We Don't Know We Don't Know: Asserting Ignorance

Abstract

The pragmatic logic of assertions shows a connection between ignorance and (informal) decidability. In it, we can express pragmatic factual ignorance and first-order ignorance as well as some of their variants. We also show how some pragmatic versions of second-order ignorance and of Rumsfeld-ignorance may be formulated. A specific variant of second-order ignorance is particularly relevant. This indicates a strong pragmatic version of ignorance of ignorance, irreducible to any previous form of ignorance, which defines limits to what can justifiably be asserted about higher-order ignorance. Finally, we relate the justified assertion of second-order ignorance (that cannot be known) with scientific assertions.

1 Introduction

Consider the following extract taken from a speech given by the former US Defense Secretary Donald Rumsfeld:

> Reports that say that something hasn't happened are always interesting to me, because as we know, there are *known knowns*; there are things we know we know. We also know there are *known unknowns*; that is to say we know there are some things we do not know. But there are also *unknown unknowns*—the ones we don't know we don't know [17].

In 2013, Rumsfeld was awarded with the Foot in Mouth Award for the most nonsensical remark made by a public figure by the British Plain English Campaign. The spokesman of the campaign ironically said: "We think we know what he means. But we don't know if we really know" [2].

Far from being nonsensical, Rumsfeld's statements organize knowledge and uncertainty into categories. In this paper we will focus on the class of what Rumsfeld called *unknown unknowns*, and which in the literature is usually termed fundamental or severe uncertainty[1].

[1]This is the sense of uncertainty as used for instance by Keynes: "By 'uncertain' knowledge I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty [...] The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years [...] About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know". ([14], pp. 113–114). Uncertainty means incompleteness of knowledge or information, while ignorance is assumed to be the total absence of knowledge.

Nor should the term "severe uncertainty" be confounded with probabilistic notions of risk, which can be characterised as the product of the probability of an event and the severity of its consequences ([15]). In the Rumsfeld taxonomy, these are *known unknowns*. Severe uncertainty, in contrast, exemplifies cases of uncertainty in which it is practically impossible to come up with a meaningful probabilistic risk assessment of an event. Such cases could refer to events unfolding in deep time, or loss of predictability in complex systems. When statistical assessments of an occurrence of an event as well as the identity of the event itself are both unknown, we are ignorant of the event.

Different forms and levels of such ignorance are possible. In this paper we will focus on Rumsfeld-ignorance and second-order ignorance. Although neither can consistently be known (on this, see [9]), what we will do in this paper is to show that it is nevertheless possible to justifiably assert these forms of ignorance. In order to show this, we investigate, first, the two forms of ignorance in the pragmatic framework of the logic of assertions, which is then extended with an epistemic modality.

2 Forms of Ignorance

Logical analysis reveals that various levels of ignorance are available. Kit Fine's [9] proposal classifies forms of ignorance that include higher-order ignorance:

(1) One is *ignorant that* α: $\neg K\alpha$

(2) It is *epistemically possible* that α: $\neg K\neg\alpha$

(3) One is *ignorant of the fact that* α: $\alpha \wedge \neg K\alpha$

(4) One is first-order ignorant *whether* α: $I\alpha = \neg K\alpha \wedge \neg K\neg\alpha$[2]

(5) One is *Rumsfeld-ignorant* of α: $I\alpha \wedge \neg K(I\alpha)$

(6) One is *second-order ignorant* whether α: $II\alpha$.

Statements (1) and (2) are standard statements: formula in (1) can be read as "It is not known that α" and formula in (2) as "For all that is known, α is possible". Formula in (2) is thus the dual of $K\alpha$. The statement (3) represents the Hintikka–Moore sentences, the knowledge of which was shown to lack any model in [13]. Such formulas have also been instantiated in the Church–Fitch knowability paradox, in order to show that the formula in (3) cannot be known ([11], [16]). In Fine's presentation, this was interpreted to mean that factual ignorance cannot be known.

Fine then focuses on higher-order forms of ignorance. Assuming the modal system $\mathsf{S4}$, he shows that:

(i) The formula in (6) implies the formula in (4), i.e. second-order ignorance implies first-order ignorance. For assume that you are not first-order ignorant: $\neg(\neg K\alpha \wedge \neg K\neg\alpha)$. This means that $K\alpha \vee K\neg\alpha$. Let us now assume $K\alpha$. In $\mathsf{S4}$, we derive $KK\alpha$ from $K\alpha$. Thus one knows that one is not first-order ignorant

[2] On this, see [18].

(Lemma 3, [9]). Similarly for $K\neg\alpha$.

(ii) The formula in (6) implies the formula in (5), i.e. second-order ignorance implies Rumsfeld-ignorance. This is because by (i) we have that second-order ignorance implies first-order ignorance, and "ignorance *whether* one is ignorant whether α implies ignorance *that* one is ignorant whether α" [9].

(iii) The formula in (5) implies the formula in (4), i.e. Rumsfeld-ignorance implies second-order ignorance $I\alpha \wedge \neg K(I\alpha)$. For assume that one is Rumsfeld-ignorant. By definition, one does not know $I\alpha$. Since one is first-order ignorant, then one does not know that one is not first-order ignorant.

(iv) Rumsfeld-ignorance cannot be known. Now assume that one knows to be Rumsfeld-ignorant. This means that $K(I\alpha \wedge \neg K(I\alpha))$. From the former it follows that $K(I\alpha)$ and $K\neg K(I\alpha)$. By factivity of knowledge, we get $\neg K(I\alpha)$. This is a contradiction.

