

Learning controllers for performance through LMI regions

Andrea Bisoffi¹, Claudio De Persis¹ and Pietro Tesi²

Abstract—In an open-loop experiment, an input sequence is applied to an unknown linear time-invariant system (in continuous or discrete time) affected also by an unknown-but-bounded disturbance sequence (with an energy or instantaneous bound); the corresponding state sequence is measured. The goal is to design directly from the input and state sequences a controller that enforces a certain performance specification on the transient behaviour of the unknown system. The performance specification is expressed through a subset of the complex plane where closed-loop eigenvalues need to belong, a so called LMI region. For this control design problem, we provide here convex programs to enforce the performance specification from data in the form of linear matrix inequalities (LMI). For generic LMI regions, these are sufficient conditions to assign the eigenvalues within the LMI region for all possible dynamics consistent with data, and become necessary and sufficient conditions for special LMI regions. In this way, we extend classical model-based conditions from a seminal work in the literature to the setting of data-driven control from noisy data. Through two numerical examples, we investigate how these data-based conditions compare with each other.

I. INTRODUCTION

Whenever it is challenging or cumbersome to derive a model for a process to be controlled or to identify unambiguously its parameters, a viable alternative is to bypass these two steps altogether and, from data collected on the process, design directly a (feedback) controller [20]. Direct data-driven control was conceived within the discipline of system identification, and is enjoying renewed popularity thanks a fundamental result by Willems et al. [22, Thm. 1] for linear systems and noiseless data, see also [8], [10]. A natural continuation within linear systems has been how to handle the realistic case of noisy data, whose induced uncertainty has been addressed via tools from robust control. With bounded noise and noisy input-state data points collected in an open-loop experiment, one ends up with a *set* of dynamical matrices (A, B) consistent with data and wants to design a controller that guarantees certain properties of the closed-loop system for all such (A, B) . (Necessary and sufficient conditions (typically in the form of convenient convex programs) were given in the cases of stabilization [10], [9], linear quadratic regulation [15], [11], [23], and dynamic performance [1], [2], [21]. Imposing a certain performance specification for the process to be controlled is, in applications, as relevant as stabilization; however, for

these data-based control designs, dynamic performance has been less investigated than stabilization and, to the best of our knowledge, only in terms of quadratic [1], [2], \mathcal{H}_2 [21], [2] or \mathcal{H}_∞ performance [21].

An alternative method to impose performance specifications is by imposing that the closed-loop eigenvalues belong to specific subsets of the complex plane. Indeed, some salient characteristics of the closed-loop transient response (in continuous time) depend on these subsets of the complex plane: e.g., convergence rate is greater than $\ell > 0$ if all eigenvalues have real part less than $-\ell$, damping ratio is greater than $\cos \theta$ if eigenvalues are within a cone with vertex in 0 and aperture 2θ , and eigenvalues in the intersection of these two sets with suitable ℓ and θ achieve a fast response with limited overshoot, see Example 1 later. The fundamental work in [5] showed that for a certain subset of the complex plane and a given model (A, B) , finding the feedback gain K by which all eigenvalues of $A + BK$ belong to that subset is equivalent to solving a linear matrix inequality (LMI), see [5, Thm. 2.2] recalled later in Fact 1; such subsets take thus the name of LMI regions. Notably, the open left halfplane and the open unit disk are LMI regions, and a large number of subsets of the complex plane can be expressed as LMI regions, see Fig. 1; moreover, the intersection of LMI regions is also *equivalently* associated with the conjunction of the respective LMIs, see [5, Cor. 2.3] recalled later in Fact 2, to the effect that LMI regions are dense in the set of convex regions that are symmetric with respect to the real axis [5, §II.C]. In the approach for performance by LMI regions, appealing features are then that they equivalently give rise to convex inequalities and can express, or at least approximate closely, the subsets of the complex plane relevant for control purposes. Finally, the approach with LMI regions does not exclude using also \mathcal{H}_2 and \mathcal{H}_∞ approaches [5, §III]; however, with respect to them, it implements in an easy way performance specifications by linking the desired characteristics of the time response to regions of the complex plane that are well known to a control engineer familiar with frequency methods and loop shaping. Indeed, the approach by LMI regions has been used effectively in experimental applications [16], [17], [19], [7].

All these positive features in the model-based case appear promising and have motivated us to study how to impose performance specification through LMI regions also in the data-based case. In this case we need to assign the eigenvalues in LMI regions for all matrices (A, B) consistent with data. From a conceptual viewpoint, this is similar in nature to [6] that investigates robustness of pole clustering in LMI regions with respect to complex unstructured and

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¹A. Bisoffi and C. De Persis are with ENTEG and J.C. Willems Center for Systems and Control, University of Groningen, 9747 AG Groningen, The Netherlands {a.bisoffi, c.de.persis}@rug.nl

²P. Tesi is with DINFO, University of Florence, 50139 Florence, Italy {pietro.tesi}@unifi.it

real structured uncertainty (such as parameter uncertainty); here we consider robustness with respect to a different type of uncertainty, namely, that induced by noisy data. Our contribution is that we provide sufficient conditions to design a controller enforcing robust eigenvalue assignment in spite of noisy data for generic LMI regions and their intersections, under noise models with an energy bound on the whole noise sequence of the experiment and with an instantaneous bound on each noise element of the sequence; moreover, we obtain that these sufficient conditions become also necessary for special LMI regions; finally, all these results hold both for continuous and discrete time and are given in terms of convenient linear matrix inequalities. The proposed data-driven approach based on LMI regions, which has not been investigated so far, constitutes an alternative method to \mathcal{H}_2 and \mathcal{H}_∞ approaches to guarantee performance. In a nutshell, the approach features an experiment for data collection, performance specifications are intuitively expressed as LMI regions, and the proposed convex programs design the controller to enforce the specification automatically; we then believe that the approach has the potential to incentivize these new data-based techniques among control engineers.

Structure: In Section II, we report the notions needed from the model-based setting in [5]. In Section III we formulate the data-based problem. In Section IV, we give sufficient conditions for generic LMI regions, whereas, for special LMI regions, necessary and sufficient conditions are given in Section V. Other relevant conditions for generic LMI regions are in Section VI. How all these conditions compare is investigated numerically in Section VII.

Notation: $\mathbb{N}_{\geq 1}$ denotes the natural numbers $1, 2, \dots$; \mathbb{R} denotes the real numbers; \mathbb{C} denotes the complex numbers. For $z \in \mathbb{C}$, \bar{z} denotes the complex conjugate of z . Given $n \in \mathbb{N}_{\geq 1}$, I_n (or I) denotes an identity matrix of dimension n (or of suitable dimension). For a matrix with complex entries, the hermitian operator is $\text{He } A := A + A^H$; for a matrix with real entries, the transposition operator is $\text{Tr } A := A + A^T$. For symmetric matrices A and C , we sometimes abbreviate a symmetric matrix $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ as $\begin{bmatrix} A & * \\ B & C \end{bmatrix}$ or $\begin{bmatrix} A & B^T \\ * & C \end{bmatrix}$. Positive definiteness (semidefiniteness, respectively) of a symmetric matrix A is indicated as $A \succ 0$ ($A \succeq 0$, respectively). For a $A = A^T \succeq 0$, $A^{1/2}$ denotes the unique positive semidefinite root of A . The Kronecker product is denoted by \otimes and the standard properties of the Kronecker product we use can be found in [13, §4.2].

II. REVIEW ON LMI REGIONS

This section recalls the notions we need from [5].

Definition 1: [5, Def. 2.1] A subset \mathcal{S} of the complex plane is called an LMI region if for some $s \in \mathbb{N}_{\geq 1}$, there exists a symmetric matrix $\alpha \in \mathbb{R}^{s \times s}$ and a matrix $\beta \in \mathbb{R}^{s \times s}$ such that

$$\mathcal{S} = \{z \in \mathbb{C} : \alpha + z\beta + \bar{z}\beta^T \prec 0\} \quad (1)$$

where the matrix $\alpha + z\beta + \bar{z}\beta^T$ is Hermitian. (α, β) are called data of \mathcal{S} .

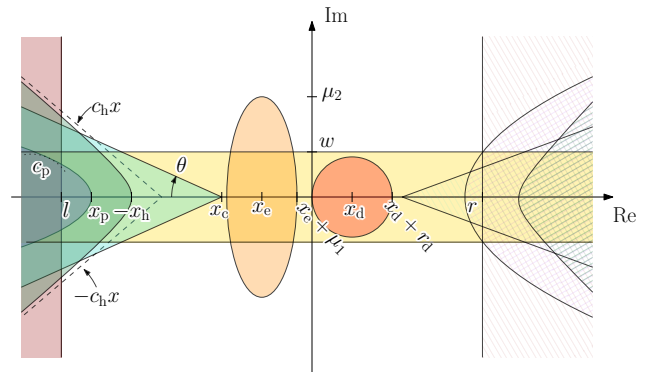


Fig. 1. Regions of the complex plane that can be represented as LMI regions for $s = 2$: facing left (filled) and facing right (hatched). The only constrained parameters of these regions are: $r_d > 0$ (radius of disk), $w > 0$ (semewidth of horizontal strip), $\mu_1 > 0$ and $\mu_2 > 0$ (semiaxes of ellipse), $c_p > 0$ (curvature of parabola), $x_h > 0$ and $c_h > 0$ (vertex and angular coefficient of asymptotes of hyperbola), $\theta \in (0, \pi/2)$ (semiaperture of cone). All other parameters (l, r, x_d, x_e, x_p, x_c) are free.

A generic $s \in \mathbb{N}_{\geq 1}$ is possible; however, $s = 2$ is a convenient trade-off between tractability and expressivity since it allows expressing a plethora of well-known quadratic curves. Most common ones are shown in Fig. 1. By expressing z in (1) in terms of its real and imaginary parts, we write (1) for $s = 2$ as

$$\begin{aligned} \mathcal{S} = \{x + jy \in \mathbb{C} : \\ & \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} + (x + jy) \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} + (x - jy) \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} \prec 0\} \\ = \{x + jy \in \mathbb{C} : & \alpha_{11} + 2x\beta_{11} < 0, \\ & \alpha_{11}\alpha_{22} - \alpha_{12}^2 + 2x(\alpha_{11}\beta_{22} + \alpha_{22}\beta_{11} - \alpha_{12}(\beta_{12} + \beta_{21})) \\ & - x^2((\beta_{12} + \beta_{21})^2 - 4\beta_{11}\beta_{22}) - y^2(\beta_{12} - \beta_{21})^2 > 0\}. \end{aligned} \quad (2)$$

The α and β corresponding to the regions in Fig. 1 are reported in the appendix, in Table III.

The next definition expresses in short that eigenvalues of a matrix belong to a certain LMI region.

