$ to its limit $\boldsymbol{u}$ in the topology of $\mathcal{C}([0,T],\mathcal{H})$. The proof of Theorem 3.3 is over. □

4. The Contraction Semigroup

We now turn our attention to the physically more relevant case where all the parameters in play, except the coupling constants p and q, are strictly positive, that is,

$$\alpha, \hat\alpha, \beta, \hat\beta, \gamma, \hat\gamma, \eta > 0.$$

The physical meaning of the conditions $\gamma, \hat\gamma > 0$ can be interpreted within the theory of thermoelastic stability (see [18]).

We make the following hypotheses:

H1. Both MGT equations are in the subcritical regime, namely,

$$\min\{\alpha, \hat\alpha\} > \max\Big\{\frac{\gamma}{\beta}, \frac{\hat\gamma}{\hat\beta}\Big\}.$$

H2. For some $\varepsilon > 0$, the coupling parameters satisfy the condition

$$|\hat\alpha - \alpha| \geq \begin{cases} \dfrac{1+\varepsilon}{\alpha\beta - \gamma}\,\eta(p-\alpha q)^2 & \text{if } \hat\alpha \geq \alpha, \\[2mm] \dfrac{1+\varepsilon}{\hat\alpha\hat\beta - \hat\gamma}\,\dfrac{(p-\hat\alpha q)^2}{\eta} & \text{if } \alpha \geq \hat\alpha. \end{cases}$$

Remark 4.1. A particular instance complying with (H2) is when

$$\alpha = \hat\alpha \qquad \text{and} \qquad p - \alpha q = 0.$$

In this case, system (3.1) takes the simpler form

$$\begin{cases} \dddot{u} + \alpha\ddot{u} - \beta\dot{u}_{xx} - \gamma u_{xx} = -\alpha q\dot\theta_x - q\ddot\theta_x, \\ \dddot\theta + \alpha\ddot\theta - \hat\beta\dot\theta_{xx} - \hat\gamma\theta_{xx} = -\eta\alpha q\dot{u}_x - \eta q\ddot{u}_x, \end{cases} \tag{4.1}$$

and when performing the multiplications as in the previous proof, the contributions of the coupling terms cancel each other.

Theorem 4.2. *Let* (H1)-(H2) *hold. Then, there exist a constant* $\delta > 0$ *and an equivalent norm* $|\cdot|_{\mathcal{H}}$ *in* $\mathcal{H}$ *such that, for all sufficiently regular initial data, the energy*

$$\mathsf{E}(t) = \frac{1}{2}|\boldsymbol{u}(t)|^2_{\mathcal{H}}$$

fulfills the differential inequality

$$\frac{d}{dt}\mathsf{E} + \delta\Big[\|\dot{u}_x\|^2 + \|\dot\theta_x\|^2\Big] \leq 0. \tag{4.2}$$

In particular, with respect to this norm, $S(t)$ is a contraction semigroup.

Proof. On account of the symmetry of the system, it is enough to prove the result when $\hat{\alpha} \geq \alpha$. Indeed, the other case is merely obtained by renaming p and q as p/η and q/η, respectively. Accordingly, we set

$$\begin{aligned}\varkappa &= \alpha\beta - \gamma > 0,\\ \kappa &= \alpha\hat{\beta} - \hat{\gamma} > 0,\\ \omega &= \alpha(\hat{\alpha} - \alpha) \geq 0,\end{aligned}$$

and (H2) reads

$$\omega \geq \frac{1+\varepsilon}{\varkappa}\,\eta\alpha(p - \alpha q)^2. \tag{4.3}$$

The equivalent norm of $\boldsymbol{u} = (u, v, w, \theta, \phi, \psi) \in \mathcal{H}$ is given by

$$\begin{aligned}|\boldsymbol{u}|^2_{\mathcal{H}} = {} & \eta\alpha\|w + \alpha v\|^2 + \eta\gamma\|v_x + \alpha u_x\|^2 + \eta\varkappa\|v_x\|^2\\ & + \alpha\|\psi + \alpha\phi\|^2 + \hat{\gamma}\|\phi_x + \alpha\theta_x\|^2 + \kappa\|\phi_x\|^2 + \omega\alpha\|\phi\|^2\\ & - 2\eta\alpha(p - \alpha q)\langle v_x, \phi\rangle.\end{aligned}$$

The equivalence follows from Lemma 3.6, since here we have $r = m = n = 0$. For the same reason, the energy identity (3.11) now becomes

$$\frac{d}{dt}\mathsf{E} + \eta\alpha\varkappa\|\dot{u}_x\|^2 + \alpha\kappa\|\dot{\theta}_x\|^2 + \omega\|\ddot{\theta}\|^2 = -2\eta\alpha(p - \alpha q)\langle \dot{u}_x, \ddot{\theta}\rangle. \tag{4.4}$$

If $\omega = 0$, hence $p - \alpha q = 0$, the result is already proved. If $\omega > 0$, by means of (4.3) we get

$$\begin{aligned}-2\eta\alpha(p - \alpha q)\langle \dot{u}_x, \ddot{\theta}\rangle &\leq \frac{\eta^2\alpha^2(p - \alpha q)^2}{\omega}\|\dot{u}_x\|^2 + \omega\|\ddot{\theta}\|^2\\ &\leq \frac{1}{1+\varepsilon}\eta\alpha\varkappa\|\dot{u}_x\|^2 + \omega\|\ddot{\theta}\|^2.\end{aligned}$$

Plugging the latter inequality into the energy identity, we are done. □

5. Exponential Stability

The positivity of the parameters, along with (H1)-(H2), are actually strong enough to drive the solutions to zero exponentially fast.

