REPRODUCING KERNEL HILBERT SPACES OF POLYANALYTIC FUNCTIONS OF INFINITE ORDER

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CONTENTS

ABSTRACT. In this paper we introduce reproducing kernel Hilbert spaces of polyanalytic functions of infinite order. First we study in details the counterpart of the Fock space and related results in this framework. In this case the kernel function is given by $e^{z\overline{w}+\overline{z}w}$ which can be connected to kernels of polyanalytic Fock spaces of finite order. Segal-Bargmann and Berezin type transforms are also considered in this setting. Then, we study the reproducing kernel Hilbert spaces of complex-valued functions with reproducing kernel $\dfrac{1}{(1-z\overline{w})(1-\overline{z}w)}$ and $\dfrac{1}{1-2\mathrm{Re}\,z\overline{w}}$. The corresponding backward shift operators are introduced and investigated.

1. INTRODUCTION AND PRELIMINARY RESULTS

Polyanalytic functions were introduced in 1908 by Kolossov to solve problems in elasticity theory, see [27]. For a general introduction to this topic see [7, 12, 13]. In more recent times, this function theory was studied by several authors from different perspectives, see [2, 3, 6, 18, 32] and the references therein. Polyanalytic functions are used also to study sampling and interpolation problems on Fock spaces using time frequency analysis techniques such as short-time Fourier transform (STFT) or Gabor transforms, see [1]. In the next parts of this introduction, we collect some basic definitions and explain what we mean by polyanalytic functions of infinite order in our setting. Some important facts that will be needed in the sequel will be also revised. Then, we will explain the general construction of the kernels associated to the reproducing kernel Hilbert spaces of polyanalytic functions of infinite order, which will be studied in this paper. We conclude by describing the contents of the paper.

1.1. **Definitions.** A complex valued function $f : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ which belongs to the kernel of a power $n \geq 1$ of the classical Cauchy-Riemann operator $\dfrac{\partial}{\partial \overline{z}}$, that is

$$\frac{\partial^n}{\partial \overline{z}^n} f(z) = 0, \quad \forall z \in \Omega,$$

is called a polyanalytic function of order n. An interesting fact regarding these functions is that any polyanalytic function of order n can be decomposed in terms of n analytic functions so that we have a decomposition of the following form

$$(1.1) \qquad\qquad f(z) = \sum_{k=0}^{n-1} \overline{z}^k f_k(z),$$

for which all f_k are analytic functions on Ω. In particular, expanding each analytic component using the series expansion theorem lead to an expression of this form

$$(1.2) \qquad\qquad f(z) = \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \overline{z}^k z^j a_{k,j},$$

where $(a_{k,j})$ are complex coefficients. In this paper, we are interested by the case where the expansion (1.2) is of infinite order, which means that we consider functions of the form

$$(1.3) \qquad\qquad f(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{z}^k z^j a_{k,j},$$

which will be called polyanalytic functions of infinite order. We note that such functions were discussed in [12, 13] in which they were mentioned as conjugate analytic functions.

For $n = 1, 2, \ldots$ we recall that polyanalytic Fock spaces of order n can be defined as follows

$$\mathcal{F}_n(\mathbb{C}) := \left\{ g \in H_n(\mathbb{C}), \quad \frac{1}{\pi} \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} dA(z) < \infty \right\}.$$

The reproducing kernel associated to the space $\mathcal{F}_n(\mathbb{C})$ is given by

$$(1.4) \qquad\qquad K_n(z, w) = e^{z\overline{w}} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \binom{n}{k+1} |z - w|^{2k},$$

for every $z, w \in \mathbb{C}$.