Definition 2: [5, p. 359] For $\mathcal{S} \subseteq \mathbb{C}$, the matrix A is \mathcal{S} -stable if all eigenvalues of A lie in \mathcal{S} .

We recall a first elegant result from [5].

Fact 1: [5, Thm. 2.2] For an LMI region $\mathcal{S} \subseteq \mathbb{C}$ with data (α, β) , the matrix A is \mathcal{S} -stable if and only if there exists a symmetric matrix P such that

$$P \succ 0, \alpha \otimes P + \beta \otimes (AP) + \beta^T \otimes (PA^T) \prec 0. \quad (3)$$

As it emerges from the proof of [5, Thm. 2.2], an LMI region \mathcal{S} is not limited to be in the left halfplane. Then, Fact 1 enables treating continuous and discrete time simultaneously. For an LMI region \mathcal{S} with data (α, β) , define its characteristic matrix $M_{\mathcal{S}}$ as

$$M_{\mathcal{S}}(A, P) := \alpha \otimes P + \beta \otimes (AP) + \beta^T \otimes (PA^T), \quad (4)$$

so that $M_{\mathcal{S}}(A, P) \prec 0$ is precisely the main condition in (3). This brings us to a second key result from [5], recalled next.

Fact 2: [5, Cor. 2.3] Given two LMI regions \mathcal{S}_1 with data (α_1, β_1) and \mathcal{S}_2 with data (α_2, β_2) , a matrix A is both \mathcal{S}_1 -stable and \mathcal{S}_2 -stable if and only if there exists a symmetric

positive definite matrix P such that $M_{S_1}(A, P) \prec 0$ and $M_{S_2}(A, P) \prec 0$.

From linear systems theory, Hurwitz and Schur stability of a matrix A correspond to eigenvalues of A lying respectively in the open left halfplane

$$\mathcal{S}_H := \{x + jy \in \mathbb{C} : x < 0\} = \{z \in \mathbb{C} : 0 + z + \bar{z} < 0\} \quad (5)$$

and in the open unit disk

$$\begin{aligned} \mathcal{S}_S &:= \{x + jy \in \mathbb{C} : x^2 + y^2 < 1\} \\ &= \{z \in \mathbb{C} : \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \bar{z} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \prec 0\} \quad (6) \end{aligned}$$

or, equivalently, to the existence of a symmetric P satisfying the Lyapunov conditions

$$(P \succ 0, AP + PA^\top \prec 0) \text{ and } (P \succ 0, APA^\top - P \prec 0);$$

these precise conditions can be obtained by using \mathcal{S}_H and \mathcal{S}_S as LMI regions and applying Fact 1 to them. On the other hand, Fact 1 alone enables considering more general subsets in the complex plane, and Fact 2 using their intersections.

A controller that assigns for a closed-loop system $\dot{x}/x^+ = A_{cl}x$ the eigenvalues of A_{cl} in a certain region of the complex plane can effectively enforce meaningful performance specifications since different regions of the complex plane for the eigenvalues of A_{cl} correspond to different transient behaviours, as we illustrate in the next example.

Example 1: For suitable parameters $\ell > 0$, $\rho > 0$, $\theta \in (0, \pi/2)$, consider the subset $\mathcal{S}(\ell, \rho, \theta)$ depicted in Fig. 2, left. $\mathcal{S}(\ell, \rho, \theta) = \{z = x + jy \in \mathbb{C} : x < -\ell\} \cap \{z = x + jy \in \mathbb{C} : x^2 + y^2 < \rho^2\} \cap \{z = x + jy \in \mathbb{C} : (\cos \theta)|y| < -(\sin \theta)x, x < 0\}$: hence, it guarantees a minimum convergence rate of ℓ (halfplane), a maximum natural frequency of ρ (disk) and a minimum damping ratio $\cos \theta$ (cone). In terms of performance, these correspond to upper bounds on the settling time, the overshoot, the frequency of oscillatory modes and the magnitude of high-frequency poles [5][12, §3.3-3.4]. By Fact 2, eigenvalues of A_{cl} are located in $\mathcal{S}(\ell, \rho, \theta)$ if there exists $P = P^\top \succ 0$ such that

$$\begin{aligned} \begin{bmatrix} \ell & 0 \\ 0 & -1 \end{bmatrix} \otimes P + \text{Tr} \left\{ \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \otimes (A_{cl}P) \right\} &\prec 0, \\ \begin{bmatrix} -\rho & 0 \\ 0 & -\rho \end{bmatrix} \otimes P + \text{Tr} \left\{ \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \otimes (A_{cl}P) \right\} &\prec 0, \\ \text{Tr} \left\{ \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \otimes (A_{cl}P) \right\} &\prec 0. \end{aligned}$$

When designing a controller $u = Kx$ for $\dot{x}/x^+ = Ax + Bu$, one considers a closed-loop matrix $A + BK$ in (3) and looks for $P = P^\top \succ 0$ and K such that

$$\alpha \otimes P + \beta \otimes ((A + BK)P) + \beta^\top \otimes (P(A + BK)^\top) \prec 0.$$

This inequality is not linear in P and K , so one uses instead

$$\text{find } P = P^\top \succ 0, Y \quad (7a)$$

$$\text{s. t. } \alpha \otimes P + \text{Tr} \{ \beta \otimes (AP + BY) \} \prec 0 \quad (7b)$$

with (7b) now linear in P and Y . From the underlying change of variables, K is YP^{-1} . In the presence of r LMI regions \mathcal{S}_i with data (α_i, β_i) , $i = 1, \dots, r$, (7) extends by Fact 2 to

$$\text{find } P = P^\top \succ 0, Y \quad (8a)$$

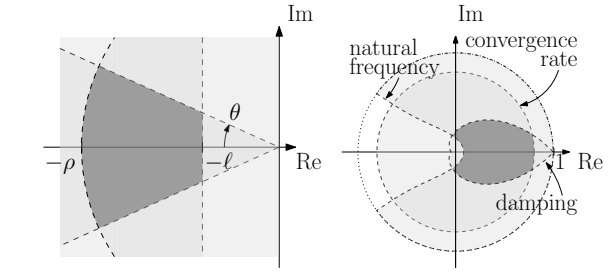


Fig. 2. (Left) Region $\mathcal{S}(\ell, \rho, \theta)$ of Example 1 enforcing a desirable transient behaviour in continuous time. (Right) The analogous region in discrete time where loci of constant natural frequency, convergence rate and damping are indicated, see Example 6.

$$\text{s. t. } \alpha_i \otimes P + \text{Tr} \{ \beta_i \otimes (AP + BY) \} \prec 0, i = 1, \dots, r. \quad (8b)$$

III. DATA-DRIVEN CONTROL BY LMI REGIONS

A. Problem formulation

Consider a linear time-invariant system

$$\dot{x}^\circ = A_\star x + B_\star u + d \quad (9)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $d \in \mathbb{R}^n$ is the disturbance and x° represents the time derivative \dot{x} of state x in continuous time or the update x^+ of the state x in discrete time. For convenience, we call x° state preview. The matrices A_\star and B_\star are *unknown* to us, and we rely instead on data collected through an experiment on the system. Specifically, we apply an input sequence $u(t_0), u(t_1), \dots, u(t_{T-1})$ and measure the corresponding state and state preview sequences $x(t_0), x(t_1), \dots, x(t_{T-1})$ and $x^\circ(t_0), x^\circ(t_1), \dots, x^\circ(t_{T-1})$, where the preview sequence in continuous time is the sequence of state derivatives $\dot{x}(t_0), \dot{x}(t_1), \dots, \dot{x}(t_{T-1})$ and in discrete time is the sequence of states $x(t_1), x(t_2), \dots, x(t_T)$. During the experiment, a disturbance sequence $d(t_0), d(t_1), \dots, d(t_{T-1})$ acts on system (9) and affects the evolution of the system. This sequence is also *unknown* to us, and due to its influence on the state evolution, we say that data are noisy. In summary, the measured sequences are collected in the matrices

$$U_0 := [u(t_0) \ \dots \ u(t_{T-1})] \quad (10a)$$

$$X_0 := [x(t_0) \ \dots \ x(t_{T-1})] \quad (10b)$$

$$X_1 := [x^\circ(t_0) \ \dots \ x^\circ(t_{T-1})] \quad (10c)$$

$$= \begin{cases} \begin{bmatrix} \dot{x}(t_0) & \dots & \dot{x}(t_{T-1}) \end{bmatrix} & \text{in cont. time} \\ \begin{bmatrix} x(t_1) & \dots & x(t_T) \end{bmatrix} & \text{in discr. time} \end{cases}$$

and the unknown disturbance sequence in $D_0 := [d(t_0) \ \dots \ d(t_{T-1})]$. The times t_0, t_1, \dots, t_{T-1} are taken as the $0, 1, \dots, T-1$ multiples of a certain period T_s ; this is a natural choice in discrete time since these times correspond to periodic sampling times, and we adopt the same choice in continuous time as well (although this is not necessary). Since the data generation mechanism is (9), the data points in the experiment satisfy

$$X_1 = A_\star X_0 + B_\star U_0 + D_0.$$

As for the disturbance sequence, we know only that, for some $p \in \mathbb{N}_{\geq 1}$ and some matrix $\Delta \in \mathbb{R}^{n \times p}$, it belongs to the set

$$\mathcal{D} := \{D \in \mathbb{R}^{n \times T} : DD^\top \preceq \Delta\Delta^\top\} \quad (11)$$

and this corresponds to knowing a bound on the energy of any disturbance sequence, and in particular of D_0 , which satisfies $D_0 \in \mathcal{D}$. The pairs (A, B) that could have generated the data points U_0, X_0, X_1 for a disturbance sequence $D \in \mathcal{D}$ correspond to the set

$$\mathcal{C} := \{(A, B) : X_1 = AX_0 + BU_0 + D, D \in \mathcal{D}\}, \quad (12)$$

and \mathcal{C} is called the set of *matrices consistent with data*. Since $D_0 \in \mathcal{D}$, we have $(A_\star, B_\star) \in \mathcal{C}$. The objective of this work is to design a linear feedback controller $u = Kx$ that assigns the eigenvalues of the closed-loop matrix $A_\star + B_\star K$ within a certain subset \mathcal{R} of the complex plane. Since A_\star and B_\star are unknown, this is achieved by imposing that K assigns the eigenvalues of $A + BK$ within \mathcal{R} for all $(A, B) \in \mathcal{C}$. As a first step, we address the case of \mathcal{R} given by a single LMI region \mathcal{S} with data (α, β) where the objective becomes

$$\text{find } P = P^\top \succ 0, K \quad (13a)$$

$$\text{s. t. } \alpha \otimes P + \beta \otimes ((A + BK)P) \\ + \beta^\top \otimes (P(A + BK)^\top) \prec 0 \quad \forall (A, B) \in \mathcal{C}. \quad (13b)$$

In words, solving this problem yields a certificate and a controller for \mathcal{S} -stability, i.e., the Lyapunov-like matrix P and the gain K . For $\alpha = 0$ and $\beta = 1$, (13) becomes a classical continuous-time stabilization problem and the corresponding $V(x) = x^\top P x$ is a *common* Lyapunov function [21, §II.B] since it accommodates all matrices (A, B) consistent with data. Analogously, the matrix P obtained as a solution to a generic (13) qualifies as a *common* Lyapunov-like matrix. As a second step, we address the case of \mathcal{R} given by the intersection of r LMI regions \mathcal{S}_i , $i = 1, \dots, r$, each with data (α_i, β_i) , that is, $\mathcal{R} := \bigcap_{i=1}^r \mathcal{S}_i$. The objective becomes

$$\text{find } P = P^\top \succ 0, K \quad (14a)$$

$$\text{s. t. } M_{\mathcal{S}_1}((A + BK), P) \prec 0, \\ \dots, M_{\mathcal{S}_r}((A + BK), P) \prec 0 \quad \forall (A, B) \in \mathcal{C} \quad (14b)$$

where the definition of characteristic matrices $M_{\mathcal{S}_1}, \dots, M_{\mathcal{S}_r}$ is in (4). (14) is the natural extension of (13) by taking into account Fact 2 for an intersection of LMI regions.

