REGULARITY FOR THE 3D EVOLUTION NAVIER-STOKES EQUATIONS UNDER NAVIER BOUNDARY CONDITIONS IN SOME LIPSCHITZ DOMAINS

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Abstract. For the evolution Navier-Stokes equations in bounded 3D domains, it is well-known that the uniqueness of a solution is related to the existence of a regular solution. They may be obtained under suitable assumptions on the data and smoothness assumptions on the domain (at least $C^{2,1}$). With a symmetrization technique, we prove these results in the case of Navier boundary conditions in a wide class of merely *Lipschitz domains* of physical interest, that we call *sectors*.

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1. Introduction

Let $T>0$ and let $\Omega\subset\mathbb{R}^3$ be a bounded domain; once forever we clarify that this means that Ω is open, nonempty and connected. The evolution 3D Navier-Stokes equations

$$u_t-\mu\Delta u+(u\cdot\nabla)u+\nabla p=f\ ,\quad \nabla\cdot u=0\ ,\qquad \text{in }\Omega\times(0,T)\,, \tag{1}$$

model the motion of an incompressible viscous fluid: u is its velocity, p its pressure, f is an external force, $\mu>0$ is the kinematic viscosity. The equations (1) are complemented with some initial and boundary conditions, the most common being the homogeneous Dirichlet conditions ($u=0$ on $\partial\Omega$), also called no-slip boundary conditions. In 1827, Navier [20] proposed conditions with friction, in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip. The homogeneous Navier boundary conditions read

$$u\cdot\nu=(\mathbf{D}u\cdot\nu)\cdot\tau=0\quad\text{on }\partial\Omega\,, \tag{2}$$

where $\mathbf{D}u=\frac{1}{2}(\nabla u+\nabla^\top u)$ is the strain tensor, ν is the outward normal vector to $\partial\Omega$ while τ is tangential. The boundary conditions (2) turn out to be appropriate in many physically relevant cases [4, 21], in particular in presence of turbulent boundary layers [12]; see Section 3 in [7] for a survey of problems in which (2) arise. The first contribution (in 1973) to (1)-(2) is due to Solonnikov-Scadilov [22]. For regularity results, see [1, 2, 5, 7, 8].

We put $Q_T:=\Omega\times(0,T)$ and we consider (1) in Q_T, complemented with (2) and initial conditions:

$$\begin{cases} u_t-\mu\Delta u+(u\cdot\nabla)u+\nabla p=f & \text{in }Q_T\,,\\ \nabla\cdot u=0 & \text{in }Q_T\,,\\ u\cdot\nu=(\mathbf{D}u\cdot\nu)\cdot\tau=0 & \text{on }\partial\Omega\times(0,T)\,,\\ u(x,y,z,0)=u_0(x,y,z) & \text{in }\Omega\,,\end{cases}\qquad \int_\Omega p(t)=0\quad\forall t\in(0,T)\,. \tag{3}$$

in which the pressure p is defined up to an additive constant so that we fixed to zero its mean value. We are interested in existence and, possibly, uniqueness of the solution of (3); it is well-known [23] that uniqueness is strictly related to the regularity of the solution. Under Dirichlet boundary conditions, this requires a C^2-boundary. Under Navier boundary conditions, Ω needs to have a $C^{2,1}$-boundary, see [2, 5, 6], because of the appearance of derivatives in (2), whose traces are defined when $\partial\Omega\in C^{2,1}$; see e.g. [26, Theorem 8.7b]. However, many domains of physical and engineering interest fail to be smooth. This is the case of a pipe bifurcation in a water grid, of a joint in a network of oil pipelines, of the section of a vein containing blood, of a half-ball representing a drop of water on an impermeable table, of a half circular cylinder modeling a road tunnel, of a bottle containing wine, see Figure 1.

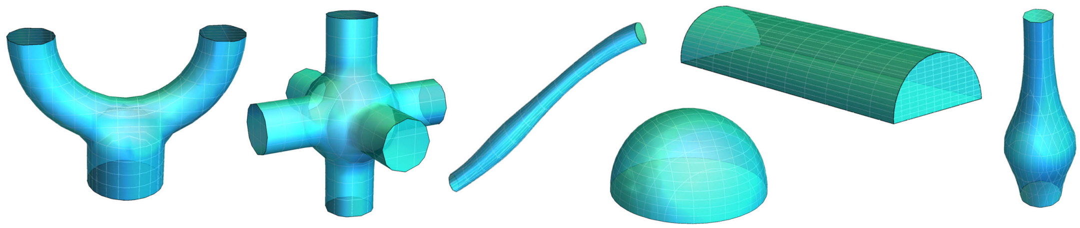

FIGURE 1. From left to right: a pipe bifurcation, a joint, a vein, a drop, a tunnel, a bottle.

The main purpose of the present paper (Theorem 1) is to prove regularity and uniqueness results for (3) in a suitable class of merely Lipschitz domains, the *sectors*, see Definition 3 below; this class includes all the domains in Figure 1. For the proofs we take advantage of the reflection method introduced in [14] for the Euler equations and subsequently applied in [3, 15] to the Navier-Stokes equations. The reflection is possible because we have Navier boundary conditions; under Dirichlet boundary conditions the same argument does not allow smooth extensions of the involved functions and vector fields. A further difference with respect to Dirichlet boundary conditions is the possible failure of the Poincaré inequality in axisymmetric domains, see [2, Lemma 3.3] and Proposition 1 below. Therefore, we provide a new variant of the needed bounds. We point out that (3) in domains where all the components of the solution vanish on a subset of positive 2D Hausdorff measure of $\partial\Omega$, e.g. rectangular parallelepipeds, Poincaré-Sobolev inequalities hold [19].

In the unforced case $f \equiv 0$ (Theorem 2) we extend classical uniqueness results for small data [13, 16] and the Leray principle [17, 18]. These results will be used in a forthcoming paper [10].

2. Main results

In order to characterize sectors, we need some definitions.

Definition 1. *We call* **face** *any bounded planar domain ω (open in $\mathbb{R}^2$) and we denote by P_ω the plane containing ω. Let P be a plane and let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that*

$$\Omega \cap P = \emptyset \qquad \text{and} \qquad \overline{\Omega} \cap P \quad \text{is the union of a finite number } h \geq 1 \text{ of (closed) faces;} \tag{4}$$

we denote by Ω_P the interior of the closure of the union between Ω and its reflection about P.

Note that if (4) holds then Ω_P is a (connected) domain and contains the h faces. Let $P_1, ..., P_m$ be m planes ($m \geq 1$) and let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that (4) holds for the m couples

$$\Omega \text{ and } P_1\,, \quad \Omega_{P_1} \text{ and } P_2\,, \quad \dots, \quad \Big(\big(\Omega_{P_1}\big)_{P_2}\cdots\Big)_{P_{m-1}} \text{ and } P_m\,;$$

then we can iteratively define the domain

$$\Omega_{P_1,...,P_m} := \bigg(\Big(\big(\Omega_{P_1}\big)_{P_2}\cdots\Big)_{P_{m-1}}\bigg)_{P_m}\,.$$

Definition 2. *We say that a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ is* **smoothly periodically extendable** *if it admits a periodic extension with $C^{2,1}$ boundary and if $\partial\Omega$ has a finite number $k \geq 2$ of faces ω_i ($i = 1, ..., k$), all lying on at most six planes $p_1, \dots, p_6$ such that:*

$$p_s \cap \Omega = \emptyset \ \ \forall s = 1, \dots, 6 \quad \text{and} \quad p_1 \parallel p_4\,,\ p_2 \parallel p_5\,,\ p_3 \parallel p_6\,,\ p_1 \perp p_2\,,\ p_1 \perp p_3\,,\ p_2 \perp p_3\,. \tag{5}$$

The extension can occur in either one, two, or three (orthogonal) directions. For a circular cylinder, there is only one direction. For a planar pipe bifurcation (see the third picture in Figure 4), there are two directions. For a 3D pipe bifurcation, as in the second picture in Figure 1, there are three directions. For a cube, one has both a 2D periodic extension (in which case the boundary of the resulting domain would be two parallel planes) and a 3D extension (in which case the extension would be the whole $\mathbb{R}^3$, with empty boundary). We point out that the number of planes is *at most* six: it is exactly six for a cube or for the joint in Figure 1, while less than six for all the other domains in Figure 1. We also emphasize that the boundary $\partial\Omega$ of any smoothly periodically extendable domain Ω may be written as

$$\partial\Omega = \overline{\bigcup_{i=1}^{k} \omega_i \cup \Gamma}\ , \tag{6}$$

for some Γ having $C^{2,1}$-regularity.