1.2. **The kernels construction: general discussion.** Consider a function $F(z_1, z_2)$ in the Hardy space of the bidisk $\mathbb{D}^2$ in $\mathbb{C}^2$. Then, $g(z) = F(z, z)$ belongs to the Bergman space of the disk $\mathbb{D}$, and the map $F \mapsto g$ is onto and contractive, but not one-to-one. For instance, the polynomials $z_1^n z_2^m$ belong to $\mathbf{H}^2(\mathbb{D}^2)$ and have the same image z^s with $n + m = s$. On the other hand the map $g(z) = F(z, \overline{z})$ is one-to-one, and its image is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{1}{(1 - z\overline{w})(1 - \overline{z}w)}$$

The corresponding reproducing kernel Hilbert space consists of polyanalytic functions of infinite order.

Motivated by the above discussion we consider a function $c(\mathbf{z}, \mathbf{w})$ positive definite in some open subset Ω of $\mathbb{C}^{2N}$, and analytic in $\mathbf{z}$ and $\overline{\mathbf{w}}$. We assume that

$$(1.5) \qquad\qquad \Omega_s = \left\{ z \in \mathbb{C}^N : (z, \overline{z}) \in \Omega \right\}$$

is open and non-empty. The function

$$k(z, w) = c((z, \overline{z}), (\omega, \overline{\omega}))$$

is then positive definite in Ω_s. The purpose of the present work is to study the corresponding reproducing kernel Hilbert spaces of polyanalytic functions of infinite order. For instance in the case of the Fock space with reproducing kernel $c(\overline{z}, \omega) = e^{\sum_{n=1}^{2N} z_n \overline{\omega_n}}$ we have the kernel

$$(1.6) \qquad k(z, w) = e^{\sum_{n=1}^{N} (z_n \overline{w_n} + \overline{z_n} w_n)}$$

while in the case of the Drury-Arveson space with reproducing kernel $\frac{1}{1 - \sum_{n=1}^{2N} z_n \overline{w_n}}$ which is positive definite in the open unit ball of $\mathbb{C}^{2N}$, the corresponding kernel is

$$(1.7) \qquad \frac{1}{1 - \sum_{n=1}^{N} (z_n \overline{w_n} + \overline{z_n} w_n)}$$

positive definite in the open ball of $\mathbb{C}^N$ centered at the origin and with radius $\frac{1}{\sqrt{2}}$. In this paper we will focus on $N = 1$.

A general family of examples correspond to

$$k(z, w) = K_1(z, w) \overline{K_2(z, w)}$$

where K_1 and K_2 are analytic kernels, or, in the matrix-valued case,

$$k(z, w) = K_1(z, w) \otimes \overline{K_2(z, w)}.$$

The structure of the paper is as follows: in Section 2 we introduce the kernel function K associated to the polyanalytic Fock space $\mathcal{SF}(\mathbb{C})$ of infinite order and we study various properties. We give a sequential characterization of the space $\mathcal{SF}(\mathbb{C})$ and, in particular, we prove that the creation and annihilation operators are adjoint of each other. We also introduce and study two backward shift operators. In Section 3 we prove that by taking the power series of the polyanalytic Fock kernels of finite order $(K_n)_{n \geq 0}$ we obtain the kernel function K multiplied up to an exponential kernel. In Section 4 and 5 we study Segal-Bargmann and Berezin type transforms and some related operators. In Section 6 we present the polyanalytic Hardy space of infinite order and we study the Gleason problem. We also prove some results on the backward shift operator in this setting. Finally, Section 7 is devoted to the case of Drury-Arveson space.

2. The polyanalytic Fock space of infinite order and associated kernel

We denote by M_z and $M_{\overline{z}}$ the multiplication operators by z and $\overline{z}$. Then, we will prove the following main result

Theorem 2.1. *The reproducing kernel Hilbert space with reproducing kernel $e^{z\overline{w} + \overline{z}w}$ is, up to a multiplicative positive factor, the only reproducing kernel Hilbert space of polyanalytic functions of infinite order, regular at the origin, and for which*

$$(2.1) \qquad \left(\frac{\partial}{\partial z} \right)^* = M_z$$

$$(2.2) \qquad \left(\frac{\partial}{\partial \overline{z}} \right)^* = M_{\overline{z}}.$$

To this end, we need the following:

Definition 2.2. *We consider the kernel function given by*

$$(2.3) \qquad K(z, w) = e^{z\overline{w} + \overline{z}w} = e^{2Re(z\overline{w})}, \quad \forall (z, w) \in \mathbb{C}^2.$$

We denote by $(\mathcal{H}(K), \langle \cdot, \cdot \rangle_{\mathcal{H}(K)})$ the reproducing kernel Hilbert space associated to the kernel function (2.3).