Theorem 5.1. *Within assumptions* (H1)-(H2)*, the energy fulfills the exponential decay estimate*

$$\mathsf{E}(t) \leq M\mathsf{E}(0)e^{-\mu t},$$

for some $M \geq 1$ and $\mu > 0$, both independent of $\mathsf{E}(0)$.

Proof. All the constants appearing in this proof are understood to be independent of the particular solution $\boldsymbol{u}(t)$. In what follows, we will exploit several times the equivalence of the two norms in $\mathcal{H}$.

We define the functional $\Lambda = \Lambda(t)$ as follows:

$$\Lambda = \frac{1}{2}\Big[\eta\|\ddot{u}\|^2 - \eta\|\dot{u}\|^2 + \eta\beta\|u_x\|^2 + \eta\beta\|\dot{u}_x\|^2 + \|\ddot{\theta}\|^2 - \|\dot{\theta}\|^2 + \hat{\beta}\|\theta_x\|^2 + \hat{\beta}\|\dot{\theta}_x\|^2$$
$$+ 2\eta\langle\ddot{u}, u\rangle + 2\eta\alpha\langle\dot{u}, u\rangle + 2\langle\ddot{\theta}, \theta\rangle + 2\hat{\alpha}\langle\dot{\theta}, \theta\rangle + 2\eta q\langle\dot{\theta}_x, u\rangle + 2\eta q\langle\dot{u}_x, \theta\rangle\rangle$$
$$+ 2\eta\gamma\langle u_x, \dot{u}_x\rangle + 2\hat{\gamma}\langle\theta_x, \dot{\theta}_x\rangle\Big].$$

It is readily seen that

$$|\Lambda(t)| \leq K\mathsf{E}(t), \tag{5.1}$$

for some $K > 0$. We now multiply the two equations of (3.1) by suitable test functions, to obtain the following set of identities:

- 1^{st} equation times ηu:

$$\frac{1}{2}\frac{d}{dt}\Big[\eta\beta\|u_x\|^2 - \eta\|\dot{u}\|^2 + 2\eta\langle\ddot{u}, u\rangle + 2\eta\alpha\langle\dot{u}, u\rangle + 2\eta q\langle\dot{\theta}_x, u\rangle\Big] + \eta\gamma\|u_x\|^2$$
$$= \eta\alpha\|\dot{u}\|^2 - \eta p\langle\dot{\theta}_x, u\rangle + \eta q\langle\dot{\theta}_x, \dot{u}\rangle.$$

- 2^{nd} equation times θ:

$$\frac{1}{2}\frac{d}{dt}\Big[\hat{\beta}\|\theta_x\|^2 - \|\dot{\theta}\|^2 + 2\langle\ddot{\theta}, \theta\rangle + 2\hat{\alpha}\langle\dot{\theta}, \theta\rangle + 2\eta q\langle\dot{u}_x, \theta\rangle\Big] + \hat{\gamma}\|\theta_x\|^2$$
$$= \hat{\alpha}\|\dot{\theta}\|^2 - \eta p\langle\dot{u}_x, \theta\rangle + \eta q\langle\dot{u}_x, \dot{\theta}\rangle.$$

- 1^{st} equation times $\eta\ddot{u}$:

$$\frac{1}{2}\frac{d}{dt}\Big[\eta\|\ddot{u}\|^2 + \eta\beta\|\dot{u}_x\|^2 + 2\eta\gamma\langle u_x, \dot{u}_x\rangle\Big] + \eta\alpha\|\ddot{u}\|^2 = \eta\gamma\|\dot{u}_x\|^2 - \eta p\langle\dot{\theta}_x, \ddot{u}\rangle - \eta q\langle\ddot{\theta}_x, \ddot{u}\rangle.$$

- 2^{nd} equation times $\ddot{\theta}$:

$$\frac{1}{2}\frac{d}{dt}\Big[\|\ddot{\theta}\|^2 + \hat{\beta}\|\dot{\theta}_x\|^2 + 2\hat{\gamma}\langle\theta_x, \dot{\theta}_x\rangle\Big] + \hat{\alpha}\|\ddot{\theta}\|^2 = \hat{\gamma}\|\dot{\theta}_x\|^2 - \eta p\langle\dot{u}_x, \ddot{\theta}\rangle - \eta q\langle\ddot{u}_x, \ddot{\theta}\rangle.$$