We are now in position to define the class of Lipschitz domains where we can obtain regularity results for (1) under the Navier boundary conditions (2).

Definition 3. *A bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ is a* **sector** *if one of the two following facts occurs:*

(A) there exists a bounded $C^{2,1}$-domain Ω_m having at least $m \geq 0$ planes of symmetry $P_1, \dots, P_m$ and such that $\Omega_m = \Omega_{P_1,\dots,P_m}$ when $m \geq 1$; if $m = 0$, then Ω has $C^{2,1}$-boundary ($\Omega_0 \equiv \Omega$);

(B) there exists a smoothly periodically extendable domain Ω^m having at least $m \geq 0$ planes of symmetry $P_1, \dots, P_m$ and such that $\Omega^m = \Omega_{P_1,\dots,P_m}$; if $m = 0$, then Ω is smoothly periodically extendable ($\Omega^0 \equiv \Omega$).

Not only the boundary of a sector satisfies (6), but each of its faces "sticks orthogonally" to the smooth part Γ. In the sequel we refer to sectors of type (A) and (B). This class of Lipschitz domains is sufficiently wide to contain most of the domains needed in physics and engineering, in particular all the domains depicted in Figure 1: while the drop is of type (A), all the other domains are of type (B). Roughly speaking, Definition 3 states that a sector reconstructs the domain Ω_m or Ω^m after a finite number m of reflections about the faces, possibly none if Ω is $C^{2,1}$ or if Ω is already smoothly periodically extendable. As a consequence, we have that $|\Omega_m| = |\Omega^m| = 2^m|\Omega|$; the difference between

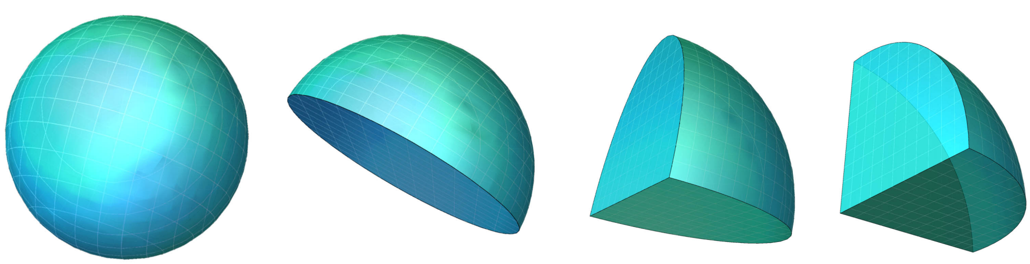

FIGURE 2. Some sectors obtained as subdomains of a sphere.

Ω_m and Ω^m is that the first has $C^{2,1}$ boundary, while the second is only Lipschitzian. Moreover, it is mandatory to specify that the planes of symmetry are *at least* m; if Ω_m is a ball or Ω^m is a circular cylinder, then they have infinitely many planes of symmetry and a sector may be half a sphere, a quarter of sphere, and so on (also for a cylinder), see Figure 2.

From a geometric point of view, smoothly periodically extendable domains do not require symmetrizations with respect to the planes in (5), for instance a straight cylinder or a cube. But from an analytic point of view, in order to implement our symmetrization technique, we need to apply the following principle:

$$
\text{(7)}\qquad \begin{array}{c}\text{to obtain the domain of periodicity } \Omega_{\mathcal{P}},\\ \text{a sector of type } (B) \text{ has to be reflected in each of the directions of periodicity,}\\ \text{except for those directions that have already been used to obtain } \Omega^m.\end{array}
$$

Since this principle is delicate, we give a detailed description with some examples in Appendix 1.

Let us now recall the usual spaces in the treatment of the Navier-Stokes equations

$$
\text{(8)}\qquad \begin{array}{c} H = \{v \in L^2(\Omega);\, \nabla\cdot v = 0,\, v\cdot\nu = 0 \text{ on } \partial\Omega\}, \quad G = \{v \in L^2(\Omega);\, \exists g \in H^1(\Omega),\, v = \nabla g\},\\ V = H \cap H^1(\Omega),\end{array}
$$

in which we denote by $v\cdot\nu$ the normal trace of v. Then $L^2(\Omega) = H \oplus G$ and $H \perp G$, where orthogonality is intended in $L^2(\Omega)$. By [23, Theorem 1.4] we know that H is a closed subspace of $L^2(\Omega)$; therefore, V is a closed subspace of $H^1(\Omega)$. When the domain is a generic A, different from Ω, we specify $H(A)$, $G(A)$, $V(A)$. We endow $H(A)$ and $V(A)$, respectively, with the scalar products and norms

$$
\text{(9)}\qquad \begin{array}{ll} (v,w)_A := \displaystyle\int_A v\cdot w\,, & \|v\|_{2,A}^2 := \displaystyle\int_A |v|^2\,,\\ (\mathbf{D}v,\mathbf{D}w)_A := \displaystyle\int_A \mathbf{D}v : \mathbf{D}w\,, & \|\mathbf{D}v\|_{2,A}^2 := \displaystyle\int_A |\mathbf{D}v|^2\,,\end{array}
$$

so that $H(A)$ and $V(A)$ are Hilbert spaces; here $\mathbf{D}v : \mathbf{D}w$ is the scalar product between matrices. Given $v = (v_1, v_2, v_3) \in L^p(A)$ with $1 \le p \le \infty$, we denote by $\|v\|_{p,A} := \left(\sum_{i=1}^3 \int_A |v_i|^p\right)^{1/p}$ its $L^p(A)$-norm.

Let us also introduce the kernel of the linear map $v \mapsto \mathbf{D}v$

$$
\mathcal{K}_\Omega := \{v \in V : \mathbf{D}v \equiv 0 \text{ in } \Omega\},
$$

and, when $\mathcal{K}_\Omega$ is not trivial, we use the decomposition

$$
\text{(10)}\qquad \forall v \in V \qquad v = \overline{v} + v_{\mathcal{K}} \qquad \text{with } v_{\mathcal{K}} \in \mathcal{K}_\Omega\,,\ \overline{v} \in \mathcal{K}_\Omega^\perp\,.
$$

The non-triviality of $\mathcal{K}_\Omega$ causes the failure of the Poincaré inequality: $\|\mathbf{D}v\|_{2,\Omega}$ *does not bound* $\|v\|_{2,\Omega}$. This is made precise in the next proposition, proved in [25], see also [2, 10] for some complements.

Proposition 1. *The dimension of the kernel $\mathcal{K}_\Omega$ depends on Ω and only three cases can occur*

$$
\dim \mathcal{K}_\Omega = \begin{cases} 0 & \text{if } \Omega \text{ is not axisymmetric,}\\ 1 & \text{if } \Omega \text{ is monoaxially symmetric,}\\ 3 & \text{if } \Omega \text{ is a ball.}\end{cases}
$$

Moreover, $\|\mathbf{D}\cdot\|_{2,\Omega}$ and $\|\nabla\cdot\|_{2,\Omega}$ are equivalent norms in $\mathcal{K}_\Omega^\perp$ and there exists $C_\Omega > 0$ such that

$$
\text{(11)}\qquad \|v\|_{2,\Omega} \le C_\Omega \begin{cases} \|\mathbf{D}v\|_{2,\Omega} & \text{if } \Omega \text{ is not axisymmetric}\\ \|v_{\mathcal{K}}\|_{2,\Omega} + \|\mathbf{D}v\|_{2,\Omega} & \text{if } \Omega \text{ is axisymmetric}\end{cases} \qquad \forall v \in V.
$$

By "monoaxially symmetric" we mean here that Ω has exactly one axis of (axial) symmetry. We also recall that $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ is called a **weak solution** of (3) if

$$
\text{(12)}\qquad \int_0^T (u(t),v)_\Omega\, \phi'(t)dt + \phi(0)(u_0,v)_\Omega = \int_0^T \left\{2\mu(\mathbf{D}u(t),\mathbf{D}v)_\Omega - (f(t),v)_\Omega + \int_\Omega \big(u(t)\cdot\nabla\big)u(t)\cdot v\right\}\phi(t)dt
$$

for all $v \in V$ and for all $\phi \in \mathcal{D}[0,T)$. In Section 3 we prove the following result.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a sector, $T > 0$, $f \in L^2(Q_T)$ and $u_0 \in H$; then* (3) *admits a (global) weak solution $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$. If $u_0 \in V$, then there exists*

$$0 < T^* = T^*\big(\Omega, \mu, \|u_0\|_{2,\Omega}, \|\mathbf{D}u_0\|_{2,\Omega}, \|f\|_{2,Q_T}\big) \leq T, \tag{13}$$

such that the weak solution u of (3) *is unique in $[0, T^*)$ and*

$$u \in L^\infty(0,T^*;V) \qquad u_t, \Delta u, \nabla p \in L^2(Q_{T^*}). \tag{14}$$