Proposition 2.3. *The function* $K : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* (2.3) *is a positive definite kernel.*

Proof. It is clear that we have

$$(2.4) \qquad\qquad K(z,w) = F(z,w)\overline{F(z,w)},$$

for every $z, w \in \mathbb{C}$ and F denotes the reproducing kernel of the classical Fock space $\mathcal{F}(\mathbb{C})$. Thus, since $\overline{F(z,w)}$ is also a positive definite kernel we can conclude, since K is a product of positive definite kernels. $\qquad\qquad\square$

We observe that the following integral representation holds

Proposition 2.4. *It holds that*

$$(2.5) \qquad\qquad \frac{1}{\pi} \int_{\mathbb{C}} K(z,w) e^{-|w|^2} dA(w) = e^{|z|^2} = \sqrt{K(z,z)}, \text{ for any } z \in \mathbb{C}.$$

Proof. We set $w = x + iy$, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and use the classical Gaussian integral

$$\int_{\mathbb{R}} e^{-at^2+bt} dt = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \quad a > 0, b \in \mathbb{C}.$$

We have

$$\begin{aligned}
\frac{1}{\pi} \int_{\mathbb{C}} K(z,w) e^{-|w|^2} dA(w) &= \frac{1}{\pi} \int_{\mathbb{C}} e^{z\overline{w}+\overline{z}w} e^{-|w|^2} dA(w) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} e^{z(x-iy)+\overline{z}(x+iy)} e^{-(x^2+y^2)} dxdy \\
&= \frac{1}{\pi} \left(\int_{\mathbb{R}} e^{x(\overline{z}+z)-x^2} dx \right) \left(\int_{\mathbb{R}} e^{yi(\overline{z}-z)-y^2} dy \right) \\
&= e^{\frac{(z+\overline{z})^2}{4}} e^{-\frac{(\overline{z}-z)^2}{4}} \\
&= e^{|z|^2} \\
&= \sqrt{K(z,z)},
\end{aligned}$$

as stated. $\qquad\qquad\square$

Next result is simple but very useful:

Proposition 2.5. *For any* $z, w \in \mathbb{C}$ *it holds that*

i) $\dfrac{\partial}{\partial z} K(z,w) = \overline{w} K(z,w)$ *and* $\dfrac{\partial}{\partial \overline{z}} K(z,w) = w K(z,w)$.

ii) $\dfrac{\partial}{\partial w} K(z,w) = \overline{z} K(z,w)$ *and* $\dfrac{\partial}{\partial \overline{w}} K(z,w) = z K(z,w)$.

Proof. It is an easy calculation based on the expression of the kernel function $K(z,w)$ given by formula (2.3). $\qquad\qquad\square$

Corollary 2.6. *For any* $z, w \in \mathbb{C}$ *and* $n = 1, 2, \dots$ *it holds that*

i) $\dfrac{\partial^n}{\partial z^n} K(z,w) = \overline{w}^n K(z,w)$ *and* $\dfrac{\partial^n}{\partial \overline{z}^n} K(z,w) = w^n K(z,w)$.

ii $\dfrac{\partial^n}{\partial w^n} K(z,w) = \overline{z}^n K(z,w)$ *and* $\dfrac{\partial^n}{\partial \overline{w}^n} K(z,w) = z^n K(z,w)$.

