Computers and Mathematics with Applications 91 (2021) 202–228

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Mathematical analysis and numerical approximation of a general linearized poro-hyperelastic model☆

Nicolás Barnafi, Paolo Zunino *, Luca Dedè, Alfio Quarteroni

MOX, Department of Mathematics, Politecnico di Milano, Piazza Leonardo da Vinci 32, Milano, 20133, Italy

ARTICLE INFO

Article history:
Available online 29 August 2020

Keywords:
Poromechanics
Finite element approximation
Inf–sup stability condition

ABSTRACT

We describe the behavior of a deformable porous material by means of a poro-hyperelastic model that has been previously proposed in Chapelle and Moireau (2014) under general assumptions for mass and momentum balance and isothermal conditions for a two-component mixture of fluid and solid phases. In particular, we address here a linearized version of the model, based on the assumption of small displacements. We consider the mathematical analysis and the numerical approximation of the problem. More precisely, we carry out firstly the well-posedness analysis of the model. Then, we propose a numerical discretization scheme based on finite differences in time and finite elements for the spatial approximation; stability and numerical error estimates are proved.

Particular attention is dedicated to the study of the saddle-point structure of the problem, that turns out to be interesting because velocities of the fluid phase and of the solid phase are combined into a single quasi-incompressibility constraint. Our analysis provides guidelines to select the componentwise polynomial degree of approximation of fluid velocity, solid displacement and pressure, to obtain a stable and robust discretization based on Taylor–Hood type finite element spaces. Interestingly, we show how this choice depends on the porosity of the mixture, i.e. the volume fraction of the fluid phase.

1. Introduction

Poromechanics addresses the behavior of saturated porous media and in particular the interaction of mechanical deformations and flow through porous materials. Since its origin in the context of civil engineering [1–4], it has been used for countless applications (see the review [5] and the references therein). More recently, these models have captured the

☆ The authors have been partially supported by the ERC Advanced Grant iHEART, Italy, "An Integrated Heart Model for the simulation of the cardiac function", 2017-2022, P.I. A. Quarteroni (ERC2016AdG, project ID: 740132).

* Corresponding author.
E-mail addresses: nicolas.barnafi@polimi.it (N. Barnafi), paolo.zunino@polimi.it (P. Zunino), luca.dede@polimi.it (L. Dedè), alfio.quarteroni@polimi.it (A. Quarteroni).

https://doi.org/10.1016/j.camwa.2020.07.025

attention of researchers interested in the behavior of highly deformable soft biological tissues [6–8]; a prominent example of application in this area is the perfusion of the heart [9–11].

Poromechanics formulations were originally developed for civil applications, which held them inadequate for biomechanics and especially for soft tissues undergoing large deformations [12–15]. This called for more general formulations, arising from the fundamental principles of continuum mechanics and thermodynamics. Thanks to their improved generality, such formulations are more flexible and applicable to a broader range of scenarios [16–18]. These models inherit desirable physical features, such as energy conservation [19], that are reflected into their mathematical properties. However, the analysis of well-posedness, stability and approximation of such new generation of poromechanics models is still largely open. The main objective of this work is to contribute to their analysis and approximation.

We follow in particular the works by Chapelle and co-workers [16,19,20], where a general thermodynamically consistent poromechanics formulation is introduced. In the original formulation [16], the authors develop their model for the general case of large deformations, which is characterized as a mixture of fluid and solid phases that simultaneously coexist at every point in the computational domain. Such model is extremely challenging from the mathematical analysis standpoint, because of the nonlinearity of the constitutive equations and the geometric nonlinearity due to large deformations. For these reasons, we focus on the linearization of the previous model, proposed by the same authors in [19,20] and derived under the assumption of small deformations. In this setting, the porosity (fluid volume fraction) is a fixed parameter of the model.

When the fluid phase is strictly incompressible and the solid phase is nearly-incompressible, the model exhibits an interesting saddle-point structure where a linear combination of the velocities of the fluid and solid phases determine the fulfillment of the quasi-incompressibility constraint. The weights of the linear combination of velocities depend on the porosity of the material. The main contribution of this work consists of showing how the approximation of the fluid and solid phases interact with the stability of the scheme. The works [19,20] looked at the problem as it was formed by coupled equations of parabolic type, which somehow put the role of the incompressibility constraint in the background. Here, we change this perspective towards a hybrid system of parabolic and hyperbolic partial differential problems. This new approach allows us to put into evidence the saddle-point nature of the problem and the role of the weighted inf–sup condition between fluid velocity, solid displacement and pressure to determine the stability of the approximation scheme. More precisely, after discretizing the problem by means of finite differences in time and finite elements in space, we address the numerical stability of a numerical scheme based on the family of Taylor–Hood finite elements [21] for the approximation of fluid velocity, solid displacement and pressure; both fluid velocity and solid displacement are required to have a degree of approximation higher than that of the pressure. Our analysis confirms that the inf–sup stability of the scheme depends on the porosity and provides guidelines to choose the polynomial order used for the approximation of the velocity and displacement in different scenarios obtained by varying this parameter. We notice that such analysis may be particularly relevant also for the fully nonlinear version of the model, where the porosity is a variable of the system.

Throughout this work, these topics are organized as follows. After presenting the model in Section 2, we address in Section 3 the semi-discrete problem, which pits into evidence the generalized saddle point structure of the problem. The well-posedness, proved using the theory of Differential Algebraic Equations [22], and the energy estimates of the semi-discrete problem are useful in Section 4, which addresses the well-posedness of the continuous problem by means of the Faedo–Galerkin method. In Section 5 we modify the fully discrete formulation used in [20] to be solved with an implicit scheme and present its a-priori error analysis validated by numerical tests, independently in space and time. The inf–sup stability of the numerical scheme is fully addressed in Section 6, where we first extend the classic divergence discrete inf–sup condition to the case in which the velocity is multiplied by a weight function, and then use this result for the stability analysis involving both fluid and solid phases. Finally, in Section 7 we present two other numerical tests: the first one is the swelling test (already addressed in [20]) and the second one is specifically designed to test the inf–sup stability of the discretization addressed in 6.

2. The mathematical model

Throughout the manuscript we consider a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ together with the classical Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$ with corresponding norms $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$. We also denote $(\cdot,\cdot)$ the L^2-inner product, and given an arbitrary Hilbert space H we denote the duality pairing with its dual space H' as $\langle\cdot,\cdot\rangle_{H',H}$. For a positive function ψ, we consider the weighted Sobolev spaces $L^2(\Omega, \psi\, dx)$ with norm $\|f\|_\psi^2 = (f, f)_\psi = \int_\Omega f^2\psi\, dx$. Also, we use the convention of denoting scalars, vectors, tensors and matrices as a, $\boldsymbol{a}$, $\boldsymbol{A}$ and $\mathbf{A}$, respectively. We finally define the Bochner spaces $L^p(0, T; X)$, $1 \le p < \infty$, and $L^\infty(0, T; X)$ for any Banach space X with norms $\left(\int_0^T \|x(s)\|_X^q\, ds\right)^{1/q}$ and $\sup_{s\in(0,T)} \|x(s)\|_X$ respectively. Weak time derivatives are considered in $W^{k,p}(0, T; X) = \left\{x \in L^p(0, T; X) : \partial_t^n x \in L^p(0, T; X)\, \forall n \in \mathbb{N}, n \le k\right\}$, $1 \le p \le \infty$.

The strong formulation of the poroelastic model is given by the following system of equations, with the primary variables being the fluid velocity $\boldsymbol{v}_f$, pressure p, displacement $\boldsymbol{y}_s$ and solid velocity $\boldsymbol{v}_s$:

$$\begin{aligned} \rho_f\phi\partial_t\boldsymbol{v}_f - 2\mu_f\,\mathrm{div}(\phi\boldsymbol{\varepsilon}(\boldsymbol{v}_f)) + \phi\nabla p + \phi^2\boldsymbol{\kappa}_f^{-1}(\boldsymbol{v}_f - \boldsymbol{v}_s) &= \rho_f\phi f + \theta\boldsymbol{v}_f, && \text{in } \Omega,\\ \frac{(1-\phi)^2}{\kappa_s}\partial_t p + \mathrm{div}(\phi\boldsymbol{v}_f + (1-\phi)\boldsymbol{v}_s) &= \rho_f^{-1}\theta, && \text{in } \Omega,\\ \rho_s(1-\phi)\partial_t\boldsymbol{v}_s - \mathrm{div}(\mathbb{C}_{\texttt{Hooke}}\boldsymbol{\varepsilon}(\boldsymbol{y}_s)) + (1-\phi)\nabla p - \phi^2\boldsymbol{\kappa}_f^{-1}(\boldsymbol{v}_f - \boldsymbol{v}_s) &= \rho_s(1-\phi)f, && \text{in } \Omega,\\ \boldsymbol{v}_s &= \partial_t\boldsymbol{y}_s, && \text{in } \Omega. \end{aligned} \tag{1}$$

The first equation is the conservation of momentum for the fluid phase, which turns out to be a generalized Stokes law which incorporates the Brinkman effect; the second equation represents mass conservation; the third one is the conservation of momentum of the solid phase and the last one relates solid displacement and velocity. Also, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathrm{sym}(\nabla\boldsymbol{u})$ and the relevant parameters are given by: $\phi = \phi(\boldsymbol{x})$, porosity; ρ_f, ρ_s, fluid/solid density; μ_f, fluid viscosity; $\boldsymbol{\kappa}_f$, permeability tensor; $f \in \boldsymbol{L}^2(\Omega)$, external load; $\theta \in L^2(\Omega)$, fluid source/sink; κ_s, bulk modulus and $\mathbb{C}_{\texttt{Hooke}}\boldsymbol{\tau} = 2\mu\boldsymbol{\tau} + \lambda\,\mathrm{tr}()\tau\boldsymbol{I}$, with λ, μ Lamé parameters.

The boundary conditions for this problem can be very general; for simplicity, we restrict ourselves to Dirichlet boundary conditions:

$$\boldsymbol{v}_f = \boldsymbol{v}_D \quad \text{and} \quad \boldsymbol{y}_s = \boldsymbol{y}_D \quad \text{on } \partial\Omega,$$

for given $\boldsymbol{v}_D, \boldsymbol{y}_D$ in $H^{1/2}(\partial\Omega)$. We have left aside the natural no-slip condition $\boldsymbol{v}_f = \boldsymbol{v}_s$ on the boundary. A simple weak imposition of this condition was analyzed for a monolithic solver in [20], so there is no loss of generality in our choice. We also consider the problem with homogeneous Dirichlet conditions to avoid using additional lifting terms.

Remark. The main differences between this model and the Biot model [2], the most popular linear poromechanics model, rely on the presence of the permeability in the fluid and solid momentum equations and the symmetric way in which fluid and solid velocities behave; the latter means that fluid quantities are multiplied by the fluid fraction ϕ whereas solid quantities are multiplied by the solid fraction $1-\phi$. These features will be evident during the analysis, as they yield a positive definite formulation of the semi-discrete, continuous in time problem in the framework of Differential Algebraic Equations.

2.1. Variational formulation

We consider the classical Sobolev subspaces $\boldsymbol{H}_0^1(\Omega) := \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{w} = \boldsymbol{0} \text{ on } \partial\Omega\}$ and $L_0^2(\Omega) := \{q \in L^2(\Omega) : (q, 1) = 0\}$. The weak formulation of problem (1) reads: Find $(\boldsymbol{y}_s, \boldsymbol{v}_s, \boldsymbol{v}_f, p)$ in $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$:

$$\begin{aligned} &(\rho_f\phi\partial_t\boldsymbol{v}_f, \boldsymbol{v}_f^*) + 2\mu_f(\phi\boldsymbol{\varepsilon}(\boldsymbol{v}_f), \boldsymbol{\varepsilon}(\boldsymbol{v}_f^*)) - (p, \mathrm{div}(\phi\boldsymbol{v}_f^*)) + (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f, \boldsymbol{v}_f^*) - (\theta\boldsymbol{v}_f, \boldsymbol{v}_f^*)\\ &\qquad -(p, \mathrm{div}(\phi\boldsymbol{v}_f^*)) - (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_s, \boldsymbol{v}_f^*) = (\rho_f\phi\boldsymbol{f}, \boldsymbol{v}_f^*),\\ &\left(\frac{(1-\phi)^2}{\kappa_s}\partial_t p, q\right) + (q, \mathrm{div}(\phi\boldsymbol{v}_f)) + (q, \mathrm{div}((1-\phi)\boldsymbol{v}_s)) = (\rho_f^{-1}\theta, q),\\ &(\rho_s(1-\phi)\partial_t\boldsymbol{v}_s, \boldsymbol{w}_s) + (\mathbb{C}_{\texttt{Hooke}}\boldsymbol{\varepsilon}(\boldsymbol{y}_s), \boldsymbol{\varepsilon}(\boldsymbol{w}_s)) + (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_s, \boldsymbol{w}_s)\\ &\qquad -(p, \mathrm{div}((1-\phi)\boldsymbol{w}_s)) - (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f, \boldsymbol{w}_s) = (\rho_s(1-\phi)\boldsymbol{f}, \boldsymbol{w}_s),\\ &(\partial_t\boldsymbol{y}_s, \boldsymbol{v}_s^*) - (\boldsymbol{v}_s, \boldsymbol{v}_s^*) = 0, \end{aligned} \tag{2}$$

for every test function $(\boldsymbol{w}_s, \boldsymbol{v}_s^*, \boldsymbol{v}_f^*, q)$ in $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$, with initial conditions $\boldsymbol{v}_f(0) = \boldsymbol{\xi}_f, p(0) = \xi_p, \boldsymbol{y}_s(0) = \boldsymbol{\xi}_s, \boldsymbol{v}_s(0) = \boldsymbol{\xi}_v$. By defining $\boldsymbol{\sigma}_{\texttt{vis}}(\boldsymbol{v}_f) = 2\mu_f\boldsymbol{\varepsilon}(\boldsymbol{v}_f)$, $\boldsymbol{\sigma}_{\texttt{skel}}(\boldsymbol{y}_s) = \mathbb{C}_{\texttt{Hooke}}\boldsymbol{\varepsilon}(\boldsymbol{y}_s)$ and the following Riesz operators:

$$\begin{aligned} &A_f : \boldsymbol{H}_0^1(\Omega) \to (\boldsymbol{H}_0^1(\Omega))' && \langle A_f(\cdot_1), (\cdot_2)\rangle = (\phi\boldsymbol{\sigma}_{\texttt{vis}}(\cdot_1), \boldsymbol{\varepsilon}(\cdot_2))\\ &A_s : \boldsymbol{H}_0^1(\Omega) \to (\boldsymbol{H}_0^1(\Omega))' && \langle A_f(\cdot_1), (\cdot_2)\rangle = (\phi\boldsymbol{\sigma}_{\texttt{vis}}(\cdot_1), \boldsymbol{\varepsilon}(\cdot_2))\\ &K : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega) && \langle K(\cdot_1), (\cdot_2)\rangle = (\phi^2\boldsymbol{\kappa}_f^{-1}(\cdot_1), (\cdot_2))\\ &B_\phi : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega) && \langle B_\phi(\cdot_1), (\cdot_2)\rangle = -((\cdot_2), \mathrm{div}(\phi(\cdot_1)))\\ &B_{1-\phi} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega) && \langle B_{1-\phi}(\cdot_1), (\cdot_2)\rangle = -((\cdot_2), \mathrm{div}((1-\phi)(\cdot_2))) \end{aligned}$$

problem (2) can be written in block form as:

$$\begin{bmatrix} \rho_f\phi & 0 & 0 & 0\\ 0 & \frac{(1-\phi)^2}{\kappa_s} & 0 & 0\\ 0 & 0 & 0 & \rho_s(1-\phi)\\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_t\boldsymbol{v}_f\\ \partial_t p\\ \partial_t\boldsymbol{y}_s\\ \partial_t\boldsymbol{v}_s \end{bmatrix} + \begin{bmatrix} A_f + K - \theta & B_\phi^T & 0 & -K\\ B_\phi & 0 & 0 & B_{1-\phi}\\ -K & B_{1-\phi}^T & A_s & K\\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_f\\ p\\ \boldsymbol{y}_s\\ \boldsymbol{v}_s \end{bmatrix} = \begin{bmatrix} F_f\\ F_p\\ F_s\\ \boldsymbol{0} \end{bmatrix}, \tag{3}$$

with the notation being understood. For the sake of analysis, we use $\partial_t \boldsymbol{y}_s$ instead of $\boldsymbol{v}_s$ in the Stokes equation, mass and momentum conservation to obtain the following equivalent formulation:

$$
\begin{aligned}
&(\rho_f \phi \partial_t \boldsymbol{v}_f, \boldsymbol{v}_f^*) + (\phi \boldsymbol{\sigma}_{\mathtt{vis}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_f^*)) - (\theta \boldsymbol{v}_f, \boldsymbol{v}_f^*) + (\phi^2 \boldsymbol{\kappa}_f^{-1} \boldsymbol{v}_f, \boldsymbol{v}_f^*) \\
&\qquad - (p, \operatorname{div}(\phi \boldsymbol{v}_f^*)) - (\phi^2 \boldsymbol{\kappa}_f^{-1} \partial_t \boldsymbol{y}_s, \boldsymbol{v}_f^*) = (\rho_f \phi \boldsymbol{f}, \boldsymbol{v}_f^*), \\
&\left(\frac{(1-\phi)^2}{\kappa_s} \partial_t p, q \right) + (q, \operatorname{div}(\phi \boldsymbol{v}_f)) + (q, \operatorname{div}((1-\phi)\partial_t \boldsymbol{y}_s)) = (\rho_f^{-1} \theta, q), \\
&(\rho_s (1-\phi) \partial_t \boldsymbol{v}_s, \boldsymbol{w}_s) + (\boldsymbol{\sigma}_{\mathtt{skel}}, \boldsymbol{\varepsilon}(\boldsymbol{w}_s)) + (\phi^2 \boldsymbol{\kappa}_f^{-1} \partial_t \boldsymbol{y}_s, \boldsymbol{w}_s) \\
&\qquad - (p, \operatorname{div}((1-\phi)\boldsymbol{w}_s)) - (\phi^2 \boldsymbol{\kappa}_f^{-1} \boldsymbol{v}_f, \boldsymbol{w}_s) = (\rho_s (1-\phi) \boldsymbol{f}, \boldsymbol{w}_s), \\
&(\rho_s (1-\phi) \partial_t \boldsymbol{y}_s, \boldsymbol{v}_s^*) - (\rho_s (1-\phi) \boldsymbol{v}_s, \boldsymbol{v}_s^*) = 0,
\end{aligned} \tag{4}
$$

for all test functions $(\boldsymbol{v}_f^*, q, \boldsymbol{w}_s, \boldsymbol{v}_s^*)$ in $\boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$; when written in block form, it reads

$$
\begin{bmatrix}
\rho_f \phi & 0 & -K & 0 \\
0 & \frac{(1-\phi)^2}{\kappa_s} & -B_{1-\phi} & 0 \\
0 & 0 & K & \rho_s(1-\phi) \\
0 & 0 & \rho_s(1-\phi) & 0
\end{bmatrix}
\begin{bmatrix}
\partial_t \boldsymbol{v}_f \\ \partial_t p \\ \partial_t \boldsymbol{y}_s \\ \partial_t \boldsymbol{v}_s
\end{bmatrix}
+ \begin{bmatrix}
A_f + K - \theta & B_\phi^T & 0 & 0 \\
-B_\phi & 0 & 0 & 0 \\
-K & B_{1-\phi}^T & A_s & 0 \\
0 & 0 & 0 & -\rho_s(1-\phi)
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{v}_f \\ p \\ \boldsymbol{y}_s \\ \boldsymbol{v}_s
\end{bmatrix}
=
\begin{bmatrix}
F_f \\ F_p \\ F_s \\ \boldsymbol{0}
\end{bmatrix}. \tag{5}
$$

Although at first glance formulation (5) breaks the structure of the problem, it presents the useful property that the combination of the two matrix blocks yields a generalized saddle point structure. This property would not hold with (3), and it is fundamental in proving the existence of solutions using Theorem 1. Also, we remark that our formulation differs from that proposed in [20] in the functional setting. More precisely, we look for the solid velocity in a weaker space, namely $\boldsymbol{L}^2(\Omega)$ instead of $\boldsymbol{H}_0^1(\Omega)$. Our choice is determined by the different approach to the analysis of the problem and in particular by the fact that an energy estimate for $\boldsymbol{v}_s$ in $\boldsymbol{H}_0^1(\Omega)$ would be hardly derived. Besides this technical difficulty, there is no reason to conclude that $\boldsymbol{v}_s$ and $\boldsymbol{y}_s$ shall not belong to the same functional space. As a result, in the numerical discretization of the problem we approximate both using the same finite element space that is conforming to $\boldsymbol{H}_0^1(\Omega)$.

Remark. Note that all blocks, except for A_s, depend on the porosity ϕ. Also, our formulation differs from that proposed in [20] in the choice of test functions. Indeed, they are interchanged between the displacement and solid velocity equations, and moreover we look for the solid velocity in the space $\boldsymbol{L}^2(\Omega)$ instead of $\boldsymbol{H}_0^1(\Omega)$. These choices present higher difficulties during the analysis, but in return they shed light on the well-posedness of an alternative formulation in which $\boldsymbol{v}_s$ would no longer a variable.