(v) Second-order ignorance cannot be known. Fine argues that if one knew one were second-order ignorant of whether α, one would know that one was Rumsfeld-ignorant of α by (ii), which contradicts (iv). Statement (v) is a theorem of S4 stating a logical impossibility of coming to know second-order ignorance.

So in addition to the factual ignorance of the Hintikka–Moore sentences or the knowability paradox, Rumsfeld-ignorance and second-order ignorance cannot be known. In what follows we want to point out that two forms of ignorance in (iv) and (v) can nevertheless be consistently and justifiably asserted. In other words, we can study propositions that express various forms of ignorance that may be justifiably asserted even though they cannot be known. For instance, assertions that scientists put forth tend to concern various levels of ignorance and be ones that do not become or need not become known, not even in the long historical perspective, but nevertheless are, or ought to be, ones that may be consistently and justifiably asserted.

In order to provide a formal analysis of how asserting ignorance works we will rely on Logic for Pragmatics (also known as Pragmatic Logic) ([8], $\mathcal{L}^P$ for short), which is a logical system developed for the purpose of a pragmatic analysis of acts of assertion and their extensions.

3 Pragmatic logic for assertions

In their logical system named Logic for Pragmatics ($\mathcal{L}^P$), Dalla Pozza & Garola ([8]) proposed a framework for an analysis of assertions by introducing connectives that work according to pragmatic rules of interpretation. The proposal contained a pragmatic interpretation of intuitionistic propositional logic in terms of an illocutionary logic for assertions, taking into account both Dummett's work and the theory of illocutionary forces ([1]). In $\mathcal{L}^P$, propositions are either *true* or *false*, while the judgements that come to be expressed as assertions are *justified* (J) or *unjustified* (U). Assertions in $\mathcal{L}^P$ are logical entities without reference to speaker's intentions or beliefs ([8]). Sharing the general properties of assertions, assertions in $\mathcal{L}^P$ cannot fall under the scope of truth-functional connectives ([12]).

The language of $\mathcal{L}^P$ is composed of two sets of formulas: *radical* and *sentential*. Every sentential formula contains at least one radical formula as its proper sub-formula. Radical formulas are semantically interpreted by assigning them a truth value. Sentential formulas are pragmatically evaluated by assigning them a justification value in $\{J, U\}$, defined in terms of an intuitive notion of a proof. The pragmatic language $\mathcal{L}^P$ is the following:

- *Descriptive signs*: Propositional letters $p, q, r, \ldots$;
- *Logical signs for radical formulas*: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$;
- *Logical signs for sentential formulas*: the sign of pragmatic illocutionary force $\vdash$;
- *Pragmatic connectives*: $\sim$ is pragmatic negation, $\cap$ is pragmatic conjunction, $\cup$ is pragmatic disjunction, $\supset$ is pragmatic implication, $\equiv$ is pragmatic equivalence;
- *Formation Rules* (FRs):
  - *Radical formulas* (RF), recursively defined by the following FRs:

    FR1 (Atomic formulas): Every propositional letter is a RF.

    FR2 (Molecular formulas):

    (i) Let α be a RF. Then $\neg\alpha$ is a RF;

(ii) Let α_1 and α_2 be RF. Then $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$, $\alpha_1 \rightarrow \alpha_2$ and $\alpha_1 \leftrightarrow \alpha_2$ are RF.

– *Sentential formulas* (SF), recursively defined by the following FRs:

FR3 (Elementary formulas): Let α be a RF. Then $\vdash \alpha$ is a SF.

FR4 (Complex formulas):

(i) Let δ be a SF. Then $\sim \delta$ is a SF.

(ii) Let δ_1 and δ_2 be SF. Then $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, and $\delta_1 \equiv \delta_2$ are SFs.

Every radical formula of $\mathcal{L}^P$ has a truth value. Every sentential formula has a justification value that is defined in terms of the intuitive notion of proof, which depends on the truth-value of its subformula radicals. Pragmatic connectives have a meaning explicated by a variant of the BHK (Brouwer–Heyting–Kolmogorov) interpretation of intuitionistic logical constants. The illocutionary force of assertions plays an essential role in determining the *pragmatic* component of the meaning of an elementary expression, together with the *semantic* component, namely, the meaning of p as interpreted in a specific semantics.

A *pragmatic interpretation* of $\mathcal{L}^P$ is an ordered pair $\langle \{J, U\}, \pi_\sigma \rangle$, where $\{J, U\}$ is the set of justification values and π_σ is the pragmatic evaluation function that accords with justification rules:

Justification Rules are rules that regulate the pragmatic evaluation π_σ, by specifying the justification-conditions for sentential formulas in terms of the mapping σ of assignments of truth-values to their radical sub-formulas:

JR0 Elementary formulas are justified by means of conclusive evidence for p.

JR1 Let α be a radical formula. Then $\pi_\sigma(\vdash \alpha) = J$ iff a proof exists that α is true, i.e. σ assigns the value *true* to α. Likewise, $\pi_\sigma(\vdash \alpha) = U$ iff no proof exists that α is *true*.

JR2 Let δ be a sentential formula. Then, $\pi_\sigma(\sim \delta) = J$ iff a proof exists that δ is *unjustified*, i.e. that $\pi_\sigma(\delta) = U$.

JR3 Let δ_1 and δ_2 be sentential formulas. Then:

(i) $\pi_\sigma(\delta_1 \cap \delta_2) = J$ iff $\pi_\sigma(\delta_1 = J)$ and $\pi_\sigma(\delta_2 = J)$;

(ii) $\pi_\sigma(\delta_1 \cup \delta_2) = J$ iff $\pi_\sigma(\delta_1 = J)$ or $\pi_\sigma(\delta_2 = J)$;

(iii) $\pi_\sigma(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi_\sigma(\delta_2) = J$ whenever $\pi_\sigma(\delta_1) = J$.