Proof. We will prove i), the other statements follow similar arguments. We apply Proposition 2.5 and get

$$\frac{\partial}{\partial z} K(z,w) = \overline{w} K(z,w).$$

Thus, we apply a second time the complex derivative with respect to the variable z, we use again Proposition 2.5 and get

$$\frac{\partial^2}{\partial z^2} K(z, w) = \overline{w}^2 K(z, w).$$

Then, we repeat the same calculation n-times and obtain

$$\frac{\partial^n}{\partial z^n} K(z, w) = \overline{w}^n K(z, w).$$

$\square$

Now, let us consider the commutator operators given by

$$(2.6) \qquad \left[\frac{\partial}{\partial z}, M_z\right] := \frac{\partial}{\partial z} M_z - M_z \frac{\partial}{\partial z}$$

and

$$(2.7) \qquad \left[\frac{\partial}{\partial \overline{z}}, M_{\overline{z}}\right] := \frac{\partial}{\partial \overline{z}} M_{\overline{z}} - M_{\overline{z}} \frac{\partial}{\partial \overline{z}}.$$

Then, we can prove the following

Proposition 2.7. *For any $z, w \in \mathbb{C}$ we have*

$$\left[\frac{\partial}{\partial z}, M_z\right] K(z, w) = K(z, w)$$

and

$$\left[\frac{\partial}{\partial \overline{z}}, M_{\overline{z}}\right] K(z, w) = K(z, w).$$

Proof. We have

$$\frac{\partial}{\partial z} M_z K(z, w) = \frac{\partial}{\partial z} (z K(z, w))$$
$$= z \frac{\partial}{\partial z} K(z, w) + K(z, w)$$
$$= z \overline{w} K(z, w) + K(z, w).$$

On the other hand

$$M_z \frac{\partial}{\partial z} K(z, w) = M_z \left(\overline{w} K(z, w)\right)$$
$$= z \overline{w} K(z, w),$$

hence, we obtain

$$\left[\frac{\partial}{\partial z}, M_z\right] K(z, w) = K(z, w).$$

In a similar way we can prove that

$$\left[\frac{\partial}{\partial \overline{z}}, M_{\overline{z}}\right] K(z, w) = K(z, w).$$

$\square$

Thanks to the reproducing kernel property we have this more general result

Theorem 2.8. *For any $f \in \mathcal{H}(K)$, the following identities hold*

$$\left[\frac{\partial}{\partial z}, M_z\right] f = f$$

and

$$\left[\frac{\partial}{\partial \overline{z}}, M_{\overline{z}}\right] f = f.$$

Proof. Let $f \in \mathcal{H}(K)$, we know that

$$f(z) = < f, K_z >_{\mathcal{H}(K)}, \text{ for any } z \in \Omega.$$

Thus, for any $z \in \Omega$ we apply Proposition 2.7 and get

$$\left[\frac{\partial}{\partial z}, M_z\right] f(z) = < f, \left[\frac{\partial}{\partial z}, M_z\right] K_z >_{\mathcal{H}(K)}$$
$$= < f, K_z >_{\mathcal{H}(K)}$$
$$= f(z).$$

Hence, it follows that

$$\left[\frac{\partial}{\partial z}, M_z\right] f = f.$$

In the same way we can prove that

$$\left[\frac{\partial}{\partial \overline{z}}, M_{\overline{z}}\right] f = f.$$

$\square$

As a consequence of Proposition 2.5 we can study more properties of the kernel function in (2.3).

