ON THE LOGICAL PHILOSOPHY OF ASSERTIVE GRAPHS

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Abstract. The logic of Assertive Graphs (AGs) is a modification of Peirce's logic of Existential Graphs (EGs), which is intuitionistic and which takes assertions as its explicit object of study. In this paper we extend AGs into a classical graphical logic of assertions (ClAG) whose internal logic is classical. The characteristic feature is that both AGs and ClAG retain deep-inference rules of transformation. Unlike classical EGs, both AGs and ClAG can do so without explicitly introducing polarities of areas in their language. We then compare advantages of these two graphical approaches to the logic of assertions with a reference to a number of topics in philosophy of logic and to their deep-inferential nature of proofs.

1. Introduction

The notion of assertion plays an essential role in logic. It is a key ingredient in most logical systems, either implicitly or explicitly. Frege's ideographical language of the *Begriffsschrift* introduced a specific sign designating assertion, '$\vdash$'. It expresses the acknowledgement of the truth of the content of the assertion ([5]).

In Peirce's graphical logic of Existential Graphs (EGs) which he introduced in 1896 there is no explicit sign for assertion. Yet the notion of assertion surfaces virtually everywhere across his logical writings. The reason is that making an assertion signals liability that the utterer of the logical statement bears on the truth of the proposition ([28]). Peirce incorporated assertion as an implicit sign *embedded* in the notation of the *Sheet of Assertion* (SA) ([3]). It is an embedded sign, since SA represents both logical truth and the assertoric nature of graphical logical formulas scribed upon the sheet.

In intuitionistic logic, an explicit notion of assertion became commonplace in analyzing inference and proofs ([1, 11, 15, 22]), and to explicate certain topics in the philosophy of logic such as the meaning of logical constants ([14, 18, 20, 34]). The idea of the notion of assertion thus appears robustly invariant across a range of logical theories, logical methods, and logical notations.

The prevalence of assertions in a number of logical theories suggests that there are also systems of logical graphs which take this embedded or implicit nature of assertions as an explicit object of study. Indeed the result is the system of Assertive Graphs (AGs) first introduced in [6]. The internal logic of AGs is intuitionistic. In the present paper, we supply the set of rules of AGs with a graphical rule, termed the rule of Elimination of Coinciding Corners, which transforms the behaviour of

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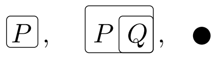
the system from intuitionistic to a classical one (ClAG). We then compare the two systems, intuitionistic and classical AGs, and discuss their impact to a number of topics in the philosophy of logic, especially as concerns inferentialism and the meaning of logical constants as represented in this graphical fashion.

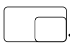
AGs provide a graphical (or, as some may prefer to say, diagrammatic) notation that serves as an alternative to performing logical inferences by its explicitly graphical rules of transformations. That notation and the rules explicitly encode assertions as their objects of transformation. The construction of AGs is inspired by Peirce's original method of Existential Graphs (EGs), which pioneered the emergence of graphical logics and systems of proofs ([28]).

For example, instead of "cuts" (ovals surrounding graphs), the languages of AGs and ClAGs use *boxes* (which do *not* signify negations) around assertions, *corners* (which signify conditionals $P \to Q$), and *blots* (which signify absurdities $\perp$):



All these are scribed on the sheet of assertion (SA). The sheet is also at once both a sign of assertion and a blank space signifying the top element ($\top$). It also denotes continuous connection between the subgraphs.

The characteristic feature of AGs is its lack of cuts. In EGs, cuts are needed in order to represent negations and conditionals, and they signify both parentheses and negation. The absence of ovals is, in turn, a result of the specific design feature of the notation of AGs, in which *polarities* of the areas of the graphs scribed on the SA need not be explicitly represented or taken into account in the application of the rules of transformation. For that reason there is also no need to introduce a separate sign for negations, either, which Peirce in his theory of EGs needed and which he further derived from the notion of the scroll that represents material conditional (see [2]). The absence of cuts means that the graphical formulas of AGs appear to be simpler and easier to read or comprehend than those of EGs, as they need not have multiple nestings of ovals.

Peirce's basic systems of EGs were its *alpha* and *beta* parts. They roughly correspond to (classical) propositional logic and (fragments of) first-order logic, respectively.[1] The difference between positive and negative areas is a crucial element in all of these systems. In AGs, however, explicitly denoting polarities is not needed. Unlike in EGs, there are no nested ovals in the language of AGs. This is due to two specific features of AGs: First, as noted there is the absence of cuts surrounding instances of graphs. Instead, graphs of AGs may be surrounded by *boxes*, , but they do not signify negations. Boxes may always be eliminated in favour of the blank sheet of assertion. In AGs, negation is defined as an implication of absurdity. Thus the conditional sign is represented in a graphically specific manner by "corners" . Thus the representation of a conditional is not a nesting of two boxes but a primitive sign, a *corner*, which may be thought of as resulting from a welding of two adjacent sides of two nested boxes. It is such design features that not only improve the readability of formulas in AGs but also contribute to the logical and philosophical meaning of the system.

[1]See e.g. [2, 29, 32, 33, 36] for these systems and the explanations of what the qualification 'roughly' means. Peirce also initiated the development of the *gamma* part of the theory of logical graphs, in which we find graphical modal (propositional and quantificational) logic, graphical epistemic logic and graphical higher-order logic, among others (see [26]).

The next section briefly introduces the basic language of AGs. Section 3 presents the intuitionistic basis of AGs, which in Section 4 is then extended into the classical variant (ClAG). Section 5 discusses main conceptual novelties of both AGs and ClAGs and compares the two systems and their impact to philosophy of logic, such as inferentialism and the meaning of logical constants. It is also shown how the deep-inference nature of transformations emerges from the transformation rules, which we take to mean that the graphical approach along the lines suggested here is the true logic of deep inference also for assertions. Section 6 concludes.