Collecting these identities, we obtain

$$\frac{d}{dt}\Lambda + \eta\alpha\|\ddot{u}\|^2 + \eta\gamma\|u_x\|^2 + \hat{\alpha}\|\ddot{\theta}\|^2 + \hat{\gamma}\|\theta_x\|^2 = \mathfrak{I},$$

where

$$\mathfrak{I} = \eta\alpha\|\dot{u}\|^2 + \eta\gamma\|\dot{u}_x\|^2 + \hat{\alpha}\|\dot{\theta}\|^2 + \hat{\gamma}\|\dot{\theta}_x\|^2 - \eta p\langle\dot{\theta}_x, \ddot{u}\rangle - \eta p\langle\dot{u}_x, \ddot{\theta}\rangle$$
$$- \eta p\langle\dot{\theta}_x, u\rangle - \eta p\langle\dot{u}_x, \theta\rangle + \eta q\langle\dot{\theta}_x, \dot{u}\rangle + \eta q\langle\dot{u}_x, \dot{\theta}\rangle.$$

Recalling the Poincaré inequality, we now estimate $\mathfrak{I}$ as

$$\mathfrak{I} \leq \frac{1}{2}\Big[\eta\alpha\|\ddot{u}\|^2 + \eta\gamma\|u_x\|^2 + \hat{\alpha}\|\ddot{\theta}\|^2 + \hat{\gamma}\|\theta_x\|^2\Big] + Q\Big[\|\dot{u}_x\|^2 + \|\dot{\theta}_x\|^2\Big],$$

for some $Q > 0$ sufficiently large. We end up with the differential inequality

$$\frac{d}{dt}\Lambda + \varpi\Big[\|\ddot{u}\|^2 + \|u_x\|^2 + \|\ddot{\theta}\|^2 + \|\theta_x\|^2\Big] \leq Q\Big[\|\dot{u}_x\|^2 + \|\dot{\theta}_x\|^2\Big], \tag{5.2}$$

having set

$$\varpi = \frac{1}{2}\min\{\eta\alpha, \eta\gamma, \hat{\alpha}, \hat{\gamma}\}.$$

At this point, we introduce for $\nu > 0$ the energy-like functional

$$\mathsf{F}(t) = \mathsf{E}(t) + \nu\Lambda(t).$$

In view of (5.1), up to taking ν small enough, we have the controls

$$\frac{1}{2}\mathsf{E}(t) \le \mathsf{F}(t) \le 2\mathsf{E}(t).$$

Moreover, collecting (4.2) and (5.2),

$$\frac{d}{dt}\mathsf{F} + (\delta - Q\nu)\Big[\|\dot{u}_x\|^2 + \|\dot{\theta}_x\|^2\Big] + \nu\varpi\Big[\|\ddot{u}\|^2 + \|u_x\|^2 + \|\ddot{\theta}\|^2 + \|\theta_x\|^2\Big] \le 0.$$

Up to further reducing ν, we conclude that

$$\frac{d}{dt}\mathsf{F} + \mu\mathsf{F} \le 0,$$

for some $\mu > 0$. Hence, the Gronwall lemma gives

$$\mathsf{F}(t) \le F(0)e^{-\mu t},$$

implying the desired conclusion with $M = 4$. □

6. The Case $\alpha = \hat{\alpha}$. Back to the Thermoelastic Model

We finally reconsider the particular instance of system (4.1), but assuming now that

$$\alpha = \frac{\gamma}{\beta} > \frac{\hat{\gamma}}{\hat{\beta}}. \tag{6.1}$$

Thus,

$$\varkappa = \alpha\beta - \gamma = 0 \qquad \text{and} \qquad \kappa = \alpha\hat{\beta} - \hat{\gamma} > 0,$$

meaning that the second MGT equation in (4.1) is subcritical, whereas the first one is critical.[1] Observe that this situation is not covered by Theorem 5.1. Nonetheless, via a different argument, we will prove that exponential stability still occurs.

Since $\alpha\beta = \gamma$, by setting

$$z = \dot{u} + \alpha u,$$

we rewrite (4.1) as

$$\begin{cases} \ddot{z} - \beta z_{xx} = -\alpha q\dot{\theta}_x - q\ddot{\theta}_x, \\ \dddot{\theta} + \alpha\ddot{\theta} - \hat{\beta}\dot{\theta}_{xx} - \hat{\gamma}\theta_{xx} = -\eta q\dot{z}_x. \end{cases} \tag{6.2}$$

At this point, we forget for a moment how we arrived at (6.2), and we view it as a system of differential equations in the variables $(z, \dot{z}, \theta, \dot{\theta}, \ddot{\theta})$, subject to the Dirichlet boundary conditions

$$z(0,t) = z(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0. \tag{6.3}$$

[1]Clearly, the same argument would work the other way around, that is, when the first equation is subcritical ($\varkappa > 0$) and the second one critical ($\kappa = 0$).