In Section 3 we extend to sectors and conditions (2) some uniqueness and regularity results for the unforced equation that, by now, are classical statements under Dirichlet boundary conditions.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a sector, assume that $f \equiv 0$ and $u_0 \in V$. There exists $C = C(\Omega, \mu, \|u_0\|_{2,\Omega}) > 0$ such that if*

$$\|\mathbf{D}u_0\|_{2,\Omega} < C, \tag{15}$$

then the solution u of (3) *satisfies $u \in L^\infty(\mathbb{R}^+;V)$, so that it is unique and global in time.*

Moreover, for any global weak solution u of (3)*, there exists $\mathcal{T} = \mathcal{T}(u) > 0$ such that*

$$u \in L^\infty(\mathcal{T},\infty;V) \qquad u_t, \Delta u, \nabla p \in L^2(\mathcal{T},\infty;L^2(\Omega)). \tag{16}$$

Remark 1. *From the proofs it is possible to infer some quantitative information on the constants T^* and C in Theorems 1 and 2. More precisely, if Ω_m or Ω^m are not axisymmetric then*

$$T^* \geq \frac{K_\Omega \mu^5}{\Big(2\mu\|\mathbf{D}u_0\|_{2,\Omega}^2 + \|f\|_{2,Q_T}^2\Big)^2}, \qquad C = \frac{\overline{K}_\Omega \mu^2}{\|u_0\|_{2,\Omega}}$$

with $K_\Omega, \overline{K}_\Omega > 0$ depending only on Ω and m, see (30) *for sectors (A); in this case T^* does not depend on $\|u_0\|_{2,\Omega}$. If Ω_m or Ω^m are axisymmetric, then the lower bound for T^* is increasing with respect to μ and decreasing with respect to $\|u_0\|_{2,\Omega}$, $\|\mathbf{D}u_0\|_{2,\Omega}$, $\|f\|_{2,Q_T}$, while C is increasing with respect to μ and decreasing with respect to $\|u_0\|_{2,\Omega}$; the dependence on Ω and m remains.*

3. Proofs

Proof of Theorem 1. The proof is split in several cases, starting from simple situations, and extending the results to all kinds of sectors in Definition 3. First we consider sectors of type (A), then we consider sectors of type (B); for both types, there are several subcases.

• **Sectors of type (A) with $m = 0$.** In this case, Ω has $C^{2,1}$-boundary and Theorem 1 is known. This result is standard under Dirichlet boundary conditions while, under Navier boundary conditions, the proof is given in [5, 8], see also below for full details.

• **Sectors of type (A) with $m = 1$.** In this case, following Definition 1, Ω has just one face ω_1 and, according to (6), its boundary satisfies $\partial\Omega = \overline{\omega_1 \cup \Gamma}$ for some Γ having $C^{2,1}$-regularity. Then we introduce an auxiliary problem on $\Omega_1 = \Omega_{P_{\omega_1}}$ and suitable functional spaces to deal with. The main point is that if a vector field $v \in V(\Omega_1)$ is symmetric with respect to the plane P_{ω_1} then it satisfies (2) on ω_1. Indeed, its normal component vanishes so that $v \cdot \nu = 0$ on ω_1; not only this gives the first condition in (2), but we also infer that the tangential derivatives of the normal component vanishes. Combined with the fact that also the normal derivatives of the tangential components of the vector vanish, this gives $(\mathbf{D}v \cdot \nu) \cdot \tau = 0$ on ω_1. Therefore, instead of the spaces H and V in (8) we consider their closed subspaces $H^{\mathcal{E}}$ and $V^{\mathcal{E}}$ of vector fields being symmetric with respect to the plane of symmetry of Ω_1.

For sake of simplicity, up to a rotation and a translation of Ω, we may assume that ω_1 lies on the plane $z = 0$. Then the symmetry of a vector field with respect to $z = 0$ can be expressed componentwise. Let $Q_T^1 := \Omega_1 \times (0,T)$, we say that a vector field $\Psi : Q_T^1 \to \mathbb{R}^3$ with components $\Psi_i = \Psi_i(x,y,z,t)$ $(i = 1,2,3)$ and a function $q : Q_T^1 \to \mathbb{R}$ are $\mathcal{E}-$symmetric if for all $(x,y,z,t) \in Q_T^1$

$$\Psi_i(x,y,z,t) = \Psi_i(x,y,-z,t) \quad (i = 1,2), \quad \Psi_3(x,y,z,t) = -\Psi_3(x,y,-z,t), \quad q(x,y,z,t) = q(x,y,-z,t).$$

We have so characterized the following closed subspaces of $H(\Omega_1)$ and $V(\Omega_1)$:

$$H^{\mathcal{E}} := \{v \in H(\Omega_1) \,:\, v \text{ is } \mathcal{E}-\text{symmetric}\} \qquad V^{\mathcal{E}} := \{v \in V(\Omega_1) \,:\, v \text{ is } \mathcal{E}-\text{symmetric}\}\,. \tag{17}$$

We endow $H^{\mathcal{E}}$ and $V^{\mathcal{E}}$, respectively, with the scalar products and norms in (9).

Given a vector field $\Psi : Q_T \to \mathbb{R}^3$ and a function $p : Q_T \to \mathbb{R}$, we symmetrize it in Ω_1 by defining a vector field $\widehat{\Psi} : Q_T^1 \to \mathbb{R}^3$ with scalar components $\widehat{\Psi}_i(x,y,z,t)$ $(i = 1,2,3)$ and a function $\widehat{p} : Q_T^1 \to \mathbb{R}$ where

$$\widehat{\Psi}_i(x,y,z,t) := \begin{cases} \Psi_i(x,y,z,t) & \text{in } \Omega \\ \Psi_i(x,y,-z,t) & \text{in } \Omega_1 \setminus \Omega \end{cases} \ (i = 1,2), \quad \widehat{\Psi}_3(x,y,z,t) := \begin{cases} \Psi_3(x,y,z,t) & \text{in } \Omega \\ -\Psi_3(x,y,-z,t) & \text{in } \Omega_1 \setminus \Omega. \end{cases} \tag{18}$$

$$\widehat{p}(x,y,z,t) := \begin{cases} p(x,y,z,t) & \text{in } \Omega \\ p(x,y,-z,t) & \text{in } \Omega_1 \setminus \Omega. \end{cases}$$

Let $\widehat{f}$ and $\widehat{u}_0$ be the resulting $\mathcal{E}-$symmetric fields of f and u_0; then $\widehat{f} \in L^2(Q_T^1)$ and $\widehat{u}_0 \in H^{\mathcal{E}}$. We denote by $(3)_1$ the problem (3) with Q_T^1, Ω_1, $\widehat{f}$, $\widehat{u}_0$, $\widehat{u}$, $\widehat{p}$ instead of Q_T, Ω, f, u_0, u, p. In doing so, we set the Navier-Stokes problem in a domain with $C^{2,1}-$boundary. With an abuse of notation, we then drop $\widehat{\cdot}$ in the symmetric extensions of the functions involved in $(3)_1$; the distinction will be clear since we specify the domain Ω or Ω_1 in all the scalar products and norms.

In the space $V^{\mathcal{E}}$, we consider the following Stokes eigenvalue problem

$$\begin{cases} -\Delta e + \nabla p = \lambda e & \text{in } \Omega_1\,, \\ \nabla \cdot e = 0 & \text{in } \Omega_1\,, \\ e \cdot \nu = (\mathbf{D}e \cdot \nu) \cdot \tau = 0 & \text{on } \partial\Omega_1\,. \end{cases} \tag{19}$$

Here and in the sequel, we denote by Δu both the Laplacian of u and the Stokes operator (its projection onto $H^{\mathcal{E}}$), without distinguishing the notations; what we mean will be clear from the context. Since $V^{\mathcal{E}}$ is a separable Hilbert space and the Stokes operator is linear, compact, self-adjoint and positive, all the eigenvalues of (19) have finite multiplicity and can be ordered in an increasing divergent sequence $\{\lambda_k\}_{k\in\mathbb{N}_+}$, in which the eigenvalues are repeated according to their multiplicity. In the case where $\dim \mathcal{K}_{\Omega_1} \neq 0$ problem (19) admits zero as eigenvalue with multiplicity one or three, see Proposition 1. Up to normalization, the set of eigenfunctions $\{e_k\}_{k\in\mathbb{N}_+}$ is a complete orthonormal system in $H^{\mathcal{E}}$ and complete orthogonal in $V^{\mathcal{E}}$.