Proposition 2.9. *For any $z, w \in \mathbb{C}$ we have*

$$(2.8) \qquad \Delta_z K(z, w) = 4|w|^2 K(z, w)$$

and

$$(2.9) \qquad \Delta_w \Delta_z K(z, w) = 16 \left(1 + w\overline{z} + \overline{w}z + |w|^2 |z|^2\right) K(z, w) = 16|1 + \overline{w}z|^2 K(z, w).$$

Proof. We note that using (2.3) and Proposition 2.5 we have

$$\frac{\partial^2}{\partial \overline{z} \partial z} K(z, w) = \overline{w} \frac{\partial}{\partial \overline{z}} K(z, w)$$
$$= \overline{w} w K(z, w)$$
$$= |w|^2 K(z, w).$$

Since $\dfrac{\partial^2}{\partial \overline{z} \partial z} = \dfrac{1}{4}\Delta_z$ we conclude that

$$(2.10) \qquad \Delta_z K(z, w) = 4|w|^2 K(z, w).$$

Furthermore, by taking the derivative with respect to w we get

$$\frac{\partial}{\partial w} \Delta_z K(z, w) = 4\overline{w} \frac{\partial}{\partial w} (w K(z, w))$$
$$= 4\overline{w}(K(z, w) + w\overline{z} K(z, w))$$
$$= 4(1 + w\overline{z})\overline{w} K(z, w).$$

Then, we apply the derivative with respect to $\overline{w}$, develop the computations and get

$$\frac{\partial^2}{\partial\overline{w}\partial w}\Delta_z K(z,w) = 4(1+w\overline{z})\frac{\partial}{\partial\overline{w}}\left(\overline{w}K(z,w)\right)$$
$$= 4(1+w\overline{z})(1+\overline{w}z)K(z,w)$$
$$= 4|1+\overline{w}z|^2 K(z,w),$$

from which we conclude that

$$\Delta_w\Delta_z K(z,w) = 16|1+\overline{w}z|^2 K(z,w).$$

$\square$

Corollary 2.10. *Let $\Omega = \mathbb{D}$ denote the unit disk, we consider the operator*

$$T_{z,w} := \frac{\Delta_w\Delta_z}{16|1+\overline{w}z|^2}, \quad \forall(z,w) \in \Omega \times \Omega.$$

Then, we have

(2.11) $$T_{z,w}K(z,w) = K(z,w), \quad \forall(z,w) \in \Omega \times \Omega.$$

Proof. It is a direct consequence of Proposition 2.9. $\square$

Remark 2.11. From (2.10) we deduce that if w is a fixed parameter, then the kernel function $K(z,w)$ can be seen as an eigenfunction of the Laplace operator Δ_z with eigenvalue given by $4|w|^2$.

Definition 2.12. *The polyanalytic Fock space $\mathcal{SF}(\mathbb{C})$ of infinite order is the set of functions of the form*

(2.12) $$f(z) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z),$$

satisfying the conditions

    i) $f_n \in \mathcal{F}(\mathbb{C})$ for any $n \geq 0$;

    ii) $\|f\|^2_{\mathcal{SF}(\mathbb{C})} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{\mathcal{F}(\mathbb{C})} < \infty.$

Then, we consider the scalar product on $\mathcal{SF}(\mathbb{C})$ given by

(2.13) $$\langle f, g\rangle_{\mathcal{SF}(\mathbb{C})} := \sum_{k=0}^{\infty} k!\langle f_k, g_k\rangle_{\mathcal{F}(\mathbb{C})},$$

for any $f = \sum_{k=0}^{\infty} \overline{z}^k f_k$ and $g = \sum_{k=0}^{\infty} \overline{z}^k g_k$ with $f_k, g_k \in \mathcal{F}(\mathbb{C})$ for every $k \geq 0$.

Proposition 2.13. *A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ belongs to $\mathcal{SF}(\mathbb{C})$ if and only if f is of the form*

$$f(z) = \sum_{(m,n)\in\mathbb{N}^2} z^m \overline{z}^n \alpha_{m,n},$$

with $(\alpha_{m,n}) \subset \mathbb{C}$ and such that

(2.14) $$\|f\|^2_{\mathcal{SF}(\mathbb{C})} = \sum_{(m,n)\in\mathbb{N}^2} m!n!|\alpha_{m,n}|^2 < \infty.$$

Moreover, if for any $(m,n) \in \mathbb{N}^2$ we set $\phi_{m,n}(z,\overline{z}) = \dfrac{z^m \overline{z}^n}{\sqrt{m!n!}}$ then, the family of functions $\{\phi_{m,n}\}_{m,n\geq 0}$ form an orthonormal basis of $\mathcal{SF}(\mathbb{C})$.