3. Analysis of the semi-discrete problem

In this section, we analyze a semi-discrete, continuous in time, version of (2). We follow an approach similar to the one used in [23]. For this, consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of simplices K of characteristic size h and $\mathbb{P}_k(K)$ the polynomials of degree $k \geq 1$ in K to define $X_h^k = \{q \in C(\overline{\Omega}) : q|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\}$. With them, we define the following discrete spaces:

$$
\begin{aligned}
&V_{f,h} = \boldsymbol{H}_0^1(\Omega) \cap [X_h^{k+1}]^d, \qquad V_{s,h} = \boldsymbol{H}_0^1(\Omega) \cap [X_h^{k+1}]^d, \\
&Q_{p,h} = L_0^2(\Omega) \cap X_h^k, \qquad Q_{v,h} = \boldsymbol{L}^2(\Omega) \cap [X_h^k]^d,
\end{aligned}
$$

which are conforming and satisfy the discrete inf–sup condition described later in Section 6. Then, the semi-discrete problem reads: Find $(\boldsymbol{v}_{f,h}, p_h, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h})$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$ such that

$$
\begin{aligned}
&(\rho_f \phi \partial_t \boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*) + (\phi \boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}_{f,h}), \boldsymbol{\varepsilon}(\boldsymbol{v}_{f,h}^*)) - (\theta \boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*) + (\phi^2 \boldsymbol{\kappa}_f^{-1} \boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*) \\
&\qquad - (p_h, \operatorname{div}(\phi \boldsymbol{v}_{f,h}^*)) - (\phi^2 \boldsymbol{\kappa}_f^{-1} \partial_t \boldsymbol{y}_{s,h}, \boldsymbol{v}_{f,h}^*) = (\rho_f \phi \boldsymbol{f}, \boldsymbol{v}_{f,h}^*), \\
&\left(\frac{(1-\phi)^2}{\kappa_s} \partial_t p_h, q_h \right) + (q_h, \operatorname{div}(\phi \boldsymbol{v}_{f,h})) + (q_h, \operatorname{div}((1-\phi)\partial_t \boldsymbol{y}_{s,h})) = (\rho_f^{-1} \theta, q_h), \\
&(\rho_s (1-\phi) \partial_t \boldsymbol{v}_{s,h}, \boldsymbol{w}_{s,h}) + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_{s,h}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{s,h})) + (\phi^2 \boldsymbol{\kappa}_f^{-1} \partial_t \boldsymbol{y}_{s,h}, \boldsymbol{w}_{s,h}) \\
&\qquad - (p_h, \operatorname{div}((1-\phi)\boldsymbol{w}_{s,h})) - (\phi^2 \boldsymbol{\kappa}_f^{-1} \boldsymbol{v}_{f,h}, \boldsymbol{w}_{s,h}) = (\rho_s (1-\phi) \boldsymbol{f}, \boldsymbol{w}_{s,h}), \\
&((1-\phi) \partial_t \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h}^*) - ((1-\phi) \boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}^*) = 0,
\end{aligned} \tag{6}
$$

for any test functions $(\boldsymbol{v}^*_{f,h}, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}^*_{s,h})$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$, and for given initial conditions $\boldsymbol{v}_{f,h}(0) = \Pi_{f,h}\boldsymbol{\xi}_f$, $p_h(0) = \Pi_{p,h}\xi_p$, $\boldsymbol{y}_{s,h}(0) = \Pi_{s,h}\boldsymbol{\xi}_s$, $\boldsymbol{v}_{s,h}(0) = \Pi_{v,h}\boldsymbol{\xi}_v$; here, $\Pi_{(\cdot),h}$ denotes the L^2 projection to the corresponding discrete space. From now on it makes no contribution to specify the h subindex, and we will thus omit it on the remaining of this section.

For the analysis of problem (6) we make use of the following result from the theory of Differential Algebraic Equations [22].

Theorem 1. *Let $L : [0,T] \to \mathbb{R}^N$ and $\boldsymbol{E}, \boldsymbol{H}$ in $\mathbb{R}^{N\times N}$ be given arrays. Then, the differential algebraic equation given by*

$$\boldsymbol{E}\frac{d\boldsymbol{X}}{dt}(t) + \boldsymbol{H}\boldsymbol{X}(t) = L(t), \quad t > 0$$

has at least one solution $\boldsymbol{X} : [0,T] \to \mathbb{R}^N$ for any initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0$ if $s\boldsymbol{E} + \boldsymbol{H}$ is invertible for some $s \neq 0$.

Finally, we will make use of Korn's inequality [24]:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega)} \geq \alpha_k |\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \tag{7}$$

for some positive constant α_k and the following assumptions.

(H**1**) The porosity ϕ is such that $\phi, 1/\phi, (1-\phi)$ and $(1-\phi)^{-1}$ belong to $W^{s,r}(\Omega)$ with $s > d/r$ and there exist two positive constants $\underline{\phi}$ and $\overline{\phi}$ such that $0 < \underline{\phi} \leq \phi \leq \overline{\phi} < 1$ almost everywhere in Ω.

(H**2**) The stress tensors $\boldsymbol{\sigma}_{\mathtt{skel}}$ and $\boldsymbol{\sigma}_{\mathtt{vis}}$ give rise to continuous elliptic bilinear forms:

$$\exists C_{\mathtt{skel}} > 0 \quad : \quad (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{w}_s), \boldsymbol{\varepsilon}(\boldsymbol{w}_s)) \geq C_{\mathtt{skel}} \|\boldsymbol{\varepsilon}(\boldsymbol{w}_s)\|^2_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{w}_s \in \boldsymbol{H}^1_0(\Omega),$$
$$\exists C_{\mathtt{vis}} > 0 \quad : \quad (\phi\boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}^*_f), \boldsymbol{\varepsilon}(\boldsymbol{v}^*_f)) - (\theta \boldsymbol{v}^*_f, \boldsymbol{v}^*_f) \geq \underline{\phi}\, C_{\mathtt{vis}} \|\boldsymbol{\varepsilon}(\boldsymbol{v}^*_f)\|^2_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v}^*_f \in \boldsymbol{H}^1_0(\Omega).$$

(H**3**) The permeability tensor is symmetric and positive:

$$\exists C_k > 0 : (\phi \boldsymbol{\kappa}_f^{-1} \boldsymbol{v}^*_f, \boldsymbol{v}^*_f) \geq C_k \|\boldsymbol{v}^*_f\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}^*_f \in \boldsymbol{H}^1_0(\Omega).$$

From these hypotheses, we obtain the relevant ellipticity estimates, which we collect in the following lemma to be used later in both the well-posedness analysis and the energy estimate. We point out that the hypothesis (H**2**) poses a hard restriction on the parameter θ. We set such a strong requirement for the sake of simplicity as it will be used in what follows to straightforwardly prove the existence and the stability of solutions. However, it can be relaxed by means of a more refined approach to the analysis that exploits an exponential scaling of the velocity, namely $\boldsymbol{v}_{f,\lambda} = \exp\{-\lambda t\}\, \boldsymbol{v}_f$. Choosing λ sufficiently large would make such requirement unnecessary, but the analysis of the problem would turn out to be more involved.

Lemma 1. *Under hypotheses (H**1**), (H**2**) and (H**3**) there exist two positive constants α_f, α_s such that:*

$$(\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{w}_s), \boldsymbol{\varepsilon}(\boldsymbol{w}_s)) \geq \alpha_s \|\boldsymbol{w}_s\|^2_{\boldsymbol{H}^1(\Omega)} \qquad \forall \boldsymbol{w}_s \in \boldsymbol{H}^1_0(\Omega),$$
$$(\phi\boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}^*_f), \boldsymbol{\varepsilon}(\boldsymbol{v}^*_f)) + ([\phi\boldsymbol{\kappa}_f^{-1} - \theta \boldsymbol{I}]\boldsymbol{v}^*_f, \boldsymbol{v}^*_f) \geq \alpha_f \|\boldsymbol{v}^*_f\|^2_{\boldsymbol{H}^1(\Omega)} \qquad \forall \boldsymbol{v}^*_f \in \boldsymbol{H}^1_0(\Omega).$$

Proof. The result is a direct application of Korn's inequality with hypotheses (H**1**), (H**2**) and (H**3**), with $\alpha_s = C_{\mathtt{skel}}\alpha_k$ and $\alpha_f = \min\{\underline{\phi}\, C_{\mathtt{vis}}\alpha_k, C_k\}$. □

3.1. Existence and uniqueness

Problem (5) can be cast into the framework of Theorem 1 by defining the following operators:

$$E := \begin{bmatrix} \rho_f\phi & 0 & -K & 0 \\ 0 & \frac{(1-\phi)^2}{\kappa_S} & -B_{1-\phi} & 0 \\ 0 & 0 & K & \rho_s(1-\phi) \\ 0 & 0 & \rho_s(1-\phi) & 0 \end{bmatrix} \quad \text{and} \quad H := \begin{bmatrix} A_f + K - \theta & B^T_\phi & 0 & 0 \\ -B_\phi & 0 & 0 & 0 \\ -K & B^T_{1-\phi} & A_s & 0 \\ 0 & 0 & 0 & -\rho_s(1-\phi) \end{bmatrix}.$$

Then, identifying each operator with its induced matrix in boldface as $\boldsymbol{A}_f, \boldsymbol{A}_s, \boldsymbol{K}, \boldsymbol{B}_\phi, \boldsymbol{B}_{1-\phi}$. We also define $\boldsymbol{M}_{(\zeta)}$ the weighted mass matrix related to the inner product $(\zeta \boldsymbol{z}, \boldsymbol{z}^*)$ and the mass matrices $\boldsymbol{A}_v, \boldsymbol{A}_p$ associated to $\boldsymbol{v}_s$ and p, which give:

$$\boldsymbol{E} + \boldsymbol{H} = \begin{bmatrix} \boldsymbol{A}_f + \boldsymbol{M}_{(\rho_f\phi)} + \boldsymbol{K} - \boldsymbol{M}_{(\theta)} & \boldsymbol{K} & \boldsymbol{0} & \boldsymbol{B}^T_\phi \\ \boldsymbol{K} & \boldsymbol{A}_s + \boldsymbol{K} & \boldsymbol{M}_{(\rho_s(1-\phi))} & \boldsymbol{B}^T_{1-\phi} \\ \boldsymbol{0} & -\boldsymbol{M}_{(\rho_s(1-\phi))} & \boldsymbol{A}_v & \boldsymbol{0} \\ \boldsymbol{B}_\phi & \boldsymbol{B}_{1-\phi} & \boldsymbol{0} & -\boldsymbol{A}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & -\boldsymbol{C} \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_f + \boldsymbol{M}_{(\rho_f\phi)} + \boldsymbol{K} - \boldsymbol{M}_{(\theta)} & \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{K} & \boldsymbol{A}_s + \boldsymbol{K} & \boldsymbol{M}_{(\rho_s(1-\phi))} \\ \boldsymbol{0} & -\boldsymbol{M}_{(\rho_s(1-\phi))} & \boldsymbol{A}_v \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_\phi & \boldsymbol{B}_{1-\phi} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{C} = \boldsymbol{A}_p.$$

We first show the ellipticity of $\boldsymbol{A}$ and the inf–sup condition of $\boldsymbol{B}$ (Section 6) to then use Theorem 11 from Appendix A.

Lemma 2. *The matrix* $\boldsymbol{A}$ *is positive definite.*

Proof. We proceed directly from the bilinear forms testing against the solution, using the inequality $2(\phi^2\boldsymbol{\kappa}_f^{-1}u, v) \leq (\phi^2\boldsymbol{\kappa}_f^{-1}u, u) + (\phi^2\boldsymbol{\kappa}_f^{-1}v, v)$ and hypotheses (*H*1), (*H*2). We define $\mathcal{A}(\cdot,\cdot)$ the bilinear form associated to matrix $\boldsymbol{A}$:

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s), (\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s)) = (\rho_f\phi\boldsymbol{v}_f, \boldsymbol{v}_f) + (\phi\boldsymbol{\sigma}_{\texttt{vis}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_f)) + ([\phi^2\boldsymbol{\kappa}_f^{-1} - \theta\boldsymbol{I}]\boldsymbol{v}_f, \boldsymbol{v}_f) - (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{y}_s, \boldsymbol{v}_f)\\
&\quad +(\rho_s(1-\phi)\boldsymbol{v}_s, \boldsymbol{y}_s) + (\boldsymbol{\sigma}_{\texttt{skel}}, \boldsymbol{\varepsilon}(\boldsymbol{y}_s)) + (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{y}_s, \boldsymbol{y}_s) - (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f, \boldsymbol{y}_s)\\
&\quad -(\rho_s(1-\phi)\boldsymbol{y}_s, \boldsymbol{v}_s) + (\rho_s(1-\phi)\boldsymbol{v}_s, \boldsymbol{v}_s)\\
&\quad = (\rho_f\phi\boldsymbol{v}_f, \boldsymbol{v}_f) + (\phi\boldsymbol{\sigma}_{\texttt{vis}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_f)) + (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f, \boldsymbol{v}_f) - (\theta\boldsymbol{v}_f, \boldsymbol{v}_f)\\
&\quad +(\boldsymbol{\sigma}_{\texttt{skel}}, \boldsymbol{\varepsilon}(\boldsymbol{w}_s)) + (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{y}_s, \boldsymbol{y}_s) + (\rho_s(1-\phi)\boldsymbol{v}_s, \boldsymbol{v}_s) - 2(\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{y}_s, \boldsymbol{v}_f)\\
&\quad \forall(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s) \in \boldsymbol{H}_0^1(\Omega)\times\boldsymbol{H}_0^1(\Omega)\times\boldsymbol{L}^2(\Omega);
\end{aligned}$$

then we obtain

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s), (\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s))\\
&\quad \geq (\rho_f\phi\boldsymbol{v}_f, \boldsymbol{v}_f) + (\phi\boldsymbol{\sigma}_{\texttt{vis}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_f)) - (\theta\boldsymbol{v}_f, \boldsymbol{v}_f) + (\boldsymbol{\sigma}_{\texttt{skel}}, \boldsymbol{\varepsilon}(\boldsymbol{y}_s)) + C_v\|\boldsymbol{v}_s\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad \geq \alpha_f\|\boldsymbol{v}_f\|^2_{\boldsymbol{H}^1(\Omega)} + C_s\|\boldsymbol{y}_s\|^2_{\boldsymbol{H}^1(\Omega)} + C_v\|\boldsymbol{v}_s\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad \forall(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s) \in \boldsymbol{H}_0^1(\Omega)\times\boldsymbol{H}_0^1(\Omega)\times\boldsymbol{L}^2(\Omega). \quad \square
\end{aligned}$$

Lemma 3. *The matrices* $\boldsymbol{B}, \boldsymbol{C}$ *are such that* $\ker\boldsymbol{B}^T \cap \ker\boldsymbol{C} = \{\boldsymbol{0}\}$.

Proof. From Theorem 9 we have that $\boldsymbol{B}$ is surjective and thus $\boldsymbol{B}^T$ is injective, which yields the result. □

Remark 3.1. Note that although $\boldsymbol{C}$ is a mass matrix, usually the constant κ_s is very large, which makes the matrix $\boldsymbol{E}+\boldsymbol{H}$ positive semi-definite in practice and may produce numerical instabilities. This motivates the use of $\boldsymbol{B}$ for the proof instead, which gives the same result regardless of the problem parameters.

We can now state the existence result.

Lemma 4. *There exists at least one solution to problem* (6).

Proof. It follows from Lemmas 2 and 3 which enable Theorem 11 from Appendix A. □

To prove uniqueness, we consider the problem with null initial data $\boldsymbol{X}_0$ and forcing terms $L(t)$; because of the linearity we only need to prove that this problem has unique (null) solution. We will make use of the identity $\partial_t(f^2) = 2f\partial_t f$, the notation $c(\boldsymbol{x}, \boldsymbol{y}) = (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{x}, \boldsymbol{y})$, Young's inequality $2\,ab \leq a^2 + b^2$ and the following result regarding norm equivalence, recalling the definition of the weighted norm $\|\boldsymbol{v}\|^2_\zeta = \int \boldsymbol{v}^2\,\zeta\,dx$:

Lemma 5. *The following inequalities hold for* t *in* $[0, T]$ *almost everywhere:*

$$\begin{aligned}
\sqrt{\rho_f\underline{\phi}}\|\boldsymbol{v}_f(t)\|_{\boldsymbol{L}^2(\Omega)} &\leq \|\boldsymbol{v}_f(t)\|_{\rho_f\phi} \leq \sqrt{\rho\overline{\phi}}\|\boldsymbol{v}_f(t)\|_{\boldsymbol{L}^2(\Omega)},\\
\sqrt{\rho_s(1-\overline{\phi})}\|\boldsymbol{v}_s(t)\|_{\boldsymbol{L}^2(\Omega)} &\leq \|\boldsymbol{v}_s(t)\|_{\rho_s(1-\phi)} \leq \sqrt{\rho_s(1-\underline{\phi})}\|\boldsymbol{v}_s(t)\|_{\boldsymbol{L}^2(\Omega)},\\
\sqrt{\kappa_s^{-1}(1-\overline{\phi})^2}\|p(t)\|_{L^2(\Omega)} &\leq \|p(t)\|_{(1-\phi)^2/\kappa_s} \leq \sqrt{\kappa_s^{-1}(1-\underline{\phi})^2}\|p(t)\|_{L^2(\Omega)}.
\end{aligned}$$

Proof. We use the following: $\|\psi\|^2_{L^2(\Omega)} = \int_\Omega \psi^2(\rho_f^{-1}\phi^{-1})(\rho_f\phi)\,dx \leq \rho_f^{-1}\underline{\phi}^{-1}\|\psi\|^2_{\rho_f\phi}$. All inequalities are proved analogously. □

Theorem 2. *There exists a unique solution* $(\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)$ *in* $L^2(0,T;V_{f,h})\times L^\infty(0,T;Q_{p,h})\times L^\infty(0,T;V_{s,h})\times L^\infty(0,T;Q_{v,h})$ *of problem* (6).

Proof. We test system (4) with the solution as $(\boldsymbol{v}_f(t), p(t), \partial_t\boldsymbol{y}_s(t), \boldsymbol{v}_s(t))$ and sum the first three equations to obtain the following:

$$\begin{aligned}
&\frac{1}{2}\partial_t\left((\rho_f\phi\boldsymbol{v}_f(t), \boldsymbol{v}_f(t)) + \left(\frac{(1-\phi)^2}{\kappa_s}p(t), p(t)\right) + (\rho_s(1-\phi)\boldsymbol{v}_s(t), \boldsymbol{v}_s(t))\right)\\
&+(\phi\boldsymbol{\sigma}_{\texttt{vis}}(\boldsymbol{v}_f(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}_f(t))) + c(\boldsymbol{v}_f(t), \boldsymbol{v}_f(t)) - (\theta(t)\boldsymbol{v}_f(t), \boldsymbol{v}_f(t)) - 2c(\partial_t\boldsymbol{y}_s(t), \boldsymbol{v}_f(t))\\
&\qquad +(\boldsymbol{\sigma}_{\texttt{skel}}(\boldsymbol{y}_s(t)), \boldsymbol{\varepsilon}(\partial_t\boldsymbol{y}_s(t))) + c(\partial_t\boldsymbol{y}_s(t), \partial_t\boldsymbol{y}_s(t)) = 0.
\end{aligned} \tag{8}$$

As in the existence proof, we use Young's inequality with $c(\boldsymbol{x},\boldsymbol{y})$ and hypothesis $(H\mathbf{2})$ to obtain

$$\begin{aligned}0 &\geq \frac{1}{2}\partial_t\left((\rho_f\phi\boldsymbol{v}_f(t),\boldsymbol{v}_f(t)) + \left(\frac{(1-\phi)^2}{\kappa_s}p(t),p(t)\right) + (\rho_s(1-\phi)\boldsymbol{v}_s(t),\boldsymbol{v}_s(t))\right)\\ &\quad + (\boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}_f(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}_f(t))) - (\theta(t)\boldsymbol{v}_f(t),\boldsymbol{v}_f(t)) + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_s(t)),\boldsymbol{\varepsilon}(\partial_t\boldsymbol{y}_s(t)))\\ &\geq \frac{1}{2}\partial_t\left((\rho_f\phi\boldsymbol{v}_f(t),\boldsymbol{v}_f(t)) + \left(\frac{(1-\phi)^2}{\kappa_s}p(t),p(t)\right) + (\rho_s(1-\phi)\boldsymbol{v}_s(t),\boldsymbol{v}_s(t)) + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_s(t)),\boldsymbol{\varepsilon}(\boldsymbol{y}_s(t)))\right)\\ &\quad + (\boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}_f(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}_f(t))) - (\theta(t)\boldsymbol{v}_f(t),\boldsymbol{v}_f(t)).\end{aligned}$$

Integrating in time in $(0,s)$ and using Lemma 5, we obtain the following inequality for a general positive constant C:

$$\begin{aligned}0 &\geq (\rho_f\phi\boldsymbol{v}_f(s),\boldsymbol{v}_f(s)) + \left(\frac{(1-\phi)^2}{\kappa_s}p(s),p(s)\right) + (\rho_s(1-\phi)\boldsymbol{v}_s(s),\boldsymbol{v}_s(s))\\ &\quad + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_s(s)),\boldsymbol{\varepsilon}(\boldsymbol{y}_s(s))) + \alpha_f\int_0^s \|\boldsymbol{v}_f(s)\|^2_{\boldsymbol{H}^1(\Omega)}\,ds\\ &\geq C\left(\|\boldsymbol{v}_f(t)\|^2_{L^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{v}_f(t)\|^2_{L(0,s;\boldsymbol{H}^1(\Omega))}\right) \geq 0,\end{aligned}$$

which holds for every $t > 0$. We thus conclude that

$$\|\boldsymbol{v}_f\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} = \|\boldsymbol{v}_f\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} = \|p\|_{L^\infty(0,T;L^2(\Omega))} = \|\boldsymbol{v}_s\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} = \|\boldsymbol{y}_s\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} = 0.$$

In particular, we get that all functions $(\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)$ are zero in the $L^\infty(0,T;L^2(\Omega))$ topology. □