The system also observes the *Soundness Criterion* (SC), which is the following:

(SC) Let be α a RF. Then $\pi_\sigma(\vdash \alpha) = J$ implies that $\sigma(\alpha) = 1$.

SC states that if an assertion is justified, then the content of assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behavior.

Definition 1: A formula δ is pragmatically valid or P-VALID (respectively, INVALID or P-INVALID), if for every Tarskian semantic interpretation σ and for every pragmatic justification function π_σ, it follows that $\pi_\sigma(\delta) = J$ (respectively, $\pi_\sigma(\delta) = U$).

$\mathcal{L}^P$ has a classical fragment ($\mathcal{CL}^P$), which is made up of all the SFs without any pragmatic connectives. Axioms of $\mathcal{CL}^P$ are the following:

A1 $\vdash (\alpha_1 \to (\alpha_2 \to \alpha_1))$.

A2 $\vdash ((\alpha_1 \to (\alpha_2 \to \alpha_3)) \to ((\alpha_1 \to \alpha_2) \to (\alpha_1 \to \alpha_3)))$.

A3 $\vdash (\neg\alpha_2 \to \neg\alpha_1) \to ((\neg\alpha_2 \to \alpha_1) \to \alpha_2)$.

The rule of Modus Ponens for $\mathcal{CL}^P$ is:

(MPP) If $\vdash \alpha_1, \vdash (\alpha_1 \to \alpha_2)$, then $\vdash \alpha_2$

The semantic rules are standard Tarskian rules and they specify the truth-conditions, only for radical formulas, by the assignment function σ. These rules regulate the semantic interpretation of $\mathcal{L}^P$. Let α_1, α_2 be radical formulas and 0 = 'false' and 1 = 'true'. Then:

(i) $\sigma(\neg\alpha_1) = 1$ iff $\sigma(\alpha_1) = 0$.

(ii) $\sigma(\alpha_1 \wedge \sigma_2) = 1$ iff $\sigma(\alpha_1) = 1$ and $\sigma(\alpha_2) = 1$.

(iii) $\sigma(\alpha_1 \vee \sigma_2) = 1$ iff $\sigma(\alpha_1) = 1$ or $\sigma(\alpha_2) = 1$.

(iv) $\sigma(\alpha_1 \to \sigma_2) = 1$ iff $\sigma(\alpha_1) = 0$ or $\sigma(\alpha_2) = 1$.

The intuitionistic fragment of $\mathcal{L}^P$, $\mathcal{IL}^P$, is composed of complex formulas with atomic radicals ([8]). Axioms of $\mathcal{IL}^P$ are the following:

A1' $\delta_1 \supset (\delta_2 \supset \delta_1)$.

A2' $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_2))$.

A3' $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$.

A4' $(\delta_1 \cap \delta_2) \supset \delta_1$; $(\delta_1 \cap \delta_2) \supset \delta_2$.

A5' $\delta_1 \supset (\delta_1 \cup \delta_2)$; $\delta_2 \supset (\delta_1 \cup \delta_2)$.

A6' $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$.

A7' $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim \delta_2)) \supset (\sim \delta_1))$.

A8' $\delta_1 \supset ((\sim \delta_1) \supset \delta_2)$.

The rule of Modus Ponens for $\mathcal{IL}^P$ is:

(MPP') If $\delta_1, \delta_1 \supset \delta_2$, then δ_2,

where δ_1, δ_2 contain atomic radicals.

It is worth noting that the justification rules do not always allow the determination of the justification value of a complex sentential formula. This happens when all the justification values of its components are known. For instance, $\pi_\sigma(\delta) = J$ implies $\pi_\sigma(\sim \delta) = U$ while the converse does not hold, and $\pi_\sigma(\sim \delta) = J$ implies $\pi_\sigma(\delta) = U$ but not the converse. In addition, a formula δ is *p-valid* (respectively *invalid* or *p-invalid*), if for every π and σ, the formula $\pi_\sigma(\delta) = J$ (respectively, $\pi_\sigma(\delta) = U$). Hence, no principle analogous to truth-functionality in classical connectives holds for the pragmatic connectives in $\mathcal{L}^P$. Pragmatic connectives are governed by partial functions of justification.

The *modal* (semantic) *projection* $()^*$ of pragmatic assertions is provided by the following translation in the modal system $\mathsf{S4}$:

$(\vdash \alpha)^*$	$\Box\alpha$
$(\sim \delta)^*$	$\Box\neg(\delta)^*$
$(\delta_1 \cap \delta_2)^*$	$(\delta_1)^* \wedge (\delta_2)^*$
$(\delta_1 \cup \delta_2)^*$	$(\delta_1)^* \vee (\delta_2)^*$
$(\delta_1 \supset \delta_2)^*$	$\Box((\delta_1)^* \to (\delta_2)^*)$

Sentential and radical formulas are related by *bridge principles* that connect sentential and radical formulas (for details, see [8]):

(a) $(\vdash \neg\alpha) \supset (\sim (\vdash \alpha))$

(b) $((\vdash \alpha_1) \cap (\vdash \alpha_2)) \equiv (\vdash (\alpha_1 \wedge \alpha_2))$

(c) $((\vdash \alpha_1) \cup (\vdash \alpha_2)) \supset (\vdash (\alpha_1 \vee \alpha_2))$

(d) $(\vdash (\alpha_1 \to \alpha_2)) \supset (\vdash \alpha_1 \supset \vdash \alpha_2)$.