Introducing the product Hilbert space

$$\mathcal{V} = H^1 \times H \times H^1 \times H^1 \times H,$$

the following well-posedness theorem holds.

Theorem 6.1. *There exists an equivalent norm $|\cdot|_{\mathcal{V}}$ in $\mathcal{V}$ under which the boundary value problem* (6.2)-(6.3) *generates a contraction semigroup of bounded linear operators*

$$T(t) : \mathcal{V} \to \mathcal{V}.$$

Proof. The proof basically recasts the ones of Theorems 3.3 and 4.2. For any initial datum

$$\boldsymbol{z}_0 = (z_0, a_0, \theta_0, \phi_0, \psi_0) \in \mathcal{V},$$

we consider (in a Galerkin scheme) the solution

$$\boldsymbol{z}(t) = (z(t), \dot{z}(t), \theta(t), \dot{\theta}(t), \ddot{\theta}(t)),$$

satisfying the initial condition $\boldsymbol{z}(0) = \boldsymbol{z}_0$, which belongs to $\mathcal{C}([0,T], \mathcal{V})$ for every $T > 0$. All we need to show is a contractive estimate for the related energy. To this end, arguing as in Lemma 3.6, an equivalent norm for $\boldsymbol{z} = (z, a, \theta, \phi, \psi) \in \mathcal{V}$ is given by

$$|\boldsymbol{z}|^2_{\mathcal{V}} = \eta\alpha\|a\|^2 + \eta\gamma\|z_x\|^2 + \alpha\|\psi + \alpha\phi\|^2 + \hat{\gamma}\|\phi_x + \alpha\theta_x\|^2 + \kappa\|\phi_x\|^2.$$

Then, defining the energy

$$\mathsf{E}_0(t) = \frac{1}{2}|\boldsymbol{z}(t)|^2_{\mathcal{V}},$$

by the same calculations leading to (4.4), we find the equality

$$\frac{d}{dt}\mathsf{E}_0 + \alpha\kappa\|\dot{\theta}_x\|^2 = 0,$$

which establishes the desired result. □

The contraction semigroup $T(t)$ turns out to be exponentially stable as well.

Theorem 6.2. *The energy $\mathsf{E}_0(t)$ fulfills the exponential decay estimate*

$$\mathsf{E}_0(t) \leq M\mathsf{E}_0(0)e^{-\mu t},$$

for some $M \geq 1$ and $\mu > 0$, both independent of $\mathsf{E}_0(0)$.

The proof of Theorem 6.2 is based on linear semigroup techniques, and is postponed to the last Section 7.

Remark 6.3. Quite interestingly, we note that (6.2) is exactly the one-dimensional version of system (2.4). Accordingly, Theorems 6.1 and 6.2 give a detailed description of the evolution of the MGT-thermoelastic model. With respect to the model (2.2) without energy dissipation, this shows that exponential stability occurs provided that viscous friction is introduced in one of the two equations only (no matter which one).

We can now complete our analysis on the original semigroup $S(t)$ on $\mathcal{H}$ generated by (6.2), with the position $z = \dot{u} + \alpha u$.

Corollary 6.4. *Within assumption* (6.1), *the semigroup $S(t)$ remains exponentially stable on $\mathcal{H}$, that is, the energy $\mathsf{E}(t)$ fulfills the exponential decay estimate of Theorem 5.1.*

Proof. In light of the exponential decay of $\mathsf{E}_0(t)$, in order to prove the sought estimate for $\mathsf{E}(t)$ it is enough showing the exponential decay of the missing quantity $\|\dot u_x\|$. To this end, we write the differential equation for u

$$\dot u + \alpha u = z,$$

where now z is given, and known to decay exponentially to zero in H^1. Then,

$$u(t) = u(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} z(s)ds,$$

which readily implies the exponential decay of $\|u_x\|$. Since

$$\|\dot u_x\| = \|z_x - \alpha u_x\| \le \|z_x\| + \alpha\|u_x\|,$$

the exponential decay of $\|\dot u_x\|$ follows. □

Remark 6.5. As a concluding comment, we note that if instead of (6.1) we assume

$$\alpha = \frac{\gamma}{\beta} = \frac{\hat\gamma}{\hat\beta},$$

meaning that both the MGT equations are in the critical regime, then system (4.1) becomes

$$\begin{cases} \ddot z - \beta z_{xx} = -q\dot\zeta_x, \\ \ddot\zeta - \hat\beta\zeta_{xx} = -\eta q \dot z_x, \end{cases}$$

upon setting $\zeta = \dot\theta + \alpha\theta$. This is nothing but the system of thermoelasticity without energy dissipation (2.2).

7. Proof of Theorem 6.2

The main difficulty in the proof of Theorem 6.2 arises in handling the Dirichlet boundary condition for θ. For the analogous problem with the Neumann boundary condition, the result can be found in [31], via semigroup techniques. Here also, we follow a semigroup approach.