Proof. Let

$$f(z) = \sum_{(m,n)\in\mathbb{N}^2} \overline{z}^n z^m \alpha_{m,n},$$

with $(\alpha_{n,m}) \subset \mathbb{C}$.

Setting $f_n(z) = \sum_{m=0}^{\infty} z^m \alpha_{m,n}$, it is clear that $f(z) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z)$. Moreover, we have

$$||f||_{\mathcal{SF}(\mathbb{C})}^2 = \sum_{n=0}^{\infty} n! ||f_n||_{\mathcal{F}(\mathbb{C})}^2 = \sum_{(m,n)\in\mathbb{N}^2} m! n! |\alpha_{m,n}|^2.$$

Therefore, f belongs to the space $\mathcal{SF}(\mathbb{C})$ if and only if

$$||f||_{\mathcal{SF}(\mathbb{C})}^2 = \sum_{(m,n)\in\mathbb{N}^2} m! n! |\alpha_{m,n}|^2 < \infty.$$

On the other hand, easy computations lead to

$$\langle f, \phi_{m,n} \rangle_{\mathcal{SF}(\mathbb{C})} = \sqrt{n! m!} \alpha_{m,n}, \quad \forall m, n \geq 0.$$

If $\langle f, \phi_{m,n} \rangle_{\mathcal{SF}(\mathbb{C})} = 0$ for any $m, n \geq 0$, then we have $\alpha_{m,n} = 0$ for any $m, n \geq 0$. We note also that

$$\langle \phi_{m,n}, \phi_{m,n} \rangle_{\mathcal{SF}(\mathbb{C})} = 1 \text{ and } \langle \phi_{m,n}, \phi_{p,q} \rangle_{\mathcal{SF}(\mathbb{C})} = 0 \text{ whenever } (m,n) \neq (p,q).$$

In particular, this shows that $\{\phi_{m,n}\}_{m,n\geq 0}$ form an orthonormal basis of $\mathcal{SF}(\mathbb{C})$. This ends the proof. $\square$

Example 2.14. We recall the complex Hermite polynomials introduced in [26]

$$(2.15) \qquad H_{m,n}(z,\overline{z}) := \sum_{k=0}^{\min(m,n)} (-1)^k k! \binom{m}{k}\binom{n}{k} z^{m-k}\overline{z}^{n-k}.$$

It is easy to prove that $H_{m,n}$ belong to $\mathcal{SF}(\mathbb{C})$.

We now provide a sequential charachterization of the space $\mathcal{H}(K)$

Theorem 2.15. *We have*

$$\mathcal{H}(K) = \mathcal{SF}(\mathbb{C}).$$

Moreover, it holds that

$$(2.16) \qquad K(z,w) = \sum_{m,n=0}^{\infty} \phi_{m,n}(z,\overline{z})\overline{\phi_{m,n}(w,\overline{w})}, \quad \text{for any } z, w \in \mathbb{C}.$$

Proof. Since $(\phi_{m,n})_{m,n\geq 0}$ is an orthonormal basis of the space $\mathcal{SF}(\mathbb{C})$, the associated reproducing kernel is given by the convergent series

$$\sum_{m,n=0}^{\infty} \phi_{m,n}(z,\overline{z})\overline{\phi_{m,n}(w,\overline{w})} < \infty, \text{ for any } z, w \in \mathbb{C}.$$