It is worth noting that (a)–(d) give the formal relations between classical truth-functional and pragmatic connectives. The clause (a) states that the assertion of a negated proposition entails the pragmatic negation of the assertion, (b) tells that the conjunction of assertions is equivalent to the assertion of the conjuncts, (c) states that a disjunction of assertions implies the assertion of the disjuncts, and (d) indicates that truth-conditional implication implies pragmatic implication.

4 Proof, Knowledge and Assertion

As outlined above, the descriptive part $\mathcal{L}$ of $\mathcal{L}^P$ is identified with the language of classical propositional logic (the set of radical formulas).

The set of sentences is a set of assertions. The intuitionistic logic of $\mathcal{L}^P$ is, for instance, represented as the fragment built up from elementary sentences with atomic radicals by means of pragmatic connectives. Hence no classical connective falls under the scope of assertions.

In $\mathcal{L}^P$, there are no assertions the contents of which would be *modal* propositions. In the last section we gave just a projection of them. In order to overcome the limitation, we introduce $\mathcal{L}^P_{\Box,K}$, a pragmatic language for assertions with *modal* propositional contents. In particular, the descriptive part $\mathcal{L}_{\Box,K}$ of $\mathcal{L}^P_{\Box,K}$ is the *fusion* ([3]) $\mathcal{L}_\Box \oplus \mathcal{L}_K$ of two modal languages, $\mathcal{L}_\Box$ and $\mathcal{L}_K$, endowed with two independent 'boxes', $\Box$ and K, which are interpreted as "It is proved that" and "It is known that", respectively.

The fusion $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two modal languages, $\mathcal{L}_1$ and $\mathcal{L}_2$, endowed with two independent boxes $\Box_1$ and $\Box_2$ is the *smallest* modal language generated by both boxes. Note that the fusion of modal languages is commutative.

We obtain a language that allows us to combine alethic and epistemic features within a classically understood framework. In what follow we explain the details.

4.1 $\mathcal{L}^P_{\Box,K}$ and its semantics

The set of radical formulas and the set of sentential formulas of $\mathcal{L}^P_{\Box,K}$ are defined recursively through the following formation rules, respectively:

- $\alpha := p \,|\, \top \,|\, \bot \,|\, \neg\alpha \,|\, \alpha_1 \wedge \alpha_2 \,|\, \alpha_1 \vee \alpha_2 \,|\, \alpha_1 \rightarrow \alpha_2 \,|\, \alpha_1 \leftrightarrow \alpha_2 \,|\, \Box\alpha \,|\, K\alpha.$

- $\delta := \vdash \alpha \,|\, \sim \delta \,|\, \delta_1 \cap \delta_2 \,|\, \delta_1 \cup \delta_2 \,|\, \delta_1 \supset \delta_2 \,|\, \delta_1 \equiv \delta_2 \,.$

In order to define a pragmatic interpretation of $\mathcal{L}^P_{\Box,K}$, we have to semantically interpret $\mathcal{L}^P_{\Box,K}$. This amounts to an interpretation of its descriptive part $\mathcal{L}_{\Box,K}$.

The semantics of the fusion $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two modal languages, $\mathcal{L}_1$ and $\mathcal{L}_2$, endowed with two independent boxes $\Box_1$ and $\Box_2$, is given within the class of frames of the form $\langle W, R_1, R_2 \rangle$. In this triplet, $\langle W, R_1 \rangle$ and $\langle W, R_2 \rangle$ are frames for $\Box_1$ and $\Box_2$, respectively. The axiomatic presentation through Hilbert calculus is obtained by merging the axioms and inference rules of both logics. Some *Bridge Principles* (PBs) are, then, added, namely axioms that logically connect the independent boxes, such as $\Box_1\alpha \rightarrow \Box_2\alpha$.

We assume $\mathcal{L}_{\Box,K}$ to be the *fusion* $\mathcal{L}_\Box \oplus \mathcal{L}_K$ of $\mathcal{L}_\Box$ and $\mathcal{L}_K$ endowed with $\Box$ and K. It is natural to consider relational structures of the form $\langle W, R_\Box, R_K \rangle \in \mathcal{C}$, in which $\mathcal{C}$ is the class of frames with W a set of possible worlds, and $R_\Box, R_K \subseteq W \times W$ the binary accessibility relations on W such that $R_\Box$ is reflexive and transitive while R_K is reflexive, transitive (and possibly symmetric). In this way, $\Box$ is an S4 alethic modality, and K may be an epistemic modality of a system between S4 and S5[3].

In addition, we introduce the following *Bridge Principle* (BP):

[3]Notice that if K is considered as a **S5** modality, then ignorance can be known in virtue of the epistemic version of the modal axiom 5.

(BP) $\Box\alpha \to \neg K\neg\alpha$.

This can intuitively be read as "If it is the case that α is proved to be true, then it is not the case that α is known to be false". (BP) provides a logical connection between $\Box$ and K, which turns out to be equivalent to the condition that $R_\Box \subseteq R_K$ ([3]). Namely, (BP) is valid (on an appropriate frame), if and only if $R_\Box \subseteq R_K$.

A way to make clear the idea behind (BP) is to consider its equivalent formulation in terms of conjunction:

(BP') $\neg(\Box\alpha \wedge K\neg\alpha)$.

The principle (BP') identifies the relation that expresses a minimal condition holding between proof and knowledge, according to the pre-theoretical insights. That is, there must be a logical incompatibility between the proof that α is true and the knowledge that α is false.