Without loss of generality, we put $q = \eta = 1$, and we rewrite (6.2)-(6.3) in the abstract form

$$\frac{d}{dt}\boldsymbol{z}(t) = \mathbb{A}\boldsymbol{z}(t),$$

where $\mathbb{A}$ is the linear operator on $\mathcal{V}$ acting as

$$\mathbb{A}\begin{pmatrix} z \\ a \\ \theta \\ \phi \\ \\ \end{pmatrix} = \begin{pmatrix} a \\ \beta D^2 z - \alpha D\phi - D\psi \\ \phi \\ \\ -\alpha\psi + \hat\beta D^2\phi + \hat\gamma D^2\theta - Da \end{pmatrix},$$

with domain

$$\mathfrak{D}(\mathbb{A}) = \big\{\boldsymbol{z} \in \mathcal{V} : \mathbb{A}\boldsymbol{z} \in \mathcal{V}\big\}.$$

Here and in what follows, the symbol D stands for the space derivative with respect to x. In order to prove the exponential stability of the contraction semigroup $T(t)$, whose infinitesimal generator is the linear operator $\mathbb{A}$, we will make use of the following abstract result due to Prüss [30].

Theorem 7.1. *A contraction semigroup $T(t) = e^{t\mathbb{A}}$ of bounded linear operators on $\mathcal{V}$ is exponentially stable if and only if*

- **(i)** *the resolvent set $\rho(A)$ of the (complexification of) the operator $\mathbb{A}$ contains the imaginary axis $i\mathbb{R}$; and*
- **(ii)** *the following relation holds:*

$$\limsup_{|\lambda|\to\infty} \|(i\lambda - \mathbb{A})^{-1}\| < \infty,$$

where the norm is taken in the space of bounded linear operators on $\mathcal{V}$.

Defining the space $H^{-1} = H^{-1}(0,\pi)$, the dual space of H^1, and denoting its norm by $\|\cdot\|_{-1}$, we will make use of a well-known fact.

Lemma 7.2. *For any $a \in H^1$, we have the equality*

$$\|Da\|_{-1}^2 = \|a - \tilde{a}\|^2 = \|a\|^2 - \pi|\tilde{a}|^2,$$

where $\tilde{a}$ is the average of a, namely,

$$\tilde{a} = \frac{1}{\pi}\int_0^\pi a(x)dx.$$

We have now all the ingredients to proceed with our proof. We will reach the desired conclusion by showing points (i) and (ii) of Theorem 7.1. This will be done in the next two lemmas.

Lemma 7.3. *Point* (i) *of Theorem 7.1 holds true.*

Proof. Let $\lambda \in \mathbb{R}$ be arbitrarily fixed. Since $T(t)$ is a contraction semigroup, if $i\lambda \in \sigma(\mathbb{A})$, the spectrum of $\mathbb{A}$, then by the Hille-Yosida theorem it belongs to its boundary $\partial\sigma(\mathbb{A})$ (see [26]). Besides, the only elements in $\sigma(\mathbb{A}) \cap \partial\sigma(\mathbb{A})$ are approximate eigenvalues (see [32, Theorem 5.1-D]). Accordingly, in order to show that $i\lambda \in \rho(\mathbb{A})$, it is enough showing that there is no sequence $\boldsymbol{z}_n \in \mathfrak{D}(\mathbb{A})$ of unit norm such that the convergence

$$i\lambda\boldsymbol{z}_n - \mathbb{A}\boldsymbol{z}_n \to 0$$

holds in $\mathcal{V}$. This can be done by contradiction. The computations are pretty standard, and left to the interested reader. ☐

The proof of the last lemma, instead, is a little bit more tricky.

Lemma 7.4. *Point* (ii) *of Theorem 7.1 holds true.*

Proof. We argue by contradiction, assuming the existence of a sequence $\lambda_n \in \mathbb{R}$, with $|\lambda_n| \to \infty$, and a sequence of vectors $\boldsymbol{z}_n = (z_n, a_n, \theta_n, \phi_n, \psi_n) \in \mathfrak{D}(\mathbb{A})$, with

$$|\boldsymbol{z}_n|_{\mathcal{V}}^2 = \alpha\|a_n\|^2 + \gamma\|Dz_n\|^2 + \alpha\|\psi_n + \alpha\phi_n\|^2 + \hat{\gamma}\|D\phi_n + \alpha D\theta_n\|^2 + \kappa\|D\phi_n\|^2 = 1,$$

for which

$$i\lambda_n\boldsymbol{z}_n - \mathbb{A}\boldsymbol{z}_n \to 0 \quad \text{in } \mathcal{V}. \tag{7.1}$$