3.2. Stability analysis of the semi-discrete problem

In this section we prove that the solution of the semi-discrete problem (6) is upper bounded with respect to the data, which is equivalent to the well-posed in the sense of Hadamard [25]. This result will be used in Section 4 for the proof of existence of solutions of the continuous problem. For this, we proceed as in Section 3.1 but using non null data instead:

$$\begin{aligned}&\frac{1}{2}\partial_t\left((\rho_f\phi\boldsymbol{v}_f(t),\boldsymbol{v}_f(t)) + \left(\frac{(1-\phi)^2}{\kappa_s}p(t),p(t)\right) + (\rho_s(1-\phi)\boldsymbol{v}_s(t),\boldsymbol{v}_s(t)) + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_s(t)),\boldsymbol{\varepsilon}(\boldsymbol{y}_s(t)))\right)\\ &\qquad + (\boldsymbol{\sigma}_{\mathtt{vis}}(\boldsymbol{v}_f(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}_f(t))) - (\theta(t)\boldsymbol{v}_f(t),\boldsymbol{v}_f(t))\\ &\quad \leq (\rho_f\phi\boldsymbol{f}(t),\boldsymbol{v}_f(t)) + \frac{1}{\rho_f}(\theta(t),p(t)) + (\rho_s(1-\phi)\boldsymbol{f}(t),\boldsymbol{v}_s(t)).\end{aligned} \tag{9}$$

Throughout this section we denote with $c = c(\rho_f,\rho_s,\phi,\kappa_s,\alpha_s,\alpha_f)$ a data dependent constant used for lower bounds and with $C = C(\rho_f,\rho_s,\phi,\kappa_s,\lambda,\mu,\mu_f)$ another one for upper bounds. We will make use of Young's generalized inequality for every $\epsilon > 0$: $(a,b)_X \leq \frac{\epsilon}{2}\|a\|^2_X + \frac{1}{2\epsilon}\|b\|^2_X$. Consider $\epsilon > 0$, then from (9) we first expand the right hand side (r.h.s):

$$\begin{aligned}\text{r.h.s} &\leq \frac{1}{\epsilon}\left(\|\boldsymbol{f}(t)\|^2_{\rho_f\phi} + \|\theta(s)\|^2_{L^2(\Omega)} + \|\boldsymbol{f}(t)\|^2_{\rho_s(1-\phi)}\right) + \epsilon\left(\|\boldsymbol{v}_f(t)\|^2_{\rho_f\phi} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\rho_s(1-\phi)}\right)\\ &\leq \frac{C}{\epsilon}\left(\|\boldsymbol{f}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\theta(s)\|^2_{L^2(\Omega)}\right) + C\epsilon\left(\|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned} \tag{10}$$

Integrating in time in $(0,t)$, the left hand side (l.h.s) of (9) with hypothesis $(H\mathbf{2})$ and Lemma 5 becomes

$$\begin{aligned}\int_0^t \text{l.h.s} &\geq \left(\|\boldsymbol{v}_f(s)\|^2_{\rho_f\phi} + \|p(s)\|^2_{\frac{(1-\phi)^2}{\kappa_s}} + \|\boldsymbol{v}_s(s)\|^2_{\rho_s(1-\phi)} + (\boldsymbol{\sigma}_{\mathtt{skel}}(\boldsymbol{y}_s(s)),\boldsymbol{\varepsilon}(\boldsymbol{y}_s(s)))\right)\Bigg|_{s=0}^{s=t}\\ &\qquad + \alpha_f\int_0^t \|\boldsymbol{v}_f(s)\|^2_{\boldsymbol{H}^1(\Omega)}\,ds\\ &\quad \geq c\left(\|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \int_0^t \|\boldsymbol{v}_f(s)\|^2_{\boldsymbol{H}^1(\Omega)}\,ds\right)\\ &\qquad - C\left(\|\boldsymbol{v}_f(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right).\end{aligned} \tag{11}$$

Using the right hand side upper bound (10) and the left hand side lower bound (11) on estimate (9) we obtain:

$$\begin{aligned} c\left(\|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \int_0^t \|\boldsymbol{v}_f(s)\|^2_{\boldsymbol{H}^1(\Omega)}\,ds\right) \\ \leq \frac{C}{\epsilon}\int_0^t \left(\|\boldsymbol{f}(s)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\theta(s)\|^2_{L^2(\Omega)}\right) ds \\ + C\left(\|\boldsymbol{v}_f(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right) \\ + C\epsilon\int_0^t \left(\|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)}\right) ds. \end{aligned} \tag{12}$$

Taking the supremum of t in $(0, T)$ and using the upper bound $\int_0^T \varphi(s)\,ds \leq T\,|\varphi|_\infty$ we obtain the following estimate:

$$\begin{aligned} &(c - CT\epsilon)\left(\|\boldsymbol{v}_f\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|p\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{v}_s\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\right) \\ &+c\left(\|\boldsymbol{y}_s\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{v}_f\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\right) \leq \frac{C}{\epsilon}\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\theta\|^2_{L^2(0,T;L^2(\Omega))}\right) \\ &+C\left(\|\boldsymbol{v}_f(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right) \end{aligned}$$

where we choose $\epsilon = \frac{c}{2CT}$, thus obtaining the following estimate.

$$\begin{aligned} \|\boldsymbol{v}_f\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{v}_f\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{y}_s\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} + \|p\|^2_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{v}_s\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\ \leq \tilde{C}T\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\theta\|^2_{L^2(0,T;L^2(\Omega))}\right) \\ + \tilde{C}\left(\|\boldsymbol{v}_f(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right), \end{aligned} \tag{13}$$

where $\tilde{C} = 2\max\{C, C^2\}c^{-1}$. Now we extend the previous estimate to include time derivatives, which will be useful later when we apply the Faedo–Galerkin method to show the existence of solutions at the continuous level. First from the fluid equation in (6) we obtain the following bound for every test function $\boldsymbol{v}_f^*$ in $\boldsymbol{H}_0^1(\Omega)$:

$$(\rho_f\phi\partial_t\boldsymbol{v}_f(t), \boldsymbol{v}_f^*) \leq C\left(\|\boldsymbol{f}(t)\|_{(\boldsymbol{H}_0^1(\Omega))'} + \|p(t)\|_{L^2(\Omega)} + \|\partial_t\boldsymbol{y}_s(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|_{\boldsymbol{H}^1(\Omega)}\right)\|\boldsymbol{v}_f^*\|_{1,\Omega}.$$

Thus, since for all S in $(\boldsymbol{H}_0^1(\Omega))'$ we have $\|S\|_{(\boldsymbol{H}_0^1(\Omega))'} = \sup_{v\in\boldsymbol{H}_0^1(\Omega), v\neq 0}\frac{S(v)}{\|v\|_{\boldsymbol{H}_0^1(\Omega)}}$, using Lemma 5, $\partial_t\boldsymbol{y}_s = \boldsymbol{v}_s$, taking the supremum on $\|\boldsymbol{v}_f^*\|_{\boldsymbol{H}^1(\Omega)} = 1$ and then squares on both sides we get

$$\|\rho_f\phi\partial_t\boldsymbol{v}_f(t)\|^2_{(\boldsymbol{H}_0^1(\Omega))'} \leq C\left(\|\boldsymbol{f}(t)\|^2_{(\boldsymbol{H}_0^1(\Omega))'} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{H}^1(\Omega)}\right). \tag{14}$$

Similarly, from the solid momentum we get for every test function $\boldsymbol{w}_s$ in $\boldsymbol{H}_0^1(\Omega)$ that

$$\begin{aligned} &(\rho_s(1-\phi)\partial_t\boldsymbol{v}_s(t), \boldsymbol{w}_s) \\ &\quad\leq C\left(\|\boldsymbol{y}_s(t)\|_{\boldsymbol{H}^1(\Omega)} + \|\partial_t\boldsymbol{y}_s(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|_{L^2(\Omega)} + \|\boldsymbol{f}(t)\|_{(\boldsymbol{H}_0^1(\Omega))'}\right)\|\boldsymbol{w}_s\|_{\boldsymbol{H}^1(\Omega)}, \end{aligned}$$

and taking the supremum on $\|\boldsymbol{w}_s\|_{\boldsymbol{H}^1(\Omega)} = 1$ we obtain

$$\|\rho_s(1-\phi)\partial_t\boldsymbol{v}_s(t)\|^2_{(\boldsymbol{H}_0^1(\Omega))'} \leq C(\|\boldsymbol{y}_s(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{f}(t)\|^2_{(\boldsymbol{H}_0^1(\Omega))'}). \tag{15}$$

From the mass conservation equation, we obtain for every test function q in $H^1(\Omega)$ that

$$\left(\frac{(1-\phi)^2}{\kappa_s}\partial_t p(t), q\right) \leq C\left(\|\theta(t)\|_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|_{\boldsymbol{L}^2(\Omega)}\right)\|q\|_{H^1(\Omega)},$$

thus taking supremum on $\|q\|_{H^1(\Omega)}$ we obtain

$$\|(1-\phi)^2\kappa_s^{-1}\partial_t p(t)\|^2_{(H^1(\Omega))'} \leq C\left(\|\theta(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}_f(t)\|^2_{\boldsymbol{L}^2(\Omega)}\right). \tag{16}$$

As $\boldsymbol{v}_s = \partial_t\boldsymbol{y}_s$, we analogously get for $\boldsymbol{v}_s^*$ in $\boldsymbol{L}^2(\Omega)$ that

$$\|(1-\phi)\partial_t\boldsymbol{y}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} = \|(1-\phi)\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{v}_s(t)\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{17}$$

Finally, using estimates (13), (14), (15), (16) and (17), weighted by positive constants $\alpha_1, \alpha_2, \alpha_3$ and α_4 respectively combined with (13) and $\|f(t)\|_{(\boldsymbol{H}_0^1(\Omega))'} \leq \|f(t)\|_{\boldsymbol{L}^2(\Omega)}$ we get the following estimate:

$$
\begin{aligned}
&\alpha_1\|\rho_f\phi\partial_t \boldsymbol{v}_f\|_{L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')} + \alpha_2\|\rho_s(1-\phi)\partial_t \boldsymbol{v}_s\|_{L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')}\\
&+ \alpha_3\|(1-\phi)^2\kappa_s^{-1}\partial_t p\|_{L^2(0,T;(H^1(\Omega))')} + \alpha_4\|(1-\phi)\partial_t \boldsymbol{y}_s\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\
&+ (1-[\alpha_1+\alpha_2+\alpha_3]C)\|\boldsymbol{v}_f\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + (1-\alpha_1 C)\|\boldsymbol{v}_f\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + (1-\alpha_2 C)\|\boldsymbol{y}_s\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))}\\
&+ (1-[\alpha_1+\alpha_2]C)\|p\|^2_{L^2(0,T;L^2(\Omega))} + (1-[\alpha_1+\alpha_2+\alpha_3+\alpha_4]C)\|\boldsymbol{v}_s\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\
&\qquad\leq \overline{C}\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\theta\|^2_{L^2(0,T;L^2(\Omega))}\right)\\
&\qquad\quad + \tilde{C}\left(\|\boldsymbol{v}_f(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_s(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_s(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right),
\end{aligned}
\tag{18}
$$

where $\overline{C} = \max\{\tilde{C}T, C\}$. Choosing $(\alpha_i)_{i=1}^4$ such that $1-[\alpha_1+\alpha_2+\alpha_3+\alpha_4]C \geq 1/2$, $1-[\alpha_1+\alpha_2+\alpha_3]C \geq 1/2$, $1-\alpha_1 C \geq 1/2$, $1-\alpha_2 C \geq 1/2$ and $1-[\alpha_1+\alpha_2]C \geq 1/2$, i.e $\alpha_i = 1/8C$ for all i, we can give a complete energy estimate, which we state in the following theorem (we restore the subindex h for readability).

Theorem 3. *There exists unique solution to problem (6) which satisfies the following a priori estimate:*

$$
\begin{aligned}
&\|\rho_f\phi\partial_t \boldsymbol{v}_{f,h}\|_{L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')} + \|\rho_s(1-\phi)\partial_t \boldsymbol{v}_{s,h}\|_{L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')}\\
&+ \|(1-\phi)^2\kappa_s^{-1}\partial_t p_h\|_{L^2(0,T;(H^1(\Omega))')} + \|(1-\phi)\partial_t \boldsymbol{y}_{s,h}\|^2_{L^2(0,T;(\boldsymbol{H}_0^1(\Omega))')}\\
&+ \|\boldsymbol{v}_{f,h}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{v}_{f,h}\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{y}_{s,h}\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} + \|p_h\|^2_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{v}_{s,h}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\
&\qquad\leq \overline{C}\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\theta\|^2_{L^2(0,T;L^2(\Omega))}\right)\\
&\qquad\quad + \tilde{C}\left(\|\boldsymbol{v}_{f,h}(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p_h(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_{s,h}(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{y}_{s,h}(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right).
\end{aligned}
\tag{19}
$$

4. Analysis of the continuous problem

In this section we prove that there exists a unique solution of problem (2). For this we use a Faedo–Galerkin argument, which consists in proving that a discrete solution converges to a limit that solves the continuous problem. Typical Faedo–Galerkin schemes use the finite dimensional spaces generated by the eigenvectors of the problem [26], but other discrete constructions, such as Galerkin schemes are acceptable [27], the latter being the approach we use. Here we recall that a sequence $f_n|_{n=1}^\infty$ in $L^2(I,X)$, with $I \subset \mathbb{R}$ and X Banach space, converges weakly to f in $L^2(I,X)$, written $f_n \rightharpoonup f$, if and only if

$$
\int_0^T (\Theta, f_n)_X \to \int_0^T (\Theta, f)_X \quad \forall \Theta \in L^2(I,X').
$$

A sequence $f_n'|_{n=1}^\infty$ in $L^2(I,X')$ converges weakly to f' in $L^2(I,X')$ (or weakly*), written $f_n' \overset{*}{\rightharpoonup} f'$, if and only if

$$
\int_0^T (f_n', x) \to \int_0^T (f', x) \quad \forall x \in L^2(I,X),
$$

and further note that weak convergence implies weak convergence in the dual space thanks to the Riesz isometry. We will make use of the Banach–Alaoglu–Bourbaki Theorem, which states that the closed ball is weakly compact [28]. The Faedo–Galerkin technique, used in the following Lemma, which consists in **(i)** obtaining an estimate which gives the inclusion of the solution in a closed ball, **(ii)** using such inclusion to apply the Banach–Alaoglu–Bourbaki [28, Theorem 3.16] theorem to extract a weakly (or weakly*) convergent subsequence and **(iii)** proving that the limit function is a solution of the problem.

Lemma 6. *There exists a solution $(\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)$ in $\boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ to problem (2) that satisfies the energy estimate (19).*

Proof. Consider a solution $(\boldsymbol{v}_{f,h}, p_h, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h})$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$ of problem (6), then in virtue of estimate (19) we use the Banach–Alaoglu–Bourbaki theorem to obtain a subsequence $(\boldsymbol{v}_{f,h'}, p_{h'}, \boldsymbol{y}_{s,h'}, \boldsymbol{v}_{s,h'})|_{h'>0}$, in which we replace h' with h for simplicity, such that:

$$
\begin{aligned}
&\boldsymbol{v}_{f,h} \rightharpoonup \boldsymbol{v}_f \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)), \quad \partial_t \boldsymbol{v}_{f,h} \rightharpoonup \partial_t \boldsymbol{v}_f \in L^2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\boldsymbol{y}_{s,h} \rightharpoonup \boldsymbol{y}_s \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)), \quad \partial_t \boldsymbol{y}_{s,h} \rightharpoonup \partial_t \boldsymbol{y}_s \in L^2(0,T;(\boldsymbol{L}^2(\Omega))'),\\
&\boldsymbol{v}_{s,h} \rightharpoonup \boldsymbol{v}_s \in L^2(0,T;\boldsymbol{L}^2(\Omega)), \quad \partial_t \boldsymbol{v}_{s,h} \rightharpoonup \partial_t \boldsymbol{v}_s \in L^2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&p_h \rightharpoonup p \in L^2(0,T;L^2(\Omega)).
\end{aligned}
\tag{20}
$$

We obtain convergence of all linear forms (see Appendix B) and thus the limit functions are a solution of the following problem:

$$\begin{aligned}(\phi\partial_t \boldsymbol{v}_f, \boldsymbol{v}_{f,h}^*) + (\phi\boldsymbol{\sigma}_{\mathtt{vis}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_f^*)) + ([\phi^2\boldsymbol{\kappa}_f^{-1} - \theta\boldsymbol{I}]\boldsymbol{v}_f, \boldsymbol{v}_{f,h}^*) - (\phi^2\boldsymbol{\kappa}_f^{-1}\partial_t\boldsymbol{y}_s, \boldsymbol{v}_{f,h}^*) &= (\phi\boldsymbol{f}, \boldsymbol{v}_{f,h}^*),\\ \left(\frac{(1-\phi)^2}{\kappa_s}p, q_h\right) + \int_0^t (\operatorname{div}(\phi\boldsymbol{v}_f(s)), q_h)\,ds + (\operatorname{div}((1-\phi)(\boldsymbol{y}_s - \Pi_{V_{s,h}}\boldsymbol{y}_s(0))), q_h)\,ds &= (\rho_f^{-1}\theta, q_h)\\ ((1-\phi)\partial_t\boldsymbol{v}_s, \boldsymbol{w}_{s,h}) + (\boldsymbol{\sigma}_{\mathtt{skel}}, \boldsymbol{\varepsilon}(\boldsymbol{w}_{s,h})) + (\phi^2\boldsymbol{\kappa}_f^{-1}\partial_t\boldsymbol{y}_s, \boldsymbol{w}_{s,h}) - (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f, \boldsymbol{w}_{s,h}) &= ((1-\phi)\boldsymbol{f}, \boldsymbol{w}_{s,h}),\\ ((1-\phi)\partial_t\boldsymbol{y}_s, \boldsymbol{v}_{s,h}^*) - ((1-\phi)\boldsymbol{v}_s, \boldsymbol{v}_{s,h}^*) &= 0,\end{aligned} \tag{21}$$

for all test functions $(\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*)$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$. Finally, as we are using conforming approximations, and thus for every test function $(\boldsymbol{v}_f^*, q, \boldsymbol{w}_s, \boldsymbol{v}_s^*)$ in $\boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ there exists a sequence of functions $(\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*)$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$, such that $(\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*) \to (\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*)$ strongly in h. We thus obtain that (21) also holds for all test functions $(\boldsymbol{v}_f^*, q, \boldsymbol{w}_s, \boldsymbol{v}_s^*)$ in $\boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$, which proves the existence. Finally, the energy estimate (19) uses only the regularity of the continuous functions, which concludes the proof. □

Remark. The conservation of mass is satisfied in integral form

$$\frac{(1-\phi)^2}{\kappa_s}p(t) = \frac{(1-\phi)^2}{\kappa_s}\Pi_{Q_{p,h}}p(0) + \rho_f^{-1}\theta - \int_0^t \operatorname{div}(\phi\boldsymbol{v}_f(s))\,ds + \operatorname{div}\big((1-\phi)(\boldsymbol{y}_s(t) - \Pi_{V_{s,h}}\boldsymbol{y}_s(0))\big)$$

as an equality in $L^2(\Omega)$; however, the corresponding differential form

$$\partial_t p + \operatorname{div}(\phi\boldsymbol{v}_f + (1-\phi)\boldsymbol{v}_s) = \rho_f^{-1}\theta,$$

is only satisfied in $(H^1(\Omega))'$. Indeed, the term $\operatorname{div}\boldsymbol{v}_s$ belongs to $(H^1(\Omega))'$, and no extra regularity can be obtained a priori for the solid velocity. In such cases, p is also referred to as a mild solution. It is also possible to write the problem for a pressure in $L^2(0,T;(H^1(\Omega))')$, and as $\partial_t p$ was shown to be in $L^2(0,T;(H^1(\Omega))')$ as well, we would have higher regularity in time by lowering the spatial regularity. In other words, p belongs to $C(0,T;(H^1(\Omega))') \cap L^2(0,T;L^2(\Omega))$ due to the continuous embedding $H^1(0,T;X) \subset C([0,T];X)$, where X is an arbitrary Banach space and $C(I,X)$ is the space of continuous functions from $I \subset \mathbb{R}$ to X.

We now verify that the solutions constructed in Lemma 6 are consistent with the initial conditions of the discrete problem (6).