For the statements on weak and regular solutions, in particular for the regularity results, we consider the eigenvectors $\{e_k\}_{k=1}^{\infty} \subset V^{\mathcal{E}}$ of (19) and the n^{th}-order approximation of $(3)_1$, that is,

$$\begin{cases} (u_t^n(t), e_k)_{\Omega_1} - \mu(\Delta u^n(t), e_k)_{\Omega_1} = -\big((u^n(t)\cdot\nabla)u^n(t), e_k\big)_{\Omega_1} + (f(t), e_k)_{\Omega_1} \qquad k = 1, \ldots, n, \\ u^n(0) = u_0^n \end{cases} \tag{20}$$

where $u_0^n := \sum_{k=1}^n (u_0, e_k)_{\Omega_1}\, e_k$ is the projection in $H^{\mathcal{E}}$ of u_0 onto the space spanned by $e_1, \ldots, e_n$ and $(\Delta u^n, e_k)_{\Omega_1} = -2(\mathbf{D}u^n, \mathbf{D}e_k)_{\Omega_1}$. By the theory of systems of ode's, (20) admits a unique solution

$$u^n(x,t) := \sum_{k=1}^n c_k^n(t)\, e_k(x) \tag{21}$$

with $c_k^n(t)$ being smooth coefficients. Multiplying (20) by $c_k^n(t)$ and summing for k from 1 to n we obtain

$$\frac{d}{dt}\|u^n(t)\|^2_{2,\Omega_1} + 4\mu\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} = 2(f(t), u^n(t))_{\Omega_1} \tag{22}$$

and applying the Hölder inequality

$$\begin{aligned}\frac{d}{dt}\|u^n(t)\|^2_{2,\Omega_1} \le 2\|f(t)\|_{2,\Omega_1}\|u^n(t)\|_{2,\Omega_1} \quad &\Rightarrow \quad \|u^n(t)\|_{2,\Omega_1} \le \|u_0^n\|_{2,\Omega_1} + \int_0^t \|f(\tau)\|_{2,\Omega_1} d\tau \\ &\Rightarrow \quad \|u^n(t)\|_{2,\Omega_1} \le \|u_0\|_{2,\Omega_1} + \sqrt{T}\|f\|_{2,Q_T^1} \qquad \forall t \in [0,T]\end{aligned} \tag{23}$$

which gives a *uniform bound* for $\|u^n(t)\|_{2,\Omega_1}$. In particular, by using the decomposition (10), this gives a uniform bound for $\|u^n_{\mathcal{K}}(t)\|_{2,\Omega_1}$; in turn, since $\mathcal{K}_{\Omega_1}$ is finite dimensional by Proposition 1, this gives a uniform bound for $\|\nabla u^n_{\mathcal{K}}(t)\|_{2,\Omega_1}$. With the a priori bounds in $L^\infty(0,T;H^{\mathcal{E}})$ and $L^2(0,T;V^{\mathcal{E}})$, derived from (22)-(23), one obtains a weak solution $u \in L^\infty(0,T;H^{\mathcal{E}}) \cap L^2(0,T;V^{\mathcal{E}})$. Since u and u_0 are $\mathcal{E}-$symmetric, we infer the $\mathcal{E}-$symmetry of p through $(3)_1$, implying the zero mean value condition; moreover, u satisfies (12), i.e. the restriction of u to Ω is a weak solution of (3).

Let $u_0 \in V$ (and, also, the symmetric extension $u_0 \in V^{\mathcal{E}}$); if we multiply the equations in (20) by $\lambda_k c_k^n(t)$ and we sum over k, we get

$$\frac{d}{dt}\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} + \mu\|\Delta u^n(t)\|^2_{2,\Omega_1} = \big((u^n(t)\cdot\nabla)u^n(t), \Delta u^n(t)\big)_{\Omega_1} - \big(f(t), \Delta u^n(t)\big)_{\Omega_1}, \tag{24}$$

since $\big(u_t^n(t), -\Delta u^n(t)\big)_{\Omega_1} = \frac{d}{dt}\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1}$. Under Dirichlet boundary conditions, the regularity of weak solutions is well-established, see [16, Theorems 2-2'] or [13, Theorem 6.1]. This method cannot be directly applied to Navier boundary conditions due the already mentioned possible failure of the Poincaré inequality, see Proposition 1. Hence, we need to distinguish two cases:

<u>Case 1:</u> If Ω_1 not axisymmetric, for the nonlinear term in (24), we use the Sobolev inequality, the Poincaré inequality $(11)_1$ and the equivalence between the norms $\|\nabla\cdot\|_{2,\Omega_1}$ and $\|\mathbf{D}\cdot\|_{2,\Omega_1}$

$$\|v\|_{6,\Omega_1} \le C_1\|\mathbf{D}v\|_{2,\Omega_1} \qquad \forall v \in V^{\mathcal{E}} \tag{25a}$$

$$\|\nabla w\|_{3,\Omega_1} \le \overline{C}_2(\|\Delta w\|^{1/2}_{2,\Omega_1}\|\mathbf{D}w\|^{1/2}_{2,\Omega_1} + \|\mathbf{D}w\|_{2,\Omega_1}) \le C_2\|\mathbf{D}w\|^{1/2}_{2,\Omega_1}\|\Delta w\|^{1/2}_{2,\Omega_1} \quad \forall w \in H^2(\Omega_1)\cap V^{\mathcal{E}} \tag{25b}$$

in which $C_1, C_2, \overline{C}_2 > 0$ are constants depending on the domain Ω_1, see [11, p.27]. Since $\|(v\cdot\nabla)w\|_{2,\Omega_1} \le \|v\|_{6,\Omega_1}\|\nabla w\|_{3,\Omega_1}$ for all $v, w \in H^2(\Omega_1)\cap V^{\mathcal{E}}$, we then infer

$$\begin{aligned}|\big((u^n\cdot\nabla)u^n, \Delta u^n\big)_{\Omega_1}| \le \|(u^n\cdot\nabla)u^n\|_{2,\Omega_1}\|\Delta u^n\|_{2,\Omega_1} &\le C_1C_2\|\mathbf{D}u^n\|^{3/2}_{2,\Omega_1}\|\Delta u^n\|^{3/2}_{2,\Omega_1} \\ &\le \frac{3^3C_1^4C_2^4}{2^5\mu^3}\|\mathbf{D}u^n\|^6_{2,\Omega_1} + \frac{\mu}{2}\|\Delta u^n\|^2_{2,\Omega_1},\end{aligned} \tag{26}$$

in which we used the Hölder inequality, (25a)-(25b) and the Young inequality $ab \le \frac{a^4}{4} + \frac{3}{4}b^{4/3}$ with $a = \big(\frac{3}{2\mu}\big)^{3/4}C_1C_2\|\mathbf{D}u^n\|^{3/2}_{2,\Omega_1}$ and $b = \big(\frac{2\mu}{3}\big)^{3/4}\|\Delta u^n\|^{3/2}_{2,\Omega_1}$.

We bound the last term in (24) by using the Schwartz and Young inequalities

$$|(f, \Delta u^n)_{\Omega_1}| \le \|f\|_{2,\Omega_1}\|\Delta u^n\|_{2,\Omega_1} \le \begin{cases} \dfrac{\|f\|^2_{2,\Omega_1}}{2\mu} + \dfrac{\mu}{2}\|\Delta u^n\|^2_{2,\Omega_1} \\ \dfrac{\|f\|^2_{2,\Omega_1}}{\mu} + \dfrac{\mu}{4}\|\Delta u^n\|^2_{2,\Omega_1}, \end{cases} \tag{27}$$

and, through (24)-(27), we obtain

$$\frac{d}{dt}\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} \le \gamma\|\mathbf{D}u^n(t)\|^6_{2,\Omega_1} + \frac{\|f(t)\|^2_{2,\Omega_1}}{2\mu}, \tag{28}$$

where $\gamma := \frac{3^3 C_1^4 C_2^4}{2^5 \mu^3}$. By applying Lemma 1 with $y(t) = \|\mathbf{D}u^n(t)\|^2_{2,\Omega_1}$ and $h(t) = \frac{\|f(t)\|^2_{2,\Omega_1}}{2\mu}$ we infer that

$$\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} \le \frac{1}{\sqrt{\left(\|\mathbf{D}u_0\|^2_{2,\Omega_1} + \frac{\|f\|^2_{2,Q^1_T}}{2\mu}\right)^{-2} - 2\gamma t}} := F(t) \qquad \forall t \in [0, T^*), \tag{29}$$

for some

$$T^* \ge \frac{2\mu^2}{\gamma\left(2\mu\|\mathbf{D}u_0\|^2_{2,\Omega_1} + \|f\|^2_{2,Q^1_T}\right)^2} = \frac{K_\Omega\, \mu^5}{\left(2\mu\|\mathbf{D}u_0\|^2_{2,\Omega} + \|f\|^2_{2,Q_T}\right)^2} \quad K_\Omega := \frac{2^{6-2m}}{3^3 C_1^4 C_2^4} \ \ (m = 1), \tag{30}$$

recalling that $\|\mathbf{D}u^n(0)\|^2_{2,\Omega_1} \le \|\mathbf{D}u_0\|^2_{2,\Omega_1} = 2\|\mathbf{D}u_0\|^2_{2,\Omega}$ and $\|f\|^2_{2,Q^1_T} = 2\|f\|^2_{2,Q_T}$.