Componentwise,

$$i\lambda_n z_n - a_n \to 0 \quad \text{in } H^1, \tag{7.2}$$

$$i\lambda_n a_n - \beta D^2 z_n + \alpha D\phi_n + D\psi_n \to 0 \quad \text{in } H, \tag{7.3}$$

$$i\lambda_n \theta_n - \phi_n \to 0 \quad \text{in } H^1, \tag{7.4}$$

$$i\lambda_n \phi_n - \psi_n \to 0 \quad \text{in } H^1, \tag{7.5}$$

$$i\lambda_n \psi_n + \alpha\psi_n - \hat{\beta} D^2\phi_n - \hat{\gamma} D^2\theta_n + Da_n \to 0 \quad \text{in } H. \tag{7.6}$$

Multiplying (7.1) by $\boldsymbol{z}_n$ in $\mathcal{V}$, and taking the real part, we have

$$\mathrm{Re}\langle i\lambda_n \boldsymbol{z}_n - \mathbb{A}\boldsymbol{z}_n, \boldsymbol{z}_n\rangle_{\mathcal{V}} = \alpha\kappa\|D\phi_n\|^2 \to 0,$$

yielding the convergence

$$\phi_n \to 0 \quad \text{in } H^1. \tag{7.7}$$

In turn, as $|\lambda_n| \to \infty$, we infer from (7.4) that

$$\theta_n \to 0 \quad \text{in } H^1. \tag{7.8}$$

Moreover, using (7.5),

$$\xi_n = \frac{\psi_n}{\lambda_n} \to 0 \quad \text{in } H^1.$$

Dividing (7.6) by λ_n, and then taking the product with ψ_n,

$$\frac{i\lambda_n + \alpha}{\lambda_n}\|\psi_n\|^2 + \hat{\beta}\langle D\phi_n, D\xi_n\rangle + \hat{\gamma}\langle D\theta_n, D\xi_n\rangle - \langle a_n, D\xi_n\rangle \to 0.$$

Hence, we get

$$\psi_n \to 0 \quad \text{in } H. \tag{7.9}$$

Up to now, the proof parallels the one in [31]. The new argument is the following: we multiply (7.2) by a_n, and (7.3) by z_n, so obtaining

$$i\lambda_n\langle z_n, a_n\rangle - \|a_n\|^2 \to 0,$$
$$i\lambda_n\langle a_n, z_n\rangle + \beta\|Dz_n\|^2 \to 0.$$

Adding the first equation with the complex conjugate of the second one, we end up with

$$\beta\|Dz_n\|^2 - \|a_n\|^2 \to 0.$$

On the other hand, since $|\boldsymbol{z}_n|^2_{\mathcal{V}} = 1$, exploiting (7.7)-(7.9) we also know that

$$\alpha\|a_n\|^2 + \gamma\|Dz_n\|^2 \to 1.$$

Recalling that $\alpha\beta = \gamma$, we conclude that

$$\|a_n\|^2 \to \frac{1}{2\alpha}.$$

The strategy is showing that the latter convergence leads to a contradiction. Denote by A the inverse of the Laplace-Dirichlet operator $-D^2$. It is well-known that A is a strictly positive selfadjoint operator mapping isometrically H^{-1} onto H^1. Observe first that (7.3) implies that

$$|\lambda_n|\|a_n\|_{-1} \le C, \tag{7.10}$$

for some $C > 0$. Then, a multiplication of (7.6) by ADa_n yields

$$\|Da_n\|_{-1}^2 \to 0.$$

Indeed, the only nontrivial term to control is

$$|\lambda_n\langle\psi_n, ADa_n\rangle| \le \|\psi_n\||\lambda_n|\|a_n\|_{-1} \le C\|\psi_n\| \to 0.$$

Therefore, from Lemma 7.2 we learn that

$$\|a_n\|^2 - \pi|\tilde{a}_n|^2 = \|a_n - \tilde{a}_n\|^2 \to 0.$$

Hence, up to a subsequence, there exists

$$\tilde{a} \in \mathbb{C} \qquad \text{with} \qquad |\tilde{a}|^2 = \frac{1}{2\pi\alpha}$$

such that

$$a_n \to \tilde{a} \quad \text{in } H \qquad \Rightarrow \qquad a_n \to \tilde{a} \quad \text{in } H^{-1}.$$

At the same time, we infer from (7.10) the convergence

$$a_n \to 0 \quad \text{in } H^{-1}.$$

By the uniqueness of the limit, we draw a contradiction.

Appendix: The Linear Theory of Gurtin

Given a (visco)elastic heat conductor of mass density $\rho > 0$ occupying a volume $\Omega \subset \mathbb{R}^N$ at rest, we write the equations ruling the evolution of the displacement vector $U_i = U_i(\boldsymbol{x}, t)$ and the entropy $\eta = \eta(\boldsymbol{x}, t)$, with $(\boldsymbol{x}, t) \in \Omega \times \mathbb{R}$. Introducing the velocity vector

$$u_i = \dot{U}_i,$$

these read

$$\begin{cases} \rho\dot{u}_i = \partial_j t_{ij}, \\ T_0\dot{\eta} = \partial_i q_i, \end{cases} \tag{A.1}$$