Lemma 7. *The initial condition of the previously constructed solution is the weak limit of the initial condition of the discrete solution.*

Proof. From now on we consider a function φ in $C_c^\infty([0,T])$ such that $\varphi(T) = 0$ and $\varphi(0) = 1$. With this, for a general function u in $L^2(0,T;X)$ with $\partial_t u$ in $L^2(0,T;X')$ and a function v in X we get

$$\int_0^T \langle\partial_t u, \varphi v\rangle_{X',X}\,dt = (u(0), v) - \int_0^T \partial_t\varphi(u,v)\,dt. \tag{22}$$

We now write all equations in (21) and (6) as follows:

$$\begin{aligned}\int_0^T (\phi\partial_t\boldsymbol{v}_f, \varphi\boldsymbol{v}_{f,h}^*)\,dt &= \int_0^T F_f(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s, \boldsymbol{v}_{f,h}^*)\,dt, & \int_0^T (\phi\partial_t\boldsymbol{v}_{f,h}, \varphi\boldsymbol{v}_{f,h}^*)\,dt &= \int_0^T F_f(\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h}, \boldsymbol{v}_{f,h}^*)\,dt,\\ \int_0^T \left(\frac{(1-\phi)^2}{\kappa_s}p, \varphi q_h\right)dt &= \int_0^T \varphi F_p(p, \boldsymbol{v}_f, \boldsymbol{y}_s, q_h)\,dt, & \int_0^T \left(\frac{(1-\phi)^2}{\kappa_s}p_h, \varphi q_h\right) &= \int_0^T \varphi F_p(p_h, \boldsymbol{v}_f, \boldsymbol{y}_s, q_h)\,dt,\\ \int_0^T ((1-\phi)\partial_t\boldsymbol{v}_s, \varphi\boldsymbol{w}_{s,h})\,dt &= \int_0^T F_s(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s, \boldsymbol{w}_{s,h})\,dt, & \int_0^T ((1-\phi)\partial_t\boldsymbol{v}_{s,h}, \varphi\boldsymbol{w}_{s,h})\,dt &= \int_0^T F_s(\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h}, \boldsymbol{w}_{s,h})\,dt,\\ \int_0^T ((1-\phi)\partial_t\boldsymbol{y}_s, \varphi\boldsymbol{v}_{s,h})\,dt &= \int_0^T F_v(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s, \boldsymbol{v}_{s,h}^*)\,dt, & \int_0^T ((1-\phi)\partial_t\boldsymbol{y}_{s,h}, \varphi\boldsymbol{v}_{s,h})\,dt &= \int_0^T F_v(\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}^*)\,dt,\end{aligned}$$

for all $(\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*)$ in $V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$. From them, using (22) we can take the limit of the discrete solution for every discrete test function:

$$\begin{aligned}(\boldsymbol{v}_f(0), \boldsymbol{v}_{f,h}) &= \int_0^T \partial_t\varphi(\boldsymbol{v}_f, \boldsymbol{v}_{f,h}^*)\,dt + \int_0^T \varphi F_f(\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s, \boldsymbol{v}_{f,h}^*)\,dt\\ &= \lim_{\mathfrak{h}\to 0}\int_0^T (\boldsymbol{v}_{f,\mathfrak{h}}, \partial_t\varphi\boldsymbol{v}_{f,h}^*)\,dt + \int_0^T \varphi F_f(\boldsymbol{v}_{f,\mathfrak{h}}, \boldsymbol{y}_{s,\mathfrak{h}}, \boldsymbol{v}_{s,\mathfrak{h}}, \boldsymbol{v}_{f,h}^*)\,dt = \lim_{\mathfrak{h}\to 0}(\Pi_{V_{f,\mathfrak{h}}}\boldsymbol{v}_f(0), \boldsymbol{v}_{f,h}^*).\end{aligned}$$

This ensures consistency for every $\boldsymbol{v}_{f,h}^*$ in $V_{f,h}$, and by density we obtain the consistency of the initial condition. Proceeding analogously for $\partial_t \boldsymbol{y}_s$ and $\partial_t \boldsymbol{v}_s$ gives the desired result. Note that the pressure does not require such procedure, as the initial condition appears explicitly in the integral equation. □

Corollary 1. *The previously constructed solution is unique.*

Proof. Consider two solutions with the same forcing terms and the same initial conditions. The problem that arises by considering their difference due to linearity has null datum, and using the energy estimate (19) we see that the solution is null. □

We have thus proved the following theorem, which is the main result of this section.

Theorem 4. *There exists a unique solution* $(\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)$ *in* $L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \times L^2(0,T;L_0^2(\Omega)) \times L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \times L^2(0,T;$ $\boldsymbol{L}^2(\Omega))$ *of problem (6) which satisfies the energy estimate (19) and is consistent with the initial data.*

5. Error analysis of a fully discrete formulation

In this section, we consider as in Section 3 a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of symplexes K of characteristic size h and the discrete spaces $V_{f,h}, V_{s,h}, Q_{p,h}$, with the added regularity of $Q_{v,h} = V_{s,h}$. We also define the full spaces $X = \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ and $X_h = V_{f,h} \times Q_{p,h} \times V_{s,h} \times Q_{v,h}$ with norm

$$\|(\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)\|_X^2 := \|\boldsymbol{v}_f\|_{\boldsymbol{H}^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 + \|\boldsymbol{y}_s\|_{\boldsymbol{H}^1(\Omega)}^2 + \|\boldsymbol{v}_s\|_{\boldsymbol{H}^1(\Omega)}^2,$$

and set the projections $P_{0,h}^k : L^2(\Omega) \to X_h^k$, $P_{1,h}^k : H^1(\Omega) \to X_h^k$ together with their approximation properties [27]:

- $(AP_{H,1})$: $\|\boldsymbol{v} - P_{1,h}^k \boldsymbol{v}\|_{1,\Omega} \le Ch^\ell |\boldsymbol{v}_f|_{\ell+1,\Omega}$, $1 \le \ell \le k$ for each $\boldsymbol{v}_f$ in $\boldsymbol{H}^{\ell+1}(\Omega)$.
- $(AP_{H,0})$: $\|\boldsymbol{v} - P_{1,h}^k \boldsymbol{v}\|_{0,\Omega} \le Ch^{\ell+1} |\boldsymbol{v}|_{\ell+1,\Omega}$, $1 \le \ell \le k$ for each $\boldsymbol{v}$ in $\boldsymbol{H}^{\ell+1}(\Omega)$.
- (AP_L): $\|q - P_{0,h}^k q\|_{0,\Omega} \le Ch^{\ell+1} |q|_{\ell+1,\Omega}$, $1 \le \ell \le k$ for each q in $H^{\ell+1}(\Omega)$.

We split the time interval $[0,T]$ uniformly into $t_0 = 0, t_1 = \Delta t, \dots, t_N = N\Delta t = T$ with timestep Δt and for simplicity we will use from now on the notation $\Phi^n := \Phi(t_n)$. We use a backward Euler finite difference approximation for the time derivatives:

$$\partial_t u(t_n) \approx \frac{u^n - u^{n-1}}{\Delta t} =: D_{\Delta t} u^n,$$

which gives the following fully discrete formulation proposed in [20] with a different order of test spaces: Given $(\boldsymbol{v}_{f,h}^{n-1}, p_h^{n-1}, \boldsymbol{y}_{s,h}^{n-1}, \boldsymbol{v}_{s,h}^{n-1})$ in X_h, find $(\boldsymbol{v}_{f,h}^n, p_h^n, \boldsymbol{y}_{s,h}^n, \boldsymbol{v}_{s,h}^n)$ in X_h such that

$$\begin{aligned}
&(D_{\Delta t}\boldsymbol{v}_{f,h}^n, \boldsymbol{v}_{f,h}^*)_{\rho_f\phi} + a_f(\boldsymbol{v}_{f,h}^n, \boldsymbol{v}_{f,h}^*) - (p_h, \operatorname{div}(\phi \boldsymbol{v}_{f,h}^*)) \\
&\qquad + c(\boldsymbol{v}_{f,h} - \boldsymbol{v}_{s,h}^n, \boldsymbol{v}_{f,h}^*) = (\boldsymbol{f}, \boldsymbol{v}_{f,h}^*)_{\rho_f\phi} && \forall \boldsymbol{v}_{f,h}^* \in V_{f,h}, \\
&(D_{\Delta t} p_h^n, q_h)_{\frac{(1-\phi)^2}{\kappa_S}} + (q_h, \operatorname{div}(\phi \boldsymbol{v}_{f,h}^n + (1-\phi)\boldsymbol{v}_{s,h}^n)) = (\theta, q_h)_{\frac{1}{\rho_f}} && \forall q_h \in Q_{p,h}, \\
&(D_{\Delta t}\boldsymbol{v}_{s,h}^n, \boldsymbol{w}_{s,h})_{\rho_s(1-\phi)} + a_s(\boldsymbol{y}_{s,h}^n, \boldsymbol{w}_{s,h}) - (p_h^n, \operatorname{div}((1-\phi)\boldsymbol{w}_{s,h})) \\
&\qquad - c(\boldsymbol{v}_{f,h}^n - \boldsymbol{v}_{s,h}^n, \boldsymbol{w}_{s,h}) = (\boldsymbol{f}, \boldsymbol{w}_{s,h})_{\rho_s(1-\phi)} && \forall \boldsymbol{w}_{s,h} \in V_{s,h}, \\
&a_s(D_{\Delta t}\boldsymbol{y}_{s,h}^n, \boldsymbol{v}_{s,h}^*) - a_s(\boldsymbol{v}_{s,h}^n, \boldsymbol{v}_{s,h}^*) = 0 && \forall \boldsymbol{v}_{s,h}^* \in Q_{v,h},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
a_f(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*) &= (\boldsymbol{\sigma}_{\texttt{vis}}(\boldsymbol{v}_{f,h}), \boldsymbol{\varepsilon}(\boldsymbol{v}_{f,h}^*)) - (\theta \boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*), \\
a_s(\boldsymbol{y}_{s,h}, \boldsymbol{w}_{s,h}) &= (\boldsymbol{\sigma}_{\texttt{skel}}(\boldsymbol{y}_{s,h}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{s,h})), \\
c(\boldsymbol{a}, \boldsymbol{b}) &= (\phi^2 \boldsymbol{\kappa}_f^{-1} \boldsymbol{a}, \boldsymbol{b}), \\
(a, b)_\zeta &= (\zeta a, b).
\end{aligned}$$

A fully implicit (Backward Euler) time discretization is an adequate choice for the parabolic part of the problem, namely the momentum equation of the fluid phase and the mass balance equation. However, it might not be appropriate for the momentum equation of the solid phase, as it violates the intrinsic energy conservation property of elastodynamics. In this respect, other approaches may be adopted for the solid phase, such as the classical Newmark scheme or a mid-point rule as in [19].

Remark. The equation $D_{\Delta t}\boldsymbol{y}_{s,h} = \boldsymbol{v}_{s,h}$ has been weakly enforced using the bilinear form a_s. This was also done in [20] and presents advantages during the error analysis with the cost of requiring a higher order of approximation and higher regularity assumptions in the solid velocity. The error analysis can also be carried out for the L^2 product with the strategy we use in what follows, but the convergence rates obtained that way are suboptimal.

We start by showing that problem (23) is well-posed.

Lemma 8. *There exists a unique solution* $(\boldsymbol{v}^n_{f,h}, p^n_h, \boldsymbol{y}^n_{s,h}, \boldsymbol{v}^n_{s,h})$ *in* X_h *of problem* (23).

Proof. Consider the test function $x^*_h = (\boldsymbol{v}^n_{f,h}, p^n_h, \boldsymbol{v}^n_{s,h}, \boldsymbol{y}^n_{s,h})$ and denote the right hand side generically as $F(\boldsymbol{v}^*_{f,h}, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}^*_{s,h})$, which gives

$$\Delta t^{-1}(\|\boldsymbol{v}^n_{f,h}\|^2_{\rho_f\phi} + \|p^n_h\|^2_{(1-\phi)^2/\kappa_s} + \|\boldsymbol{v}^n_{s,h}\|^2_{\rho_s(1-\phi)} + \|\boldsymbol{y}^n_{s,h}\|^2_{a_s}) + a_f(\boldsymbol{v}^n_{f,h}, \boldsymbol{v}^n_{f,h}) \leq F(\boldsymbol{v}^n_{f,h}, p^n_h, \boldsymbol{v}^n_{s,h}, \boldsymbol{y}^n_{s,h}).$$

First note that if $F = 0$, then the only solution is $x_h = 0$. We can then conclude from the discrete Fredholm Alternative Theorem that the solution is unique. The same inequality gives that $(\boldsymbol{v}^n_{f,h}, p^n_h, \boldsymbol{y}^n_{s,h}, \boldsymbol{v}^n_{s,h})$ belongs to $V_{f,h} \times Q_{p,h} \times V_{s,h} \times (\boldsymbol{L}^2(\Omega) \cap \{\boldsymbol{v}^n_{s,h}|_{\partial\Omega} = 0\} \cap [X^{k+1}_h]^d)$. Finally the last equation gives

$$\sqrt{a_s(\boldsymbol{v}^n_{s,h}, \boldsymbol{v}^n_{s,h})} = \sup_{\boldsymbol{w}_{s,h}\in V_{s,h}} \frac{a_s(\boldsymbol{v}^n_{s,h}, \boldsymbol{w}_{s,h})}{\|\boldsymbol{w}_{s,h}\|_{H^1(\Omega)}} = \sup_{\boldsymbol{w}_{s,h}\in V_{s,h}} \frac{-a_s(D_{\Delta t}\boldsymbol{y}^n_{s,h}, \boldsymbol{w}_{s,h})}{\|\boldsymbol{w}_{s,h}\|_{H^1(\Omega)}} = \sqrt{a_s(D_{\Delta t}\boldsymbol{y}^n_{s,h}, D_{\Delta t}\boldsymbol{y}^n_{s,h})},$$

which gives $\boldsymbol{v}^n_{s,h}$ in $Q_{v,h}$. □

We will use the discrete Gronwall Lemma, which we recall for reference [27].

Lemma 9 (*Discrete Gronwall Lemma*)**.** *Consider* $g_0 \geq 0$ *and a sequence* $(p_n)^\infty_{n=0}$ *such that* $p_n \geq 0$. *If* $(f_n)^\infty_{n=0}$ *is such that*

$$f_0 \leq g_0 \quad \textit{and} \quad f_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s f_s,$$

then

$$f_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) exp\left(\sum_{s=0}^{n-1} k_s\right).$$

We also make use of the following tools for the analysis of the approximation properties in time.

Lemma 10. *For any symmetric bilinear form b:*

$$b(\varphi^n, D_{\Delta t}\varphi^n) = \frac{1}{2}D_{\Delta t} b(\varphi^n, \varphi^n) + \frac{1}{2}\Delta t\, b(D_{\Delta t}\varphi^n, D_{\Delta t}\varphi^n). \tag{24}$$

Lemma 11. *The following inequality holds for a backwards difference approximation in a Hilbert space H:*

$$\|D_{\Delta t}\varphi - \partial_t\varphi\|_{\ell^\infty(0,T;H)} \leq \Delta t\|\partial_{tt}\varphi\|_{L^\infty(0,T;H)} \quad \forall \varphi \in W^{2,\infty}(0, T; H).$$

Proof. The Fundamental Theorem of Calculus gives $D_{\Delta t}\varphi^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \partial_t\varphi(s)\, ds$, and so using the monotonicity of the integral $\| \int_I \cdot\, dr\|_H \leq \int_I \| \cdot \|_H\, dr$ we obtain:

$$\begin{aligned}
\|D_{\Delta t}\varphi^n - \partial_t\varphi^n\|^2_H &= \|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (\partial_t\varphi(s) - \partial_t\varphi^n)\, ds\|^2_H \\
&\leq \left(\int_{t_{n-1}}^{t_n} \frac{1}{\Delta t}\left\|\partial_t\varphi(s) - \partial_t\varphi^n\right\|_H ds\right)^2 = \left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\left\|\int_s^{t_n}\partial_{tt}\varphi(r)\, dr\right\|_H ds\right)^2 \\
&\leq \left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\int_s^{t_n}\|\partial_{tt}\varphi(r)\|_H\ drds\right)^2 \leq \left(\frac{\|\partial_{tt}\varphi\|_{L^\infty(t_{n-1},t_n;H)}}{\Delta t}\Delta t^2\right)^2 \\
&\leq \Delta t^2\|\partial_{tt}\varphi\|^2_{L^\infty(0,T;H)}.
\end{aligned}$$

Taking the supremum on n and square root gives the conclusion. □

Corollary 2. *Consider two Hilbert spaces* $Z \subset H$ *and an interpolation operator* $I_h : H \to H_h$ *into a conforming discretization* H_h *such that*

$$\|\varphi - I_h\varphi\|_H \leq Ch^k\|\varphi\|_Z,$$

then it holds that

$$\|D_{\Delta t}I_h\varphi - \partial_t\varphi\|_{\ell^\infty(0,T;H)} \leq \max\{C\|\varphi\|_{L^\infty(0,T;Z)}, \|\partial_{tt}\varphi\|_{L^\infty(0,T;H)}\}(h^k + \Delta t) \quad \forall \varphi \in W^{2,\infty}(0, T; H).$$

Proof. It follows directly from Lemma 11 and the decomposition $\partial_t\varphi - D_{\Delta t}I_h\varphi = \partial_t\varphi - D_{\Delta t}\varphi + D_{\Delta t}\varphi - D_{\Delta t}I_h\varphi$. □

We then write problem (1) as finding $x = (\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s)$ in X such that

$$\mathcal{E}(\partial_t x, x^*) + \mathcal{H}(x, x^*) = F(x^*) \quad \forall x^* := (\boldsymbol{v}_f^*, q, \boldsymbol{w}_s, \boldsymbol{v}_s^*) \in X, \tag{25}$$

where in analogy with the notation used in Section 3.1 we define the bilinear forms

$$\begin{aligned}\mathcal{E}(\partial_t x, x^*) &:= (\partial_t \boldsymbol{v}_f, \boldsymbol{v}_f^*)_{\rho_f\phi} + (\partial_t p, q)_{(1-\phi)^2/\kappa_s} + (\partial_t \boldsymbol{v}_s, \boldsymbol{w}_s)_{\rho_s(1-\phi)} + a_s(\partial_t \boldsymbol{y}_s, \boldsymbol{v}_s^*),\\ \mathcal{H}(x, x^*) &:= a_f(\boldsymbol{v}_f, \boldsymbol{v}_f^*) + a_s(\boldsymbol{y}_s, \boldsymbol{w}_s) - a_s(\boldsymbol{v}_s, \boldsymbol{v}_s^*)_{\rho_s(1-\phi)} + c(\boldsymbol{v}_f - \boldsymbol{v}_s, \boldsymbol{v}_f^* - \boldsymbol{w}_s)\\ &\quad - (p, \operatorname{div}(\phi\boldsymbol{v}_f^* + (1-\phi)\boldsymbol{w}_s)) + (q, \operatorname{div}(\phi\boldsymbol{v}_f + (1-\phi)\boldsymbol{v}_s)),\end{aligned}$$

and set its discrete counterpart as: Given x_h^{n-1} in X_h, find $x_h^n = (\boldsymbol{v}_{f,h}^n, p_h^n, \boldsymbol{y}_{s,h}^n, \boldsymbol{v}_{s,h}^n)$ in X_h such that

$$\mathcal{E}(D_{\Delta t}x_h^n, x_h^*) + \mathcal{H}(x_h^n, x_h^*) = F(x_h^*) \quad \forall x_h^* := (\boldsymbol{v}_{f,h}^*, q_h, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*) \in X_h. \tag{26}$$

We proceed by showing the invertibility of $\mathcal{H}$, for which we add the following hypothesis, recalling that the bilinear form $c(\cdot,\cdot) = (\phi^2\boldsymbol{\kappa}_f^{-1}\cdot,\cdot)$:

(H**4**) The permeability tensor $\boldsymbol{\kappa}_f$ is large enough:

$$\exists C_{sk} : a_s(\boldsymbol{w}_s, \boldsymbol{w}_s) - c(\boldsymbol{w}_s, \boldsymbol{w}_s) \geq C_{sk}\|\boldsymbol{w}_s\|^2_{\boldsymbol{H}^1(\Omega)} \quad \forall \boldsymbol{w}_s \in \boldsymbol{H}_0^1(\Omega).$$

Theorem 5. *Under assumptions (H**1**), (H**2**), (H**3**) and (H**4**) it holds that the problem of finding x_h in X_h such that*

$$\mathcal{H}(x_h, x_h^*) = \mathcal{F}(x_h^*) \qquad \forall x_h^* \in X_h$$

is well-posed for every $\mathcal{F}$ in X_h'. Moreover, if $\tilde{x}$ is the function such that

$$\mathcal{H}(\tilde{x}, x^*) = \mathcal{F}(x^*) \quad \forall x^* \in X,$$

then, defining $Z = \boldsymbol{H}^{k+2}(\Omega) \times H^{k+1}(\Omega) \times \boldsymbol{H}^{k+2}(\Omega) \times \boldsymbol{H}^{k+2}(\Omega)$, the following holds for a positive problem dependent constant C:

$$\|\tilde{x} - x_h\|_X \leq Ch^{k+1}\|\tilde{x}\|_Z.$$

Proof. Let $V_h = V_{f,h} \times V_{s,h} \times Q_{v,h}$ and $Q_h = Q_{p,h}$ with bilinear forms

$$\begin{aligned}\mathcal{A}((\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}, \boldsymbol{v}_{s,h}), (\boldsymbol{v}_{f,h}^*, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*)) &= a_s(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}^*) + a_s(\boldsymbol{y}_{s,h}, \boldsymbol{w}_{s,h})\\ &\quad + a_s(\boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}^*) + c(\boldsymbol{v}_{f,h} - \boldsymbol{v}_{s,h}, \boldsymbol{v}_{f,h}^* - \boldsymbol{w}_{s,h})\\ \mathcal{B}_1((\boldsymbol{v}_{f,h}^*, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*), q_h) &= (q_h, \operatorname{div}(\phi\boldsymbol{v}_{f,h}^* + (1-\phi)\boldsymbol{w}_{s,h})),\\ \mathcal{B}_2((\boldsymbol{v}_{f,h}^*, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}^*), q_h) &= (q_h, \operatorname{div}(\phi\boldsymbol{v}_{f,h}^* + (1-\phi)\boldsymbol{v}_{s,h})).\end{aligned}$$

Note that, using Young's inequality we obtain the following:

$$\begin{aligned}&c(\boldsymbol{v}_{f,h} - \boldsymbol{v}_{s,h}, \boldsymbol{v}_{f,h} - \boldsymbol{w}_{s,h}) = c(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}) - c(\boldsymbol{v}_{f,h}, \boldsymbol{w}_{s,h}) - c(\boldsymbol{v}_{s,h}, \boldsymbol{v}_{f,h}) + c(\boldsymbol{v}_{s,h}, \boldsymbol{w}_{s,h})\\ &\quad \geq c(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}) - \frac{1}{2}\left(c(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}) + c(\boldsymbol{w}_{s,h}, \boldsymbol{w}_{s,h}) + c(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{f,h}) + c(\boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}) + c(\boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}) + c(\boldsymbol{w}_{s,h}, \boldsymbol{w}_{s,h})\right)\\ &\quad \geq -c(\boldsymbol{v}_{s,h}, \boldsymbol{v}_{s,h}) - c(\boldsymbol{w}_{s,h}, \boldsymbol{w}_{s,h}),\end{aligned}$$

which combined with hypothesis (H**4**) shows that $\mathcal{A}$ is elliptic, and forms $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy the hypothesis of Theorem 12 in virtue of Theorem 8. The conclusion comes then from Theorem 12 and the approximation properties $(AP_{H,1})$, $(AP_{H,0})$, (AP_L). □