2. Assertive Graphs

Here we introduce the bare essentials of the graphical system of AGs. A detailed discussion of the basic conventions, language and the system of proofs of AGs is found in [6].

2.1. **Fundamental Conventions.** Expressions of AGs are instances of graphs standing for assertions and their relations, and are constructed from primitive ones in a recursive fashion. All graph instances are those that are scribed on a *sheet of assertion* (SA). Seven fundamental conventions are needed to frame the development of the language and logic of AGs.

Convention 1: We always have a right to a blank SA.

Convention 2: We denote the assertion of a graph α by writing it enclosed within a *box*. So, if α is the proposition P, this gives us the following graph:

$$\boxed{P}$$

Since graphs are scribed on the SA, anything on the SA is an assertion. Thus a box around a graph is not necessary for that graph to be an assertion. Rather, the boxing should be understood as a deictic device that draws our attention to the graph enclosed within it, such as "This is what I say:____ "; "Look, here's____ ". The point of this comes clear as we proceed.

Convention 3: A *juxtaposition* is to assert two graphs on the SA at two different positions. The meaning of juxtaposed graphs is that their significations are to be considered *independently* of each other:

$$\boxed{P} \quad \boxed{Q}$$

Thus this graph expresses the *conjunction* of two assertions.

Since the conjunction of assertion is equivalent to the assertion of the conjuncts—a standard feature of the logic of assertions—the previous graph is also equivalent to the following:

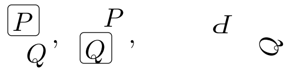
$$\boxed{P\ Q}$$

Notice that by virtue of the above conventions, the previous graph is equivalent also to any of the following three assertions:

$$\boxed{P}\ \ Q\ , \quad P\ \ \boxed{Q}\ , \quad P\ \ Q$$

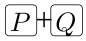
(Comma is not part of the language of graphs and does not appear on the SA, it merely means that the examples such as above are listed consecutively; in reality they are understood to be scribed on the two-dimensional SAs thrice.)

Since the SA is an unordered sheet, the following graphs are likewise examples of graphs equivalent to any of the graphs presented above in the context of Convention 3.



Commutativity and associativity result from the spatial representation of assertions and are not part of any explicit rule.

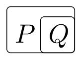
Convention 4: Two graphs juxtaposed on the SA but conceived not independently but *alternatively* asserted are connected by a *thin line* with a bar crossing it:[2]



This notation represents the *disjunction* of assertions.

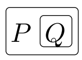
The next convention informs that the box notation is nevertheless not a superfluous design feature of these graphs, and that its importance comes from the way in which we represent conditional assertions.

Convention 5: We call *corner* the sign in which the inner and outer boxes are connected by the sharing of two adjacent sides:



This represents an implicational relation of two assertions. The graph P occurs on the *outer area* of the corner and Q occurs on the *inner area* of the corner. As these graphs are, just as existential graphs in general, read in terms of what Peirce calls an "endoporeutic principle" (R 293, [28]), namely from outside in, the graph on the outer area of the corner is the antecedent of the conditional and the graph on the inner area of the corner is its consequent. This notation is not constructed from the nesting of two boxes. The corner is a primitive sign in which the two adjacent sides of the box that demarcate the inner area of the corner are irrevocably welded with parts of the two adjacent sides of the box that demarcate the outer area of the corner.

It is thus worth noting that the graph above that expresses the implication of two assertions is not equivalent—that is, neither notationally, syntactically nor semantically equivalent—to the following:



The latter expresses an assertion of a conjunction of P with an assertion of Q.

Convention 6: An absurdity is indicated with a heavy dot '●', termed the *blot*.[3]

Convention 7: There are also features of AGs that are logically irrelevant and about which the conventions should pronounce nothing. In other words, no feature left unstated in these conventions has any significance to the meaning of the graphs.

For instance, the following graphs are not part of any of the conventions 1–6:

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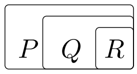
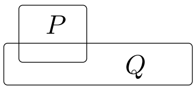
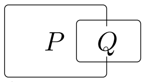
[2] The cross bar is a design feature added (i) in order to respect the history of the development of relevant notation for disjunctive assertions, and (ii) in order to not to confuse the connection with quantificational lines that may occur in relevant extensions of AGs.

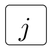
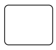
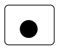
[3] The term "blot" is derived from Peirce's originals (R S-30, 1906). We should imagine it to refer to the blackening of the entire area on which that blot occurs, thus acquiring the meaning that there is no room left for any assertion on that area—that is, everything is false. For convenience, we represent such a blackening as a heavy black dot so that we need not represent very large (and possible infinite) areas of the sheet as black.

This concludes our abridged presentation of the system of conventions for AGs.

2.2. **The Language of AGs.** The set of well-formed (well-scribed) graphs of the language of AGs can be recursively defined as follows:

(1) Atomic graphs h, j, s,... (of a denumerable set); the blank   , the blot ●, and their boxings

, , , , 

scribed on the SA are well-formed graphs.