where $t_{ij} = t_{ij}(\boldsymbol{x}, t)$ is the stress tensor, $q_i = q_i(\boldsymbol{x}, t)$ is the heat flux vector, and $T_0 > 0$ is the reference temperature, assumed to be uniform. The distinctive character of the theory lies in the choice of the constitutive equations for t_{ij}, η and q_i, which in this case take the form

$$t_{ij}(t) = \int_{-\infty}^{t} \Big[c_{ijrs}(t-y)\partial_s u_r(y) + g_{ij}(t-y)\dot{\theta}(y) + h_{ijr}(t-y)\partial_r\theta(y)\Big]\,dy, \tag{A.2}$$

$$\eta(t) = \int_{-\infty}^{t} \Big[l_{ij}(t-y)\partial_j u_i(y) + a(t-y)\dot{\theta}(y) + m_i(t-y)\partial_i\theta(y)\Big]\,dy, \tag{A.3}$$

$$q_i(t) = \int_{-\infty}^{t} \Big[T_0 f_{rsi}(t-y)\partial_s u_r(y) + T_0 n_i(t-y)\dot{\theta}(y) + k_{ij}(t-y)\partial_j\theta(y)\Big]\,dy. \tag{A.4}$$

In what follows, we will set for simplicity $T_0 = 1$. Besides on the time t, all the other quantities appearing above depend on the space variable $\boldsymbol{x}$, thereafter omitted. Here, θ is the relative temperature, while

$$c_{ijrs},\, g_{ij},\, l_{ij},\, k_{ij},\, a \qquad \text{and} \qquad h_{ijr},\, f_{ijr},\, m_i,\, n_i$$

are the constitutive even and odd tensors. In particular: the fourth order tensor c_{ijrs}, the second order tensor l_{ij} and a are part of the generalized stress relaxation function; k_{ij} is the thermal conductivity tensor; and g_{ij} is the stress temperature relaxation tensor. It is natural to assume symmetry in the first two indices for h_{ijl} and f_{ijl}, to wit,

$$h_{ijr} = h_{jir}, \qquad f_{ijr} = f_{jir}.$$

In the linear theory of M.E. Gurtin, the fundamental assumption is the *invariance under temporal inversion of the production of the entropy*, which translates into the following symmetries (see Theorem 3 and Corollary 2 in [14]):

$$c_{ijrs} = c_{rsij}, \qquad l_{ij} = l_{ji}, \qquad k_{ij} = k_{ji},$$

as well as into the equalities

$$g_{ij} = -l_{ij}, \qquad h_{ijr} = f_{ijr} + \mathfrak{c}, \qquad m_i = -n_i + \mathfrak{c},$$

for a certain $\mathfrak{c} = \mathfrak{c}(\boldsymbol{x})$. Under the physically meaningful assumptions that h_{ijr}, f_{ijr} and m_i, n_i vanish at infinity (see Corollary 3 in [14] and the previous comments), the function $\mathfrak{c}$ is zero, so that

$$h_{ijr} = f_{ijr}, \qquad m_i = -n_i.$$

Choosing specific forms of the tensors above gives rise to different thermoelastic and thermoviscoelastic models.

A.1. Thermoelasticity without energy dissipation. We begin to analyze the particular case in which all the constitutive functions are independent of time. Namely, we assume that

$$c_{ijrs}(t) = c^*_{ijrs}, \qquad l_{ij}(t) = l^*_{ij}, \qquad k_{ij}(t) = k^*_{ij}, \qquad a(t) = a^*,$$

and

$$h_{ijr}(t) = h^*_{ijr}, \qquad m_i(t) = m^*_i,$$

where all the *star* objects depend only on $\boldsymbol{x}$. From our previous discussion, the other quantities of the model become

$$g_{ij}(t) = -l^*_{ij}, \qquad f_{ijr}(t) = h^*_{ijr}, \qquad n_i(t) = -m^*_i.$$

We require that c^*_{ijrs}, k^*_{ij} and a^* be positive definite. At this point, we define the thermal displacement

$$\Theta(t) = \Theta(0) + \int_0^t \theta(y)dy,$$

satisfying the relation $\dot{\Theta} = \theta$. Making the reasonable positions that

$$\partial_j U_i(-\infty) = \theta(-\infty) = \partial_i \Theta(-\infty) = 0,$$

upon plugging (A.2)-(A.4) into (A.1) we arrive at the system

$$\begin{cases} \rho \dot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s U_r - l^*_{ij} \theta + h^*_{ijr} \partial_r \Theta \big], \\ a^* \dot{\theta} = \partial_i \big[h^*_{rsi} \partial_s U_r - m^*_i \theta + k^*_{ij} \partial_j \Theta \big] - l^*_{ij} \partial_j u_i - m^*_i \partial_i \theta. \end{cases}$$

A derivation with respect to time yields

$$\text{(A.5)} \qquad \begin{cases} \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r - l^*_{ij} \dot{\theta} + h^*_{ijr} \partial_r \theta \big], \\ a^* \ddot{\theta} = \partial_i \big[h^*_{rsi} \partial_s u_r - m^*_i \dot{\theta} + k^*_{ij} \partial_j \theta \big] - l^*_{ij} \partial_j \dot{u}_i - m^*_i \partial_i \dot{\theta}. \end{cases}$$

This is known as the system of thermoelasticity without energy dissipation, or of type II in the terminology of Green and Naghdi, which has deserved much attention in recent years (see, e.g., [15, 16, 21, 25]). For the more physically relevant case of centrosymmetric materials, the tensors of odd order vanish, that is, $h^*_{rsi} = 0$ and $m^*_i = 0$. Accordingly, (A.5) reduces to system (2.2) of Section 2.