Then we integrate (24) from 0 to $t \in [0, T^*)$ and, through (26)-(27), we find $G(t) > 0$ on $[0, T^*)$ such that

$$\begin{aligned} &\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} + \frac{\mu}{4}\int_0^t \|\Delta u^n(\tau)\|^2_{2,\Omega_1} d\tau \le \|\mathbf{D}u_0\|^2_{2,\Omega_1} + \frac{1}{\mu}\int_0^t \|f(\tau)\|^2_{2,\Omega_1} d\tau + \gamma \int_0^t \|\mathbf{D}u^n(\tau)\|^6_{2,\Omega_1} d\tau \\ &\Rightarrow \int_0^t \|\Delta u^n(\tau)\|^2_{2,\Omega_1} d\tau \le \frac{4}{\mu}\left(\|\mathbf{D}u_0\|^2_{2,\Omega_1} + \frac{\|f\|^2_{2,Q^1_{T^*}}}{\mu} + \gamma\int_0^t F(\tau)^3 d\tau\right) := G(t) \qquad \forall t \in [0, T^*). \end{aligned} \tag{31}$$

Subsequently, we multiply the first equation in (20) by $\frac{d}{dt}c^n_k(t)$ and we sum for k from 1 to n, obtaining

$$\|u^n_t(t)\|^2_{2,\Omega_1} = \mu\big(\Delta u^n(t), u^n_t(t)\big)_{\Omega_1} - \big((u^n(t)\cdot\nabla)u^n(t), u^n_t(t)\big)_{\Omega_1} + \big(f(t), u^n_t(t)\big)_{\Omega_1}.$$

By proceeding as for (26), through Hölder and Young inequalities we have

$$\begin{aligned} \|u^n_t(t)\|_{2,\Omega_1} &\le \mu\|\Delta u^n(t)\|_{2,\Omega_1} + \|(u^n(t)\cdot\nabla)u^n(t)\|_{2,\Omega_1} + \|f(t)\|_{2,\Omega_1} \\ &\le \mu\|\Delta u^n(t)\|_{2,\Omega_1} + C_1C_2\|\mathbf{D}u^n(t)\|^{3/2}_{2,\Omega_1}\|\Delta u^n(t)\|^{1/2}_{2,\Omega_1} + \|f(t)\|_{2,\Omega_1} \\ &\le \|\Delta u^n(t)\|_{2,\Omega_1}(\mu + \tfrac{C_1C_2}{2}) + \tfrac{C_1C_2}{2}\|\mathbf{D}u^n(t)\|^3_{2,\Omega_1} + \|f(t)\|_{2,\Omega_1}. \end{aligned} \tag{32}$$

After squaring and integrating from 0 to $t \in [0, T^*)$, we obtain

$$\begin{aligned} \int_0^t \|u^n_t(\tau)\|^2_{2,\Omega_1} d\tau \le& 3(\mu + \tfrac{C_1C_2}{2})^2 \int_0^t \|\Delta u^n(\tau)\|^2_{2,\Omega_1} d\tau + \tfrac{3}{4}C_1^2C_2^2 \int_0^t \|\mathbf{D}u^n(\tau)\|^6_{2,\Omega_1} d\tau + 3\int_0^t \|f(\tau)\|^2_{2,\Omega_1} d\tau \\ \le& 3(\mu + \tfrac{C_1C_2}{2})^2 G(t) + \tfrac{3}{4}C_1^2C_2^2 \int_0^t F(\tau)^3 d\tau + 3\|f\|^2_{2,Q^1_{T^*}} \qquad \forall t \in [0, T^*). \end{aligned} \tag{33}$$

From (29)-(31)-(33) we infer the boundedness of u^n in $L^\infty(0, T^*; V^{\mathcal{E}})$ and the boundedness of Δu^n, u^n_t in $L^2(0, T^*; L^2(\Omega_1))$. Hence, up to a subsequence, we have weak convergence in $L^2(0, T^*; H^{\mathcal{E}})$ of u^n_t and Δu^n, respectively, to u_t and Δu. For some $v^m_k \in C^0[0, T^*]$ let $v^m(x,t) := \sum_{k=1}^m v^m_k(t)e_k(x)$; multiplying the equations in (20) by $v^m_k(t)$ and summing for k from 1 to m we get

$$\int_0^{T^*}\int_{\Omega_1} \big(u^n_t - \mu\Delta u^n + (u^n\cdot\nabla)u^n - f\big)\cdot v^m dxdt = 0 \qquad \forall n \ge m.$$

Hence, by letting $n \to \infty$, we obtain $\int_0^{T^*}\int_{\Omega_1}\big(u_t - \mu\Delta u + (u\cdot\nabla)u - f\big)\cdot v^m dxdt = 0$. Since

$$\|(u(t)\cdot\nabla)u(t)\|_{2,\Omega_1} \le \tfrac{C_1C_2}{2}(\|\Delta u(t)\|_{2,\Omega_1} + \|\mathbf{D}u(t)\|^3_{2,\Omega_1}),$$

see (32), we infer that $u_t - \mu\Delta u + (u\cdot\nabla u) - f \in L^2(Q^1_{T^*})$. Being the space of such v^m dense in $L^2(0, T^*; H^{\mathcal{E}})$, there exists a unique function p with zero mean value and with $\nabla p \in L^2(Q^1_{T^*})$ such that $u_t - \mu\Delta u + (u\cdot\nabla)u - f = -\nabla p$. Since (u,p) is $\mathcal{E}-$symmetric and sufficiently regular, u satisfies (2) and (u,p) solves (3). Moreover, it is unique on $[0,T^*)$ and satisfies (14).

Case 2: If Ω_1 is axisymmetric, we decompose $u^n = \overline{u}^n + u^n_{\mathcal{K}}$ following (10) with $V^{\mathcal{E}}$ and Ω_1 instead of V and Ω. The main difference with Case 1 is the failure of the Poincaré inequality, see Proposition 1. Hence, the estimates (25a)-(25b) become

$$\|v\|_{6,\Omega_1} \le \|\overline{v}\|_{6,\Omega_1} + \|v_{\mathcal{K}}\|_{6,\Omega_1} \le C_3\Big(\|\mathbf{D}\overline{v}\|_{2,\Omega_1} + \|v_{\mathcal{K}}\|_{2,\Omega_1}\Big) \qquad \forall v\in V^{\mathcal{E}} \tag{34a}$$

$$\begin{aligned}\|\nabla w\|_{3,\Omega_1} &\le \|\nabla \overline{w}\|_{3,\Omega_1} + \|\nabla w_{\mathcal{K}}\|_{3,\Omega_1}\\ &\le C_4\Big(\|\Delta\overline{w}\|^{1/2}_{2,\Omega_1}\|\mathbf{D}\overline{w}\|^{1/2}_{2,\Omega_1} + \|\mathbf{D}\overline{w}\|_{2,\Omega_1} + \|\nabla w_{\mathcal{K}}\|_{2,\Omega_1}\Big) \qquad \forall w\in H^2(\Omega_1)\cap V^{\mathcal{E}}\end{aligned} \tag{34b}$$

with $C_3, C_4 > 0$ depending on Ω_1. By repeated use of Hölder and Young inequalities, we bound the nonlinear term in (24)

$$\begin{aligned}\left|\big((u^n\cdot\nabla)u^n,\Delta u^n\big)_{\Omega_1}\right| &\le \|(u^n\cdot\nabla)u^n\|_{2,\Omega_1}\|\Delta u^n\|_{2,\Omega_1} \le C_3\Big(\|\mathbf{D}\overline{u}^n\|_{2,\Omega_1} + \|u^n_{\mathcal{K}}\|_{2,\Omega_1}\Big)\\ &\quad\times C_4\Big(\|\Delta\overline{u}^n\|^{1/2}_{2,\Omega_1}\|\mathbf{D}\overline{u}^n\|^{1/2}_{2,\Omega_1} + \|\mathbf{D}\overline{u}^n\|_{2,\Omega_1} + \|\nabla u^n_{\mathcal{K}}\|_{2,\Omega_1}\Big)\|\Delta\overline{u}^n\|_{2,\Omega_1}\\ &\le \frac{\mu}{2}\|\Delta\overline{u}^n\|^2_{2,\Omega_1} + \frac{C_5}{\mu^3}\Big(\|\mathbf{D}\overline{u}^n\|_{2,\Omega_1} + \|u^n_{\mathcal{K}}\|_{2,\Omega_1}\Big)^4\|\mathbf{D}\overline{u}^n\|^2_{2,\Omega_1}\\ &\quad + \frac{C_6}{\mu}\Big(\|\mathbf{D}\overline{u}^n\|_{2,\Omega_1} + \|u^n_{\mathcal{K}}\|_{2,\Omega_1}\Big)^2\big(\|\mathbf{D}\overline{u}^n\|^2_{2,\Omega_1} + \|\nabla u^n_{\mathcal{K}}\|^2_{2,\Omega_1}\big).\end{aligned} \tag{35}$$