We are now ready to address the error estimate for the fully discrete model (23). To this purpose we use the decomposition of the numerical error into the approximation error, denoted with χ, and the remaining truncation error, denoted with φ, as follows:

$$\begin{aligned}\boldsymbol{e}_f^n = \boldsymbol{v}_f(t_n) - \boldsymbol{v}_{f,h}^n &= \boldsymbol{v}_f(t_n) - I_{f,h}\boldsymbol{v}_f(t_n) + I_{f,h}\boldsymbol{v}_f(t_n) - \boldsymbol{v}_{f,h}^n &&= \chi_f^n + \varphi_{f,h}^n,\\ e_p^n = p(t_n) - p_h^n &= p(t_n) - I_{p,h}p(t_n) + I_{p,h}p(t_n) - p_h^n &&= \chi_p^n + \varphi_{p,h}^n,\\ \boldsymbol{e}_s^n = \boldsymbol{y}_s(t_n) - \boldsymbol{y}_{s,h}^n &= \boldsymbol{y}_s(t_n) - I_{f,h}\boldsymbol{y}_s(t_n) + I_{f,h}\boldsymbol{y}_s(t_n) - \boldsymbol{y}_{s,h}^n &&= \chi_s^n + \varphi_{s,h}^n,\\ \boldsymbol{e}_v^n = \boldsymbol{v}_s(t_n) - \boldsymbol{v}_{s,h}^n &= \boldsymbol{v}_s(t_n) - I_{f,h}\boldsymbol{v}_s(t_n) + I_{f,h}\boldsymbol{v}_s(t_n) - \boldsymbol{v}_{s,h}^n &&= \chi_v^n + \varphi_{v,h}^n,\end{aligned}$$

$$\delta_{\Delta t}\xi^n = D_{\Delta t}I_{\xi,h}\xi(t_n) - \frac{\partial\xi(t_n)}{\partial t}, \qquad \text{for } \xi \in \{\boldsymbol{v}_f, p, \boldsymbol{y}_s, \boldsymbol{v}_s\},$$

where all quantities are analogously defined as in the fluid case, with interpolators $I_{f,h}, I_{p,h}, I_{s,h}, I_{v,h}$ defined as follows. Set the Ritz projection $\Pi_h^{\mathcal{H}} : X \to X_h$ as:

$$\mathcal{H}(\Pi_h^{\mathcal{H}} x, x_h^*) = \mathcal{H}(x, x_h^*) \quad \forall x_h^* \in X_h,$$

which is well defined in virtue of Theorem 5, then the interpolation operators are defined as

$$(I_{f,h}\boldsymbol{v}_f(t_n), I_{p,h}p(t_n), I_{s,h}\boldsymbol{y}_s(t_n), I_{v,h}\boldsymbol{v}_s(t_n)) := \Pi_h^{\mathcal{H}}(\boldsymbol{v}_f(t_n), p(t_n), \boldsymbol{y}_s(t_n), \boldsymbol{v}_s(t_n)).$$

With these definitions we have the following corollary of Theorem 5, for which we recall that $X = \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ and $Z = \boldsymbol{H}^{k+2}(\Omega) \times H^{k+1}(\Omega) \times \boldsymbol{H}^{k+2}(\Omega) \times \boldsymbol{H}^{k+2}(\Omega)$.

Corollary 3. *If $x \in W^{2,\infty}(0,T;X) \cap L^\infty(0,T;Z)$, then the following estimate holds for a problem dependent constant C:*

$$\|D_{\Delta t}\Pi_h^{\mathcal{H}} x^n - \partial_t x\|_{\ell^\infty(0,T;X)} \leq C \max\{\|\partial_{tt}x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\} \left(h^{k+1} + \Delta t\right).$$

Proof. This is a direct application of Corollary 2 to the Ritz projector $\Pi_h^{\mathcal{H}}$. □

Then our strategy to perform the error analysis can be split into the following steps: **(i)** Derive equations for the numerical error by subtracting the fully discrete model from the continuous model, **(ii)** split the error into the approximation and truncation errors, **(iii)** use the orthogonality properties of the projector $\Pi_h^{\mathcal{H}}$ in order to eliminate the approximation error from the equations and **(iv)** recover an upper bound for the total error by triangle inequality and approximation properties. We proceed according to the described roadmap.
(i) Consider in (25) the test function $x^* = x_h^*$ and then take the difference with (26) to obtain the error equation. With $\boldsymbol{e}_x = (\boldsymbol{e}_f^n, e_p^n, \boldsymbol{e}_s^n, \boldsymbol{e}_v^n)$ we obtain:

$$\mathcal{E}(\partial_t x - D_{\Delta t}x_h^n, x_h^*) + \mathcal{H}(\boldsymbol{e}_x, x_h^*) = 0 \quad \forall x_h^* \in X_h. \tag{27}$$

(ii) In accordance with our definitions we consider $\chi_x^n = (\chi_f^n, \chi_p^n, \chi_s^n, \chi_v^n)$ and $\varphi_{x,h} = (\varphi_{f,h}^n, \varphi_{p,h}^n, \varphi_{s,h}^n, \varphi_{v,h}^n)$. Note that the time error can be written as

$$\partial_t x - D_{\Delta t}x_h^n = \partial_t x - D_{\Delta t}\Pi_h^{\mathcal{H}} x + D_{\Delta t}\Pi_h^{\mathcal{H}} x - D_{\Delta t}x_h^n = D_{\Delta t}\varphi_{x,h} - \delta_{\Delta t}x,$$

and so we can rewrite the error Eq. (27) as

$$\mathcal{E}(D_{\Delta t}\varphi_{x,h}, x_h^*) + \mathcal{H}(\chi_x^n, x_h^*) + \mathcal{H}(\varphi_{x,h}, x_h^*) = \mathcal{E}(\delta_{\Delta t}x^n, x_h^*) \quad \forall x_h^* \in X_h. \tag{28}$$

(iii) By definition we have that $\mathcal{H}(\chi_x, x_h^*) = 0$, which gives an expression more suitable for the analysis:

$$\mathcal{E}(D_{\Delta t}\varphi_{x,h}, x_h^*) + \mathcal{H}(\varphi_{x,h}, x_h^*) = \mathcal{E}(\delta_{\Delta t}x^n, x_h^*). \tag{29}$$

From here we can obtain an error estimate for the truncation error, which we give in the following lemma.

Lemma 12. *Assume that $\boldsymbol{v}_f, \boldsymbol{y}_s, \boldsymbol{v}_s$ in $W^{2,\infty}(0,T;\boldsymbol{H}^{k+2}(\Omega))$ and p in $W^{2,\infty}(0,T;H^{k+1}(\Omega))$ as well hypotheses (H1), (H2), (H3) and (H4). Then, there exists a constant $C > 0$, possibly dependent on the problem parameters, such that:*

$$\begin{aligned} &\|\varphi_{f,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{p,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{v,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{s,h}\|_{\ell^\infty(H^1(\Omega))} \\ &\quad + \Delta t(\|\varphi_{f,h}\|_{\ell^2(H^1(\Omega))} + c_c\|\varphi_{f,h} - \varphi_{v,h}\|_{\ell^2(L^2(\Omega))}) \leq CTe^T \max\{\|\partial_{tt}x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t). \end{aligned}$$

Proof. The test function $x_h^* = \varphi_{x,h}$ in (29) yields

$$\begin{aligned} &(D_{\Delta t}\varphi_{f,h}^n, \varphi_{f,h}^n)_{\rho_f\phi} + (D_{\Delta t}\varphi_{p,h}^n, \varphi_{p,h}^n)_{(1-\phi)^2/\kappa_s} + (D_{\Delta t}\varphi_{v,h}^n, \varphi_{v,h}^n)_{\rho_s(1-\phi)} + a_s(D_{\Delta t}\varphi_{s,h}^n, \varphi_{s,h}^n) \\ &\quad + a_f(\varphi_{f,h}^n, \varphi_{f,h}^n) + a_s(\varphi_{s,h}^n, \varphi_{v,h}^n) + c(\varphi_{f,h}^n - \varphi_{v,h}^n, \varphi_{f,h}^n - \varphi_{v,h}^n) - a_s(\varphi_{v,h}^n, \varphi_{s,h}^n) \\ &\quad \leq (\delta_{\Delta t}\boldsymbol{v}_{f,h}^n, \varphi_{f,h}^n)_{\rho_f\phi} + (\delta_{\Delta t}p^n, \varphi_{p,h}^n)_{(1-\phi)^2/\kappa_s} + (\delta_{\Delta t}\boldsymbol{v}_{s,h}^n, \varphi_{v,h}^n)_{\rho_s(1-\phi)} + (\delta_{\Delta t}\boldsymbol{y}_{s,h}^n, \varphi_{s,h}^n)_{\rho_s(1-\phi)}. \end{aligned} \tag{30}$$

We define

$$\begin{aligned} &\mathcal{E}_h^n = \|\varphi_{f,h}^n\|_{\rho_f\phi}^2 + \|\varphi_{p,h}^n\|_{(1-\phi)^n/\kappa_s}^2 + \|\varphi_{v,h}^n\|_{(1-\phi)\rho_s}^2 + \|\varphi_{s,h}^n\|_{a_s}^2, \\ &\|\boldsymbol{v}\|_{a_f}^2 = a_f(\boldsymbol{v}, \boldsymbol{v}), \quad \|\boldsymbol{v}\|_{a_s}^2 = a_s(\boldsymbol{v}, \boldsymbol{v}), \end{aligned}$$

and proceed by using the positivity of c as $c(x,x) \geq \alpha_c\|x\|_{0,\Omega}^2$, (24) and Corollary 3 to obtain that

$$\begin{aligned} &D_{\Delta t}\mathcal{E}_h^n + \|\varphi_{f,h}^n\|_{a_f}^2 + \alpha_c\|\varphi_{f,h}^n - \varphi_{v,h}^n\|_{L^2(\Omega)}^2 \\ &\quad + \Delta t(\|D_{\Delta t}\varphi_{f,h}^n\|_{\rho_f\phi}^2 + \|D_{\Delta t}\varphi_{p,h}^n\|_{(1-\phi)^2/\kappa_s}^2 + \|D_{\Delta t}\varphi_{v,h}^n\|_{\rho_s(1-\phi)}^2 + \|D_{\Delta t}\varphi_{s,h}^n\|_{a_s}^2) \\ &\quad \leq C \max\{\|\partial_{tt}x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t)(\|\varphi_{f,h}^n\|_{\rho_f\phi}^2 + \|\varphi_{p,h}^n\|_{(1-\phi)^n/\kappa_s} + \|\varphi_{v,h}^n\|_{(1-\phi)\rho_s} + \|\varphi_{s,h}^n\|_{a_s}) \\ &\quad \leq \frac{C^2 \max\{\|\partial_{tt}x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}^2}{2}(h^{k+1} + \Delta t)^2 + \frac{1}{2}\mathcal{E}_h^n \end{aligned} \tag{31}$$

where C denotes a general constant depending on the data. Now, we bound all the norms with discrete time derivatives on (31) by 0 from below and sum on $n = 1, \dots, m$ to get

$$\mathcal{E}_h^m + \Delta t \sum_{n=1}^{m} (\|\varphi_{f,h}^n\|_{a_f}^2 + c_c \|\varphi_{f,h}^n - \varphi_{v,h}^n\|_{L^2(\Omega)}^2) \le \frac{CT \max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}^2}{2} (h^{k+1} + \Delta t)^2 + \Delta t \sum_{n=1}^{m} \mathcal{E}_h^n,$$

and thus Lemma 9 (the discrete Gronwall Lemma), gives, for $\Delta t < 0.5$:

$$\frac{1}{2}\mathcal{E}_h^m + \Delta t \sum_{n=1}^{m} (\|\varphi_{f,h}^n\|_{a_f}^2 + c_c \|\varphi_{f,h}^n - \varphi_{v,h}^n\|_{L^2(\Omega)}^2) \le \frac{C \max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}^2 T}{2} (h^{k+1} + \Delta t)^2 e^T.$$

Rearranging terms and using norm equivalences as in Lemma 5 gives the desired result:

$$\begin{aligned}&\|\varphi_{f,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{p,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{v,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{s,h}\|_{\ell^\infty(H^1(\Omega))} \\ &\quad + \Delta t(\|\varphi_{f,h}\|_{\ell^2(H^1(\Omega))} + \|\varphi_{f,h} - \varphi_{v,h}\|_{\ell^2(L^2(\Omega))}) \le CTe^T \max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t). \qquad \square\end{aligned}$$

(iv) We conclude this section with the full error estimate.

Theorem 6. *Assume that $x \in W^{2,\infty}(0,T;X) \cap L^\infty(0,T;Z)$ as well as hypotheses (**H1**), (**H2**), (**H3**) and (**H4**). Then, there exists a constant $C(T) > 0$, possibly dependent on the problem parameters, such that:*

$$\begin{aligned}&\|\boldsymbol{e}_f\|_{\ell^\infty(L^2(\Omega))} + \|e_p\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{e}_v\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{e}_s\|_{\ell^\infty(H^1(\Omega))} \\ &\quad + \Delta t(\|\boldsymbol{e}_f\|_{\ell^2(H^1(\Omega))} + c_c\|\boldsymbol{e}_f - \boldsymbol{e}_v\|_{\ell^2(L^2(\Omega))}) \le C(T)\max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t).\end{aligned}$$

Proof. By definition from Corollary 3 we get the error estimate of the projector $\Pi_h^{\mathcal{H}}$, and thus setting again a generic parameter dependent constant C we can write

$$\begin{aligned}\|\chi_x\|_{L^2} &= \|\chi_f\|_{\boldsymbol{L}^2(\Omega)} + \|\chi_p\|_{L^2(\Omega)} + \|\chi_s\|_{\boldsymbol{L}^2(\Omega)} + \|\chi_v\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \|\chi_f\|_{\boldsymbol{H}^1(\Omega)} + \|\chi_p\|_{L^2(\Omega)} + \|\chi_s\|_{\boldsymbol{H}^1(\Omega)} + \|\chi_v\|_{\boldsymbol{L}^2(\Omega)} \le C\, h^{k+1}\|x\|_Z \end{aligned} \tag{32}$$

almost everywhere in t. The triangle inequality together with (32) give the conclusion as follows:

$$\begin{aligned}&\|\boldsymbol{e}_f\|_{\ell^\infty(L^2(\Omega))} + \|e_p\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{e}_v\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{e}_s\|_{\ell^\infty(H^1(\Omega))} + \Delta t(\|\boldsymbol{e}_f\|_{\ell^2(H^1(\Omega))} + \|\boldsymbol{e}_f - \boldsymbol{e}_v\|_{\ell^2(L^2(\Omega))}) \\ &\le \|\varphi_{f,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{p,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{v,h}\|_{\ell^\infty(L^2(\Omega))} + \|\varphi_{s,h}\|_{\ell^\infty(H^1(\Omega))} \\ &+\Delta t(\|\varphi_{f,h}\|_{\ell^2(H^1(\Omega))} + \|\varphi_{f,h} - \varphi_{v,h}\|_{\ell^2(L^2(\Omega))}) \\ &+C(\|\chi_f\|_{\boldsymbol{H}^1(\Omega)} + \|\chi_p\|_{L^2(\Omega)} + \|\chi_s\|_{\boldsymbol{H}^1(\Omega)} + \|\chi_v\|_{\boldsymbol{L}^2(\Omega)}) \\ &\le C(T)\max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t) + Ch^{k+1}\|x\|_{L^\infty(0,T;Z)} \\ &\le C(T)\max\{\|\partial_{tt} x\|_{L^\infty(0,T;X)}, \|x\|_{L^\infty(0,T;Z)}\}(h^{k+1} + \Delta t). \quad \square \end{aligned} \tag{33}$$

5.1. Numerical tests

We now set up numerical tests for estimating the rates of convergence. For this, we consider the time domain $I = (0, 1)$, the spatial domain $\Omega = (0, 1)^2$ and the following idealized parameters:

$$\rho_f = 1, \quad \boldsymbol{\kappa}_f^{-1} = \boldsymbol{I}, \quad \mu_f = 10, \quad \rho_s = 1, \quad \mu_s = 10, \quad \lambda_s = 10, \quad \kappa_s = 1, \quad \phi = 0.1.$$

For simplicity, we assume that the forcing terms on the fluid and solid equations are different, respectively $\boldsymbol{f}_f$ and $\boldsymbol{f}_s$, so that fixing a displacement, fluid velocity and pressure we recover a source θ and the load terms $\boldsymbol{f}_f, \boldsymbol{f}_s$. We thus set the Dirichlet boundary conditions according to the following manufactured analytical solution:

$$\begin{aligned}&\boldsymbol{y}_s(t,x,y) = t^2(0.5x^3\cos(4\pi y), -x^3\sin(4\pi y)), \quad \boldsymbol{v}_s(t,x,y) = 2\,t(0.5x^3\cos(4\pi y), -x^3\sin(4\pi y)), \\ &\boldsymbol{v}_f(t,x,y) = t^2(\sin^2(4\pi y), \sin^2(4\pi y)), \quad p(t,x,y) = t^2\,(1 - \sin(4\pi x)\sin(4\pi y))\,,\end{aligned}$$

Using such solution, which satisfies all the regularity requirements of the convergence theorem, we perform numerical tests in support of the theory using a polynomial order of $k = 1$. As a result, $\boldsymbol{v}_f$, $\boldsymbol{y}_s$ and $\boldsymbol{v}_s$ belong to $[X_h^2]^2$, whereas the pressure belongs to X_h^1. These tests are designed to test the convergence with respect to Δt and h, independently. First, choosing a very small Δt, we progressively decrease the mesh characteristic size such that the space approximation error dominates over the one on time as shown in Table 1. Then, for a fixed small value of h, namely using a very refined mesh, we test the convergence in time as shown in Table 2.

Table 1
Errors and convergence rates for problem (23) with $T = 1$ and $\Delta t = 10^{-4}$; dofs stands for degrees of freedom.

dofs	h	$\|\boldsymbol{e}_s\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_s$	$\|\boldsymbol{e}_v\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_v$	$\|\boldsymbol{e}_f\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_f$	$\|\boldsymbol{e}_p\|_{\ell^\infty(L^2)}$	$\mathtt{rate}_p$
1063	2.357e−01	6.742e−03	–	1.363e−01	–	6.614e−02	–	2.536e−03	–
2266	1.571e−01	3.089e−03	1.92	6.210e−02	1.94	3.296e−02	1.72	1.128e−03	2
4570	1.088e−01	1.500e−03	1.96	3.005e−02	1.97	1.662e−02	1.86	4.707e−04	2.38
10 527	7.071e−02	6.428e−04	1.97	1.286e−02	1.97	7.231e−03	1.93	1.819e−04	2.21
23 287	4.714e−02	2.965e−04	1.91	5.939e−03	1.91	3.262e−03	1.96	8.291e−05	1.94

Table 2
Errors convergence rates for problem (23) for a fixed structured mesh with 70 elements per side yielding 124 327 dofs.

Δt	$\|\boldsymbol{e}_s\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_s$	$\|\boldsymbol{e}_v\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_v$	$\|\boldsymbol{e}_f\|_{\ell^\infty(\boldsymbol{H}^1)}$	$\mathtt{rate}_f$	$\|\boldsymbol{e}_p\|_{\ell^\infty(L^2)}$	$\mathtt{rate}_p$
1.000e−03	3.614e−05	–	6.510e−03	–	2.363e−04	–	4.751e−05	–
5.000e−04	1.862e−05	0.957	3.376e−03	0.947	1.214e−04	0.961	2.427e−05	0.969
2.500e−04	9.468e−06	0.976	1.729e−03	0.965	6.171e−05	0.976	1.227e−05	0.984
1.250e−04	4.793e−06	0.982	8.822e−04	0.971	3.148e−05	0.971	6.172e−06	0.991

6. The inf–sup condition

Lemma 3 shows that the existence, uniqueness and stability of the discrete solution depend on the fulfillment of condition $\ker \boldsymbol{B}^T \cap \ker \boldsymbol{C} = \{\boldsymbol{0}\}$, where matrix $\boldsymbol{C}$ is related to the term

$$\left(\frac{(1-\phi)^2}{\kappa_s}\partial_t p_h, q_h\right)$$

and matrix $\boldsymbol{B}$ is related to

$$b((\boldsymbol{v}_f^*, \boldsymbol{w}_s), q) = -(q, \operatorname{div}(\phi \boldsymbol{v}_f^*) + \operatorname{div}((1-\phi)\boldsymbol{w}_s)).$$

As already mentioned in Remark 3.1, the coefficient κ_s is often very large. For this reason, the stability of the numerical scheme hinges, in practice, around the term $b((\boldsymbol{v}_f^*, \boldsymbol{w}_s), q)$. This implies that an inf–sup condition involving the discrete spaces $V_{f,h}$, $V_{s,h}$, $Q_{p,h}$ must be satisfied. The scope of this section is to analyze the inf–sup stability of the bilinear form b. Such form corresponds to a weak divergence operator with weights that depend on the function ϕ, that is the porosity of the material. The main question that we address here is what conditions must be satisfied by the discrete spaces $V_{f,h}$, $V_{s,h}$, given $Q_{p,h}$, in different regimes of porosity, namely when ϕ is approaching the limit cases $\phi \approx 1$ and $\phi \approx 0$ respectively. The practical relevance of this question is confirmed by Fig. 1, where we see that locking appears nearly in absence of the solid or fluid phase ($\phi \approx 1$ and $\phi \approx 0$ respectively).

We divide the work in two parts, first operating at the continuous level we generalize the classical div-stability to weighted Sobolev spaces (with an H^1 weight function) and then use this intermediate result to conclude with the inf–sup stability of the form b reported above. Second, we move at the discrete level where we prove the inf–sup condition for the specific case of the generalized Taylor–Hood-type elements.