Given the intuitive interpretation of the $\Box$, it is then possible to read classical alethic contents in an intuitionistic fashion. Such an interpretation is proposed for the following reason: since the pragmatic connectives are here interpreted intuitionistically, any pragmatic language is essentially an intuitionistic one. Classical modal propositions of assertions and their intuitionistic-like connections are defined *via* pragmatic connectives.

The semantic and the pragmatic interpretation of $\mathcal{L}^P_{\Box,K}$ are given through the following definitions.

Definition 2:[Semantic interpretation of $\mathcal{L}^P_{\Box,K}$] Let $\mathcal{C}$ be the class of frames $F = \langle W, R_\Box, R_K\rangle$ such that W is a set of possible worlds,

$R_\Box$, $R_K \subseteq W \times W$ are binary accessibility relations on W, $R_\Box$ is reflexive and transitive, R_K is reflexive, transitive (and possibly symmetric), and $R_\Box \subseteq R_K$.

Let $\mathcal{V}_F$ be the class of valuations

$$v\colon \begin{cases} PROP \to \wp(W) \\ p \mapsto v(p) \subseteq W \end{cases}$$

on a frame $F \in \mathcal{C}$, in which $PROP$ is the set of atomic propositional radicals.

Let $\mathcal{M} = \{M = \langle F, v\rangle \mid F \in \mathcal{C}, v \in \mathcal{V}_F\}$ be the class of models on a frame F.

Let $M = \langle\langle W,\, R_\Box,\, R_K\rangle\,, v\rangle \in \mathcal{M}$.

Then, a *semantic interpretation* σ_v of $\mathcal{L}^P_{\Box,K}$ on M is any function

$$\sigma_v\colon \begin{cases} (\mathcal{L}_\Box \oplus \mathcal{L}_K) \times W \to \{T, F\} \\ (\alpha, w) \mapsto \sigma_v(\alpha, w) \in \{T, F\}\,, \end{cases}$$

which satisfies the following truth-rules:

(TR1) Let $p \in PROP$ and $w \in W$. Then:

(i) $\sigma_v(\top, w) = T$

(ii) $\sigma_v(\bot, w) = F$

(iii) $\sigma_v(p, w) = T \Leftrightarrow p \in v(p)$.

(TR2) Let $\alpha, \alpha_1, \alpha_2 \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:

(i) $\sigma_v(\neg\alpha, w) = T \Leftrightarrow \sigma_v(\alpha, w) = F$

(ii) $\sigma_v(\alpha_1 \wedge \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1, w) = T$ and $\sigma_v(\alpha_2, w) = T$

(iii) $\sigma_v(\alpha_1 \vee \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1, w) = T$ or $\sigma_v(\alpha_2, w) = T$

(iv) $\sigma_v(\alpha_1 \rightarrow \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_2, w) = T$ whenever $\sigma_v(\alpha_1, w) = T$

(v) $\sigma_v(\alpha_1 \leftrightarrow \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1 \rightarrow \alpha_2, w) = T$ and $\sigma_v(\alpha_2 \rightarrow \alpha_1, w) = T$.

(TR3) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:

(i) $\sigma_v(\Box\alpha, w) = T \Leftrightarrow$ for all $v \in W$, $\sigma_v(\alpha, v) = T$ whenever $wR_{\Box}v$

(ii) $\sigma_v(K\alpha, w) = T \Leftrightarrow$ for all $v \in W$, $\sigma_v(\alpha, v) = T$ whenever $wR_K v$.

Definition 3:[Pragmatic Interpretation of $\mathcal{L}^P_{\Box,K}$] Let σ_v be a *semantic interpretation* of $\mathcal{L}^P_{\Box,K}$ on a model M. Then a *pragmatic interpretation* π_{σ_v} of $\mathcal{L}^P_{\Box,K}$ on M is any (partial) function π_{σ_v} such that

$$\pi_{\sigma_v} : \begin{cases} \mathcal{L}^P_{\Box,K} \times W \rightarrow \{J, U\} \\ (\delta, w) \mapsto \pi_{\sigma_v}(\delta, w) \in \{J, U\} \end{cases}$$

such that it satisfies the following *Justification Rules* (JRs) and the *Correctness Criterion* (CC):

(JR1*) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:

- $\pi_{\sigma_v}(\vdash \alpha, w) = J \Leftrightarrow$ a proof exists that $\sigma_v(\alpha, w) = T$.

Hence, $\pi_{\sigma_v}(\vdash \alpha, w) = U \Leftrightarrow$ no proof exists that $\sigma_v(\alpha, w) = T$.

(JR2*) Let $\delta, \delta_1, \delta_2 \in \mathcal{L}^P_{\Box,K}$ and $w \in W$. Then:

(i) $\pi_{\sigma_v}(\sim\delta, w) = J \Leftrightarrow$ a proof exists that $\pi_{\sigma_v}(\delta, w) = U$

(ii) $\pi_{\sigma_v}(\delta_1 \cap \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1, w) = J$ and $\pi_{\sigma_v}(\delta_2, w) = J$

(iii) $\pi_{\sigma_v}(\delta_1 \cup \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1, w) = J$ or $\pi_{\sigma_v}(\delta_2, w) = J$

(iv) $\pi_{\sigma_v}(\delta_1 \supset \delta_2, w) = J \Leftrightarrow$ a proof exists that $\pi_{\sigma_v}(\delta_2, w) = J$ whenever $\pi_{\sigma_v}(\delta_1, w) = J$

(v) $\pi_{\sigma_v}(\delta_1 \equiv \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1 \supset \delta_2, w) = J$ and $\pi_{\sigma_v}(\delta_2 \supset \delta_1, w) = J$

(CC) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then

- $\pi_{\sigma_v}(\vdash \alpha, w) = J \Rightarrow \sigma_v(\alpha, w) = T$.