Due to (23) and to the finite dimensionality $\dim\mathcal{K}_{\Omega_1} \le 3$ there exists $C_7 := C_7(\|u_0\|_{2,\Omega_1}, \|f\|_{2,Q^1_T})$ such that

$$\|u^n_{\mathcal{K}}(t)\|_{2,\Omega_1},\ \|\nabla u^n_{\mathcal{K}}(t)\|_{2,\Omega_1} \le C_7 \qquad \text{for a.e. } t\in[0,T].$$

Hence, (35) can be rewritten as

$$\left|\big((u^n\cdot\nabla)u^n,\Delta u^n\big)_{\Omega_1}\right| \le \frac{\mu}{2}\|\Delta\overline{u}^n\|^2_{2,\Omega_1} + \gamma\big(1 + \|\mathbf{D}\overline{u}^n\|^6_{2,\Omega_1}\big)$$

with $\gamma := \gamma(\Omega,\mu,\|u_0\|_{2,\Omega_1},\|f\|_{2,Q^1_T}) > 0$ increasing with respect to $\|u_0\|_{2,\Omega_1}$, $\|f\|_{2,Q^1_T}$ and decreasing with respect to μ. Therefore (28) becomes

$$\frac{d}{dt}\|\mathbf{D}u^n(t)\|^2_{2,\Omega_1} = \frac{d}{dt}\|\mathbf{D}\overline{u}^n(t)\|^2_{2,\Omega_1} \le \gamma\|\mathbf{D}\overline{u}^n(t)\|^6_{2,\Omega_1} + \gamma + \frac{\|f(t)\|^2_{2,\Omega_1}}{2\mu}. \tag{36}$$

Lemma 1 still holds with $y(t) = \|\mathbf{D}\overline{u}^n(t)\|^2_{2,\Omega_1}$, $h(t) = \gamma + \frac{\|f(t)\|^2_{2,\Omega_1}}{2\mu}$ and, as above, we infer the existence of

$$T^* = T^*(\Omega,\mu,\|u_0\|_{2,\Omega},\|\mathbf{D}u_0\|_{2,\Omega},\|f\|_{2,Q_T}).$$

The rest of the proof follows as in Case 1, up to the constants. We underline that the key point in this case is the estimate (23), producing the dependence on $\|u_0\|_{2,\Omega}$ on the bounding constants.

• **Sectors of type** (A) **with** $m \ge 2$. In this case Ω has m faces $\omega_1,\dots,\omega_m$ and its boundary is given by (6) for some Γ having $C^{2,1}$-regularity. For $m=2$ we define $\Omega_2 = \Omega_{P_{\omega_1},P_{\omega_2}}$ and we consider first (3) in $\Omega_{P_{\omega_1}}$, thereby reducing to a domain with a unique face containing ω_2. To this problem we apply the results proved for $m=1$ and we infer the statement for $m=2$ since $\Omega_2 = \big(\Omega_{P_{\omega_1}}\big)_{P_{\omega_2}}$. More generally, we proceed by finite induction: for $m\ge 3$ we exploit the results obtained in the case $m-1$.

• **Sectors of type** (B)**, smoothly periodically extendable in one direction.** This is the only case of sectors (B) where the $\mathcal{K}_{\Omega^m}$ may be nontrivial and, due to Proposition 1, it has at most dimension 1. Moreover, in this case, there exists a unique couple of planes in (5), say $p_1 \parallel p_4$. With no loss of generality, we assume $p_1 : z = 0$, $p_4 : z = z_0$ and $\widetilde{p}_4 : z = -z_0$ the symmetric of p_4 with respect to p_1, for some $z_0 \in \mathbb{R}^+$. We define the *cell of periodicity* $\Omega_{\mathcal{P}}$ as the result of the reflections stated in (7), see also the examples in Appendix 1.

We include the periodicity condition in the functional setting, defining

$$H^s_{\mathcal{P}}(\Omega_{\mathcal{P}}) := \left\{ u \in H^s(\Omega_{\mathcal{P}}) :\ u = \sum_{k\in\mathbb{Z}} c_k e^{\frac{i\pi k}{z_0} z},\ \sum_{k\in\mathbb{Z}} k^{2s}|c_k|^2 < \infty \right\} \quad (s = 0, 1)$$

with $H^0 = L^2$, and we obtain the spaces ($H^{\mathcal{E}}(\Omega_{\mathcal{P}})$ and $V^{\mathcal{E}}(\Omega_{\mathcal{P}})$ are as in (17), with $\Omega_{\mathcal{P}}$ replacing Ω_1)

$$H_{\mathcal{P}} := H^{\mathcal{E}}(\Omega_{\mathcal{P}}) \cap L^2_{\mathcal{P}}(\Omega_{\mathcal{P}}) \qquad V_{\mathcal{P}} := V^{\mathcal{E}}(\Omega_{\mathcal{P}}) \cap H^1_{\mathcal{P}}(\Omega_{\mathcal{P}}) \ :$$

these are Hilbert spaces endowed with the scalar products in (9) on $\Omega_{\mathcal{P}}$.

Then we define the symmetric extensions $f^{\mathcal{P}}$ and $u_0^{\mathcal{P}}$ of f and u_0 in $\Omega_{\mathcal{P}}$. We define f^1 and u_0^1 as the $\mathcal{E}-$symmetric extension of $f^0 \equiv f$ and $u_0^0 \equiv u_0$ in Ω^1, see (18) with Ω^1 instead of Ω_1. By iterating, we define f^m and u_0^m respectively the $\mathcal{E}-$symmetric extension of f^{m-1} and u_0^{m-1} on Ω^m ($m \geq 1$), see (18) with Ω^{m-1} and Ω^m instead of Ω and Ω_1; clearly, f^m and u_0^m are coherent with the symmetries of Ω^m. Following (7), if $\Omega^m \equiv \Omega_{\mathcal{P}}$ we put $f^{\mathcal{P}}$ and $u_0^{\mathcal{P}}$ respectively equal to f^m and u_0^m; if not, we define $f^{\mathcal{P}}$ and $u_0^{\mathcal{P}}$ respectively as the $\mathcal{E}-$symmetric extension of f^m and u_0^m on $\Omega_{\mathcal{P}}$, see (18) with Ω^m and $\Omega_{\mathcal{P}}$ instead of Ω and Ω_1. Clearly, taking $f \in L^2(0,T;H)$ and $u_0 \in V$ we get $f^{\mathcal{P}} \in L^2(0,T;H_{\mathcal{P}})$ and $u_0^{\mathcal{P}} \in V_{\mathcal{P}}$.

We denote by $\Gamma_1 := \partial\Omega_{\mathcal{P}} \cap (p_4 \cup \widetilde{p}_4)$ and $\Gamma_N := \partial\Omega_{\mathcal{P}} \setminus \Gamma_1$. Then we construct an auxiliary problem $(3)^{\mathcal{P}}$, considering (3) with $\Omega_{\mathcal{P}}$, $u^{\mathcal{P}}$, $p^{\mathcal{P}}$, $f^{\mathcal{P}}$, $u_0^{\mathcal{P}}$ instead of Ω, u, p, f, u_0 and by replacing (2) with the periodic boundary conditions

$$(37)\quad u^{\mathcal{P}}(x,y,-z_0,t) = u^{\mathcal{P}}(x,y,z_0,t) \quad \text{on } \Gamma_1 \times (0,T)\,, \qquad u^{\mathcal{P}}\cdot\nu = (\mathbf{D}u^{\mathcal{P}}\cdot\nu)\cdot\tau = 0 \quad \text{on } \Gamma_N \times (0,T).$$

We then introduce the related Stokes eigenvalue problem

$$(38)\qquad \begin{cases} -\Delta e^{\mathcal{P}} + \nabla p^{\mathcal{P}} = \lambda^{\mathcal{P}} e^{\mathcal{P}} & \text{in } \Omega_{\mathcal{P}}\,, \\ \nabla\cdot e^{\mathcal{P}} = 0 & \text{in } \Omega_{\mathcal{P}}\,, \\ e^{\mathcal{P}}(x,y,-z_0) = e^{\mathcal{P}}(x,y,z_0) & \text{on } \Gamma_1, \\ e^{\mathcal{P}}\cdot\nu = (\mathbf{D}e^{\mathcal{P}}\cdot\nu)\cdot\tau = 0 & \text{on } \Gamma_N. \end{cases}$$

All the eigenvalues of (38) have finite multiplicity and can be ordered in an increasing divergent sequence $\{\lambda_k^{\mathcal{P}}\}_{k\in\mathbb{N}_+}$, in which the eigenvalues are repeated according to their multiplicity. Moreover, the set of eigenvectors $\{e_k^{\mathcal{P}}\}_{k\in\mathbb{N}_+}$ forms a compete orthogonal system in $V_{\mathcal{P}}$ and $H_{\mathcal{P}}$.