6.1. The weighted inf–sup condition

In this section, we study the weighted inf–sup condition for $b(\boldsymbol{v}_f, q) = (q, \operatorname{div}(\omega \boldsymbol{v}_f))$, which is a generalized form of the classical inf–sup condition for the divergence operator. Also, from now on we will consider a general function ω such that $\omega \geq \underline{\omega} > 0$ and both ω, $1/\omega$ belong to $W^{s,r}(\Omega)$ with $s > d/r$. The result at the continuous level requires first a preliminary lemma regarding weighted Sobolev spaces.

Lemma 13. *If ω and $1/\omega$ belong to $W^{s,r}(\Omega)$ with $s > d/r$, then the application $\boldsymbol{v} \to \omega\boldsymbol{v}$ is a bijection in $\boldsymbol{H}^1(\Omega)$ and the following bounds hold:*

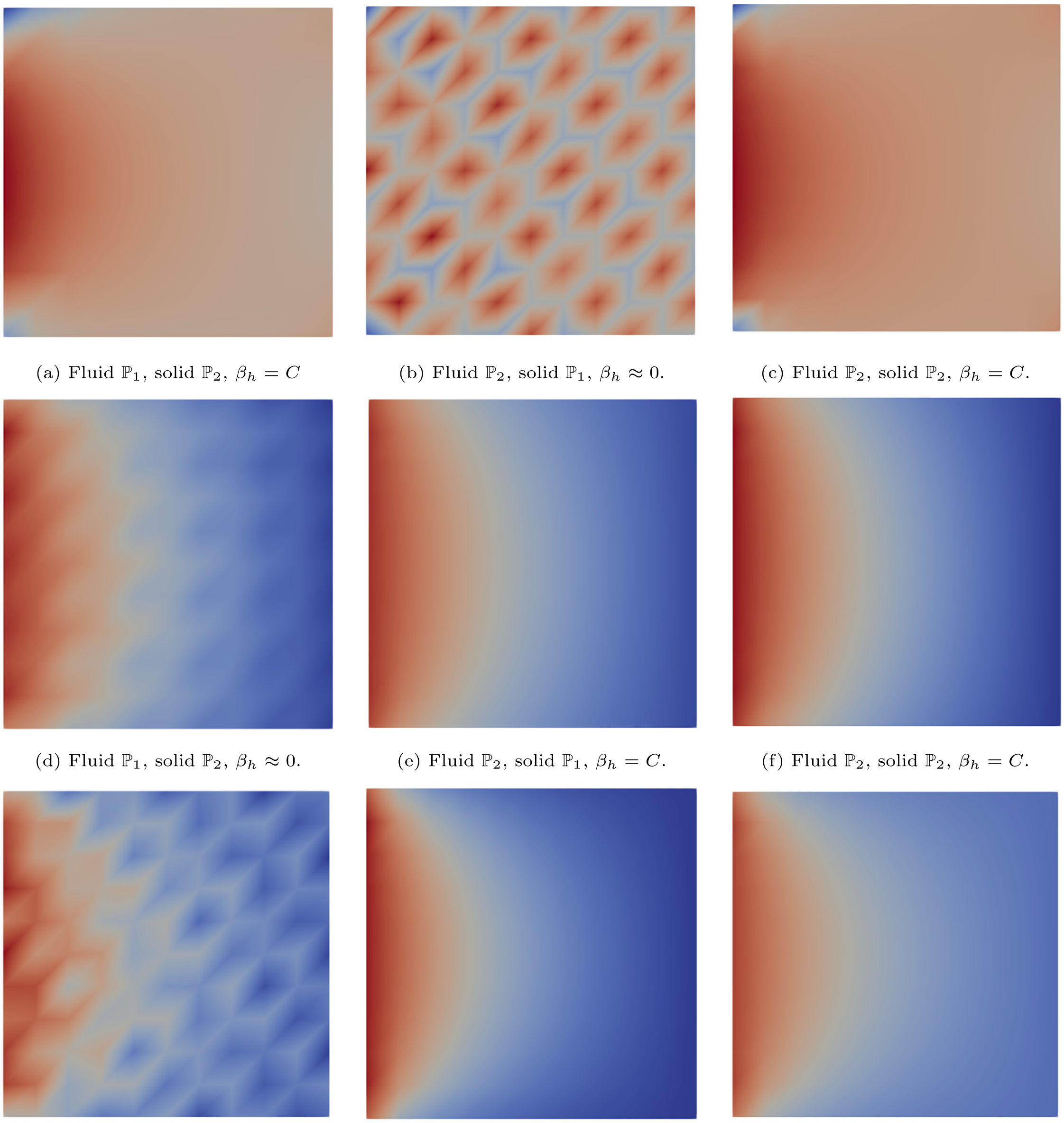
$$\frac{1}{C_{\mathtt{bij}}\|\omega^{-1}\|_{W^{s,r}(\Omega)}}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq \|\omega\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C_{\mathtt{bij}}\|\omega\|_{W^{s,r}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \tag{34}$$

for a positive constant $C_{\mathtt{bij}}$.

Proof. A direct application of the Sobolev product Theorem [29, Theorem 1.4.4.2] gives that both $\omega\boldsymbol{v}$ and $\omega^{-1}\boldsymbol{v}$ belong to $\boldsymbol{H}^1(\Omega)$ and satisfy the inequalities

$$\|\omega\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C_{\mathtt{bij}}\|\omega\|_{W^{s,r}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \qquad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C_{\mathtt{bij}}\|\omega^{-1}\|_{W^{s,r}(\Omega)}\|\omega\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)},$$

for a positive constant $C_{\mathtt{bij}}$, which states the result. □



(a) Fluid $\mathbb{P}_1$, solid $\mathbb{P}_2$, $\beta_h = C$ (b) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_1$, $\beta_h \approx 0$. (c) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_2$, $\beta_h = C$.

(d) Fluid $\mathbb{P}_1$, solid $\mathbb{P}_2$, $\beta_h \approx 0$. (e) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_1$, $\beta_h = C$. (f) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_2$, $\beta_h = C$.

(g) Fluid $\mathbb{P}_1$, solid $\mathbb{P}_2$, $\beta_h \approx 0$. (h) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_1$, $\beta_h = C$. (i) Fluid $\mathbb{P}_2$, solid $\mathbb{P}_2$, $\beta_h = C$.

Fig. 1. Comparison of the pressure in a swelling test at $T = 1.5$. First row on a solid dominant regime ($\phi = 10^{-8}$), second row on a mixed regime ($\phi = 0.5$) and third row on a fluid dominant regime ($\phi = 1 - 10^{-4}$). All tests are performed with $\mathbb{P}_1$ elements for the pressure. See Section 7.1 for a detailed description of the test case.

Remark. The hypothesis ω, $1/\omega$ in $W^{s,r}(\Omega)$ implies that ω is strictly positive.

The weighted inf–sup condition at the continuous level is then a direct consequence of the isomorphism $\omega\boldsymbol{v} \to \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$.

Lemma 14. *There exists a positive constant β which satisfies the following:*

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega q\,\mathrm{div}(\omega\boldsymbol{v})}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta\|q\|_{0,\Omega} \quad \forall q \in L_0^2(\Omega). \tag{35}$$

Proof. Using Lemma 13, we proceed as follows:

$$\sup_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1_0(\Omega),\\ v\neq 0}} \frac{-\int_\Omega q\,\mathrm{div}(\omega \boldsymbol{v})}{\|\boldsymbol{v}\|_{H^1(\Omega)}} = \sup_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1_0(\Omega),\\ v\neq 0}} \frac{-\int_\Omega q\,\mathrm{div}(\boldsymbol{v})}{\|\omega^{-1}\boldsymbol{v}\|_{H^1(\Omega)}} \geq \frac{1}{C_{\mathtt{bij}}\|\omega^{-1}\|_{W^{s,r}(\Omega)}} \sup_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1_0(\Omega),\\ v\neq 0}} \frac{-\int_\Omega q\,\mathrm{div}(\boldsymbol{v})}{\|\boldsymbol{v}\|_{H^1(\Omega)}},$$

which proves the statement. □

Now we address the discrete version of the inf–sup condition, recalling that it is not a consequence of the continuous one even though we are using conforming finite dimensional spaces. Let us define the following spaces:

$$\boldsymbol{V}_k = \boldsymbol{H}^1_0(\Omega)\cap [X_h^k]^d, \quad Q_k = L^2_0(\Omega)\cap X_h^k.$$

Our aim is to extend the proof in [30] and [31], see also [32] for an overview, for the 2D and 3D cases respectively developed for $\omega = 1$ by means of the macroelements technique, where a modified inf–sup condition at the element level will be used together with Verfürth's trick [33] and an inverse estimate to conclude the global statement. We highlight that although we do not address the approximation of ω by means of finite elements, all the forecoming analysis holds as long as its approximation is still in $W^{s,r}(\Omega)$ and strictly positive.

We start with a brief review of the relevant results from the macroelements technique [34]. A macroelement M is defined as a union of continuous elements on the mesh, and for each one of its elements there is an affine map which maps it into an element of a reference macroelement. All macroelements which can be mapped into one particular reference macroelement form an equivalence class. Let $\mathcal{M}_h$ be a macroelement partition of the mesh $\mathcal{T}_h$, which is assumed to be shape regular [27, Chapter 3.1]. For M in $\mathcal{M}_h$ we denote

$$\boldsymbol{V}^M_{k+1,0} = \boldsymbol{V}_{k+1}\cap \boldsymbol{H}^1_0(M), \quad Q^M_k = \{q|_M : q\in Q_k\}, \quad Q^M_{k,\perp} = \left\{ q\in Q^M_k : \int_M q\,\mathrm{div}(\omega \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v}\in \boldsymbol{V}^M_{k+1,0}\right\}.$$

We now focus on proving the following result.

Theorem 7. *Let $\mathcal{M}_h$ be a macroelement partition of the (shape regular) mesh $\mathcal{T}_h$ such that*

(H_M) for each M in $\mathcal{M}_h$, the space $Q^M_{k,\perp}$ is one dimensional given by constant functions.

Then, there exists a positive constant $\beta = \beta(\omega)$ such that:

$$\sup_{\substack{\boldsymbol{v}_h\in \boldsymbol{V}_{k+1}\\ \boldsymbol{v}_h\neq 0}} \frac{\int_\Omega q_h\,\mathrm{div}(\omega \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h\in Q_k. \tag{36}$$

Remark. We have simplified the original theorem by removing some hypotheses regarding the macroelements partition. These hold under the standard assumption of shape regularity of the mesh, so we removed them for the sake of clarity (see [34] for details).

In order to prove this theorem, we need the following lemmas. This first one allows us to extend an inf–sup condition from the macroelement level to the global level.

Lemma 15. *Let $\overline{\Omega} = \overline{\Omega}_1\cup\overline{\Omega}_2$ and for $i = 1, 2$ set $\boldsymbol{V}_k(\Omega_i) = \{\boldsymbol{v}_h \in \boldsymbol{V}_k : \boldsymbol{v}_h = 0 \text{ in } \Omega\setminus\Omega_i\}$. Suppose also that the following conditions hold:*

$$\sup_{\substack{\boldsymbol{v}_i\in \boldsymbol{V}_{k+1}(\Omega_i)\\ \boldsymbol{v}_i\neq 0}} \frac{\int_{\Omega_i} \mathrm{div}(\omega \boldsymbol{v}_i) q_h\,dx}{\|\boldsymbol{v}_i\|_{\boldsymbol{H}^1(\Omega_i)}} \geq \beta_i \|q_h\|_{L^2(\Omega_i)}, \quad \forall q_h\in Q_k,\ i\in\{1,2\}.$$

Then, the following global condition also holds:

$$\sup_{\substack{\boldsymbol{v}_h\in \boldsymbol{V}_{k+1}(\Omega)\\ \boldsymbol{v}_h\neq 0}} \frac{\int_{\Omega} \mathrm{div}(\omega \boldsymbol{v}_h) q_h\,dx}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega}, \quad \forall q_h\in Q_k,$$

where $\beta = 1/\sqrt{2}\min(\beta_1,\beta_2)$. If $\Omega_1\cap\Omega_2 = \emptyset$, then $\beta = \min(\beta_1,\beta_2)$.

Proof. The proof relies only on the bilinearity of the form $(\mathrm{div}(\omega \boldsymbol{v}_h), q_h)$, thus it suffices to proceed verbatim as in [30, Proposition 3.1]. □

We then show that at the macroelement level, the inf–sup condition is satisfied for the space of constants.

Lemma 16. *Let $\mathcal{E}_{\hat{M}}$ be a class of equivalent macroelements and assume that for every M the space $Q_{k,\perp}^M$ satisfies (H_M) of Theorem 7. Then there exists a positive constant $\beta_{\hat{M}}$ which depends only on the reference macroelement and the mesh regularity such that the following inequality holds:*

$$\sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_{k+1,0}^M\\ \boldsymbol{v}_h\neq 0}} \frac{\int_M q_h \operatorname{div}(\omega\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,M}} \geq \beta_{\hat{M}}\|q_h\|_{0,M} \quad \forall q_h \in Q_k^M \cap L_0^2(M).$$

Proof. See [34, Lemma 3.1]. □

We finally show that if functions which are constant at the macroelement are removed from the pressure space, the inf–sup condition holds. Defining $\Pi_0 : L^2(\Omega) \to \{q \in L^2(\Omega) : q|_M \text{ is constant } \forall M \in \mathcal{M}_h\}$ the orthogonal projector with respect to the scalar product of $L^2(\Omega)$ we get the following result.

Lemma 17. *Under hypothesis (H_M) of Theorem 7, there exists a positive constant c such that*

$$\inf_{q_h\in Q_k}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{k+1}} \frac{b(q_h,\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}^1(\Omega)}} \geq c\|(I-\Pi_0)q_h\|_{L^2(\Omega)}. \tag{37}$$

Proof. This is a direct consequence of Lemma 16. See [34, Lemma 3.2]. □

Now we proceed with Verfürth's trick [33]. We generalize it in the following lemma, which requires the definition of the L^2 projector $\Pi_0 : Q_k \to Q_{0,h}$, the last space given by the space of macroelement-wise constants:

$$Q_{0,h} = \{q \in L_0^2(\Omega) : q|_M \text{ is constant } \forall M \in \mathcal{M}_h\}.$$

Lemma 18. *Assume that there exists a linear operator $\Pi_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_k$ such that for every $\boldsymbol{v}$ in $\boldsymbol{H}_0^1(\Omega)$ there is a positive constant c which satisfies*

$$\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_{\boldsymbol{H}^r(\Omega)} \leq c\sum_{K\in\mathcal{T}_h}\left(h_K^{2(1-r)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(K)}^2\right)^{1/2}, \quad r\in\{0,1\}. \tag{38}$$

Then, there exist two positive constants c_1, c_2 such that for every q_h in Q_h the following holds:

$$\sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_{k+1}\\ v_h\neq 0}} \frac{(\operatorname{div}(\omega\boldsymbol{v}_h),q_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)}} \geq \frac{c_1}{\|\omega^{-1}\|_{W^{s,r}(\Omega)}}\|q_h\|_{L^2(\Omega)} - c_2\frac{\|\omega\|_{W^{s,r}(\Omega)}}{\|\omega^{-1}\|_{W^{s,r}(\Omega)}}\|(I-\Pi_0)q_h\|_{L^2(\Omega)}. \tag{39}$$

Proof. We use $\|\Pi_h\boldsymbol{v}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|$ with (38) to bound the inf–sup condition from below:

$$\sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_{k+1}\\ \boldsymbol{v}_h\neq 0}} \frac{(\operatorname{div}(\omega\boldsymbol{v}_h),q_h)}{\|v_h\|_{\boldsymbol{H}^1(\Omega)}} \geq \sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\\ \Pi_h\boldsymbol{v}\neq 0}} \frac{(\operatorname{div}(\omega\Pi_h\boldsymbol{v}),q_h)}{\|\Pi_h\boldsymbol{v}\|_{H^1(\Omega)}} \geq \sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\\ \Pi_h\boldsymbol{v}\neq 0}} \frac{(\operatorname{div}(\omega\Pi_h\boldsymbol{v}),q_h)}{(1+c)\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}}.$$

Then, we define $\boldsymbol{v}$ such that $\operatorname{div}(\omega\boldsymbol{v}) = q_h$, $\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\omega^{-1}\|_{W^{s,r}(\Omega)}\|q_h\|_{L^2(\Omega)}$ and proceed with integration by parts, hypothesis (38) and Cauchy–Schwarz inequality:

$$\begin{aligned}
\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\\ \Pi_h\boldsymbol{v}\neq 0}} \frac{(\operatorname{div}(\omega\Pi_h\boldsymbol{v}),q_h)}{(1+c)\|\boldsymbol{v}\|_{1,\Omega}} &\geq \frac{\|q_h\|_{0,\Omega}}{(1+c)\|\omega^{-1}\|_{W^{s,r}(\Omega)}} + \frac{(\operatorname{div}(\omega[\Pi_h\boldsymbol{v}-\boldsymbol{v}]),q_h)}{C\|\omega^{-1}\|_{W^{s,r}(\Omega)}\|q_h\|_{0,\Omega}}\\
&\geq \frac{\|q_h\|_{0,\Omega}}{(1+c)\|\omega^{-1}\|_{W^{s,r}(\Omega)}} - \frac{(\omega[\Pi_h\boldsymbol{v}-\boldsymbol{v}],\nabla q_h)}{C\|\omega^{-1}\|_{W^{s,r}(\Omega)}\|q_h\|_{0,\Omega}},\\
&\geq \frac{\|q_h\|_{0,\Omega}}{(1+c)\|\omega^{-1}\|_{W^{s,r}(\Omega)}} - \frac{c\|\omega\|_{W^{s,r}(\Omega)}\|\boldsymbol{v}\|_{1,\Omega}\sum_{K\in\mathcal{T}_h}h_K\|\nabla q_h\|_{0,K}}{C\|\omega^{-1}\|_{W^{s,r}(\Omega)}\|q_h\|_{0,\Omega}}\\
&\geq \frac{\|q_h\|_{0,\Omega}}{(1+c)\|\omega^{-1}\|_{0,\Omega}} - \frac{c\|\omega\|_{W^{s,r}(\Omega)}\sum_{K\in\mathcal{T}_h}h_K\|\nabla q_h\|_{0,K}}{C\|\omega^{-1}\|_{W^{s,r}(\Omega)}}.
\end{aligned}$$

We now use the inverse inequality [27, Proposition 6.3.2]:

$$h_K\|\nabla w_h\|_{0,K} \leq C\|w_h\|_{0,K} \quad \forall w_h \in \mathbb{P}_k(K),$$

which, considering $w_h = q_h - \Pi_0 q_h$ and setting $h = \max_K h_K$ gives the desired result. □

Then, we are ready to prove Theorem 7.

Proof of macroelement condition (Theorem 7). We first note that hypothesis (38) holds by considering the interpolation operator in $\boldsymbol{V}_{k+1}$, so we set $\Pi_h = \Pi_{\boldsymbol{V}_{k+1}}$. Then, consider the weak inf–sup from Lemmas 17 and 18. Adding (37) and (39) we obtain

$$\sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_{k+1}\\ \boldsymbol{v}_h \neq 0}} \frac{b(\boldsymbol{v}_h, q)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega}, \text{ with } \beta \geq \frac{c_1 C}{C\|\omega^{-1}\|_{W^{s,r}(\Omega)} + c_2\|\omega\|_{W^{s,r}(\Omega)}}. \quad \square \tag{40}$$

We now extend [30, Theorem 4.1], by proving that the space $Q^M_{k,\perp}$ is one dimensional, as required by condition (H_M) of Theorem 7.

Theorem 8. *Let $\{\mathcal{T}_h\}_h$ be a regular family of triangulations of Ω (as in [27, Section 3.1]), and assume that each one of them contains at least three triangles if $\Omega \subset \mathbb{R}^2$ or that every element has at least one inner vertex if $\Omega \subset \mathbb{R}^3$. Then, for $k \geq 1$ the finite element space $\boldsymbol{V}_{k+1} \times Q_k$ satisfies condition (H_M) of Theorem 7.*

Proof. We treat the two-dimensional and three-dimensional cases independently.

$\Omega \subset \mathbb{R}^2$: In this case the proof is performed as in [30] with minor modifications. We first modify the weight for the Legendre polynomials, which needs to incorporate ω, thus following the notation in the mentioned work we rewrite [30, Equation (4.2)] as

$$\int_{x_A}^0 f(x)\, d\mu_{a,x} = \int_a \omega \lambda^a_{AB}\lambda^a_{AE} f(x)\, dxdy \quad \forall f : [x_A, 0] \to \mathbb{R}.$$

The proof then follows verbatim as the original one, in which we require the strict positivity of ω in [30, Equation (4.7)] to conclude.

$\Omega \subset \mathbb{R}^3$: In this case we see again that the proof requires only the strict positivity of ω in [31, Equation (2.6)]. $\square$

Remark. The 3D proof is simpler but not sharp. In fact, the condition of the inner vertex can be weakened, but a minimal mesh has not been characterized yet as far as we know. The 2D case is instead more technical but it allows for the characterization of a minimal mesh for inf–sup stability.