In the next section we will apply this multi-modal pragmatic language in order to investigate the issue of asserting various forms of ignorance.

5 Asserting Ignorance

The informal notion of (un)decidability is the starting point for an analysis of pragmatic facets of ignorance. The pragmatic notion of the *first-level decidability* can be expressed in the following way:

(7) $((\vdash \alpha) \cup (\vdash \neg\alpha)) = J$.

This means that α or its negation can be justifiably asserted, namely that there exists a proof of α or a proof of $\neg\alpha$. The modal translation of (7) is

$$\Box\alpha \vee \Box\neg\alpha.$$

The *second-level decidability* can then be formulated as:

(8) $((\vdash \alpha) \cup ({\sim}\!\vdash \alpha)) = J.$

Formula (8) states that α can be asserted or α can not be asserted. This means that there exists a proof of α or there exists a proof that α cannot be proven. Modally speaking, (8) translates into

$$\Box\alpha \vee \Box\neg\Box\alpha.$$

The *first-level pragmatic undecidability* can then be expressed in the following way.

(9) $((\vdash \alpha) \cup (\vdash \neg\alpha)) = U.$

Formula (9) states that neither α nor $\neg\alpha$ are asserted, namely that there is no proof of α and there is no proof of $\neg\alpha$. The modal version of (9) is

$$\neg(\Box\alpha \vee \Box\neg\alpha).$$

Finally, the *second-level pragmatic undecidability* can be expressed in $\mathcal{L}^P$ as follows:

(10) $((\vdash \alpha) \cup ({\sim}\!\vdash \alpha)) = U.$

This means that it is not the case that α can be asserted or that α cannot be asserted. In other words, it is not the case that there exists a proof of α or that there exists a proof showing that α cannot be proven. The modal translation of (10) is

$$\neg(\Box\alpha \vee \Box\neg\Box\alpha).$$

It is noteworthy that second-level decidability (8) implies first-level decidability (7). This follows from the bridge principle (a) for negations. The converse does not hold. Moreover, no inferential relation is attributable either to the first-level undecidability (9) or to the second-level undecidability (10), since they are both unjustified formulas. As seen above, inferences occurring in $\mathcal{L}^P$ involve only justified assertions.

Issues regarding (un)decidability cannot be treated in $\mathcal{L}^P$. In order to provide a pragmatic treatment of ignorance in $\mathcal{L}^P$, let us consider $\mathcal{L}^P_{\Box,K}$.[4] This strategy has been pursued in [4], [5] and [6].

Following Fine's sketch of the situation, in our pragmatic framework we can express the following pragmatic version of factual ignorance:

(11) $(\vdash (\alpha \wedge \neg K\alpha)) = J.$

Formula (11) states that factual ignorance may be justifiably asserted. We can assert the Hintikka–Moore sentences, although the knowledge of them has no models. The multi-modal translation of (11) is $\Box(\alpha \wedge \neg K\alpha)$. By applying the bridge principle (b) for conjunctions to (11), we moreover have the formula

(11*) $((\vdash \alpha) \cap (\vdash \neg K\alpha)) = J.$

[4]From now on, we use $\mathcal{L}^P$ to refer both to the logic for pragmatics as well as to its extensions.

A natural reading of (11*) is the following one: There is a proof of α and there is a proof of the fact that α is not known. In order to give meaning to this expression it is important to carefully consider the characteristics of $\mathcal{L}^P$. In particular, we uncover an assumption which we are forced to bring to light, namely a difference between *pragmatic* and *epistemic* conditions. The presentation of $\mathcal{L}^P$ reveals that this pragmatic system does not specify whether there is an epistemic subject at present which has at its disposal the proofs to justifiably assert the propositional contents. Rather, the act of assertion is used here in a pragmatic, objective and impersonal way.

However, knowledge is usually something that is attributed to collections of epistemic agents. The interplay between epistemic and pragmatic agents and the conditions of justification is nonetheless far from irrelevant. There are discrepancies between epistemic and pragmatic conditions. If the pragmatic and the epistemic agent would be identical, call it S, then S has a proof that entitles S to assert α and S has a proof that entitles S to assert that it does not know α. This would be a puzzling situation. Thus we assume, on the one hand, that pragmatic and epistemic conditions for justification are not the same. This is taken into account in $\mathcal{L}^P$: In $\mathcal{L}^P$, $\vdash \alpha$ refers to an idealized subject, and is reflected by the strong notion of proof used in the pragmatic justification conditions. On the other hand, we can consider the epistemic subject occurring, either implicitly or explicitly, in the epistemic operator of the radical formula. In this case, it is taken as an empirical inquirer of the world. This means that in $\mathcal{L}^P$, an act of assertion can be justified by the existence of a proof for the propositional content, even if no one may ever come to know such proof (for a discussion of this issue in $\mathcal{L}^P$, see [7]).

From the formula (11) it is possible to infer, by applying classical transformations in the radical formula and the bridge principle (a) for negations, that

(12) $(\sim(\vdash (\alpha \to K\alpha))) = J$.

Intuitively, (12) states that it is not the case that asserting the truth of α implies knowledge of α.