A.2. MGT-thermoelasticity. Again, we assume that all the constitutive functions, with the only exception of the thermal conductivity tensor k_{ij}, are independent of time, and we rename them as before. Concerning k_{ij}, we assume the history-dependence form

$$k_{ij}(t) = k^*_{ij} \big(1 - e^{-\frac{t}{\tau}}\big) + \frac{1}{\tau} \hat{k}_{ij} e^{-\frac{t}{\tau}},$$

where $\tau > 0$ is a small relaxation parameter, and $\hat{k}_{ij} = \hat{k}_{ij}(\boldsymbol{x})$ is symmetric and positive definite, while we relax the positivity assumption on k^*_{ij}. Therefore, combining equations (A.1)-(A.4), we are led to

$$\text{(A.6)} \qquad \begin{cases} \rho \ddot{u}_i = \partial_j \big[c^*_{ijrs} \partial_s u_r - l^*_{ij} \dot{\theta} + h^*_{ijr} \partial_r \theta \big], \\ \tau a^* \dddot{\theta} + a^* \ddot{\theta} = \partial_i \big[h^*_{rsi} (\partial_s u_r + \tau \partial_s \dot{u}_r) - m^*_i (\dot{\theta} + \tau \ddot{\theta}) + k^*_{ij} \partial_j \theta + \hat{k}_{ij} \partial_j \dot{\theta} \big] \\ \qquad\qquad - l^*_{ij} (\partial_j \dot{u}_i + \tau \partial_j \ddot{u}_i) - m^*_i (\partial_i \dot{\theta} + \tau \partial_i \ddot{\theta}). \end{cases}$$

To be more precise, the second equation above is obtained by computing

$$\frac{d}{dt} \big(\dot{\eta} + \tau \ddot{\eta} \big).$$

System (A.6) is the natural extension of the one proposed in [31] for centrosymmetric materials, and can be viewed as the general model for the MGT-thermoelasticity, where the temperature obeys to an equation of MGT type. When dealing with centrosymmetric materials, $h_{ijr} = 0$ and $m_i = 0$, so that (A.6) takes the simpler form (2.3).

A.3. MGT-Thermoviscoelasticity. We have seen that MGT-thermoelasticity is derived from the Gurtin theory, assuming the dependence of the thermal conductivity tensor k_{ij} on its history in a certain form. Now we show that MGT-thermoviscoelasticity can be obtained in a similar way, by postulating this memory dependence form in the other tensors. This recalls some ideas developed in [7] and [20], where different levels of viscosity in the form of memory terms lead to different forms of damping, and where the relation between the MGT equation (2.1) and linear viscosity is investigated. We will restrict our attention to centrosymmetric materials, where odd order tensors vanish. We also suppose that *all* the constitutive functions exhibit the same history-dependence form

$$
\begin{aligned}
c_{ijrs}(t) &= c^*_{ijrs}\big(1 - e^{-\frac{t}{\tau}}\big) + \frac{1}{\tau}\hat{c}_{ijrs}e^{-\frac{t}{\tau}},\\
l_{ij}(t) &= l^*_{ij}\big(1 - e^{-\frac{t}{\tau}}\big) + \frac{1}{\tau}\hat{l}_{ij}e^{-\frac{t}{\tau}},\\
k_{ij}(t) &= k^*_{ij}\big(1 - e^{-\frac{t}{\tau}}\big) + \frac{1}{\tau}\hat{k}_{ij}e^{-\frac{t}{\tau}},\\
a(t) &= a^*\big(1 - e^{-\frac{t}{\tau}}\big) + \frac{1}{\tau}\hat{a}e^{-\frac{t}{\tau}},
\end{aligned}
$$

for a small relaxation parameter $\tau > 0$, where

$$
\hat{c}_{ijrs} = \hat{c}_{ijrs}(\boldsymbol{x}), \qquad \hat{l}_{ij} = \hat{l}_{ij}(\boldsymbol{x}), \qquad \hat{k}_{ij} = \hat{k}_{ij}(\boldsymbol{x}), \qquad \hat{a} = \hat{a}(\boldsymbol{x}).
$$

The tensors $\hat{c}_{ijrs}$, $\hat{k}_{ij}$, a^* and $\hat{a}$ are positive definite, whereas the positivity of the remaining tensors is not required. Substituting these relations into (A.1)-(A.4), we obtain our system (2.5).

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