B. Equivalent forms of set \mathcal{C}

The set \mathcal{C} introduced in (12) plays a key role in the developments and in this section we present for it three different forms equivalent to each other. The first form is

$$\mathcal{C} = \{[A \ B] = Z^\top : [I \ Z^\top] \begin{bmatrix} \mathbf{C} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I \\ Z \end{bmatrix} \preceq 0\} \quad (15a)$$

$$\mathbf{C} := -\Delta\Delta^\top + X_1 X_1^\top \quad (15b)$$

$$\mathbf{B} := - \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} X_1^\top, \mathbf{A} := \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top, \quad (15c)$$

and this form can be obtained with algebraic computations from the definition of \mathcal{C} in (12) by expressing D in (12) as $D = X_1 - AX_0 - BU_0$, substituting this D in the condition defining \mathcal{D} in (11), and collecting $[I \ A \ B] = [I \ Z^\top]$ to the left and its transpose to the right. We make the next assumption on matrix $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$.

Assumption 1: Matrix $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ has full row rank.

Full row rank of $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$ is intimately related to persistence of excitation of the input and disturbance sequences, see a detailed discussion in [3, §4.1]. The rank condition can be verified directly from data and when it does not hold, one can typically enforce it simply by collecting more data points, thereby adding columns to $\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}$. By Assumption 1, $\mathbf{A} = \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top \succ 0$. Thanks to $\mathbf{A} \succ 0$, we have the second form of the set \mathcal{C} as

$$\mathcal{C} = \{[A \ B] = Z^\top : (Z - Z_c)^\top \mathbf{A} (Z - Z_c) \preceq \mathbf{Q}\} \quad (16a)$$

$$Z_c := -\mathbf{A}^{-1} \mathbf{B}, \mathbf{Q} := \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B} - \mathbf{C}. \quad (16b)$$

We noted before $\mathbf{A} \succ 0$; the sign definiteness of \mathbf{Q} is also a structural property as claimed next.

Lemma 1: Under Assumption 1, $\mathbf{A} \succ 0$ and $\mathbf{Q} \succeq 0$.

Proof: The lemma is the same as [3, Lemma 1], so the proof is omitted. ■

With Lemma 1, we can give the third form of \mathcal{C} as

$$\mathcal{C} = \{Z_c + \mathbf{A}^{-1/2} \Upsilon \mathbf{Q}^{1/2} : \Upsilon^\top \Upsilon \preceq I\}. \quad (17)$$

The fact that the set \mathcal{C} in (16) rewrites equivalently as in (17) is straightforward for $\mathbf{A} \succ 0$ and $\mathbf{Q} \succ 0$; it is less so for $\mathbf{A} \succ 0$ and $\mathbf{Q} \succeq 0$ and the proof for this case is in [3, Prop. 1]. The third form of \mathcal{C} in (17) is the one we typically need to obtain our main results in the sequel.

Remark 1: Instead of (11), one can consider for the disturbance sequence the bound given by

$$\mathcal{D} := \{D \in \mathbb{R}^{n \times T} : [I \ D] \begin{bmatrix} R & S^\top \\ S & Q \end{bmatrix} \begin{bmatrix} I \\ D^\top \end{bmatrix} \preceq 0\}$$

with matrices R and Q symmetric and $Q \succ 0$ [1], [21], [2]. Because of $Q \succ 0$, D cannot be too “large”; so, as in (11), the knowledge of \mathcal{D} corresponds to knowing a bound on the energy of any disturbance sequence. Moreover, one can obtain a set \mathcal{C} analogous to (15), namely, \mathcal{C} as in (15a) with, instead of (15b)-(15c),

$$\mathbf{C} := R + X_1 S + S^\top X_1^\top + X_1 Q X_1^\top$$

$$\mathbf{B} := - \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} (S + Q X_1^\top), \mathbf{A} := \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} Q \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top,$$

and still conclude from Assumption 1 and $Q \succ 0$ that $\mathbf{A} = \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} Q^{1/2} Q^{1/2} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top \succ 0$. Finally, Lemma 1 remains valid for the different expressions of \mathbf{A} and, consequently, of \mathbf{Q} in (16b) and it can be proven with the very same proof strategy of [3, Lemma 1] as long as one substitutes the expression of \mathbf{Q}_p there with

$$\mathbf{Q}_p := Q^{1/2} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top \left(\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} Q \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^\top \right)^{-1} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} Q^{1/2}.$$

The rest of our result hold identically.

IV. SUFFICIENT CONDITION FOR GENERIC LMI REGIONS

In this section we consider generic LMI regions and, for them, look for data-based counterparts of Facts 1 and 2, which will result in the sufficient conditions of Theorem 1 given next and of Corollary 1 given later.

Theorem 1: *Let Assumption 1 hold and \mathcal{S} be an LMI region with data (α, β) . (13) is feasible if the next program is feasible*

$$\text{find } P = P^\top \succ 0, Y \quad (18a)$$

$$\text{s. t. } \left[\begin{array}{c} \left\{ \begin{array}{l} (\beta\beta^\top) \otimes \mathbf{Q} + \alpha \otimes P \\ + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \end{array} \right\} \star \\ I_s \otimes [\frac{P}{Y}] \end{array} \right] \begin{array}{c} \star \\ -I_s \otimes \mathbf{A} \end{array} \prec 0 \quad (18b)$$

If (18) is feasible, the controller gain in (13) is $K = YP^{-1}$.

Proof: The proof shows how to enforce the condition of Fact 1 for all matrices $(A, B) \in \mathcal{C}$. Apply Schur complement to (18b) and obtain equivalently, since $\mathbf{A} \succ 0$,

$$\begin{aligned} & (\beta\beta^\top) \otimes \mathbf{Q} + \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \\ & + (I_s \otimes [\frac{P}{Y}]^\top) (I_s \otimes \mathbf{A}^{-1}) (I_s \otimes [\frac{P}{Y}]) \prec 0. \end{aligned}$$

Equivalently, multiply both sides by any $\lambda > 0$ and then replace λP , λY with P , Y to obtain

$$\begin{aligned} & \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \\ & + \frac{1}{\lambda} (I_s \otimes ([\frac{P}{Y}]^\top \mathbf{A}^{-1/2})) (I_s \otimes (\mathbf{A}^{-1/2} [\frac{P}{Y}])) \\ & + \lambda (\beta \otimes \mathbf{Q}^{1/2}) (\beta^\top \otimes \mathbf{Q}^{1/2}) \prec 0. \end{aligned} \quad (19)$$

We use the so-called Petersen's lemma [18] in the version reported in [3, Fact 1]. The existence of $\lambda > 0$ such that (19) holds is equivalent by [3, Fact 1] to

$$\begin{aligned} & \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \\ & + \text{Tr} \left\{ (\beta \otimes \mathbf{Q}^{1/2}) \mathbf{Y}^\top (I_s \otimes (\mathbf{A}^{-1/2} [\frac{P}{Y}])) \right\} \\ & \prec 0 \quad \forall \mathbf{Y}: \mathbf{Y}^\top \mathbf{Y} \preceq I_{sn} \end{aligned} \quad (20)$$

With Υ as in (17), i.e., $\Upsilon^\top \Upsilon \preceq I_n$, consider the block-diagonal matrix $\mathbf{Y} = I_s \otimes \Upsilon$ with s blocks. $\mathbf{Y}^\top \mathbf{Y} = \begin{bmatrix} \Upsilon^\top \Upsilon & & 0 \\ & \ddots & \\ 0 & & \Upsilon^\top \Upsilon \end{bmatrix} \preceq \begin{bmatrix} I_n & & 0 \\ & \ddots & \\ 0 & & I_n \end{bmatrix} = I_{sn}$ so (20) implies for the selected block-diagonal \mathbf{Y} that

$$\begin{aligned} & 0 \succ \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \\ & + \text{Tr} \left\{ (\beta \otimes \mathbf{Q}^{1/2}) (I_s \otimes \Upsilon^\top) (I_s \otimes (\mathbf{A}^{-1/2} [\frac{P}{Y}])) \right\} \\ & = \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \} \\ & + \text{Tr} \left\{ \beta \otimes (\mathbf{Q}^{1/2} \Upsilon^\top \mathbf{A}^{-1/2} [\frac{P}{Y}]) \right\} \\ & = \alpha \otimes P + \text{Tr} \left\{ \beta \otimes \left((Z_c + \mathbf{A}^{-1/2} \Upsilon \mathbf{Q}^{1/2})^\top [\frac{P}{Y}] \right) \right\} \\ & \quad \forall \Upsilon: \Upsilon^\top \Upsilon \preceq I_n. \end{aligned} \quad (21)$$

Equivalently, we have by (17) that

$$0 \succ \alpha \otimes P + \text{Tr} \{ \beta \otimes ([A \ B] [\frac{P}{Y}]) \} \quad \forall (A, B) \in \mathcal{C}. \quad (22)$$

In summary, feasibility of (18) implies feasibility of

$$\text{find } P = P^\top \succ 0, Y \text{ subject to (22).}$$

Feasibility of this problem is equivalent to feasibility of (13) by the standard change of variables given by $Y = KP$. ■

The feasibility program in (18) is convenient since the constraint (18b) is a linear matrix inequality in the decision variables P , Y . With respect to the model-based condition in Fact 1, Theorem 1 no longer gives a necessary and sufficient condition because, from (20) to (21) in the proof, we used that for matrices $\mathbf{D} = \mathbf{D}^\top$, \mathbf{E} , \mathbf{G}

$$0 \succ \mathbf{D} + \mathbf{E}\mathbf{F}\mathbf{G} + \mathbf{G}^\top \mathbf{F}^\top \mathbf{E}^\top \quad \forall \mathbf{F}: \mathbf{F}^\top \mathbf{F} \preceq I \quad (23a)$$

implies

$$0 \succ \mathbf{D} + \mathbf{E}(I \otimes \mathbf{f})\mathbf{G} + \mathbf{G}^\top (I \otimes \mathbf{f}^\top) \mathbf{E}^\top \quad \forall \mathbf{f}: \mathbf{f}^\top \mathbf{f} \preceq I, \quad (23b)$$

but is not implied by it in general¹. The larger the number s of blocks \mathbf{f} on the diagonal is, the sparser the matrix $I \otimes \mathbf{f}$ (of unit norm) is, the more conservative it is to replace that matrix with the full matrix \mathbf{F} (of unit norm). Hence, the chances of feasibility decrease with the dimension s of the LMI region considered in Theorem 1. From Theorem 1, which is the data-based counterpart of Fact 1, we obtain the next data-based counterpart of Fact 2.