The notion of *being ignorant whether* α has the following two pragmatic versions, $I_1(\alpha)$ and $I_2(\alpha)$:

(13) $I_1(\alpha) =_{\text{def.}} ((\vdash \neg K\alpha) \cap (\vdash \neg K\neg\alpha)) = J$, which by virtue of the bridge principle (b) for conjunctions is logically equivalent to $(\vdash ((\neg K\alpha) \wedge (\neg K\neg\alpha))) = J$.

(14) $I_2(\alpha) =_{\text{def.}} ((\sim\vdash K\alpha) \cap (\sim\vdash K\neg\alpha)) = J$.

On the one hand, formula (13) expresses *first-level pragmatic ignorance* of whether α, since its intended meaning is that there is a proof that α is unknown and there is also a proof that $\neg\alpha$ is unknown. On the other hand, formula (14) indicates *second-level pragmatic ignorance.* It affirms that there exists a proof that α cannot be known and there exists a proof that $\neg\alpha$ cannot be known. One can state this by taking α in (14) to be an intrinsically unknowable proposition, that is, it can never be the case that α may be known. Applying the bridge principle (a) to (13), it follows that (13) implies (14).

Building upon (13), we can then formulate the pragmatic version of Rumsfeld-ignorance:

(15) $(\vdash (I_1\alpha \wedge \neg K(I_1\alpha)) = J$, which is equivalent to

(16) $(\vdash ((\neg K\alpha) \wedge (\neg K\neg\alpha))) \wedge (\neg K((\neg K\alpha) \wedge (\neg K\neg\alpha)))) = J$.

What Fine ([9]) proved was that in $\mathsf{S4}$, the propositional content of (16) cannot be known. Yet it follows from our pragmatic logic of assertions that in $\mathcal{L}^P$ the propositional content of (16) can be justifiably asserted. This shows that not everything that can be justifiably asserted can be known. Epistemic limitations restrict the possibilities of coming to know a propositional content. A pragmatic form of Rumsfeld-ignorance based on the second-order ignorance of whether α, as in (14), cannot be formulated in $\mathcal{L}^P$, because the assertion sign cannot be iterated and because pragmatic connectives do not operate within radical formulas [12].

Consequently, the pragmatic version of Fine's second-order ignorance, which we call the *second-order pragmatic ignorance* $(I_1 I_1\alpha)$, has the following form:

(17) $(I_1 I_1\alpha) =_{\text{def.}} (\vdash (\neg K(I_1\alpha) \wedge \neg K\neg(I_1\alpha))) = J$,

which is equivalent to

$(\vdash (\neg K((\neg K\alpha) \wedge (\neg K\neg\alpha))) \wedge (\neg K\neg((\neg K\alpha) \wedge (\neg K\neg\alpha)))) = J$.

However, because of the limitations of pragmatic operators, a second-

level pragmatic ignorance of the type $I_2 I_2 \alpha$ cannot be formulated. Second-level pragmatic ignorance does not refer to content that would be justifiably assertible.

It is natural to check, however, whether the second-level pragmatic ignorance of the form $I_1 I_2 \alpha$ or $I_2 I_1 \alpha$ can be formulated in $\mathcal{L}^P$. $I_1 I_2 \alpha$ clearly violates the syntax of $\mathcal{L}^P$ and therefore cannot be expressed in our pragmatic language. But the second-order pragmatic ignorance of the form $I_2 I_1 \alpha$ can in fact be expressed in $\mathcal{L}^P$ thus:

(18) $(I_2 I_1 \alpha) =_{\text{def.}} (\sim\vdash K(I_1\alpha) \cap (\sim\vdash K\neg(I_1\alpha))) = J,$

which is equivalent to

$((\sim\vdash K((\neg K\alpha)\wedge(\neg K\neg\alpha)))\cap(\sim\vdash K\neg((\neg K\alpha)\wedge(\neg K\neg\alpha)))) = J.$

Formula (18) affirms that there is a proof that $I_1\alpha$ cannot be known and there is a proof that also the negation of $I_1\alpha$ cannot be known. This results in a very strong version of asserting ignorance of ignorance which has not been taken into account in the literature before. In [9], it was pointed out that the second-order ignorance (equivalent in S4 to Rumsfeld-ignorance) cannot be known, so that "when one is second order ignorant one enters a black hole from which there is no epistemic escape" (p. 4031). In our approach, the formula (18) that expresses a form of pragmatic ignorance can nevertheless be justifiably asserted. It is the highest level of pragmatic ignorance that can be expressed in $\mathcal{L}^P$.

6 Uses of Asserting Ignorance: Scientific Assertions

The two forms of ignorance, namely those of Rumsfeld-ignorance (iv) and second-order ignorance (v), can be consistently and justifiably asserted. We can meaningfully put forth and discuss propositions under various forms of ignorance that may be justifiably asserted even though they cannot be known. Assertions that scientists put forth tend to concern various levels of ignorance while also being ones that have not, do not or need not become known, not even in the long historical perspective, but nevertheless are such that for the sake of progress, ought to be consistently and justifiably assertible.

What kinds of assertions are those? In frontiers of science, they abound. Can we find out what, if anything, lies beyond observable universe?[5] We have ignorance of $\alpha =$ "there exists x beyond the observable universe": we don't know it and we don't know its negation. And not only that, also we don't know that we know neither it nor it's negation: for all we know, there just aren't facts of the matter to settle upon how to handle this question. When describing scientists who are mucking about in ignorance in their daily practices, Firestein might have meant just such notion of ignorance:

> There are a lot of facts to be known in order to be a professional anything—lawyer, doctor, engineer, accountant,

[5] Take 'observable universe' as 'all signals ever emitted following the inflationary epoch'.