Following [24, Theorem 3.1], where the cube periodic problem is treated, and the passages as for sectors (A), we infer the existence of a global weak solution $u^{\mathcal{P}} \in L^2(0,T;V_{\mathcal{P}}) \cap L^\infty(0,T;H_{\mathcal{P}})$ of $(3)^{\mathcal{P}}$-(37). By taking its restriction to the original sector Ω, we obtain a weak solution of (3).

We write (24) with the norm and the scalar product of $H_{\mathcal{P}}$ and, repeating similar steps as for sectors (A) with $m = 1$, we obtain that the weak solution is unique in $[0, T^*)$. Moreover, it has the regularity $u^{\mathcal{P}} \in L^\infty(0,T^*;V_{\mathcal{P}})$ with $u_t^{\mathcal{P}}, \Delta u^{\mathcal{P}}, \nabla p^{\mathcal{P}} \in L^2(0,T^*;L^2(\Omega_{\mathcal{P}}))$.

Since $(u^{\mathcal{P}}, p^{\mathcal{P}})$ is $\mathcal{E}-$symmetric in $\Omega_{\mathcal{P}}$ and sufficiently regular, u satisfies the Navier boundary conditions on $\partial\Omega^m$. Since the data have m $\mathcal{E}-$symmetries on Ω^m, the solution $(u^{\mathcal{P}}, p^{\mathcal{P}})$ has the same symmetries and its restriction satisfies the Navier boundary conditions on $\partial\Omega$; moreover, the solution of (3) satisfies (14).

• **Sectors of type (B), smoothly periodically extendable in two or three directions.** The arguments of the previous case may be adapted to all sectors of type (B). One has to properly define the periodic cell $\Omega_{\mathcal{P}}$ following the principle (7) and the boundary conditions (37); we point out that $\Gamma_N = \emptyset$ only in the case of rectangular parallelepipeds. One has also to modify the functional setting by introducing periodic spaces in two or three dimensions and symmetric spaces with symmetries in two or three directions. The rest of the proof follows as for domains smoothly periodically extendable in one direction.

Proof of Theorem 2. Assume that Ω is a sector of type (A) with $m = 1$, the other cases being similar. Consider the approximate solution in (21), that already takes into account the reflection and the extension to Ω_1. As in the proof of Theorem 1 we distinguish two cases.

Case 1: If Ω_1 is not axisymmetric, applying $(11)_1$ to (22) with $f \equiv 0$ we obtain

$$\frac{d}{dt}\|u^n(t)\|^2_{2,\Omega_1} + \frac{4\mu}{C^2_\Omega}\|u^n(t)\|^2_{2,\Omega_1} \le 0 \quad \Rightarrow \quad \|u^n(t)\|^2_{2,\Omega_1} \le \|u_0\|^2_{2,\Omega_1} e^{-\frac{4\mu}{C^2_\Omega}t};$$

hence, integrating (22) over $[0,T]$, we have

$$(39)\quad \|u^n(T)\|^2_{2,\Omega_1} + 4\mu\int_0^T \|\mathbf{D}u^n(t)\|^2_{2,\Omega_1}dt = \|u^n(0)\|^2_{2,\Omega_1} \quad \Rightarrow \quad \int_0^\infty \|\mathbf{D}u^n(t)\|^2_{2,\Omega_1}dt = \frac{\|u^n(0)\|^2_{2,\Omega_1}}{4\mu},$$

where we let $T \to \infty$. We set $y(t) = \|\mathbf{D}u^n(t)\|^2_{2,\Omega_1}$, $\gamma = \frac{3^3C_1^4C_2^4}{2^5\mu^3} = \frac{1}{\overline{K}^2_\Omega\mu^3}$, $E = \frac{\|u^n(0)\|^2_{2,\Omega_1}}{4\mu}$ and we write (28) as $\dot{y}(t) \le \gamma y(t)^3$. Then Lemma 2-$(i)$ applies provided that

$$E = \frac{\|u^n(0)\|^2_{2,\Omega_1}}{4\mu} < \frac{\overline{K}^2_\Omega\mu^3}{\|\mathbf{D}u^n(0)\|^2_{2,\Omega_1}} \iff \|u^n(0)\|_{2,\Omega}\|\mathbf{D}u^n(0)\|_{2,\Omega} < \overline{K}_\Omega\mu^2.$$

The latter inequality is ensured by (15) with $C = \frac{\overline{K}_\Omega\mu^2}{\|u_0\|_{2,\Omega}}$. Lemma 2 gives a uniform bound for the $L^\infty(\mathbb{R}^+, V)$ norm of u^n, that also holds for the limit u; this proves the first statement in Case 1.

Case 2: If Ω_1 is axisymmetric, we rewrite (22) with $f \equiv 0$ as

$$(40)\qquad \frac{d}{dt}\big(\|\overline{u}^n(t)\|^2_{2,\Omega_1} + \|u^n_{\mathcal{K}}(t)\|^2_{2,\Omega_1}\big) + 4\mu\|\mathbf{D}\overline{u}^n(t)\|^2_{2,\Omega_1} = 0.$$

From this and the finite dimensionality of $\mathcal{K}_{\Omega_1}$, we first infer that, for all $t \ge 0$

$$(41)\quad \|u^n(t)\|_{2,\Omega_1} \le \|u^n(0)\|_{2,\Omega_1} \;\Rightarrow\; \|u^n_{\mathcal{K}}(t)\|_{2,\Omega_1} \le \|u^n(0)\|_{2,\Omega_1} \;\Rightarrow \|\nabla u^n_{\mathcal{K}}(t)\|_{2,\Omega_1} \le C_1\|u^n(0)\|_{2,\Omega_1}.$$

Then, by (40) we see that $t \mapsto \|u^n(t)\|_{2,\Omega_1}$ is non-increasing and, hence, admits a (finite) limit as $t \to \infty$. Therefore, by integrating (40) over $[0,T]$ and by letting $T \to \infty$ we obtain

$$(42)\qquad 4\mu\int_0^\infty \|\mathbf{D}\overline{u}^n(t)\|^2_{2,\Omega_1}dt = \|u^n(0)\|^2_{2,\Omega_1} - \lim_{T\to\infty}\|u^n(T)\|^2_{2,\Omega_1}.$$

Now we set $y(t) = \|\mathbf{D}\overline{u}^n(t)\|^2_{2,\Omega_1}$, $E = \frac{\|u^n(0)\|^2_{2,\Omega_1} - \lim_{T\to\infty}\|u^n(T)\|^2_{2,\Omega_1}}{4\mu}$ and we write (36) as $\dot{y}(t) \le \gamma\big(y(t)^3 + 1\big)$ with $\gamma := \gamma(\Omega, \mu, \|u_0\|_{2,\Omega})$ properly modified since $f \equiv 0$. Then Lemma 2-(ii) applies provided that

$$E = \frac{\|u^n(0)\|^2_{2,\Omega_1} - \lim_{T\to\infty}\|u^n(T)\|^2_{2,\Omega_1}}{4\mu} < \frac{1}{\gamma(\|\mathbf{D}\overline{u}^n(0)\|^2_{2,\Omega_1} + 1)}.$$

Due to (41), the latter inequality is implied by

$$(43)\qquad \|u^n(0)\|^2_{2,\Omega_1}(\|\mathbf{D}\overline{u}^n(0)\|^2_{2,\Omega_1} + 1) < \frac{4\mu}{\gamma} \iff \|\mathbf{D}\overline{u}^n(0)\|^2_{2,\Omega} < \frac{1}{2}\left(\frac{4\mu}{\gamma\|u^n(0)\|^2_{2,\Omega_1}} - 1\right),$$

where the last right hand side term is positive since $\frac{1}{\gamma} > \frac{1}{\gamma(y_0+1)}$. Hence, (15) with $C = \sqrt{\frac{\mu}{\gamma\|u_0\|^2_{2,\Omega}} - \frac{1}{2}}$ implies (43). Summarizing, by Lemma 2 we have a uniform upper bound for the $L^\infty(\mathbb{R}^+, V)$ norm of $\overline{u}^n$. Combined with (41), this gives a uniform upper bound for $u^n \in L^\infty(\mathbb{R}^+, V)$ and the first statement follows also in Case 2.