Corollary 1: *Let Assumption 1 hold and \mathcal{S}_i , for $i = 1, \dots, r$, be an LMI region with data (α_i, β_i) . (14) is feasible if the next program is feasible*

$$\text{find } P = P^\top \succ 0, Y \quad (24a)$$

$$\text{s. t. } \left[\begin{array}{c} \left\{ \begin{array}{l} (\beta_i \beta_i^\top) \otimes \mathbf{Q} + \alpha_i \otimes P \\ + \text{Tr} \{ \beta_i \otimes (Z_c^\top [\frac{P}{Y}]) \} \end{array} \right\} \star \\ I_s \otimes [\frac{P}{Y}] \end{array} \right] \begin{array}{c} \star \\ -I_s \otimes \mathbf{A} \end{array} \prec 0$$

for $i = 1, \dots, r$. (24b)

If (24) is feasible, the controller gain in (14) is $K = YP^{-1}$.

Proof: (14) is equivalent, by definition of characteristic matrix in (4), to

$$\begin{aligned} & \text{find } P = P^\top \succ 0, K \\ & \alpha_1 \otimes P + \text{Tr} \{ \beta_1 \otimes ((A+BK)P) \} \prec 0 \quad \forall (A, B) \in \mathcal{C} \\ & \quad \vdots \\ & \alpha_r \otimes P + \text{Tr} \{ \beta_r \otimes ((A+BK)P) \} \prec 0 \quad \forall (A, B) \in \mathcal{C}. \end{aligned}$$

(24) implies feasibility of this program by Theorem 1. ■

V. NECESSARY AND SUFFICIENT CONDITION FOR SPECIAL LMI REGIONS

We have seen in Section IV sufficient conditions for data-driven stabilization within generic LMI regions and their intersections. In this section we show that necessary and sufficient conditions can be found for special LMI regions and their intersections. We will show that these are vertical halfplanes, disks centered on the real axis and intersections of such halfplanes and disks, and that these regions can inner-approximate subsets of the complex plane of practical interest.

¹Take $\mathbf{D} = -I$, $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{G} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. $\mathbf{D} + \mathbf{E} \begin{bmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{f} \end{bmatrix} \mathbf{G} + \mathbf{G}^\top \begin{bmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{f} \end{bmatrix} \mathbf{E}^\top = \mathbf{D} \prec 0$ for all $\mathbf{f} \in \mathbb{R}$. On the other hand, take $\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which satisfies $\mathbf{F}^\top \mathbf{F} \preceq I$; for such \mathbf{F} , $\mathbf{D} + \mathbf{E}\mathbf{F}\mathbf{G} + \mathbf{G}^\top \mathbf{F}^\top \mathbf{E}^\top = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, which is not negative definite. This shows that (23b) does not imply (23a).

In Section IV, the source of conservatism leading to sufficient conditions was substituting the block diagonal uncertainty $I \otimes \mathbf{f}$ in (23b) with the full uncertainty \mathbf{F} in (23a). A special case when no conservatism is introduced is when $I \otimes \mathbf{f} = 1 \otimes \mathbf{f}$, and this occurs for β of rank 1 as we now show. By considering at (21) in the proof of Theorem 1 and setting $\mathbf{D} := \alpha \otimes P + \text{Tr} \{ \beta \otimes (Z_c^\top [\frac{P}{Y}]) \}$, we have

$$0 \succ \mathbf{D} + \text{Tr} \left\{ \beta \otimes \left(\mathbf{Q}^{1/2} \Upsilon^\top \mathbf{A}^{-1/2} \left[\frac{P}{Y} \right] \right) \right\} \quad \forall \Upsilon: \Upsilon^\top \Upsilon \preceq I; \quad (25)$$

hence, if $\beta = \eta \cdot 1 \cdot \gamma^\top$ for some vectors η and γ in \mathbb{R}^s , (25) becomes that for all Υ such that $\Upsilon^\top \Upsilon \preceq I$,

$$0 \succ \mathbf{D} + \text{Tr} \left\{ (\eta \otimes \mathbf{Q}^{1/2}) (1 \otimes \Upsilon^\top) (\gamma^\top \otimes \mathbf{A}^{-1/2} \left[\frac{P}{Y} \right]) \right\},$$

where $1 \otimes \Upsilon^\top$ appears as we intended to show. This discussion is summarized in the next assumption.

Assumption 2: Let \mathcal{S} be an LMI region with data (α, β) . For $\beta \in \mathbb{R}^{s \times s}$, there exist η and γ in \mathbb{R}^s such that $\beta = \eta \gamma^\top$, i.e., $\begin{bmatrix} \beta_{11} & \dots & \beta_{1s} \\ \vdots & & \vdots \\ \beta_{s1} & \dots & \beta_{ss} \end{bmatrix} = \begin{bmatrix} \eta_1 \gamma_1 & \dots & \eta_1 \gamma_s \\ \vdots & & \vdots \\ \eta_s \gamma_1 & \dots & \eta_s \gamma_s \end{bmatrix}$.

A simple exemplification of this assumption follows.

Example 2: An open disk with center $(x_d, 0)$ and radius $r_d > 0$ has data (α_d, β_d) with $\alpha_d := \begin{bmatrix} -r_d & x_d \\ x_d & -r_d \end{bmatrix}$ and $\beta_d := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, and satisfies Assumption 2 with $\eta_d := \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\gamma_d := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For an LMI region \mathcal{S} satisfying Assumption 2, the program (13) is reformulated equivalently as in the next result.

Theorem 2: Let Assumption 1 hold and \mathcal{S} be an LMI region with data (α, β) satisfying Assumption 2, i.e., $\beta = \eta \gamma^\top$ for some η and γ in \mathbb{R}^s . Then, (13) is feasible if and only if the next program is feasible

$$\text{find } P = P^\top \succ 0, Y \quad (26a)$$

$$\text{s. t. } \left[\begin{array}{c} \left\{ \begin{array}{l} (\eta \otimes I_n) \mathbf{Q} (\eta^\top \otimes I_n) + \alpha \otimes P \\ + \text{Tr} \left\{ (\eta \otimes I_n) Z_c^\top (\gamma^\top \otimes [\frac{P}{Y}]) \right\} \end{array} \right\} \star \\ \gamma^\top \otimes [\frac{P}{Y}] \\ -\mathbf{A} \end{array} \right] \prec 0 \quad (26b)$$

If (26) is feasible, the controller gain in (13) is $K = Y P^{-1}$.

Proof: By Schur complement and $\mathbf{A} \succ 0$ by Assumption 1, (26b) is equivalent to

$$0 \succ \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I_n) Z_c^\top (\gamma^\top \otimes [\frac{P}{Y}]) \right\} + (\eta \otimes I_n) \mathbf{Q} (\eta^\top \otimes I_n) + (\gamma \otimes [\frac{P}{Y}]^\top) \mathbf{A}^{-1} (\gamma^\top \otimes [\frac{P}{Y}])$$

Equivalently, multiply by any $\lambda > 0$ both sides and then replace λP , λY with P , Y to obtain

$$0 \succ \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I) Z_c^\top (\gamma^\top \otimes [\frac{P}{Y}]) \right\} + \lambda (\eta \otimes I_n) \mathbf{Q}^{1/2} \mathbf{Q}^{1/2} (\eta^\top \otimes I_n) + \frac{1}{\lambda} (\gamma \otimes [\frac{P}{Y}]^\top) \mathbf{A}^{-1/2} \mathbf{A}^{-1/2} (\gamma^\top \otimes [\frac{P}{Y}]). \quad (27)$$

By Petersen's lemma [18] in the version [3, Fact 1], the existence of $\lambda > 0$ such that (27) holds is equivalent to

$$0 \succ \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I_n) Z_c^\top (\gamma^\top \otimes [\frac{P}{Y}]) \right\} + \text{Tr} \left\{ (\eta \otimes I_n) \mathbf{Q}^{1/2} \Upsilon^\top \mathbf{A}^{-1/2} (\gamma^\top \otimes [\frac{P}{Y}]) \right\}$$

$$= \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I_n) \left(Z_c^\top + \mathbf{Q}^{1/2} \Upsilon^\top \mathbf{A}^{-1/2} \right) (\gamma^\top \otimes [\frac{P}{Y}]) \right\} \quad \forall \Upsilon: \Upsilon^\top \Upsilon \preceq I_n.$$

This condition is equivalent, by (17), to

$$\begin{aligned} 0 \succ & \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I_n) [A \ B] (\gamma^\top \otimes [\frac{P}{Y}]) \right\} \\ = & \alpha \otimes P + \text{Tr} \left\{ (\eta \otimes I_n) (1 \otimes [A \ B]) (\gamma^\top \otimes [\frac{P}{Y}]) \right\} \\ = & \alpha \otimes P + \text{Tr} \left\{ (\eta \gamma^\top) \otimes ([A \ B] [\frac{P}{Y}]) \right\} \\ = & \alpha \otimes P + \text{Tr} \left\{ \beta \otimes ([A \ B] [\frac{P}{Y}]) \right\} \quad \forall (A, B) \in \mathcal{C}. \end{aligned}$$

To find $P = P^\top \succ 0$ and Y subject to this condition is equivalent to (13) by the standard change of variables $Y = KP$. \blacksquare

Parallel to Corollary 1, we have the next result for the intersection of LMI regions satisfying Assumption 2.