6.2. The inf–sup condition for the poromechanics problem

In this section we show that the discretization based on Taylor–Hood type finite elements is robust and stable. For this, we write approximation spaces as

$$\boldsymbol{V}^k_{f,h} = \boldsymbol{H}^1_0(\Omega) \cap [X^k_h]^d, \quad \boldsymbol{V}^k_{s,h} = \boldsymbol{H}^1_0(\Omega) \cap [X^k_h]^d,$$
$$\boldsymbol{Q}^k_{v,h} = \boldsymbol{L}^2(\Omega) \cap [X^k_h]^d, \quad Q^k_{p,h} = L^2_0(\Omega) \cap X^k_h.$$

Theorem 9. *Consider ϕ such that $(H\mathbf{1})$ holds, then the bilinear form $b : (\boldsymbol{V}^{k_f}_{f,h} \times \boldsymbol{V}^{k_s}_{s,h} \times \boldsymbol{Q}^{k_v}_{v,h}) \times Q^{k_p}_{p,h} \to \mathbb{R}$ given by*

$$b((\boldsymbol{v}_{f,h}, \boldsymbol{w}_{s,h}, \boldsymbol{v}_{s,h}), q^h) = (q^h, \operatorname{div}(\phi \boldsymbol{v}_{f,h}) + \operatorname{div}((1-\phi)\boldsymbol{w}_{s,h}))$$

satisfies the discrete inf–sup condition for a constant $\beta = \beta(\phi)$ given by

$$\sup_{(\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}) \in \boldsymbol{V}_{k_f} \times \boldsymbol{V}_{k_s}} \frac{b((\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h}), q_h)}{\|(\boldsymbol{v}_{f,h}, \boldsymbol{y}_{s,h})\|_{\boldsymbol{V}_{k_f} \times \boldsymbol{V}_{k_s}}} \geq \beta\, \|q_h\|_{0,\Omega} \qquad \forall q_h \in Q^p_{k_p}, \tag{41}$$

whenever the fluid velocity space or the displacement space are approximated with a degree higher than that of the pressure, i.e $\max\{k_f, k_s\} > k_p \geq 1$, for every $k_v \geq 1$. If both spaces present a higher degree of approximation, i.e $\min\{k_f, k_s\} > k_p$, then the inf–sup condition is uniformly independent of ϕ.

Proof. We consider three cases: div-stability in fluid/pressure, in displacement/pressure and in both fluid and displacement.

- Case $k_f > k_p = k_s$.
  In this case we consider $\boldsymbol{y}_{s,h} = \boldsymbol{v}_{s,h} = 0$ and conclude from Theorem 7 with $\omega = \phi$ and $\beta = \beta(\phi)$ as in (40). Note that $\beta \to 0$ as $\phi \to 0$, and remains otherwise constant.
- Case $k_s > k_p = k_f$.
  In this case we consider $\boldsymbol{v}_{f,h} = \boldsymbol{v}_{s,h} = 0$ and conclude from Theorem 7 with $\omega = 1 - \phi$ and $\beta = \beta(1-\phi)$ as in (40). Note that $\beta \to 0$ as $\phi \to 1$, and remains otherwise constant.
- Case $\min\{k_f, k_s\} > k_p$.
  In this case we consider a function $\boldsymbol{z}_h$ in $V^{k_f}_{f,h} \cap V^{k_s}_{s,h}$ and impose $\boldsymbol{v}_{f,h} = \boldsymbol{w}_{s,h} = \boldsymbol{z}_h$ to arrive at the well-known divergence form which is inf–sup stable, thus giving $\beta \geq C$, with C independent of ϕ. $\square$

6.3. Computation of the inf–sup constant

In this section we study the dependence of the inf–sup constant with respect to the porosity ϕ. The computation of the inf–sup constant of the divergence operator is a difficult task that has been widely studied by the Spectral Theory community. The point of departure is its connection with an eigenvalue problem, initially studied by E. and F. Cosserat [35,36], known as the Cosserat eigenvalue problem. It has been studied for many simple geometries (see [37] and references therein), but an efficient algorithm for computing the inf–sup was only recently developed [38]. We extend this approach to our problem by recasting the computation of the inf–sup constant as a generalized eigenvalue problem and performing numerical experiments. This eigenvalue problem depends on the isomorphism used to map $\boldsymbol{H}_0^1(\Omega)$ into $(\boldsymbol{H}_0^1(\Omega))'$, and as we show, using the isomorphisms induced by the problem better reflects instabilities seen in numerical tests (for example, Fig. 1). We will make use of the following lemma.

Lemma 19. *Let H be a Hilbert space. Then, the spaces $(H \times H)'$ and $H' \times H'$ are isometric (we consider only norms of ℓ^2 type). More explicitly, if τ in $(H \times H)'$ and φ, ψ in H' are such that $\tau = \tau(\varphi, \psi)$, then*

$$\frac{1}{\sqrt{2}}\|(\varphi, \psi)\|_{H'\times H'} \leq \|\tau\|_{(H\times H)'} \leq \|(\varphi, \psi)\|_{H'\times H'}.$$

Proof. Given φ, ψ in H', we consider the linear application $\tau : H' \times H' \to (H \times H)'$ given by $[\tau(\varphi, \psi)](x, y) = \varphi(x) + \psi(y) \quad \forall x, y \in H$. It suffices to show that τ is an isomorphism. First note that

$$|[\tau(\varphi, \psi)](x, y)| \leq \|(\varphi, \psi)\|_{H'\times H'}\|(x, y)\|_{H\times H},$$

thus $\|\tau(\varphi, \psi)\|_{(H\times H)'} \leq \|(\varphi, \psi)\|_{H'\times H'}$. For the inverse inequality we proceed as follows:

$$\|\tau(\varphi, \psi)\|_{(H\times H)'} = \sup_{\substack{x,y\in H\\ x,y\neq 0}} \frac{|[\tau(\varphi, \psi)](x, y)|}{\|(x, y)\|_{H\times H}} \geq \begin{cases} \|\varphi\|_{H'} & \text{if } x \text{ attains the norm of } \varphi, \\ \|\psi\|_{H'} & \text{if } y \text{ attains the norm of } \psi. \end{cases}$$

The last part gives $\|\tau(\varphi, \psi)\|^2_{(H\times H)'} \geq \frac{1}{\sqrt{2}}\|(\varphi, \psi)\|^2_{H'\times H'}$, which concludes the proof. □

We now proceed to construct the eigenvalue problem, for which we consider the spaces $\boldsymbol{H} = \boldsymbol{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$, and two bilinear forms $n_i : \boldsymbol{H} \times \boldsymbol{H} \to \mathbb{R}, i \in \{1, 2\}$ with induced operators N_i such that $\boldsymbol{H}_i := (\boldsymbol{H}, n_i(\cdot, \cdot))$ is a Hilbert space and the norms induced by n_i are equivalent to the norm in $\boldsymbol{H}_0^1(\Omega)$. These operators give the following characterization of the dual norm.

Lemma 20. *In the previous context, for any function φ in $W^{s,r}(\Omega)$ the following equality holds:*

$$\|\varphi\nabla q\|^2_{(\boldsymbol{H}^1(\Omega))'} = -(q, \operatorname{div}\varphi N_i^{-1}\varphi\nabla q)_{0,\Omega}, \quad i \in \{1, 2\}.$$

Proof. We use the Riesz Representation Theorem [28, Theorem 4.11] with the explicit operators N_i, thus obtaining:

$$\begin{aligned}\|\varphi\nabla q\|^2_{(\boldsymbol{H}^1(\Omega))'} &= \|N_i^{-1}\varphi\nabla q\|^2_{(\boldsymbol{H},n_i)} = n_i(N_i^{-1}\varphi\nabla q, N_i^{-1}\varphi\nabla q) \\ &= \langle\varphi \nabla q, N_i^{-1}\varphi \nabla q\rangle_{\boldsymbol{H}^{-1}\times \boldsymbol{H}_0^1} = -(q, \operatorname{div}\varphi N_i^{-1}\varphi \nabla q)_{0,\Omega}. \quad □\end{aligned}$$

Now we are in position to find the eigenvalue problem associated to the inf–sup constant.

Theorem 10. *The problem of finding the inf–sup constant of the bilinear form (36) is equivalent to finding the smallest λ in $\mathbb{R}$, $\boldsymbol{v}, \boldsymbol{y}$ in $\boldsymbol{H}_0^1(\Omega)$ and p in $L_0^2(\Omega)$ such that*

$$\begin{aligned} -N_1\boldsymbol{v} + \phi \nabla p &= 0, \\ \operatorname{div}(\phi\boldsymbol{v} + (1-\phi)\boldsymbol{y}) &= \lambda p, \\ -N_2\boldsymbol{v} + (1-\phi) \nabla p &= 0. \end{aligned} \tag{42}$$

Proof. We first define the operator $T : Q \to (\boldsymbol{H} \times \boldsymbol{H})'$ given by

$$T[q](\boldsymbol{v}, \boldsymbol{y}) = \langle\phi \nabla q, \boldsymbol{v}\rangle_{\boldsymbol{H}'\times \boldsymbol{H}} + \langle(1-\phi) \nabla q, \boldsymbol{y}\rangle_{\boldsymbol{H}'\times \boldsymbol{H}},$$

which thanks to Lemma 19 is defined with the norm

$$\|T[q]\|^2 := \|(\phi \nabla q, (1-\phi) \nabla q)\|^2_{\boldsymbol{H}'\times \boldsymbol{H}'} = \|\phi \nabla q\|^2_{\boldsymbol{H}'} + \|(1-\phi) \nabla q\|^2_{\boldsymbol{H}'}.$$

We then rewrite the inf–sup condition as follows by using Lemma 20:

$$\beta^2 = \inf_{\substack{q\in Q\\ q\neq 0}} \sup_{\substack{\boldsymbol{v},\boldsymbol{y}\in\boldsymbol{H}\\ (\boldsymbol{v},\boldsymbol{y})\neq\boldsymbol{0}}} \left(\frac{(\mathrm{div}(\phi\boldsymbol{v}+(1-\phi)\boldsymbol{y}),q)}{\|(\boldsymbol{v},\boldsymbol{y})\|_{\boldsymbol{H}_1\times\boldsymbol{H}_2}\|q\|_Q}\right)^2 = \inf_{\substack{q\in Q\\ q\neq 0}} \sup_{\substack{\boldsymbol{v},\boldsymbol{y}\in\boldsymbol{H}\\ (\boldsymbol{v},\boldsymbol{y})\neq\boldsymbol{0}}} \left(\frac{-T[q](\boldsymbol{v},\boldsymbol{y})}{\|(\boldsymbol{v},\boldsymbol{y})\|_{\boldsymbol{H}_1\times\boldsymbol{H}_2}\|q\|_Q}\right)^2 = \inf_{\substack{q\in Q\\ q\neq 0}} \frac{\|T[q]\|^2}{\|q\|_Q^2}$$

$$= \inf_{\substack{q\in Q\\ q\neq 0}} \frac{\|\phi\nabla q\|^2_{\boldsymbol{H}'}+\|(1-\phi)\nabla q\|^2_{\boldsymbol{H}'}}{\|q\|_Q^2} = \inf_{\substack{q\in Q\\ q\neq 0}} \frac{-(q,\mathrm{div}(\phi N_1^{-1}\phi\nabla q+(1-\phi)N_2^{-1}(1-\phi)\nabla q))}{\|q\|_Q^2}.$$

Defining the operator $\mathcal{S}(q) := \mathrm{div}(\phi N_1^{-1}\phi\nabla q+(1-\phi)N_2^{-1}(1-\phi)\nabla q)$ and $\lambda := \beta^2$ we prove our claim. □

We present some numerical tests to investigate the dependence of the inf–sup constant on the parameter ϕ in Fig. 2. The experiments were performed with the SciPy library [39], which contains a wrapper for the implicitly restarted Arnoldi method in ARPACK [40]. To avoid rescaling the pressure, on the unit square $\Omega = (0,1)^2$ the experiments were performed with $\boldsymbol{v} = 0$ on $x_1 = 0$ and $\boldsymbol{y} = 0$ on $x_0 = 0$. The dependence on ϕ was then tested for $N_1 = N_2 = \Delta^{-1}$ for an extension of the results regarding the divergence operator, and then to better understand the results on Fig. 1 with $N_1 = (2\,\mathrm{div}\,\mu_f\boldsymbol{\varepsilon}(\cdot))^{-1}$ and $N_2 = (\mathrm{div}\,\mathbb{C}_{\mathrm{Hooke}}\boldsymbol{\varepsilon}(\cdot))^{-1}$, the diffusive operators associated to the fluid and solid momenta, respectively, with two different sets of parameters. The numerical tests confirm that when the operators N_1 and N_2 are the same and equal to the Laplace operator, the stability behavior of the problem with respect to the fluid phase and the solid phase is symmetric. Instead, if the operators N_1 and N_2 are chosen as in the poromechanics problem, then we observe that the stability properties are dominated by the fluid phase. This behavior becomes even more evident when realistic parameters are used, in which case we observe that the stability properties of the chosen spaces $V_{f,h}^{k+1}\times V_{s,h}^{k}\times Q_{v,h}^{k}\times Q_{p,h}^{k}$ are equivalent in practice to those of $V_{f,h}^{k+1}\times V_{s,h}^{k+1}\times Q_{v,h}^{k}\times Q_{p,h}^{k}$. Still, this scenario shows that considering both the fluid velocity and the solid displacement belonging to a finite element space of higher order than the one for the pressure provides a stable approximation. This can be seen in subfigure (f), where the minimum of the $P_2 - P_2$ curve (green, $V_{f,h}^2 \times V_{s,h}^2$) is more than an order of magnitude bigger than that of the fluid-stable regime ($P_2 - P_1$ blue curve, $V_{f,h}^2 \times V_{s,h}^1$). Moreover, we notice that the minimum of the $P_2 - P_2$ curve roughly equals the maximum of the solid-stable regime ($P_1 - P_2$ orange curve, $V_{f,h}^1 \times V_{s,h}^2$).

7. Numerical tests

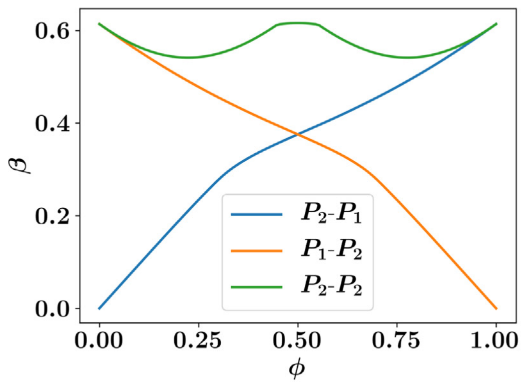
In this section, we present some numerical tests related to problem (2). The first one is a classical benchmark known as the swelling test. The second one shows a spatially dependent porosity which explores the inf–sup stability with respect to the dominant phase (solid or fluid), and the last one is a preliminary result regarding the modeling of blood perfusion in the human left ventricle with an idealized geometry.

7.1. Swelling test

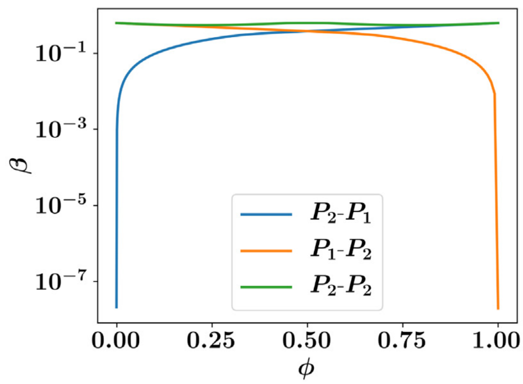
This test studies the behavior of a 2D slab in absence of volume forces. The slab is subject to an external pressure $\sigma_f\boldsymbol{n} = -p_{\mathrm{ext}}\boldsymbol{n}$, $p_{\mathrm{ext}}(t) = 10^3(1-\exp(4t^2))$ on the left and null stress on the right. Above and below it uses a no-slip boundary condition $\boldsymbol{v}_f = \boldsymbol{v}_s$, which we impose weakly with a constant $\gamma = 2\,10^5$ (more details in [19]). The boundary conditions for the solid are: sliding on the bottom and left sides, the external pressure also acts on the solid phase through $\sigma_s\boldsymbol{n} = -p_{\mathrm{ext}}\boldsymbol{n}$ on the left and the rest of the boundary is of null traction type (see Fig. 3). The results are obtained with the following parameters: $\rho_f = \rho_s = 1000$, $\mu_f = 0.035$, $\lambda_s = 711$, $\mu_s = 4066$, $\kappa_s = 2\cdot 10^8$, $\partial\Omega = 2\cdot 10^5$, $\boldsymbol{\kappa}_f^{-1} = 10^7\boldsymbol{I}$, all in SI units with $|\Omega| = 10^{-4}$ discretized with 12 elements per side. The finite element spaces used are $V_{f,h}^2$, $Q_{p,h}^1$, $V_{s,h}^2$, $Q_{v,h}^1$ for the fluid velocity, pressure, displacement and solid velocity respectively.

7.2. Inf–sup stability test

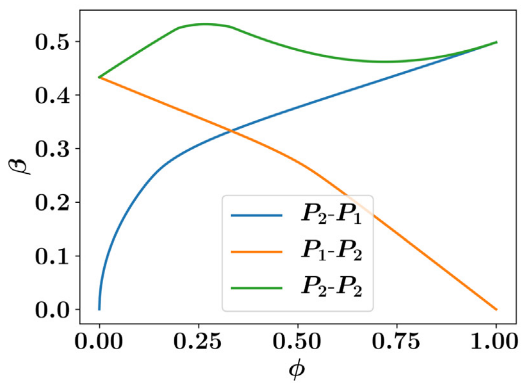
This test shows how the poromechanics problem can exhibit different stability behaviors in the same domain. We use a setting similar to the swelling test, the differences being **(i)** in the fluid, in which we impose a quadratic flow with a peak value of 0.01 on the left instead of a Neumann condition; **(ii)** in the parameters: $\lambda = \mu = 0.035$, $\boldsymbol{\kappa}_f^{-1} = 10^4\boldsymbol{I}$; and **(iii)** in the porosity function, given by $\phi \approx I_{\{y\leq 0.5\}}$ (not exactly as it must be strictly contained in [0, 1]). In Fig. 4 we show the pressure field, which is unstable only when the corresponding phase is not discretized appropriately. In Fig. 4(a) fluid and displacement are discretized with $\mathbb{P}_1$ elements (same as pressure), thus both regions show unstable behavior. In Fig. 4(b), only the fluid is unstable and thus we see instabilities where the fluid is dominant (below). Fig. 4(c) is the opposite of 4(b), and as expected when both physics are approximated with $\mathbb{P}_2$ elements we see stable pressure (Fig. 4(d)).



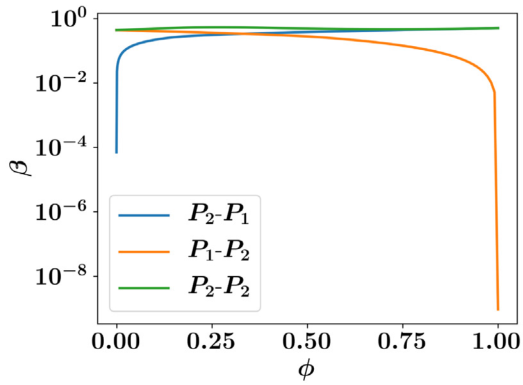
(a) $N_1 = N_2 = \Delta^{-1}$.



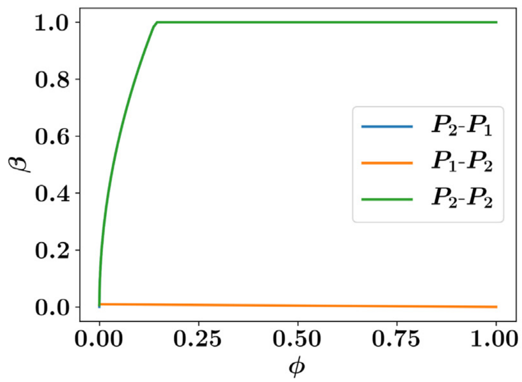
(b) $N_1 = N_2 = \Delta^{-1}$, semilog y axis.



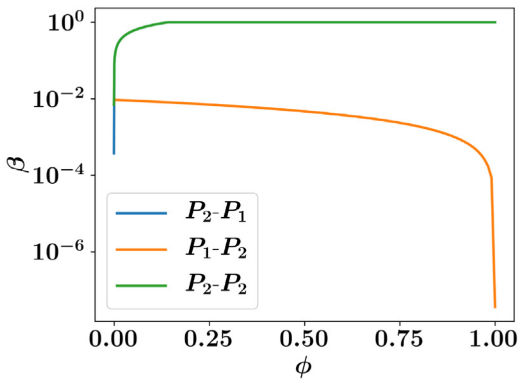
(c) $N_1 = (2\mu_f \operatorname{div} \boldsymbol{\varepsilon}(\cdot))^{-1}, N_2 = (\operatorname{div} \mathbb{C}_{\text{Hooke}}\boldsymbol{\varepsilon}(\cdot))^{-1}$.



(d) $N_1 = (2\mu_f \operatorname{div} \boldsymbol{\varepsilon}(\cdot))^{-1}, N_2 = (\operatorname{div} \mathbb{C}_{\text{Hooke}}\boldsymbol{\varepsilon}(\cdot))^{-1}$, semilog y axis.



(e) $N_1 = (2\mu_f \operatorname{div} \boldsymbol{\varepsilon}(\cdot))^{-1}, N_2 = (\operatorname{div} \mathbb{C}_{\text{Hooke}}\boldsymbol{\varepsilon}(\cdot))^{-1}$, physical parameters.