More precisely, for any $(z, w) \in \mathbb{C}^2$ we have the equalities

$$
\begin{aligned}
\sum_{m,n=0}^{\infty} \phi_{m,n}(z, \bar{z})\overline{\phi_{m,n}(w, \bar{w})} &= \sum_{m,n=0}^{\infty} \frac{z^m \bar{z}^n \bar{w}^m w^n}{m!n!} \\
&= \left(\sum_{m=0}^{\infty} \frac{z^m \bar{w}^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n \bar{z}^n}{n!} \right) \\
&= e^{z\bar{w}} e^{w\bar{z}} \\
&= e^{z\bar{w}+w\bar{z}} \\
&= K(z, w).
\end{aligned}
$$

$\square$

Theorem 2.16. *It holds that*

(2.17)
$$
\langle \frac{\partial}{\partial z} f, g \rangle_{\mathcal{SF}(\mathbb{C})} = \langle f, M_z g \rangle_{\mathcal{SF}(\mathbb{C})},
$$

moreover

(2.18)
$$
\langle \frac{\partial}{\partial \bar{z}} f, g \rangle_{\mathcal{SF}(\mathbb{C})} = \langle f, M_{\bar{z}} g \rangle_{\mathcal{SF}(\mathbb{C})}.
$$

Proof. Let $f = \sum_{k=0}^{\infty} \bar{z}^k f_k$ and $g = \sum_{k=0}^{\infty} \bar{z}^k g_k$ in $\mathcal{SF}(\mathbb{C})$ that belongs to the domains of the creation and annihilation operators. Firstly, we note that we have

$$
M_z(g) = \sum_{k=0}^{\infty} \bar{z}^k M_z(g_k).
$$

Then, it follows that

$$
\begin{aligned}
\langle \frac{\partial}{\partial z} f, g \rangle_{\mathcal{SF}(\mathbb{C})} &= \sum_{k=0}^{\infty} k! \langle \frac{\partial}{\partial z} f_k, g_k \rangle_{\mathcal{F}(\mathbb{C})} \\
&= \sum_{k=0}^{\infty} k! \langle f_k, (\frac{\partial}{\partial z})^* g_k \rangle_{\mathcal{F}(\mathbb{C})} \\
&= \sum_{k=0}^{\infty} k! \langle f_k, M_z(g_k) \rangle_{\mathcal{F}(\mathbb{C})} \\
&= \langle f, M_z(g) \rangle_{\mathcal{SF}(\mathbb{C})}.
\end{aligned}
$$

Moreover, since

$$
\frac{\partial}{\partial \bar{z}}(f)(z) = \sum_{h=0}^{\infty} (h+1) \bar{z}^h f_{h+1},
$$

and

$$
M_{\bar{z}}(g)(z) = \sum_{h=1}^{\infty} \bar{z}^h g_{h-1},
$$

it follows that

$$\langle \frac{\partial}{\partial \bar{z}} f, g \rangle_{\mathcal{SF}(\mathbb{C})} = \sum_{k=0}^{\infty} k! \langle (k+1) f_{k+1}, g_k \rangle_{\mathcal{F}(\mathbb{C})}$$

$$= \sum_{k=0}^{\infty} (k+1)! \langle f_{k+1}, g_k \rangle_{\mathcal{F}(\mathbb{C})}$$

$$= \sum_{k=1}^{\infty} k! \langle f_k, g_{k-1} \rangle_{\mathcal{F}(\mathbb{C})}$$

$$= \langle f, M_{\bar{z}}(g) \rangle_{\mathcal{SF}(\mathbb{C})}.$$

$\square$

Remark 2.17. We shall see later that that we have

$$\mathcal{F}(\mathbb{C}) \subset \mathcal{SF}(\mathbb{C}) \subset L^2(\mathbb{C}, d\mu_\beta), \quad \beta > 2.$$

The previous inclusions are strict. It is well known that the classical Fock space $\mathcal{F}(\mathbb{C})$ is the only space of entire functions on which the creation and annihilation operators are adjoints of each others and satisfy the classical commutation rules. Of course, this is not true anymore on $L^2(\mathbb{C}, d\mu)$, see Proposition 7.2 in [31]. However, the previous theorem shows that the result still holds in the subspace $\mathcal{SF}(\mathbb{C})$ of the space of polyanalytic functions of infinite order.