(2) If H is a well-formed graph, then also its boxing scribed on the SA is a well-formed graph:

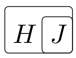
(3) If H is a well-formed graph and J is a well-formed graph, also the scribing of them at two different positions on the SA is a well-formed graph:

$H \quad J$

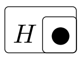
(4) If H and J are well-formed graph, also their connection with a line + is a well-formed graph:

$H + J$

(5) If H and J are well-formed graphs, then also the cornering, namely a graph in which J appears in the corner of H, scribed on the SA, is a well-formed graph:

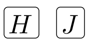
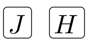
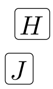
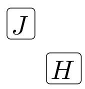


The set of these five rules define the formation rules for the language of AGs. By a graph we in what follows mean a well-formed graph. Some examples of non-well-formed (or rather non-well-scribed) graphs were given in connection to Convention 7 above. Clearly for instance the following is a well-formed graph whenever H is a well-formed graph:

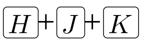
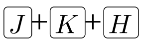
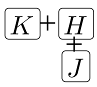


The meaning of this is to deny what H asserts. That is, the negation of an assertion is represented as the implication of the *absurdum* from that assertion.

It follows from the fact that the SA is an unordered, open-compact space that the properties of commutativity, associativity and adjunction naturally follow from the properties of that space, and that no separate rules are needed to pronounce the equivalence of assertions such as

 and  and  and ,

nor the equivalence of the assertions such as

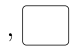
 and  and ,

and so on. Notice that when graphs are connected with the line + as in the last example, the ordering of these alternating connections is immaterial. Graphical notation is topological, and the points at which the line is connected to the graph are not fixed in any way. As long as the alternate connections are not disturbed, any disjunct thus connected can freely move along the SA, including passing through other disjuncts. In the above graph, for example, H, J and K are all connected to each other.

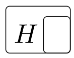
2.3. **Proofs in AGs.** The system of the logic of AGs is defined by graphical axioms and rules of transformation on the graphs of AGs. Here is a concise presentation of these graphical axioms and rules.

2.3.1. *Axioms of AGs.* The blank space indicates a tautology and can appear anywhere on the SA.

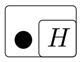
Axiom I: (The blank SA):



Axiom II: (Any graph implies a blank):



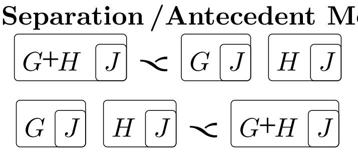
Axiom III: (*Ex falso*):



We use the sign '⤙' to denote the derivability relation for graphical expressions of the language of AGs. Simply put, G ⤙ H means that a graph G can be transformed into a graph H according to the rules of transformation.[4]

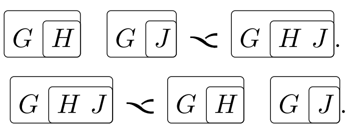
2.3.2. *Rules of Transformation.* The justification of the following set of transformation rules for AGs is quite straightforward.[5]

The Rule of Antecedent Separation /Antecedent Merging (As/Am):



That is, the disjunction of the antecedents of a cornering can be split into the juxtaposition of two cornerings with one (and not the same) of the two disjuncts as the antecedent and with the same consequent as the initial graph. Conversely, any two cornerings with the same consequent can be merged into a cornering with the disjunction of the antecedents of the initial graph and with the same consequent.

Consequent Merging/Consequent Separation (Cm/Cs):



[4]The sign '⤙' is Peirce's original and favourite design for logical consequence relation.

[5]That is, they are sound and complete, which can be shown by a Lindenbaum–Tarski construction as the underlying algebraic theory (Heyting algebra) is a variety and defines a congruence relation. A similar (though by no means identical) graphical intuitionistic system is [27], with more on e.g. admissible rules. The set of rules for AGs differs from graphical intuitionistic system in order to compensate for the lack of polarities—admittedly additional rules have to be introduced to do that and also because there are more logical primitives in AGs.

That is, the consequents of two cornerings with the same antecedent can be merged into the consequent of a cornering with the same antecedent as the initial cornerings. Conversely, the juxtaposed consequents in a cornering can be split into the juxtaposition of two cornerings with the same antecedent and with one (and not the same) of the two consequents.

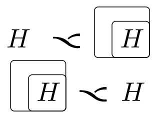
Rules of Disjunct Contradiction (DC):

$$H \prec H{+}\bullet$$

$$H{+}\bullet \prec H$$

That is, any graph is equivalent to that graph disjuncted with the blot (*absurdum*). The graphs here can just as well be boxed.

Cornering Rules (CR/UCR):



That is, any graph is equivalent to the cornering with that graph in its consequent and with a blank as its antecedent. In words, this captures that if H is scribed on the SA then H follows from the assertion of a tautology. If H follows from the assertion of a tautology, then H holds. This latter clause is called the uncornering rule (UCR).

Iteration/Deiteration Rule (It/DeIt):

To define the rule of iteration and its converse deiteration we first define the context of graphs. A *graphical context* is of the form K{ }, in which K is any graph-instance of the language of AGs, graph-instances enclosed within { } are said to be in the nest of K, and a single slot { } is the empty context. Let K$\{H\}$ be the graph obtained from K{ } by substituting H for that slot. The two rules then are:

Iteration (It): If a graph G occurs on the SA or anywhere in the nest of graphs K, it may be scribed on any area (which itself is not part of G) which (i) is the same area on which G occurs or (ii) is in the nest of $\{G\}$:

(i) $\mathrm{K}\{G\} \prec \mathrm{K}\{GG\}$. (ii) $\mathrm{K}\{GH\{J\}\} \prec \mathrm{K}\{GH\{GJ\}\}$.

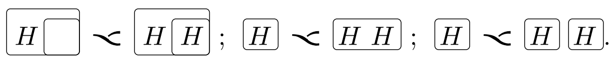
The converse of (It) is deiteration (DeIt).

Deiteration (DeIt): Anything that is the result is iteration may be deiterated, thus:

(i) $\mathrm{K}\{GG\} \prec \mathrm{K}\{G\}$. (ii) $\mathrm{K}\{GH\{GJ\}\} \prec \mathrm{K}\{GH\{J\}\}$.