Corollary 2: Let Assumption 1 hold and \mathcal{S}_i , for $i = 1, \dots, r$, be an LMI region with data (α_i, β_i) satisfying Assumption 2, i.e., $\beta_i = \eta_i \gamma_i^\top$ for some η_i and γ_i in \mathbb{R}^s . (14) is feasible if and only if the next program is feasible

$$\text{find } P = P^\top \succ 0, Y \quad (28a)$$

$$\text{s. t. } \left[\begin{array}{c} \left\{ \begin{array}{l} (\eta_i \otimes I_n) \mathbf{Q} (\eta_i^\top \otimes I_n) + \alpha_i \otimes P \\ + \text{Tr} \left\{ (\eta_i \otimes I_n) Z_c^\top (\gamma_i^\top \otimes [\frac{P}{Y}]) \right\} \end{array} \right\} \star \\ \gamma_i^\top \otimes [\frac{P}{Y}] \\ -\mathbf{A} \end{array} \right] \prec 0 \quad (28b)$$

for $i = 1, \dots, r$.

If (28) is feasible, the controller gain in (14) is $K = Y P^{-1}$.

Motivated by Theorem 2 and Corollary 2, we examine the subsets of the complex plane to which LMI regions satisfying Assumption 2 give rise. A generic LMI region with $s = 1$ and data (α, β) has trivially β of rank 1, and is $\mathcal{S} = \{z = x + jy \in \mathbb{C}: \alpha_{11} + \beta_{11}(z + \bar{z}) = \alpha_{11} + x2\beta_{11} < 0\}$, which can express vertical halfplanes, besides being possibly \emptyset or \mathbb{C} . For $s = 2$, the subsets of \mathbb{C} that can be expressed with β of rank 1 are determined in the next lemma.

Lemma 2: Let \mathcal{S} be an LMI region with $s = 2$ and data (α, β) satisfying Assumption 2, i.e., $\beta = \eta \gamma^\top$ for some η and β in \mathbb{R}^2 . Then, a nontrivial \mathcal{S} (i.e., different from \emptyset and \mathbb{C}) can only be: a vertical strip, a vertical halfplane, a disk centered on the real axis or an intersection of the last two.

Proof: Specialize the expression in (2) of a generic LMI region with $s = 2$ for the \mathcal{S} considered in the statement with $\eta := \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$, $\gamma := \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$ and $\beta := \begin{bmatrix} \eta_1 \gamma_1 & \eta_1 \gamma_2 \\ \eta_2 \gamma_1 & \eta_2 \gamma_2 \end{bmatrix}$. Then, \mathcal{S} rewrites after some algebraic computations as the set of points $x + jy \in \mathbb{C}$ such that

$$0 > \alpha_{11} + 2x(\eta_1 \gamma_1) \quad (29a)$$

$$0 < \alpha_{11} \alpha_{22} - \alpha_{12}^2 + 2x(\alpha_{11} \eta_2 \gamma_2 + \alpha_{22} \eta_1 \gamma_1 - \alpha_{12}(\eta_1 \gamma_2 + \eta_2 \gamma_1)) - (x^2 + y^2)(\eta_1 \gamma_2 - \eta_2 \gamma_1)^2. \quad (29b)$$

If $\eta_1 \gamma_2 - \eta_2 \gamma_1 = 0$, \mathcal{S} is constituted from (29) by two affine inequalities in the real part x , hence a nontrivial \mathcal{S} can be a vertical halfplane or a vertical strip depending on the values of α , η , γ . Otherwise, divide by $(\eta_1 \gamma_2 - \eta_2 \gamma_1)^2$ in (29b) and complete the square, which can be done for each α ,

η, γ with $\eta_1\gamma_2 - \eta_2\gamma_1 \neq 0$; after some computations, (29) rewrites equivalently as

$$\begin{aligned} 0 &> \alpha_{11} + 2x(\eta_1\gamma_1), \sigma > (x_0 - x)^2 + y^2 \\ x_0 &:= (\alpha_{11}\eta_2\gamma_2 + \alpha_{22}\eta_1\gamma_1 - \alpha_{12}(\eta_1\gamma_2 + \eta_2\gamma_1)) \\ &\quad \cdot (\eta_1\gamma_2 - \eta_2\gamma_1)^{-2} \\ \sigma &:= (\alpha_{11}^2\eta_2^2\gamma_2^2 + \alpha_{22}^2\eta_1^2\gamma_1^2 + \alpha_{12}^2 4\eta_1\eta_2\gamma_1\gamma_2 \\ &\quad + \alpha_{11}\alpha_{22}(\eta_1^2\gamma_2^2 + \eta_2^2\gamma_1^2) - 2\alpha_{11}\alpha_{12}\eta_2\gamma_2(\eta_1\gamma_2 + \eta_2\gamma_1) \\ &\quad - 2\alpha_{12}\alpha_{22}\eta_1\gamma_1(\eta_1\gamma_2 + \eta_2\gamma_1)) \cdot (\eta_1\gamma_2 - \eta_2\gamma_1)^{-4} \end{aligned}$$

If σ is nonpositive, \mathcal{S} is an empty set. Otherwise, a nontrivial \mathcal{S} can be a disk centered on the real axis or an intersection of it with a halfplane. ■

Let us exemplify Lemma 2 and Theorem 2 on the important special cases of Hurwitz and Schur stability.

Example 3: We have shown in (5) how the condition for Hurwitz stability can be expressed through an LMI region \mathcal{S}_H with $\alpha = 0$ and $\beta = 1$. Such LMI region satisfies Assumption 2 trivially with, e.g., $\eta = \gamma = 1$. Theorem 2 claims then that, under Assumption 1, (13) is feasible if and only if there exist $P = P^\top \succ 0$ and Y such that

$$\left[\begin{array}{c|c} \text{Tr} \{ Z_c^\top \left(\begin{bmatrix} P \\ Y \end{bmatrix} \right) \} + \mathbf{Q} & \star \\ \hline \begin{bmatrix} P \\ Y \end{bmatrix} & -\mathbf{A} \end{array} \right] \prec 0.$$

This claim is precisely the same as [3, Thm. 2] by taking into account [3, Remark 3 and Eq. (38)].

Example 4: We have shown in (6) how the condition for Schur stability can be expressed through an LMI region \mathcal{S}_S with $\alpha = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$. As shown in Example 2, such LMI region satisfies Assumption 2 with, e.g., $\eta = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Theorem 2 claims then that, under Assumption 1, (13) is feasible if and only if there exist $P = P^\top \succ 0$ and Y such that, after some computations,

$$\left[\begin{array}{c|c|c} -P & -[P]^\top Z_c & [P]^\top \\ \hline -Z_c^\top [P] & -P + \mathbf{Q} & 0 \\ \hline [P] & 0 & -\mathbf{A} \end{array} \right] \prec 0 \iff \left[\begin{array}{c|c|c} -P + \mathbf{Q} & -Z_c^\top [P] & 0 \\ \hline -[P]^\top Z_c & -P & [P]^\top \\ \hline 0 & [P] & -\mathbf{A} \end{array} \right] \prec 0.$$

This claim is precisely the same as [3, Thm. 1] by taking into account [3, Remark 3 and Eq. (37)].

As shown in the examples, (26) is a necessary and sufficient condition to solve (13) for the open left halfplane and the open unit disk. On the other hand, Lemma 2 improves on expressivity with respect to these two special cases by showing that we have equivalence with (26) for arbitrary intersections of halfplanes left or right of any real part, and disks with any center on the real axis and any radius. When considering the regions expressed by $s = 1$ and $s = 2$, one can expect that increasing s further yields more complex subsets of the complex plane, but finding their analytic expressions seems hardly tractable (numerical investigations aside), so the case $s > 2$ is an open question. Although Assumption 2 limits expressivity in the case $s = 2$, we would like to show that by intersection of halfplanes and disks we can still obtain inner-approximations of subsets of the complex plane as in the next example. Imposing eigenvalues within such an intersection for all matrices consistent with data is then without conservatism by virtue of Corollary 2.

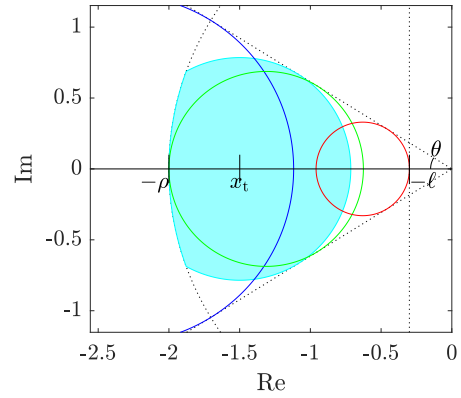


Fig. 3. Finding largest intersection of two disks inner-approximating $\mathcal{S}(\ell, \rho, \theta)$ as explained in Example 5, for $\ell = 0.3$, $\rho = 2$, $\theta = \pi/5.7$.

Example 5: Example 1 presented a relevant region for performance specifications and we show here a way to find an inner-approximation made only of intersections of disks. Instead of examining all cases, we consider $\theta \in (0, \pi/4)$ and $\rho > \ell(3 + 2\sqrt{2}) \simeq 5.83\ell$ since the first condition excludes insufficiently damped responses and the second one expresses that we do not need the magnitude ρ of high-frequency closed-loop poles to be too close to the dominant closed-loop convergence rate ℓ (e.g., $\rho = 10\ell$ would allow a decade between the two). Abbreviate $\sin \theta$ and $\cos \theta$ with s_θ and c_θ . A circle centered on the negative real axis tangent to the cone $\{x + jy \in \mathbb{C} : c_\theta |y| = -s_\theta |x|, x < 0\}$ can be parametrized by its center x_t and has equation $\{x + jy \in \mathbb{C} : (x - x_t)^2 + y^2 = s_\theta^2 x_t^2\}$. The blue, green and red circles in Fig. 3 are examples of such tangent circles. Under the previous conditions on ℓ, ρ, θ , the green circle internally tangent to $\{x + jy \in \mathbb{C} : x^2 + y^2 = \rho^2\}$ has right end less than $-\ell$, hence the green disk has the largest intersection with $\mathcal{S}(\ell, \rho, \theta)$ among all other disks between it and the red disk. As a second extreme, the blue disk has largest intersection with $\mathcal{S}(\ell, \rho, \theta)$ among all other disks with center farther to the left. Then, the disk with the largest intersection with $\mathcal{S}(\ell, \rho, \theta)$ is between these two extremes (blue and green), and we need to consider surfaces like the cyan one. A cyan surface has a break point with real part $\frac{\rho^2 + x_t^2 c_\theta^2}{2x_t} \in [-\rho, x_t]$ and its half area

$$\begin{aligned} &\frac{\pi\rho^2}{2} - \frac{1}{4} \sqrt{(x_t^2(1 + s_\theta)^2 - \rho^2)(\rho^2 - (1 - s_\theta)^2 x_t^2)} \\ &- \frac{\rho^2}{2} \arccos \frac{x_t^2 c_\theta^2 + \rho^2}{2\rho x_t} + \frac{s_\theta^2 x_t^2}{2} \arccos \frac{x_t^2(1 - s_\theta^2) - \rho^2}{2s_\theta x_t^2} \end{aligned}$$

can be computed with an integral. To find the best inner-approximation, we maximize this area with respect to x_t , which has lower and upper bounds $-\rho/c_\theta$ and $-\rho/(1 + s_\theta)$ (corresponding to the real part of centers of blue and green circles).