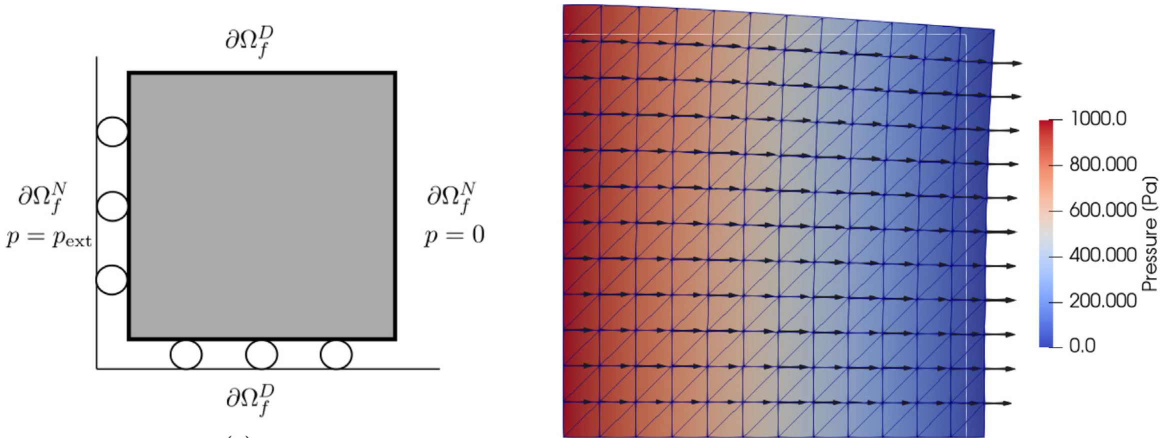


(f) $N_1 = (2\mu_f \operatorname{div} \boldsymbol{\varepsilon}(\cdot))^{-1}, N_2 = (\operatorname{div} \mathbb{C}_{\text{Hooke}}\boldsymbol{\varepsilon}(\cdot))^{-1}$, semilog y axis, physical parameters.

Fig. 2. Inf–sup constant β with respect to the porosity. Images (a), (b), (c) and (d) have all parameters set to 1, instead (e) and (f) use a realistic parameters. The code $P_a - P_b$ on the plots stands for a fluid–solid-pressure discretization with elements $\mathbb{P}_a - \mathbb{P}_b - \mathbb{P}_1$.

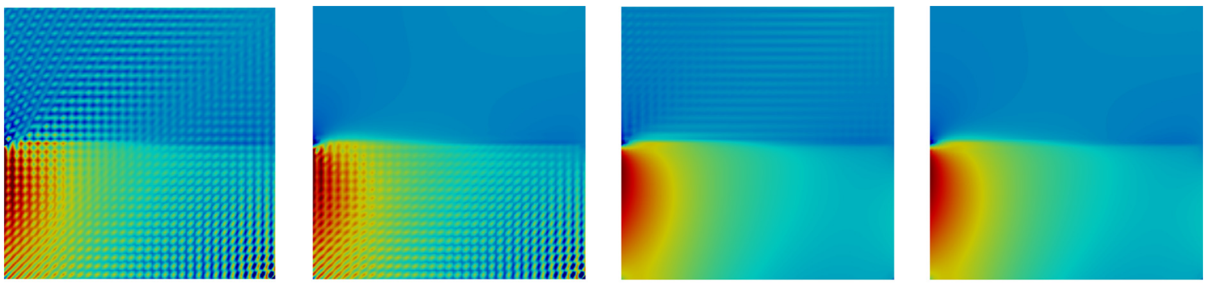
8. Conclusions

In this work we presented a complete mathematical and numerical analysis of the linearized poromechanics problem first addressed in [20]. For the well-posedness analysis we have combined the theory of Differential Algebraic Equations with the Faedo–Galerkin technique. We remark that the analysis presented here features a relaxation of the constant porosity condition used in [20].



(a) (b)

Fig. 3. (a) Boundary conditions for the swelling test, (b) results at time $t = 1$.



(a) $\mathbb{P}_1 \times \mathbb{P}_1$ (b) $\mathbb{P}_1 \times \mathbb{P}_2$ (c) $\mathbb{P}_2 \times \mathbb{P}_1$ (d) $\mathbb{P}_2 \times \mathbb{P}_2$

Fig. 4. Pressure of inf–sup test for all combinations of fluid/displacement finite element spaces.

We have discretized the problem with the backward Euler scheme in time and Taylor–Hood-type finite elements in space which require fluid velocity, displacement and pressure to be approximated by $\mathbb{P}_{k+1} - \mathbb{P}_{k+1} - \mathbb{P}_k$, thus leaving solid velocity unconstrained. The pressure and the velocities of the fluid and solid phases are coupled by a quasi-incompressibility constraint that has been thoroughly analyzed, shedding light on properties of the model that were not completely understood yet. In particular, we show that equal order approximation of the previous variables is not stable. Only the $k+1/k$-th order approximation of velocities and pressure is always stable in practice. Interestingly, our analysis shows that, depending on the porosity, the approximation of the fluid or solid velocities can be selectively degraded to the polynomial order used for the pressure. These findings are confirmed by the numerical tests which complement the ones previously performed in [20] with this model.

Appendix A. Saddle point problems

In this appendix we present some well-known results regarding saddle point problems which we use throughout the analysis. The first one is a discrete invertibility result, for reference see [41].

Theorem 11. *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices such that $\boldsymbol{A}$ is positive definite, $\boldsymbol{C}$ is positive semidefinite and* $\ker \boldsymbol{C} \cap \ker \boldsymbol{B}^T = \{\boldsymbol{0}\}$. *Then, the matrix $\boldsymbol{M}$ defined as*

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & -\boldsymbol{C} \end{bmatrix}$$

is invertible.

The next one is a generalization of the Ladyzhenskaya–Babuška–Brezzi condition, which we adapt from [42].

Theorem 12. *Consider bilinear continuous forms $\mathcal{A}: V \times V \to \mathbb{R}$, $\mathcal{B}_1 : V \times Q \to \mathbb{R}$ and $\mathcal{B}_2 : V \times Q \to \mathbb{R}$ for Hilbert spaces V, Q. Under the following hypotheses:*

$$\sup_{v\in V} \frac{\mathcal{A}(v, v^*)}{\|v\|_V} \geq \alpha \|v^*\|_V \quad \forall v \in V, \quad \sup_{v\in V} \mathcal{A}(v, v^*) > 0 \quad \forall v^* \in V,$$
$$\sup_{v\in V} \frac{\mathcal{B}_1(v, q)}{\|v\|_V} \geq \beta_1 \|q\|_Q \quad \forall q \in Q, \quad \sup_{v\in V} \frac{\mathcal{B}_2(v, q)}{\|v\|_V} \geq \beta_2 \|q\|_Q \quad forallq \in Q,$$

the following problem has a unique solution: Find (u, p) in $V \times Q$ such that

$$\begin{aligned} \mathcal{A}(u, v) + \mathcal{B}_1(v, p) &= \mathcal{F}(v) \quad \forall v \in V, \\ \mathcal{B}_2(u, q) &= \mathcal{G}(q) \quad \forall q \in Q. \end{aligned} \tag{A.1}$$

Assuming that (u_h, p_h) is the solution to a conforming Galerkin scheme in spaces $V_h \times Q_h$, then the following convergence estimate holds for constants C_1, C_2, C_3, C_4 depending on the ellipticity constant of $\mathcal{A}$, the inf–sup constants of $\mathcal{B}_1, \mathcal{B}_2$ and the continuity constants of all bilinear forms:

$$\|u - u_h\|_V \leq C_1 \inf_{v_h\in V_h} \|u - v_h\|_V + C_2 \inf_{q_h\in Q_h} \|p - q_h\|_Q,$$
$$\|p - p_h\|_Q \leq C_3 \inf_{v_h\in V_h} \|u - v_h\|_V + C_4 \inf_{q_h\in Q_h} \|p - q_h\|_Q.$$

Appendix B. Limit of bilinear forms

In this appendix we present the limits of all bilinear forms used for the Faedo–Galerkin technique. Consider the test functions φ in $C_c^\infty(0, T)$ (compactly supported functions in $(0, T)$), $\boldsymbol{v}^*_{f,h}$ in $V_{f,h}$, q_h in $Q_{p,h}$, $\boldsymbol{w}_{s,h}$ in $V_{s,h}$ and $\boldsymbol{v}^*_{s,h}$ in $Q_{v,h}$. With them, we use the weak convergence results from Theorem 3 and extract convergent subsequences as in (20) to proceed as follows:

(i) Limit of the fluid equation terms:

$$\int_0^T (\phi\partial_t \boldsymbol{v}_{f,h}, \varphi(t)\boldsymbol{v}^*_{f,h})\, dt \to \int_0^T (\phi\partial_t \boldsymbol{v}_f(t), \varphi(t)\boldsymbol{v}^*_{f,h})\, dt \quad \text{as } \partial_t \boldsymbol{v}_{f,h} \text{ converges in } L^2(0, T; (\boldsymbol{H}^1_0(\Omega))'),$$
$$\int_0^T (\boldsymbol{\sigma}_{\texttt{vis}(\boldsymbol{v}_{f,h}(t))}, \varphi(t)\boldsymbol{\varepsilon}(\boldsymbol{v}^*_{f,h}))\, dt \to \int_0^T (\boldsymbol{\sigma}_{\texttt{vis}(\boldsymbol{v}_f(t))}, \varphi(t)\boldsymbol{\varepsilon}(\boldsymbol{v}^*_{f,h}))\, dt \quad \text{as } \boldsymbol{v}_{f,h} \text{ converges in } L^2(0, T; \boldsymbol{H}^1_0(\Omega)),$$
$$\int_0^T (\phi^2[\boldsymbol{\kappa}_f^{-1} - \theta \boldsymbol{I}]\boldsymbol{v}_{f,h}(t), \varphi(t)\boldsymbol{v}^*_{f,h})\, dt \to \int_0^T (\phi^2[\boldsymbol{\kappa}_f^{-1} - \theta \boldsymbol{I}]\boldsymbol{v}_f(t), \varphi(t)\boldsymbol{v}^*_{f,h})\, dt \quad \text{as } \boldsymbol{v}_{f,h} \text{ converges in } L^2(0, T; \boldsymbol{L}^2(\Omega)),$$
$$\int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_{s,h}(t), \varphi(t)\boldsymbol{v}^*_{f,h}) \to \int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_s(t), \varphi(t)\boldsymbol{v}^*_{f,h})\, dt \quad \text{as } \boldsymbol{v}_{s,h} \text{ converges in } L^2(0, T; \boldsymbol{L}^2(\Omega)),$$

(ii) Limit of the mass conservation terms, understood in integral form $\frac{(1-\phi)^2}{\kappa_s} p(t) = \frac{(1-\phi)^2}{\kappa_s} \Pi_{Q_{p,h}} p(0) + \rho_f^{-1}\theta - \int_0^t \mathrm{div}(\phi \boldsymbol{v}_f(s))\, ds + \mathrm{div}((1-\phi)[\boldsymbol{y}_s(t) - \Pi_{V_{s,h}}\boldsymbol{y}_s(0)])$:

$$\int_0^T \left(\frac{(1-\phi)^2}{\kappa_s} p_h, \varphi(t)q_h\right)\, dt \to \int_0^T \left(\frac{(1-\phi)^2}{\kappa_s} p, \varphi(t)q_h\right)\, dt \quad \text{as } p_h \text{ converges in } L^2(0, T; L^2(\Omega)),$$
$$\int_0^T \int_0^t (\mathrm{div}(\phi \boldsymbol{v}_{f,h}(s)), \varphi(t)q_h)\, ds\, dt \to \int_0^T \int_0^t (\mathrm{div}(\phi \boldsymbol{v}_f(s)), \varphi(t)q_h)\, ds\, dt \quad \text{as } \boldsymbol{v}_{f,h} \text{ converges in } L^2(0, T; \boldsymbol{H}^1_0(\Omega)),$$
$$\int_0^T (\mathrm{div}((1-\phi)\boldsymbol{y}_{s,h}(t)), \varphi(t)q_h)\, dt \to \int_0^T (\mathrm{div}((1-\phi)\boldsymbol{y}_s(t)), \varphi(t)q_h)\, dt \quad \text{as } \boldsymbol{y}_{s,h} \text{ converges in } L^2(0, T; \boldsymbol{H}^1_0(\Omega)),$$

where for the second term there is an extra intermediate step. We define the functional

$$F_{q_h(t)}(\boldsymbol{v}_{f,h}) = \int_0^t (\mathrm{div}(\phi \boldsymbol{v}_{f,h}(s)), q_h(t))\, ds,$$

which is bounded by using hypothesis $(H\mathbf{1})$ and Cauchy–Schwartz:

$$\begin{aligned} F_{q_h(t)}(\boldsymbol{v}_{f,h}) &\leq C_\phi \int_0^T \|\boldsymbol{v}_{f,h}\|_{\boldsymbol{H}^1(\Omega)} \|q_h(t)\|_{L^2(\Omega)}\, ds \\ &\leq C_\phi \|\boldsymbol{v}_{f,h}\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \|q_h(t)\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

The result is an application of weak convergence to the functional $F_{q_h(t)}$.

(iii) Limit of the solid equation terms:

$$\int_0^T ((1-\phi)\partial_t \boldsymbol{v}_{s,h}(t), \varphi(t)\boldsymbol{w}_{s,h})\,dt \to \int_0^T ((1-\phi)\partial_t \boldsymbol{v}_s(t), \varphi(t)\boldsymbol{w}_{s,h})\,dt \quad \text{as } \partial_t \boldsymbol{v}_{s,h} \text{ converges in } L^2(0,T;(\boldsymbol{H}_0^1(\Omega))'),$$

$$\int_0^T (\mathbb{C}_{\mathtt{Hooke}}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}(t)), \varphi(t)\boldsymbol{w}_{s,h})\,dt \to \int_0^T (\mathbb{C}_{\mathtt{Hooke}}\boldsymbol{\varepsilon}(\boldsymbol{y}_s(t)), \varphi(t)\boldsymbol{w}_{s,h})\,dt \quad \text{as } \boldsymbol{y}_{s,h} \text{ converges in } L^2(0,T;\boldsymbol{H}_0^1(\Omega)),$$

$$\int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_{s,h}(t), \varphi(t)\boldsymbol{w}_{s,h}) \to \int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_s(t), \varphi(t)\boldsymbol{w}_{s,h})\,dt \quad \text{as } \boldsymbol{v}_{s,h} \text{ converges in } L^2(0,T;\boldsymbol{L}^2(\Omega)),$$

$$\int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_{f,h}(t), \varphi(t)\boldsymbol{w}_{s,h}) \to \int_0^T (\phi^2\boldsymbol{\kappa}_f^{-1}\boldsymbol{v}_f(t), \varphi(t)\boldsymbol{w}_{s,h})\,dt \quad \text{as } \boldsymbol{v}_{f,h} \text{ converges in } L^2(0,T;\boldsymbol{H}_0^1(\Omega)).$$

(iv) Limit of the solid velocity terms:

$$\int_0^T ((1-\phi)\partial_t\boldsymbol{y}_{s,h}(t), \varphi(t)\boldsymbol{v}_{s,h})\,dt \to \int_0^T ((1-\phi)\partial_t\boldsymbol{y}_s(t), \varphi(t)\boldsymbol{v}_{s,h})\,dt \quad \text{as } \partial_t\boldsymbol{y}_{s,h} \text{ converges in } L^2(0,T;\boldsymbol{L}^2(\Omega)),$$

$$\int_0^T ((1-\phi)\boldsymbol{v}_{s,h}(t), \varphi(t)\boldsymbol{v}_{s,h})\,dt \to \int_0^T ((1-\phi)\boldsymbol{v}_s(t), \varphi(t)\boldsymbol{v}_{s,h})\,dt \quad \text{as } \boldsymbol{v}_{s,h} \text{ converges in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

Finally, all time integrals in $(0,T)$ can be removed due to the fact that φ belongs to $C_c^\infty(0,T)$ and that $C_c^\infty(0,T)\otimes X$ is dense in $L^2(0,T;X)$ [27].

References

[1] M.A. Biot, General theory of three-dimensional consolidation, J. Appl. Phys. 12 (2) (1941) 155–164.
[2] M.A. Biot, Theory of elasticity and consolidation for a porous anisotropic solid, J. Appl. Phys. 26 (2) (1955) 182–185.
[3] M.A. Biot, General solutions of the equations of elasticity and consolidation for a porous material, J. Appl. Mech. 23 (1) (1956) 91–96.
[4] K. Terzaghi, Theoretical Soil Mechanics, John Wiley, New York, 1948.
[5] R. deBoer, Trends in Continuum Mechanics of Porous Media, Vol. 18, Springer Science & Business Media, 2005.
[6] M. Yang, L.A. Taber, The possible role of poroelasticity in the apparent viscoelastic behavior of passive cardiac muscle, J. Biomech. 24 (7) (1991) 587–597.
[7] L. Taber, P. Rew, Poroelastic plate and shell theories, Mech. Poroelast. Media (1996) 323–337.
[8] A. Goriely, M.G.D. Geers, G.A. Holzapfel, J. Jayamohan, A. Jérusalem, S. Sivaloganathan, W. Squier, J.A.W. van Dommelen, S. Waters, E. Kuhl, Mechanics of the brain: Perspectives, challenges, and opportunities, Biomech. Model. Mechanobiol. 14 (5) (2015) 931–965.
[9] M.P. Nash, P.J. Hunter, Computational mechanics of the heart. From tissue structure to ventricular function, J. Elasticity 61 (1–3) (2000) 113–141.
[10] R. Chabiniok, V.Y. Wang, M. Hadjicharalambous, L. Asner, J. Lee, M. Sermesant, E. Kuhl, A.A. Young, P. Moireau, M.P. Nash, D. Chapelle, D.A. Nordsletten, Multiphysics and multiscale modelling, data–model fusion and integration of organ physiology in the clinic: Ventricular cardiac mechanics, Interface Focus 6 (2) (2016).
[11] D. Chapelle, J.-F. Gerbeau, J. Sainte-Marie, I.E. Vignon-Clementel, A poroelastic model valid in large strains with applications to perfusion in cardiac modeling, Comput. Mech. 46 (1) (2010) 91–101.
[12] S. Badia, A. Quaini, A. Quarteroni, Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction, J. Comput. Phys. 228 (21) (2009) 7986–8014.
[13] M. Bukač, I. Yotov, R. Zakerzadeh, P. Zunino, Effects of poroelasticity on fluid-structure interaction in arteries: A computational sensitivity study, Model. Simul. Appl. 14 (2015) 197–220.
[14] M.H. Armstrong, A. Buganza Tepole, E. Kuhl, B.R. Simon, J.P. Vande Geest, A finite element model for mixed porohyperelasticity with transport, swelling, and growth, PLoS One 11 (4) (2016).
[15] R. Zakerzadeh, P. Zunino, A computational framework for fluid–porous structure interaction with large structural deformation, Meccanica 54 (1–2) (2019) 101–121.
[16] D. Chapelle, P. Moireau, General coupling of porous flows and hyperelastic formulations - From thermodynamics principles to energy balance and compatible time schemes, Eur. J. Mech. B 46 (2014) 82–96.
[17] A.-T. Vuong, L. Yoshihara, W.A. Wall, A general approach for modeling interacting flow through porous media under finite deformations, Comput. Methods Appl. Mech. Engrg. 283 (2015) 1240–1259.
[18] O. Coussy, Poromechanics, John Wiley & Sons, 2004.
[19] B. Burtschell, D. Chapelle, P. Moireau, Effective and energy-preserving time discretization for a general nonlinear poromechanical formulation, Comput. Struct. 182 (2017) 313–324.
[20] B. Burtschell, P. Moireau, D. Chapelle, Numerical analysis for an energy-stable total discretization of a poromechanics model with inf-sup stability, Acta Math. Appl. Sin. 35 (1) (2019) 28–53.
[21] C. Taylor, P. Hood, A numerical solution of the Navier–Stokes equations using the finite element technique, Comput. & Fluids 1 (1) (1973) 73–100.
[22] S. Campbell, Singular Systems of Differential Equations, Vol. 1, Pitman London, 1980.
[23] I. Ambartsumyan, E. Khattatov, I. Yotov, P. Zunino, A Lagrange multiplier method for a Stokes–Biot fluid–poroelastic structure interaction model, Numer. Math. 140 (2) (2018) 513–553.
[24] S. Brenner, R. Scott, The Mathematical Theory of Finite Element Methods, Texts in Applied Mathematics, Springer New York, 2007.
[25] J. Hadamard, Sur les problèmes aux dérivées partielles et leur signification physique, Princeton Univ. Bull. (1902) 49–52.
[26] L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, American Mathematical Society, 1998.
[27] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Vol. 23, Springer Science & Business Media, 2008.
[28] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science & Business Media, 2010.
[29] P. Grisvard, Elliptic Problems in Nonsmooth Domains, SIAM, 2011.
[30] D. Boffi, Stability of higher order triangular Hood–Taylor methods for the stationary Stokes equations, Math. Models Methods Appl. Sci. 4 (02) (1994) 223–235.

[31] D. Boffi, Three-dimensional finite element methods for the Stokes problem, SIAM J. Numer. Anal. 34 (2) (1997) 664–670.
[32] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, Springer Berlin Heidelberg, 2013.
[33] R. Verfürth, Error estimates for a mixed finite element approximation of the Stokes equations, RAIRO Anal. Numér. 18 (2) (1984) 175–182.
[34] R. Stenberg, Analysis of mixed finite element methods for the Stokes problem: A unified approach, Math. Comp. 42 (165) (1984) 9–23.
[35] E. Cosserat, F. Cosserat, Sur la deformation infiniment petite d'un ellipsoïde elastique, Gauthier-Villars, 1898.
[36] E. Cosserat, F. Cosserat, Sur les equations de la theorie de l'elasticité, 1898.
[37] M. Costabel, M. Crouzeix, M. Dauge, Y. Lafranche, The inf-sup constant for the divergence on corner domains, Numer. Methods Partial Differential Equations 31 (2) (2015) 439–458.
[38] D. Gallistl, Rayleigh–Ritz approximation of the inf-sup constant for the divergence, Math. Comp. 88 (315) (2018) 73–89.
[39] E. Jones, T. Oliphant, P. Peterson, et al., SciPy: Open source scientific tools for Python, 2001.
[40] R. Lehoucq, D. Sorensen, C. Yang, ARPACK Users' Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods, Vol. 6, SIAM, 1998.
[41] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, Acta Numer. 14 (2005) 1–137.
[42] C. Bernardi, C. Canuto, Y. Maday, Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem, SIAM J. Numer. Anal. 25 (6) (1988) 1237–1271.