> teacher. But with science there is one important difference. The facts serve mainly to access the ignorance ... Scientists don't concentrate on what they know, which is considerable but minuscule, but rather on what they don't know ... Science traffics in ignorance, cultivates it, and is driven by it. Mucking about in the unknown is an adventure; doing it for a living is something most scientists consider a privilege [10].

When you don't know something, to learn to know what is it that you don't know is important, and not to engage in Meno's paradox. It is to ignite the process of designing methods of coming to learn something, at the presence of Rumsfeld-ignorance.

It is such attitude that leads to the pragmatic form of ignorance: even though we cannot know the propositional content of Rumsfeld-ignorance (in $\mathsf{S4}$), it does not mean that the content cannot be justifiably asserted and thus fruitfully and meaningfully entertained. As we have seen, an analysis how this can happen is achieved in the pragmatic logic of assertions. In fact, we are compelled to make such assertions in order to ever learning to know something that we are ignorant of while knowingly being ignorant. Firestein continues:

> Working scientists don't get bogged down in the factual swamp because they don't care all that much for facts. It's not that they discount or ignore them, but rather that they don't see them as an end in themselves. They don't stop at

> the facts; they begin there, right beyond the facts, where the facts run out. Facts are selected, by a process that is a kind of controlled neglect, for the questions they create, for the ignorance they point to. [...] Being a scientist requires having faith in uncertainty, finding pleasure in mystery, and learning to cultivate doubt.

As far as the second-order ignorance is concerned, we saw that it is possible that there are proofs of both not being able to know whether we are ignorant and whether we are not ignorant. In this case, inquiry is conducted just the same. For we have just shown that this type of second-order ignorance can also be justifiably asserted. And that is all that is needed for the investigation to proceed. Firestein's recent investigation may testify to this approach:

> In 1959 MIT visual scientist Gerry Letvin published a paper titled "What the Frog's Eye Tells the Frog's Brain". This deceptively simple question has driven research in sensory systems for more than 50 years. It may be the wrong question. Similarly, the pioneering work of Hubel and Wiesel [ldots] has driven a research program in all sensory systems to uncover the neural "maps" of the world created by the brain. This idea may also be wrong. These scientific programs ... have been called into question because of developments in our most ancient and curious sensory system, olfaction. Smell is a high dimensional stimulus, as opposed

> to vision, hearing, touch, etc., which are all low dimensional stimuli. The initial program of applying strategies used in visual, auditory and somatosensory systems to olfaction has revealed that these won't work and that there must be a different neural strategy at work. *The interesting thing is we currently have no idea what that strategy may look like.*[6]

The example above expresses a scientific issue that can be justifiably asserted, albeit it cannot be known under that state of ignorance.

Finally, full second-*level* pragmatic ignorance ($I_2 I_2 \alpha$) is not formalizable in the pragmatic logic of assertions, and therefore there is no justified assertion of the propositional contents of such second-level pragmatic ignorance, either. Does this impossibility result then curtail the open-ended nature of scientific inquiry somehow? No, because what is important is that the forms of the kind ($I_2 I_1 \alpha$) are formalizable in $\mathcal{L}^P$, even though ($I_2 I_2 \alpha$) and indeed ($I_1 I_2 \alpha$) are not. The former tells that we can have a proof that ignorance of α cannot be known while having a proof that the negation of that ignorance also cannot be known. This much can be justifiably asserted about α. It may be that we cannot know to be ignorant and we cannot know not to be ignorant. But also here, learning how to come to gain information and cultivate doubt serve as an antidote to ignorance of ignorance—the kind of unconscious ignorance—just as well. We are compelled to make certain assertions in science whose propositional content may be

[6] "Modulation in the Periphery: What is the nose telling the Brain?" Stuart Firestein (Columbia), March 2019.

unknowable, in order for us to ever be able to learn something that we may be ignorant of, despite the fact that we may remain forever ignorant of whether the veil of ignorance is ever capable of being lifted.

In sum, the pragmatic logic of assertions helps to analyze epistemic situations which reach an impasse in S4. Far from showing absolute limitations to what can be known, Fine's analysis is an artefact of the formal system in which that analysis is carried out. Our pragmatic logic of assertions provides a possible way out.

7 Conclusion

Summarizing, the novelties of our pragmatic approach to ignorance are the following. The pragmatic treatment of ignorance shows that there is a connection between ignorance and the informal notion of (un)decidability. In $\mathcal{L}^P$, we have expressed pragmatic versions of factual ignorance (12) and first-order ignorance of whether α (13), as well as variants of the latter at the second-order level (14). Moreover, we have shown how pragmatic second-order ignorance and pragmatic versions of Rumsfeld-ignorance may be formulated. Second-order pragmatic ignorances of the form $I_1 I_1 \alpha$ and $I_2 I_1 \alpha$, expressed by (17) and (18) respectively, may also be formulated, while such is the case neither for $I_1 I_2 \alpha$ nor $I_2 I_2 \alpha$. Formula (18) is new and it indicates a very strong pragmatic version of ignorance of ignorance, irreducible to any of the previous ones. Differently put, (18) defines the limits of what can be justifiably asserted about ignorance of ignorance. We have also shown how the pragmatic version of Rumsfeld-ignorance, grounded on I_1 (15), can be formulated, unless grounded on I_2. Finally, we briefly discussed how the justified assertion of what cannot be known is manifested in scientific discourse. We have pointed out that the pragmatic logic of assertions can be used to analyze situations in which one has to meaningfully refer to the propositional content of what is unknown, without adopting strategies that may block the way of inquiry.

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