Clearly, anything that is the result of deiteration can also be iterated.

The following are examples showing an application of the rule of iteration in AGs:



In the first of these examples, we iterate H that lies on the antecedent area of the cornering by copying and pasting it into the consequent area. In the second example, we iterate the occurrence of H on the same area in which it occurs. In the third example, we likewise iterate an occurrence of $\boxed{\mathrm{H}}$ on that same area. These three examples are all reversible by the application of the rule of deiteration.

Conjunction Elimination (CE):

$$G \;\; H \prec G$$
$$G \;\; H \prec H$$

That is, from the scribing of independently asserted G and H on the SA (excluding those cases in which G and H rest on the antecedent area of a cornering, unless at least one of them is a blot), it is possible to derive one of these assertions.

Naturally, the graphs could just as well be boxed:

$$\boxed{G} \;\; \boxed{H} \prec \boxed{G}$$
$$\boxed{G} \;\; \boxed{H} \prec \boxed{H}$$

Notice that commutativity comes for free from the spatial and non-linear nature of the language of AGs and thus the second clauses in the above two pairs of rules are completely redundant.

Disjunction Introduction (DI):

$$G \prec G{+}H$$
$$G \prec H{+}G$$

That is, from the scribing of G on the SA it is possible to derive a disjunction of that graph with a graph H. The disjunction, as denoted by the connecting line, means "to be alternatively asserted". In a similar vein, from an assertion of G it is possible to derive the assertion of G or the assertion of H:

$$\boxed{G} \prec \boxed{G}{+}\boxed{H}$$
$$\boxed{G} \prec \boxed{H}{+}\boxed{G}$$

Again, disjuncts have no priority ordering on the topology of the SA.

Insertion in the Antecedent (InsA):

This rule also works with the contexts K{ } of graphs. The applicability of the rule of insertion in the antecedent (InsA) below is restricted to the antecedents of the cornerings whose immediate context in K (that is, the area on which the cornering is placed) is not an antecedent of another cornering.[6] Then:

$$\mathrm{K}\{\boxed{G\,\boxed{H}}\} \prec \mathrm{K}\{\boxed{G\ J\,\boxed{H}}\}$$

That is, in any unoccupied position in the area of the antecedent of the cornering, which itself does not reside, as its immediate context, within an antecedent of a cornering,[7] it is possible to insert any graph.

Deletion from the Consequent (DelC):

$$\boxed{G\,\boxed{H\ J}} \prec \boxed{G\,\boxed{H}}$$

That is, it is possible to delete any graph from the consequent of a cornering.

This completes the system of transformation rules for AGs.

A *derivation* of a graph H from G in AGs is a finite sequence of graphs $H_0, \ldots, H_n$ such that $H_n = H$, and each H_i is either an axiom, or G, or derived from previous graphs by a rule of transformation. The graph H is *derivable* from G in AGs, notated $G \prec_{\mathrm{AG}} H$, if there is a derivation of H from G in AGs. The rules of

[6]An exception is when the consequent of the cornering whose antecedent we are to insert is occupied by the blot (and the antecedent J to be inserted is not the blot ●).

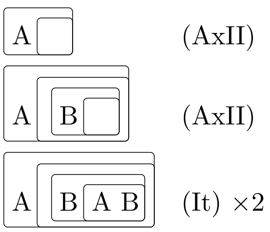
[7]This restriction to the applicability of the rule (InsA) is important, as the rule would otherwise permit derivation of invalid principles.

consequent merging (Cm) and separation (Cs) are derivable in the system of AGs and are thus not necessary to be included in the above list. But for instance (Cs) is a useful rule when showing that the deduction theorem holds ([27]).

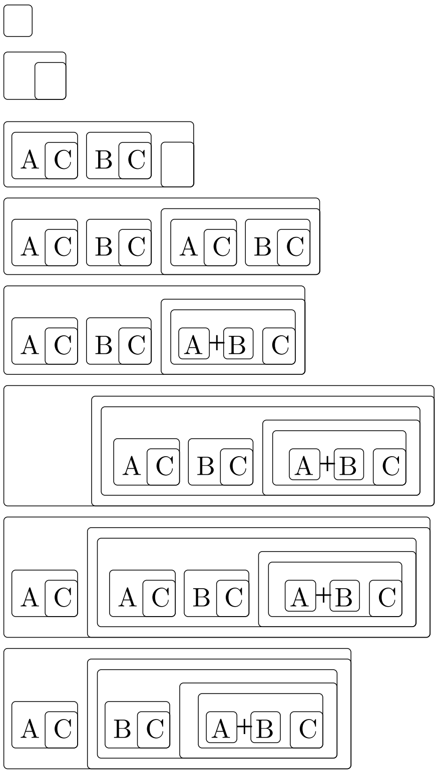
3. Assertive Graphs for Intuitionistic Logic

With these axioms and rules we can express all intuitionistically valid principles in AGs. Let us consider an axiomatic system for intuitionistic propositional logic [35, p. 68]. All these axioms can be derived in AGs. Each step in the following derivations takes place according to the derivability relation. The rules applied are tagged on the outcomes of the respective graphical transformations.