By Theorem 1, any local weak solution u can be globally extended to $u \in L^2(\mathbb{R}^+; V^{\mathcal{E}}) \cap L^\infty(\mathbb{R}^+; H^{\mathcal{E}})$. By the lower semicontinuity of the norm with respect to weak convergence, (39) and (42) give

$$\int_0^\infty \|\mathbf{D}u(t)\|^2_{2,\Omega_1} dt < \infty \,,$$

which yields the existence of $\mathcal{T} > 0$ such that $\|\mathbf{D}u(\mathcal{T})\|_{2,\Omega} < C(\Omega, \mu, \|u(\mathcal{T})\|_{2,\Omega})$, that is, (15) translated at initial time $\mathcal{T}$; therefore, the first statement applies and (16) holds.

4. Appendix 1: the reflection principle and two calculus lemmas

To understand the properties of sectors, in Figure 3 we give some examples of Lipschitz domains not fulfilling Definition 3. The domain on the left is 2/5 of a torus, but it does not generate the full torus because it is not the 2^m−th part of the torus. The next two domains do not satisfy condition (4) since $\Omega \cap P \neq \emptyset$. The domain on the right generates a periodically extendable domain which is not smooth.

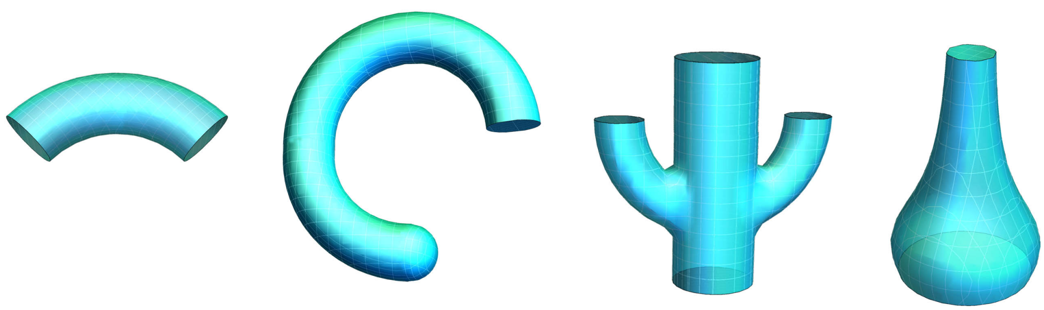

Figure 3. Some Lipschitz domains that *are not sectors*, according to Definition 3.

Then we illustrate how to apply the principle (7) for sectors of type (B). As explained in the proof of Theorem 1, we need to reflect also a cylinder or a cube with respect to some of the planes in (5), with a number of reflections $j \in \{1, 2, 3\}$ depending on the directions where the domain is periodically extendable. Hence, it is a double cylinder or eight times the cube that we treat as a smoothly periodically extendable domain; in doing so, Ω^{m+j} will be the cell of periodicity used in the proof. In Figure 4 we represent sectors of type (B): from left to right, they have to be reflected, respectively, in one, two or three directions, yielding domains Ω^{1+0} ($m = 1$, $j = 0$), Ω^{1+2} ($m = 1$, $j = 2$) and Ω^{0+3} ($m = 0$, $j = 3$); then they become smoothly periodically extendable.

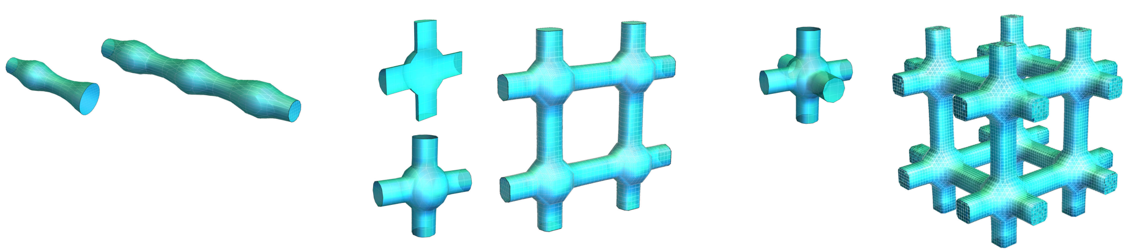

Figure 4. Sectors of type (B).

Apart for the drop of water, which is of type (A) and becomes a ball Ω_1 after one reflection, all the other domains in Figure 1 are of type (B). The pipe bifurcation and the vein become periodically extendable if reflected once, yielding Ω^{1+0}. The tunnel becomes periodically extendable with two reflections, yielding Ω^{1+1}. In general, we obtain the periodic cell Ω^{m+j} either directly ($j=0$) or after one/two/three reflections of Ω^m with respect to one/two/three planes among the p_i's ($i=1,\cdots,6$) in (5); hence, $|\Omega^{m+j}|=2^j|\Omega^m|$.

Finally, we state two calculus lemmas used to bound the time for uniqueness and regularity of the solution of (3).

Lemma 1. *Let $\gamma, T>0$, let $h\in L^1(0,T)$, and assume that $y\in \mathrm{Lip}_{\mathrm{loc}}[0,T)$ satisfies*

$$y(t)>0 \text{ in } [0,T)\,,\qquad \dot y(t)\le \gamma y(t)^3+h(t) \text{ a.e. in } [0,T)\,,\qquad \lim_{t\to T} y(t)=+\infty\,.$$

Then $y(t)\le \left(\dfrac{1}{(y(0)+\|h\|_{L^1(0,T)})^{-2}-2\gamma t}\right)^{1/2}$ *for all $t\in[0,T)$ and* $T\ge \dfrac{1}{2\gamma(y(0)+\|h\|_{L^1(0,T)})^2}$.

Proof. The result is a generalization of Bellman-Gronwall-Bihari inequality, for details see [9, Corollary 1-i)].

Lemma 2. *Let $\gamma>0$, $y\in \mathrm{Lip}_{\mathrm{loc}}[0,\infty)\cap L^1[0,\infty)$ with $y(t)>0$ in $[0,\infty)$, let $y_0:=y(0)$ and $E:=\int_0^\infty y(t)dt$. If one of the following conditions occurs*

(i) $\dot y(t)\le \gamma y(t)^3$ *a.e. in* $[0,\infty)$ *and* $E<\dfrac{1}{\gamma\, y_0}$,

(ii) $\dot y(t)\le \gamma\big(y(t)^3+1\big)$ *a.e. in* $[0,\infty)$ *and* $E<\dfrac{1}{\gamma(y_0+1)}$,

then there exists $K:=K\big(\gamma,y_0,E\big)>0$ such that $y(t)\le K$ for all $t\ge 0$.

Proof. (i) Let $T\in\big(0,\frac{1}{2\gamma y_0^2}\big)$ and F the solution of the differential equation

$$\begin{cases}\dot F(t)=\gamma F(t)^3 & t\in(0,T]\\ F(0)=y_0,\end{cases}$$

so that $y(t)\le F(t)$ for all $t\in[0,T]$. If $E<\int_0^T F(t)dt$ then there exists $t^*\in\big(0,T\big)$, satisfying $E=\int_0^{t^*}F(t)dt$; hence, by [16, Lemma 5] we find $y(t)\le F(t^*)$ for all $t\ge 0$. The thesis follows computing explicitly $F(t)=\frac{y_0}{\sqrt{1-2\gamma y_0^2 t}}$ for $t\in\big(0,\frac{1}{2\gamma y_0^2}\big)$, $\int_0^T F(t)dt=\frac{1-\sqrt{1-2\gamma y_0^2 T}}{\gamma y_0}$ with $T\in\big(0,\frac{1}{2\gamma y_0^2}\big)$, $t^*=\frac{E}{2y_0}\big(2-E\gamma y_0\big)$ and $F(t^*)=\frac{y_0}{1-E\gamma y_0}$.

(ii) We observe that

$$\dot y(t)\le \gamma\big(y(t)^3+1\big)\le \gamma\big(y(t)+1\big)^3 \text{ a.e. in } [0,\infty)$$

and we apply (i) to the function $y(t)+1$.

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