Example 6: The discrete-time analogue of the continuous-time $\mathcal{S}(\ell, \rho, \theta)$ is in Fig. 2, right, and can not be represented by an LMI region since loci with constant natural frequency equal to ρ and constant damping θ are obtained through complex exponentials [12, §8.2.3].

On the other hand, one can still give an inner-approximation through a disk as we will do in Section VII-B.

VI. ALTERNATIVE SUFFICIENT CONDITIONS

In this section we briefly present an approach alternative to Theorem 1 to obtain sufficient conditions for \mathcal{S} -stability of all matrices consistent with data, and an approach that starts from a disturbance model based on an instantaneous bound rather than a bound on the energy of the disturbance sequence. Both approaches rely on the S-procedure.

A. Sufficient condition alternative to Theorem 1

Unlike Section IV, we work directly with the first form of the set \mathcal{C} of matrices consistent with data, see (15), and use the same strategy of replacing a block diagonal uncertainty with a full uncertainty. We obtain then the next result.

Proposition 1: *Let \mathcal{S} be an LMI region with data (α, β) . (13) is feasible if the next program is feasible*

$$\text{find } P = P^\top \succ 0, Y, \tau \geq 0 \quad (30a)$$

$$\text{s. t. } \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes \begin{bmatrix} P \\ Y \end{bmatrix}^\top \\ \beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix} & 0 \end{bmatrix} - \tau \begin{bmatrix} I_s \otimes \mathbf{C} & I_s \otimes \mathbf{B}^\top \\ I_s \otimes \mathbf{B} & I_s \otimes \mathbf{A} \end{bmatrix} \prec 0. \quad (30b)$$

If (30) is feasible, the controller gain in (13) is $K = YP^{-1}$.

Proof: (30b) implies by the S-procedure and (30a) that

$$\begin{aligned} & \begin{bmatrix} I_{sn} & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes \begin{bmatrix} P \\ Y \end{bmatrix}^\top \\ \beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} I_{sn} \\ Z \end{bmatrix} \prec 0 \\ \forall Z: & \begin{bmatrix} I_s \otimes I_n & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} I_s \otimes \mathbf{C} & I_s \otimes \mathbf{B}^\top \\ I_s \otimes \mathbf{B} & I_s \otimes \mathbf{A} \end{bmatrix} \begin{bmatrix} I_s \otimes I_n \\ Z \end{bmatrix} \preceq 0. \end{aligned}$$

Since Z is a full uncertainty, the last condition implies that

$$\begin{bmatrix} I_{sn} & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes \begin{bmatrix} P \\ Y \end{bmatrix}^\top \\ \beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} I_{sn} \\ Z \end{bmatrix} \prec 0 \quad (31)$$

$$\forall Z = I_s \otimes Z:$$

$$\begin{bmatrix} I_s \otimes I_n & I_s \otimes Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} I_s \otimes \mathbf{C} & I_s \otimes \mathbf{B}^\top \\ I_s \otimes \mathbf{B} & I_s \otimes \mathbf{A} \end{bmatrix} \begin{bmatrix} I_s \otimes I_n \\ Z \end{bmatrix} \succ 0.$$

The condition

$$\begin{aligned} 0 & \succeq \begin{bmatrix} I_s \otimes I_n & I_s \otimes Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} I_s \otimes \mathbf{C} & I_s \otimes \mathbf{B}^\top \\ I_s \otimes \mathbf{B} & I_s \otimes \mathbf{A} \end{bmatrix} \begin{bmatrix} I_s \otimes I_n \\ Z \end{bmatrix} \\ & = I_s \otimes \mathbf{C} + (I_s \otimes Z^\top)(I_s \otimes \mathbf{B}) + (I_s \otimes \mathbf{B}^\top)(I_s \otimes Z) \\ & \quad + (I_s \otimes Z^\top)(I_s \otimes \mathbf{A})(I_s \otimes Z) \\ & = I_s \otimes (\mathbf{C} + Z^\top \mathbf{B} + \mathbf{B}^\top Z + Z^\top \mathbf{A} Z) \end{aligned}$$

is equivalent to

$$0 \succeq \mathbf{C} + Z^\top \mathbf{B} + \mathbf{B}^\top Z + Z^\top \mathbf{A} Z = \begin{bmatrix} I_n & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix}.$$

Hence, (31) is equivalent to

$$\begin{aligned} & \alpha \otimes P + \text{Tr} \left\{ Z^\top (\beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix}) \right\} \prec 0 \\ & \forall Z = I_s \otimes Z: \begin{bmatrix} I_n & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix} \preceq 0 \\ \iff & \alpha \otimes P + \text{Tr} \left\{ (I_s \otimes Z^\top) (\beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix}) \right\} \prec 0 \\ & \forall Z: \begin{bmatrix} I_n & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix} \preceq 0 \end{aligned}$$

$$\begin{aligned} \iff & 0 \succ \alpha \otimes P + \text{Tr} \left\{ (I_s \otimes [A \ B]) (\beta \otimes \begin{bmatrix} P \\ Y \end{bmatrix}) \right\} \\ & = \alpha \otimes P + \text{Tr} \left\{ \beta \otimes \left([A \ B] \begin{bmatrix} P \\ Y \end{bmatrix} \right) \right\} \forall (A, B) \in \mathcal{C} \end{aligned}$$

by the form of \mathcal{C} in (15). \blacksquare

In the same way as Corollary 1 from Theorem 1, one can obtain conditions for intersections of LMI regions from Proposition 1 easily, and it is thus omitted.

B. Sufficient conditions for alternative disturbance model

Unlike Section III, we consider a disturbance model involving not the energy of the disturbance sequence as (11), but an instantaneous bound on the disturbance as

$$\mathcal{D}_1 := \{d \in \mathbb{R}^n : dd^\top \preceq \epsilon I\}, \quad (32)$$

which expresses the bound $|d|^2 \leq \epsilon$. We refer the reader to [4] for a discussion on the implications of instantaneous and energy bounds on disturbance for data-driven control. The set of matrices consistent with a *single* data point $i = 0, \dots, T-1$ is now

$$\mathcal{C}_i := \{(A, B) : x^\circ(t_i) = Ax(t_i) + Bu(t_i) + d, d \in \mathcal{D}_1\} \quad (33a)$$

$$= \{[A \ B] = Z^\top : \begin{bmatrix} I_n & Z^\top \\ & Z \end{bmatrix} \begin{bmatrix} \mathbf{c}_i & \mathbf{b}_i^\top \\ \mathbf{b}_i & \mathbf{a}_i \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix} \preceq 0\} \quad (33b)$$

$$\mathbf{c}_i := -\epsilon I + x^\circ(t_i)x^\circ(t_i)^\top \quad (33c)$$

$$\mathbf{b}_i := -\begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix} x^\circ(t_i)^\top, \mathbf{a}_i := \begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix} \begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix}^\top, \quad (33d)$$

cf. (12) and (15). Hence, the set of matrices consistent with all data points is

$$\mathcal{I} := \bigcap_{i \in \mathbb{I}} \mathcal{C}_i, \quad \mathbb{I} := \{0, 1, \dots, T-1\}. \quad (34)$$

Remark 2: *We parallel here the observations in Remark 1. Instead of (32), one can consider for the disturbance d the instantaneous bound given by*

$$\mathcal{D}_i := \{d \in \mathbb{R}^n : [I \ d] \begin{bmatrix} r & s^\top \\ s & q \end{bmatrix} \begin{bmatrix} I \\ d^\top \end{bmatrix} \preceq 0\}$$

with matrix r symmetric and scalar $q > 0$. Moreover, one can obtain a set \mathcal{C}_i analogous to (33), namely, \mathcal{C}_i as in (33b) with, instead of (33c) and (33d),

$$\begin{aligned} \mathbf{c}_i & := r + x^\circ(t_i)s + s^\top x^\circ(t_i)^\top + q x^\circ(t_i)x^\circ(t_i)^\top \\ \mathbf{b}_i & := -\begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix} (s + qx^\circ(t_i)^\top), \mathbf{a}_i := q \begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix} \begin{bmatrix} x(t_i) \\ u(t_i) \end{bmatrix}^\top \end{aligned}$$

The rest of our results hold identically.

Analogously to (13), we would like to solve

$$\text{find } P = P^\top \succ 0, K \quad (35a)$$

$$\begin{aligned} \text{s. t. } & \alpha \otimes P + \beta \otimes ((A + BK)P) \\ & + \beta^\top \otimes (P(A + BK)^\top) \prec 0 \quad \forall (A, B) \in \mathcal{I}. \end{aligned} \quad (35b)$$

With an approach similar to Section IV, we can obtain the next result for the disturbance model with the instantaneous bound in (32).

Proposition 2: Let \mathcal{S} be an LMI region with data (α, β) . (35) is feasible if the next program is feasible

$$\text{find } P = P^\top \succ 0, Y, \tau_0 \geq 0, \dots, \tau_{T-1} \geq 0 \quad (36a)$$

$$\text{s. t. } \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes [P_Y]^\top \\ \beta \otimes [P_Y] & 0 \end{bmatrix} - \sum_{i \in \mathbb{I}} \tau_i \begin{bmatrix} I_s \otimes \mathbf{c}_i & I_s \otimes \mathbf{b}_i^\top \\ I_s \otimes \mathbf{b}_i & I_s \otimes \mathbf{a}_i \end{bmatrix} \prec 0. \quad (36b)$$

If (36) is feasible, the controller gain in (35) is $K = YP^{-1}$.