(1) $(A \wedge B) \rightarrow A$. This is justified by the transformation rules (AxII), and (It) in AGs (or, alternatively, by (AxI), (InsA) and (It)).
(2) $(A \wedge B) \rightarrow B$. Like (1).
(3) $A \rightarrow (B \rightarrow (A \wedge B))$. This principle can be justified in the following way:

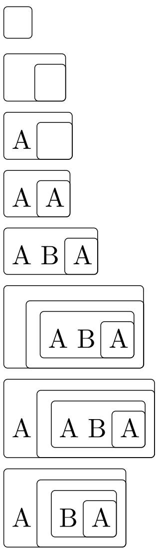


(4) $A \rightarrow (A \vee B)$. This is justified by (AxI), (It) and (DI) in AGs.
(5) $B \rightarrow (A \vee B)$. This is justified by (AxI), (It) and (DI) in AGs.
(6) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$. This intuitionistically valid formula (6) can be derived in AGs in the following way:



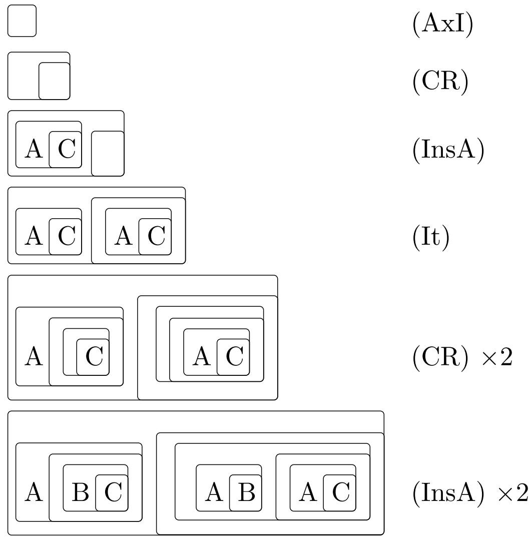
(AxI)
(CR)
(InsA)
(It) ×2
(Am)
(CR)
(InsA)
(DeIt)

(7) $A \to (B \to A)$. This can be derived in AGs as follows:



(AxI)
(CR)
(InsA)
(It)
(InsA)
(CR)
(InsA)
(DeIt)

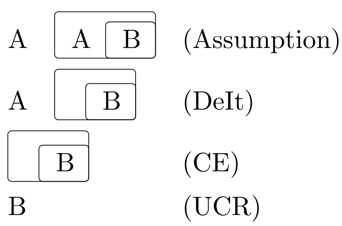
(8) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$. The validity of this formula can be shown in AGs in the following way:



In the last step, insertion is applied to antecedents whose immediate contexts are consequent areas of a cornering.

(9) $\bot \to A$. This is the axiom (AxIII) of AGs.

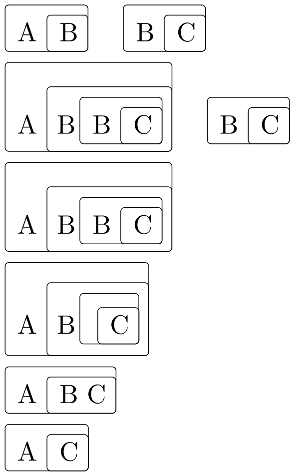
(10) From A and $A \to B$, derive B. This is justified in AGs as follows:



Notice that everything that is alternatively asserted on the SA is juxtaposed and hence is a conjunctive assertion.

Hence all the axioms and rules that completely characterize intuitionistic logic can be derived in the logic of AGs.

In order to familiarize ourselves with AGs, let us check how to derive [A C] from [A B] and [B C]:



(Assumption)

(It)

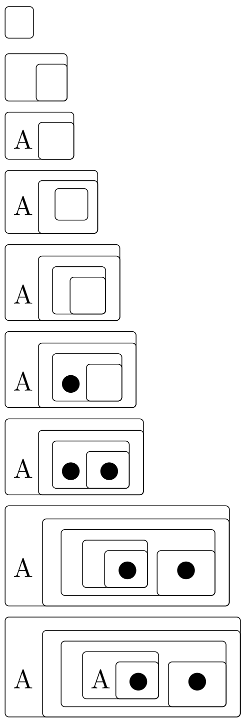
(CE)

(DeIt)

(UCR)

(DelC)

Here is how the principle of the introduction of a double negation is justified:



(AxI)

(CR)

(InsA)

(AxI)

(CR)

(InsA)

(It)

(CR)

(It)

That is, $(A \to ((A \to \bot) \to \bot))$ comes out as a theorem of AGs.

Notice that given the intuitionistic behaviour of AGs, the equivalent principles such as the elimination of the double negation, the law of the excluded middle and Peirce's Law cannot be proven. That we can derive the negation of A from the triple negation of A can, by contrast, be proven in AGs (see [6] in which this is proven for intuitionistic system of logical graphs).

In the next section, we show how to extend AGs to deal with these classically valid principles. The extension can be achieved without changing anything in the set of conventions or in the language of AGs. In particular, we can dispense both with introducing the 'cuts' as well as the negative and positive polarities of areas that would ensue from having them, despite the fact that such notions are characteristic features of the way in which classical alpha and the beta parts of the method of EGs were set up in Peirce's original graphical notation.

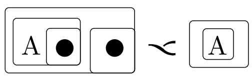
4. Assertive Graphs for Classical Logic

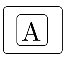
It is well-known that having principles such as Peirce's Law or the full rule of double negation in the system of axioms of the intuitionistic calculus ([35]) would lead to a classical calculus. Hence, in order to develop a classical version of AGs, which we call ClAGs, we will make a modification to the system of rules of AGs: we add a new graphical rule that enables us to *eliminate coinciding corners occupied by blots*.

The derivability '$\prec$' below means derivability in this new system ClAG. If there is a risk of confusion, we can denote it by $\prec_{\mathrm{ClAG}}$. In order to produce a classical graphical logic of AGs, the previous system is extended by the following rule.

We add the following rule of *Eliminating Coinciding Corners* (ECC):

Eliminating Coinciding Corners (ECC):



That is, if an absurdity follows from a proposition from which an absurdity follows, then we can infer that proposition. Notice that by Convention 2,  and A are exactly the same proposition.

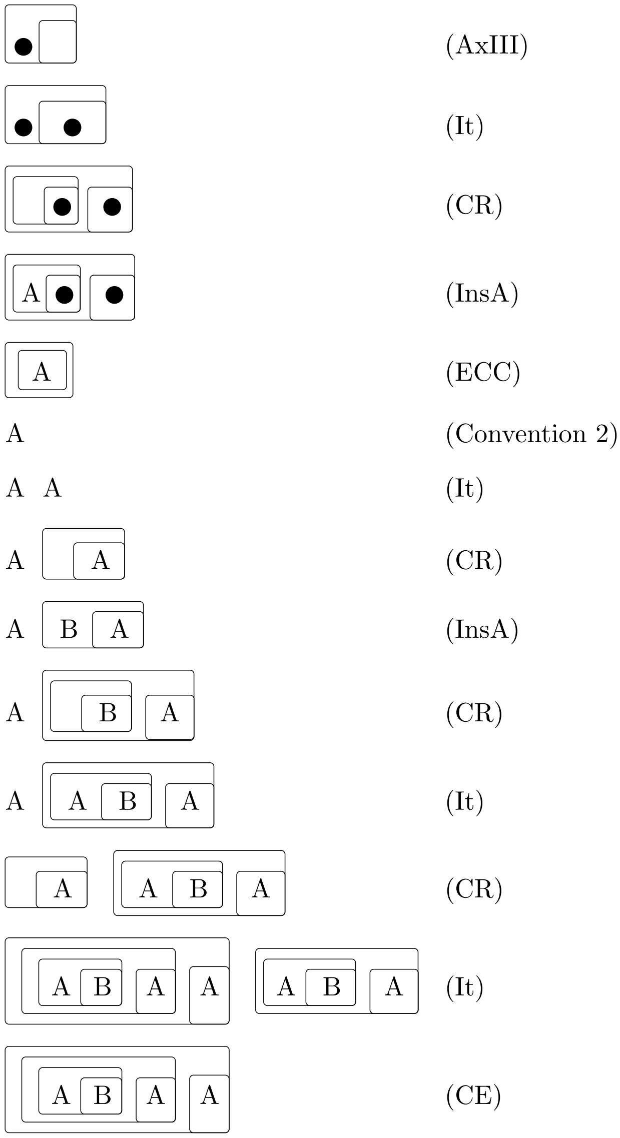
The rationale behind the rule (ECC) lies in the graphical fact that whenever there are no intermediate graphs (other than the blanks) between two identical corners occupied by a blot, then these corners annihilate each other.

For example, as an imaginary exercise think of the blot to blacken out the entire content of the corner which it occupies, so that when two such adjacent blackened corners touch upon each other, they would at once cancel each other out. What would remains from this is the proposition A resting upon the antecedent of an antecedent is surrounded by two boxes, which by virtue of Convention 2 means nothing and can as well be omitted, leaving A to rest on the SA.

Let us derive in ClAG a classical principle which is invalid in intuitionistic logic, namely Peirce's Law:

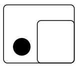
$$((A \to B) \to A) \to A).$$

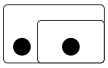
The derivation is as follows.

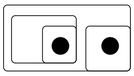


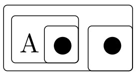
In the first application of the rule (InsA), inserting A to the antecedent of the cornering on the antecedent of another cornering is legitimate because A ($\neq$ ●) implies the blot.

The law of excluded middle (LEM), equivalent to Peirce's Law, can also now be derived; the derivation goes on as follows:

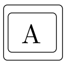

(AxIII)


(It)


(CR)

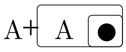


(InsA)



(ECC)

A (Convention)



(DI)

Here the application of (ECC) becomes admissible by first applying (CR) to the blot ●. Application of (ECC) then creates an assertion of A on the SA, and thus permits the disjunct, which here is the negation of A, to be added on the SA as an assertion alternative to the assertion of A.

The addition of the rule (ECC) to the calculus of AGs now results in a graphical system ClAG in which classical principles are derivable. The important and distinctive characteristics of ClAG is that it is classical without having to add any new signs to the vocabulary of AGs. In particular, to get a classical graphical logic, we need not introduce negations as cuts and the ensuing negative and positive polarities of areas as in the ordinary theory of EGs. Hence we can also avoid promulgating abundance of nests of cuts and the need to discern distinctions between negative and positive areas of graphs either by the counting of cuts or by the shading of areas.

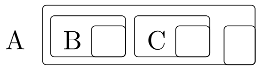
In sum, we can have a graphical logic of assertions which behaves like classical logic does and which can do so without having to add any new conventions to the system or to modify them in any way. In particular, we dispense with the conventions familiar from the original theory of EGs where one defines areas of cuts, then conceives cuts as negations, and then takes the difference between negative and positive polarities of these areas to be the fundamental quality on which rules of transformation are to operate.[8]

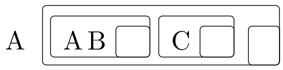
It has been noted in [25] that the family of these graphical systems of logic shows a close proximity to systems of *deep inference* (e.g. [8, 9, 17]). Here we can add another remark on the analytic power of the proposed graphical approach, namely that its tranformation rules are not merely those of deep inference but that they *analyse* the nature of deep inference. Now typical deep inference rules presented in the literature are the *switch* and *medial* [10]. They can be expressed in the classical language of EGs are follows: From A ((B) (C)) infer ((A B) (C)) (*Switch*), and from ((A B) (C D)) infer ((A) (C)) ((B) (D)) (*Medial*). (Here we use the standard EG syntax for alpha graphs and its standard rules of transformations.) The former results by applying iteration rule once to A and then erasing a copy of

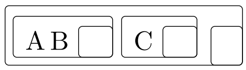
[8]For additional comparisons between the notations employed in AGs, intuitionistic logic of graphs and classical AGs, and especially in relation to how conditionals are expressed in them, see [30].

it. Medial is seen to be, in turn, a consequence of applying, first, iteration to the entire graph, and then erasing B, D, A and C, respectively.