Proof: (36b) implies by the S-procedure and (36a) that

$$\begin{bmatrix} I_{sn} & Z^\top \end{bmatrix} \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes [P_Y]^\top \\ \beta \otimes [P_Y] & 0 \end{bmatrix} \begin{bmatrix} I_{sn} \\ Z \end{bmatrix} \prec 0$$

$$\forall Z: [I_s \otimes I_n \quad Z^\top] \cdot \begin{bmatrix} I_s \otimes \mathbf{c}_i & I_s \otimes \mathbf{b}_i^\top \\ I_s \otimes \mathbf{b}_i & I_s \otimes \mathbf{a}_i \end{bmatrix} [\star]^\top \preceq 0, i \in \mathbb{I},$$

where for any matrices G and $H = H^\top$, we use the notational shorthand $G \cdot H[\star]^\top = GHG^\top$. Since Z is a full uncertainty, the last condition implies

$$\begin{bmatrix} I_{sn} & Z^\top \end{bmatrix} \begin{bmatrix} \alpha \otimes P & \beta^\top \otimes [P_Y]^\top \\ \beta \otimes [P_Y] & 0 \end{bmatrix} \begin{bmatrix} I_{sn} \\ Z \end{bmatrix} \prec 0 \quad (37)$$

$$\forall Z = I_s \otimes Z:$$

$$[I_s \otimes I_n \quad I_s \otimes Z^\top] \cdot \begin{bmatrix} I_s \otimes \mathbf{c}_i & I_s \otimes \mathbf{b}_i^\top \\ I_s \otimes \mathbf{b}_i & I_s \otimes \mathbf{a}_i \end{bmatrix} [\star]^\top \preceq 0, i \in \mathbb{I}.$$

Analogously to the proof of Proposition 1, the condition

$$0 \succeq [I_s \otimes I_n \quad I_s \otimes Z^\top] \cdot \begin{bmatrix} I_s \otimes \mathbf{c}_i & I_s \otimes \mathbf{b}_i^\top \\ I_s \otimes \mathbf{b}_i & I_s \otimes \mathbf{a}_i \end{bmatrix} [\star]^\top, i \in \mathbb{I}$$

is equivalent to

$$0 \succeq \mathbf{c}_i + Z^\top \mathbf{b}_i + \mathbf{b}_i^\top Z + Z^\top \mathbf{a}_i Z = [I_n \quad Z^\top] \begin{bmatrix} \mathbf{c}_i & \mathbf{b}_i^\top \\ \mathbf{b}_i & \mathbf{a}_i \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix}, i \in \mathbb{I};$$

(37) is thereby equivalent to

$$\alpha \otimes P + \text{Tr} \left\{ (I_s \otimes Z^\top) (\beta \otimes [P_Y]) \right\} \prec 0$$

$$\forall Z: [I_n \quad Z^\top] \begin{bmatrix} \mathbf{c}_i & \mathbf{b}_i^\top \\ \mathbf{b}_i & \mathbf{a}_i \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix} \preceq 0, i \in \mathbb{I}$$

$$\iff \alpha \otimes P + \text{Tr} \left\{ \beta \otimes ([A \ B] [P_Y]) \right\} \prec 0 \quad \forall (A, B) \in \mathcal{I}$$

by the definition of \mathcal{I} in (34) and \mathcal{C}_i in (33). \blacksquare

In the same way as Corollary 1 from Theorem 1, one can obtain conditions for intersections of LMI regions from Proposition 2 easily, and it is thus omitted.

VII. NUMERICAL INVESTIGATION

For continuous and discrete time, we illustrate our findings and compare their feasibility.

A. Continuous time

The following elements constitute our setting.

1) We consider the *dynamical system*

$$\dot{x} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 \\ 0 & 0 & -0.4 & 2 & 0 \\ 0.4 & 0.4 & -0.4 & -1.4 & 0 \\ 0 & -0.03 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + d \quad (38)$$

taken from [12, §9.5] (representing a digital tape transport). This model is used *only* to generate data points and simulate the closed-loop response since no data-driven design relies on knowing it.

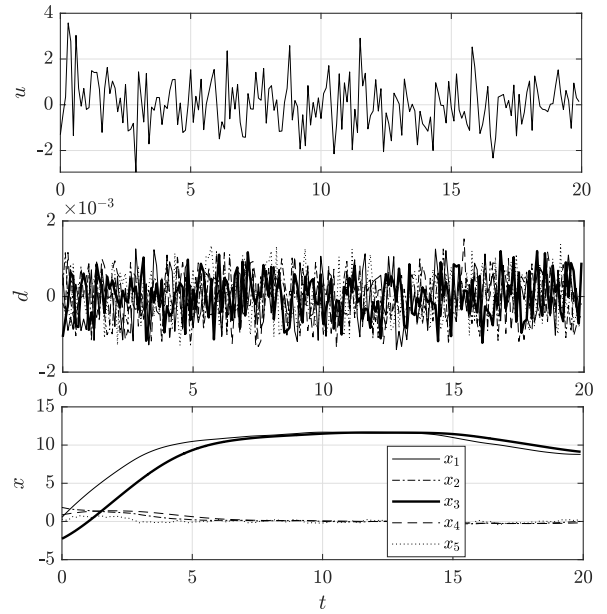


Fig. 4. Experiment for data collection. The evolution of d is not known and is reported only for completeness.

2) We know that the disturbance satisfies the squared norm bound $|d|^2 \leq \epsilon$. This can be embedded in the *disturbance model* \mathcal{D} in (11) with $\Delta = \sqrt{T\epsilon}I$. Alternatively, it can be natively captured by the disturbance model \mathcal{D}_i in (32). We use $\epsilon = 2.5 \cdot 10^{-6}$, so that $|d| \leq \sqrt{\epsilon} = 1.58 \cdot 10^{-3}$.

3) The *performance specification* is given by a region $\mathcal{S}(\ell, \rho, \theta)$, see Example 1, with parameters $\ell = 0.3$, $\rho = 2$, $\theta = \pi/5.7$. As indicated in Example 5, this region can be inner-approximated by the intersection of the disks $\{x + jy \in \mathbb{C}: x^2 + y^2 = \rho^2\}$ and $\{x + jy \in \mathbb{C}: (x - x_t)^2 + y^2 = s_\theta^2 x_t^2\}$ with $x_t = -1.4992$ (computed numerically).

4) A single *experiment for data collection* is performed on (38) under these conditions. The signals are in Fig. 4. The input is obtained by interpolating linearly a sequence that is the realization of a Gaussian variable with mean zero and unit variance. The disturbance is obtained in the same way from a realization of a random variable uniformly distributed in $|d| \leq 1.58 \cdot 10^{-3}$ and is reported only for completeness since it is not accessible. These continuous-time signals are then sampled with $T_s = 0.1$ to obtain $T = 200$ data points of the data matrices in (10).

In the setting of the previous points, we compare on the same data set the designs proposed in the previous sections:

- sufficient conditions in Corollary 1 and Proposition 1 for performance specification $\mathcal{S}(\ell, \rho, \theta)$ and disturbance model \mathcal{D} in (11),
- necessary and sufficient condition in Corollary 2 for the inner-approximation of $\mathcal{S}(\ell, \rho, \theta)$ and \mathcal{D} in (11),
- sufficient condition in Proposition 2 for $\mathcal{S}(\ell, \rho, \theta)$ and disturbance model \mathcal{D}_i in (32),
- model-based condition in (8) for $\mathcal{S}(\ell, \rho, \theta)$.

Note that Propositions 1-2 need to be easily extended for the intersection of the LMI regions composing $\mathcal{S}(\ell, \rho, \theta)$.

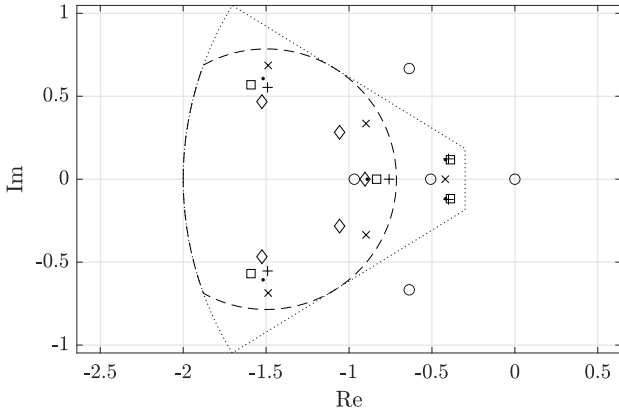


Fig. 5. Eigenvalue locations: open loop (o), model-based (x), data-based with Corollary 1 (□), with Corollary 2 (◇), with Proposition 1 (+), with Proposition 2 (·). The region $\mathcal{S}(\ell, \rho, \theta)$ is dotted, and its inner-approximation dashed.

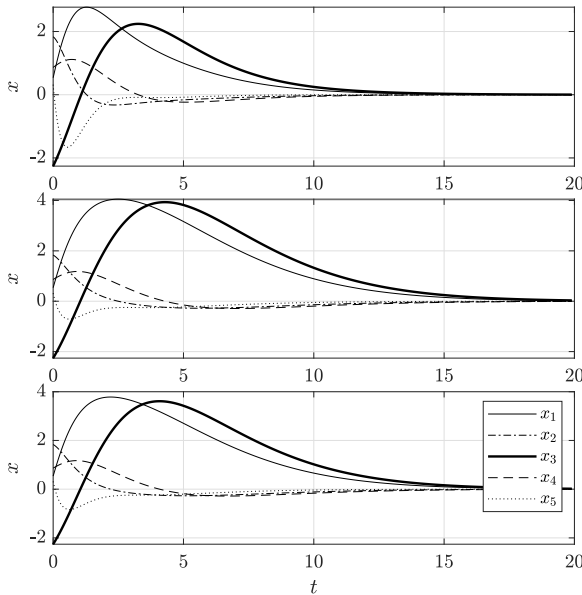


Fig. 6. Closed-loop response of (38) with controller: model-based (top), data-based and Corollary 1 (middle), data-based and Proposition 2 (bottom).

All controller designs are obtained by YALMIP [14] and MOSEK ApS in MATLAB[®] R2019b.

The resulting controller designs in terms of eigenvalues are in Fig. 5. All methods manage to move the eigenvalues into the desired $\mathcal{S}(\ell, \rho, \theta)$ or its inner-approximation, and the eigenvalue locations imposed by the different methods appear comparable. To appreciate differences, we show in Fig. 6 the time response of (38) with $d = 0$ in closed loop with a controller designed model-based or data-based with Corollary 1 or Proposition 2. All responses are consistent with the specification imposed by $\mathcal{S}(\ell, \rho, \theta)$: e.g., the exhibited convergence rates of around 15 time units are consistent with $\ell = 0.3$. The model-based solution, which does not need to robustly stabilize a set of consistent matrices, shows on the other hand a smaller overshoot.

Next, we would like to examine which of the conditions

ϵ	Prop. 2	Prop. 1	Cor. 1	Cor. 2
$2.5 \cdot 10^{-4}$	x	x	x	x
$1 \cdot 10^{-4}$	✓	x	x	x
$2.5 \cdot 10^{-5}$	✓	✓	x	x
$1 \cdot 10^{-5}$	✓	✓	✓	x
$2.5 \cdot 10^{-6}$	✓	✓	✓	✓

TABLE I

FOR ϵ , FEASIBILITY (✓) OR INFEASIBILITY (x) OF METHODS.

can withstand the largest disturbance bound ϵ and summarize the results in Table I. For the specific system and performance specification, the method with largest robustness to the disturbance bound ϵ is given by Proposition 2; this can be expected for the reasons discussed in [4] when the disturbance model is natively a norm bound $|d|^2 \leq \epsilon$. In particular, a disturbance model \mathcal{D} obtained from this norm bound induces a set \mathcal{C} that is at least as large as the set \mathcal{I} induced by \mathcal{D}_i as in Section VI-B. Moreover, the set \mathcal{I} has the desirable property that it shrinks with an increasing T ; $T = 200$ is enough to make Proposition 2 more competitive than Proposition 1, and a larger T could withstand a larger ϵ at the expense of a higher computational cost due to the presence of T extra variables $\tau_0, \dots, \tau_{T-1}$, see [4]. For methods working with \mathcal{C} , Proposition 1 and Corollary 1 are obtained in a conceptually similar way, but the former seems to have an edge over the latter. Finally, although no conservatism is introduced by Corollary 2 for the special LMI regions satisfying Assumption 2, conservatism is introduced by inner-approximating the performance specification by special LMI regions, and this source of conservatism appears to be actually more significant.