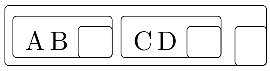
These typical deep-inference rules are thus not primitive rules of the graphical system. They can be shown to result from the application of some more fundamental rules of transformation. In ClAG, for instance, the switch rule emerges as follows:

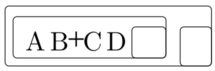


 (It)

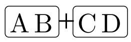
 (CE)

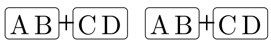
This analysis of the rule (*switch*) can be executed in the standard logic of EGs as well. The rule (*Medial*), in turn, emerges from an application of the following rules of ClAG:

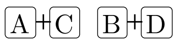


 (Am)

A B+C D (ECC)

 (Convention 2)

 (It)

 (CE) ×2

A+C B+D (Convention 2)

(Convention 2 is applied here for clarity.)

Thus our graphical notation is not only an application of, or just an alternative notation to, standard systems of deep inference. It also can analyse the very nature of such deep inferences by showing that the standard deep inference rules are not primitive.

5. Remarks on the Logical Philosophy of Assertive Graphs

In this section we outline of a number of further novelties of AGs and ClAG that have some untapped logical and philosophical significance. We list eight of them here.[9] In what follows, by AGs we mean both AGs and ClAGs, unless otherwise specified.

(1) Working with the derivations of formulas as expressed by these graphical logics of assertion is relatively uncomplicated, despite the fact that there are significantly more rules than the ordinary classical EG calculus has. The primary reason is that we do not need the insertion and erasure rules as in EGs, because in AGs we need not count the areas in which subgraphs are placed in order to ascertain

[9] We have preliminarily discussed, in another context, some of them in [12].

whether they occupy positive or negative areas (namely whether scribed within an even or odd number of cuts). By contrast, EGs promulgate cuts and they may for that reason quickly become cluttered, making it harder to determine whether the graphs are enclosed within an odd or even number of cuts. What was designed as an aid in observing the polarities, namely adding a shading to distinguish between negative and positive areas, and which Peirce indeed in 1911 came to propose (R 376), may only help in the comprehension of polarities. Shading does not, for instance, aid in the comprehension of which subgraphs count as conditionals.

In AGs, in contrast there are no such insertion and erasure rules. They are supplanted by rules that under certain conditions permit either insertions of graphs to the antecedents areas of cornerings or removals of graphs from the consequent areas of cornerings. As the cornering remains a primitive sign also in the classical rendering of the AGs, conditional structures occurring in the graphs are not assimilated with other primitives and for that reason do not become cluttered as easily as when expressed by cuts.

Whehter or to what extent the logical language of ClAGs is actually easier to comprehend than that of EGs need to be experimentally verified in future studies.

(2) There is an important difference in the meaning of the SA in AGs when contrasted with the meaning of the SA in EGs. In the latter, the blank sheet means "all truths", while in AGs it means "all assertions". The latter is justified by the presence of an intuitive notion of a proof, construction or empirical verification. When the content of the assertion is atomic, it implies that the assertion is justified by an empirical verification and not by a logical demonstration, since there is no demonstration for atomic formulas by rules of proof.

(3) From the more general philosophical point of view, the question of the meaning of logical constants can in the light of AGs be put in a novel perspective. The graphical approach possesses certain distinctive characteristics that tell against matching it with the prevailing proposals. For one, graphical notations do not agree well with inferentialism, although one might sense some similarities. The main idea in AGs is *not* to provide a proof-theoretic semantics that could justify the meaning of its logical constancy within an inferentialist framework. As explained in Section 2, graphical systems arise from systems of conventions that concern significations of basic notions, logical constants and logical operations. The list of conventions begins with the all-important first convention, which characterizes logical availability of the sheet of assertion. Noticeably, the SA itself is an assertion. One can always draw a box around a blank anywhere on the SA. Thus the system of conventions precedes what the system of well-formed graphs is as well as what the rules of transformation are. This order of preference—conventions before permissions—agrees with Peirce's original presentation of the method of EGs.

(4) AGs suggest that one can associate an assertion-based interpretation to both classical and intuitionistic interpretations of logical constancy (on the assertion-based interpretation of logical constants, see e.g. [11]). For this reason, and unlike what happens under standard inferentialist accounts, the Tonk problem of [31] does not arise. The reason why Tonk is not an issue in AGs can be seen already from the point of view of the notation of the standard logic of EGs. In the graphical approach which operates on a two-dimensional SA, it is impossible to conjure up a rule that would graphically depict both an erasure of a conjunct and an introduction of a disjunct on that sheet. The impossibility in question is a notational,

topological impossibility. On the one hand, juxtaposition is an assertion of two or more graphs *independently* of each other: assertions occur in the context of the blank which is represented by the continuity of the SA between them. On the other hand, a disjunctive assertion is an assertion of two or more *alternatively* asserted graphs, namely graphs that are enclosed within a cut *together* as well as *individually*: .

Likewise, it is impossible to create such constants by the manipulation of the rules that AGs have at its disposal. One cannot have a graphical sign standing for an operator such as Tonk that would represent *both* independent assertions (namely, the blank , in order to permit their erasure), *and* alternative assertions (namely, the crossed line +, in order to permit their introduction). The deep reason is that the *concepts* of independently and alteratively scribed assertions are parts of the *significations* of these operations. They are laid out in systems of conventions characterising graphical languages, not in systems of permissive rules of inference.