B. Discrete time

The following elements constitute our setting where we mention only those different than Section VII-A.

1) We consider the *dynamical system*

$$x^+ = (I - \frac{1}{2}L)x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + d, L := \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \quad (39)$$

where L is the Laplacian matrix of an underlying digraph.

2) We consider the same *disturbance models* as in Section VII-A, now with $\epsilon = 1 \cdot 10^{-5}$ and $|d| \leq \sqrt{\epsilon} = 3.16 \cdot 10^{-3}$.

3) The *performance specification* is given by a disk with center $(\frac{0.04+0.9}{2}, 0)$ and radius $\frac{-0.04+0.9}{2}$, which is a disk contained in the performance region described in Example 6 and in Figure 2, right.

4) A single *experiment for data collection* is performed on (39) under these conditions. For $T = 200$, the signals are in Fig. 7. The input is the realization of a Gaussian variable with mean zero and unit variance, the disturbance is the realization of a random variable uniformly distributed in $|d| \leq 3.16 \cdot 10^{-3}$ and is reported only for completeness since it is not accessible.

Within this setting, we consider the same designs as in Section VII-A and compare them for the disk giving the

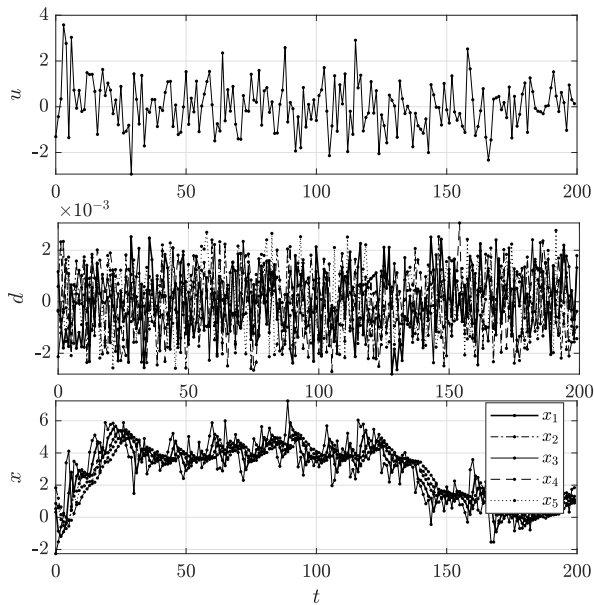


Fig. 7. Experiment for data collection. The evolution of d is not known and is reported only for completeness.

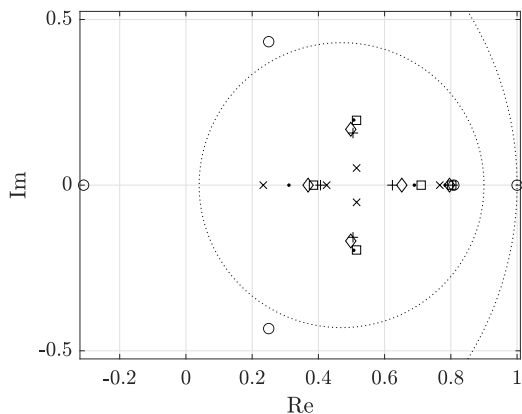


Fig. 8. Eigenvalue locations: open loop (\circ), model-based (\times), data-based with Corollary 1 (\square), with Corollary 2 (\diamond), with Proposition 1 ($+$), with Proposition 2 (\cdot). The unit and performance-specification disks are dotted.

performance specification in discrete time. The resulting controller designs in terms of eigenvalues are in Fig. 8. All methods manage to move the eigenvalues into the desired disk, and the eigenvalue locations imposed by the different methods appear comparable. In Fig. 9, the time responses of (39) with $d = 0$ in closed loop with a controller designed model-based or data-based with Corollary 1 or Proposition 2 are consistent with the specification and show a smaller overshoot in the model-based case.

Next, we examine different values of ϵ as in Section VII-A, and summarize the results in Table II. The conclusions are analogous to those in Section VII-A, except for the fact that now we no longer use Corollary 2 on an inner-approximation. Indeed, when applied on a performance specification given by disk, Corollary 2 gives a necessary and sufficient condition unlike the sufficient conditions of Corollary 1 and Proposition 1, and is then able to withstand a larger ϵ .

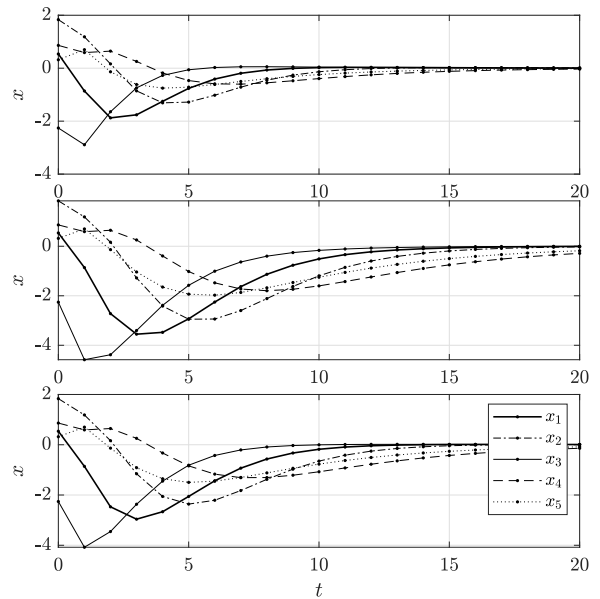


Fig. 9. Closed-loop response of (39) with controller: model-based (top), data-based and Corollary 1 (middle), data-based and Proposition 2 (bottom).

ϵ	Prop. 2	Prop. 1	Cor. 1	Cor. 2
$2.5 \cdot 10^{-4}$	\times	\times	\times	\times
$1 \cdot 10^{-4}$	\checkmark	\times	\times	\times
$5 \cdot 10^{-5}$	\checkmark	\times	\times	\checkmark
$2.5 \cdot 10^{-5}$	\checkmark	\checkmark	\times	\checkmark
$1 \cdot 10^{-5}$	\checkmark	\checkmark	\checkmark	\checkmark

TABLE II

FOR ϵ , FEASIBILITY (\checkmark) OR INFEASIBILITY (\times) OF METHODS.

APPENDIX

The analytic expressions of the LMI regions with $s = 2$ in Fig. 1 are listed in Table III: for convenience, the last column is $2 \cdot \beta$ and, for an angle θ , $\sin \theta$ or $\cos \theta$ are abbreviated as s_θ or c_θ . We emphasize that the vertical halfplanes of the two initial rows could be obtained also for $s = 1$ with, respectively, $\alpha = -l$ and $\beta = 1/2$ and $\alpha = r$ and $\beta = -1/2$.

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Description	$S = \{z = x + jy \in \mathbb{C} :$	α	2β
Vertical halfplane left of l	$x < l$	$\begin{bmatrix} -l & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Vertical halfplane right of r	$x > r$	$\begin{bmatrix} r & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$
Open disk with center $(x_d, 0)$, radius $r_d > 0$	$(x - x_d)^2 + y^2 < r_d^2$	$\begin{bmatrix} -r_d & x_d \\ x_d & -r_d \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$
Vertical strip with extremes $l < r$	$l < x < r$	$\begin{bmatrix} -r & 0 \\ 0 & l \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Horizontal strip with semiwidth $w > 0$	$y^2 < w^2$	$\begin{bmatrix} -w & 0 \\ 0 & -w \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$
Ellipsoid with center $(x_e, 0)$, semiaxes $\mu_1 > 0$ and $\mu_2 > 0$	$\frac{(x-x_e)^2}{\mu_1^2} + \frac{y^2}{\mu_2^2} < 1$	$\begin{bmatrix} -\mu_1^2 & x_e\mu_2 \\ x_e\mu_2 & -\mu_2^2 \end{bmatrix}$	$\begin{bmatrix} 0 & \mu_1 - \mu_2 \\ -\mu_1 - \mu_2 & 0 \end{bmatrix}$
Left parabola with vertex $(x_p, 0)$, curvature $c_p > 0$	$x < x_p - (c_p/2)y^2$	$\begin{bmatrix} -1 & 0 \\ 0 & -x_p \end{bmatrix}$	$\begin{bmatrix} 0 & \sqrt{\frac{c_p}{2}} \\ -\sqrt{\frac{c_p}{2}} & 1 \end{bmatrix}$
Right parabola with vertex $(x_p, 0)$, curvature $c_p > 0$	$x > x_p + (c_p/2)y^2$	$\begin{bmatrix} -1 & 0 \\ 0 & x_p \end{bmatrix}$	$\begin{bmatrix} 0 & \sqrt{\frac{c_p}{2}} \\ -\sqrt{\frac{c_p}{2}} & -1 \end{bmatrix}$
Left hyperbola with vertex $(-x_h, 0)$, asymptotes $\pm c_h x$, $x_h > 0$, $c_h > 0$	$y^2 < c_h^2(x^2 - x_h^2), x < 0$	$\begin{bmatrix} 0 & c_h x_h \\ c_h x_h & 0 \end{bmatrix}$	$\begin{bmatrix} c_h & 1 \\ -1 & c_h \end{bmatrix}$
Right hyperbola with vertex $(x_h, 0)$, asymptotes $\pm c_h x$, $x_h > 0$, $c_h > 0$	$y^2 < c_h^2(x^2 - x_h^2), x > 0$	$\begin{bmatrix} 0 & c_h x_h \\ c_h x_h & 0 \end{bmatrix}$	$\begin{bmatrix} -c_h & 1 \\ -1 & -c_h \end{bmatrix}$
Left cone with vertex $(x_c, 0)$, semiaperture $\theta \in (0, \pi/2)$	$c_\theta y < s_\theta(x_c - x), x < x_c$	$-s_\theta x_c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} s_\theta & c_\theta \\ -c_\theta & s_\theta \end{bmatrix}$
Right cone with vertex $(x_c, 0)$, semiaperture $\theta \in (0, \pi/2)$	$c_\theta y < s_\theta(x - x_c), x > x_c$	$s_\theta x_c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$-\begin{bmatrix} s_\theta & -c_\theta \\ c_\theta & s_\theta \end{bmatrix}$

TABLE III
EXPRESSIONS OF THE LMI REGIONS WITH $s = 2$ IN FIG. 1.

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