(5) While formulas in AGs are graphical (diagrammatic), they are not *representational* in the sense in which graphical formulas could be scribed on the sheet of assertion independently of their significations. The meaning of the logical constants in AGs is explicated by means of the notion of assertion, not the other way around. The meaning of the sheet of assertion is fundamental to the logical behaviour of other logical constants. The insight that choices of a certain piece of notation have important logical consequences played a fundamental role already in Peirce's development of the algebra of logic, which antedated the development of non-linear languages and graphical methods of logic.

(6) In order to justify logical inference, inferentialism requires logical constants to have various assertive uses. This is a common feature of both AGs and ClAGs. Any graph laid upon the SA is an assertion, because the conventions for the SA mean that the sheet is a constant that is also an embedded sign of assertion [3]. When justification of a proposition can be provided by virtue of a presence of a proof (or by virtue of a verification, construction, transformation etc.), then the proposition in question can be asserted. Assertion can thus mean either an act of acknowledgement of the truth of proposition, or the results of such an act. For instance, according to Heyting's original interpretation ([19]), assertions are understood in the first sense and not in the mathematical sense, as they are determinations of something that has empirical content. In the intuitionistic case, truth of the asserted content needs to be epistemically constrained. In the classical case, truth transgresses epistemic limitations of an ideal agent.

(7) Inferences in AGs and ClAG are to be viewed not as static, step-wise procedures, but as *acts* of asserting the truth of the conclusion whenever the truth of the premise(s) is asserted. However, unlike what is commonly the case in logical systems, inferences in AGs and ClAG are justified by transformation rules which may be applied deeply inside a graphical formula. As noted in previous section, graphical systems are systems of deep inference and in fact analyse the basic rules proposed as fundamental rules of deep inference. For instance, while introduction and elimination rules in systems such as natural deduction can be applied only to formulas expressing the main connectives, this need not be the case when the rules of AGs or ClAG are applied in any position of an area of graph.

The box-notation is thus an aid in both diagrammatically expressing and recognizing relevant information for the justification of inferences. Graphical notation makes explicit the *ambient space* within which graphs are located. The notation carries along the relevant context, or the 'information resources' of assertions by which these deep inferences are performed.

(8) The deep-inference feature (7) is a new aspect of assertive logics, adding to the repertoire of logically analysing assertions. A further novelty is thus that inferences in AGs and ClAG may occur without the contribution of *complete* assertions. It suffices that assertions convey a limited amount of information. Due to non-linear and multi-dimensional syntax, AGs and ClAGs can deal with all necessary forms of deep-inference. In natural deduction, primitive operations are the introduction and elimination rules. In applying the rules of AGs, however, one needs only a certain fixed amount of information about logical expressions to be at one's disposal, called *locality* in deep inference. Limitations to local contexts are computationally beneficial when the task is to process complex formulas. To locally calculate and process formulas, new primitive and symmetric operations are used in AGs, namely those of merging/separating, cornering/uncornering, and iteration/deiteration. These are added to the operations of introduction and elimination, and they enrich the set of basic rules so that only local information about the structure of formulas is needed in order to process them.

The goal of 'natural' forms of reasoning is to produce assertions that use only limited information resources. AGs and ClAGs can handle inferential connections when limited resources are at our disposal. Therefore, AGs may assume a new role in identifying not only some computational but also realistic and bounded cognitive and linguistic processes that reflect what is going on in transformations in logical graphs.

6. Conclusions

Graphical systems of logic provide new notational resources for deep-inferential systems of logic ([25]). Here we have presented some further thoughts on such approaches in terms of graphical logics that can explicitly deal with assertions. The resulting system, AGs, appears to be the first logic of assertions that also has the nature of deep inference.[10] Guided by the interpretation that logical constants are acts of assertions, AGs preserve the original idea that can be traced back to Peirce's logic of existential graphs.

By adding a rule (ECC), AGs result in a classical version ClAG. ClAG retains the basic requirements of being a logic of assertions. Like the alpha part of EGs, ClAG behaves classically but it does so by dispensing with cuts and an explicit representation of polarities. Our graphical approach to assertions may thus be a useful alternative to current logical theories of assertion, for instance because it needs no additional *ad hoc* sign of assertion and can thus avoid problems concerning the interplay of assertional signs with logical operations. What is more, and contrary to what Dummett had claimed ([15, 16]), for instance, nothing in the solution of these problems depends on the underlying logic being an intuitionistic one, since we have shown that such features can be preserved in ClAG, which is the classical version of AGs. Indeed we further noticed that ClAG does not run into same problems as

[10] This is not to claim that AGs would be the first graphical deep-inference system, as systems whose notation is explicitly diagrammatic may also have that property, among them not only the logic of existential graphs but also for example spider diagrams [21], of which there is now a considerable and growing body of research available [13].

inferentialistic approaches often do, which gives these proposed graphicalizations some added philosophical and conceptual credence.

Neither AGs nor ClAGs are particularly simple, however, as several new rules need to be introduced and conditionals and disjunctions defined as new primitives. There is nevertheless a trade-off between analytic virtues and the power of a system as an efficient calculus. Finally, among the further work that remains to be done is to test these graphicalizations against current arguments that bear on questions such as truth-conditional vs. non-truth-conditional theories of assertions, the norm vs. the Peircean responsibility (or liability) theory of assertions ([7]), as well as those related to commitment and self-representation of assertions. As inferential connections in intuitionistic graphs inherit certain key features of deep and natural reasoning that can produce inferences with limited information resources, they may be useful in theories of computation and cognition that work well with non-linear diagrammatic representations.

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