Definition 2.18. *Let $f, g \in \mathcal{SF}(\mathbb{C})$ and let $f(z) = \sum_{n=0}^{\infty} \bar{z}^n f_n(z)$ and $g(z) = \sum_{m=0}^{\infty} z^m g_m(\bar{z})$. Then, we define two backward shift operators R_∞ and L_∞ with respect to the variables z and $\bar{z}$ respectively given by*

$$(2.19) \qquad R_\infty(f)(z, \bar{z}) = \sum_{n=0}^{\infty} \bar{z}^n R_0(f_n)(z) = \frac{1}{z} \left(f(z, \bar{z}) - \sum_{n=0}^{\infty} \bar{z}^n f_n(0) \right), \quad z \in \mathbb{C}$$

and

$$(2.20) \qquad L_\infty(g)(z, \bar{z}) = \frac{1}{\bar{z}} \left(g(z, \bar{z}) - \sum_{m=0}^{\infty} z^m g_m(0) \right), \quad z \in \mathbb{C}.$$

It turns out that both the backward shift operators R_∞ and L_∞ define two contractions on the polyanalytic Fock space of infinite order $\mathcal{SF}(\mathbb{C})$. Indeed, the following result holds:

Proposition 2.19. *For any $f \in \mathcal{SF}(\mathbb{C})$, we have*

$$(2.21) \qquad ||R_\infty(f)||^2_{\mathcal{SF}(\mathbb{C})} \leq ||f||^2_{\mathcal{SF}(\mathbb{C})} - \sum_{n=0}^{\infty} n! |f_n(0)|^2 \leq ||f||^2_{\mathcal{SF}(\mathbb{C})}.$$

and

$$(2.22) \qquad ||L_\infty(g)||^2_{\mathcal{SF}(\mathbb{C})} \leq ||g||^2_{\mathcal{SF}(\mathbb{C})} - \sum_{n=0}^{\infty} n! |g_n(0)|^2 \leq ||g||^2_{\mathcal{SF}(\mathbb{C})}.$$

Proof. We will prove the result for R_∞. Indeed, if we consider

$$f(z) = \sum_{n=0}^{\infty} \bar{z}^n f_n(z), \quad f_n \in \mathcal{F}(\mathbb{C}),$$

we have

$$R_\infty(f)(z) = \sum_{n=0}^{\infty} \bar{z}^n R_0(f_n)(z).$$

Thus, using the fact R_0 is a contraction on the Fock space (see [9]) we deduce

$$||R_\infty(f)||^2_{\mathcal{SF}(\mathbb{C})} = \sum_{n=0}^{\infty} n!||R_0(f_n)||^2_{\mathcal{F}(\mathbb{C})}$$

$$\leq \sum_{n=0}^{\infty} n!(||f_n||^2_{\mathcal{F}(\mathbb{C})} - |f_n(0)|^2)$$

$$= ||f||^2_{\mathcal{SF}(\mathbb{C})} - \sum_{n=0}^{\infty} n!|f_n(0)|^2 \leq ||f||^2_{\mathcal{SF}(\mathbb{C})}$$

and this ends the proof. We note that the argument follows in a similar way for L_∞ using the fact that we can write f also in the form

$$f(z) = \sum_{m=0}^{\infty} z^m f_m(\bar{z}).$$

$\square$

Now, we consider other two operators:

Definition 2.20. *Let* $f, g \in \mathcal{SF}(\mathbb{C})$ *and let* $f(z) = \displaystyle\sum_{n=0}^{\infty} \bar{z}^n f_n(z)$ *and* $g(z) = \displaystyle\sum_{m=0}^{\infty} z^m g_m(\bar{z})$. *Then, we define two operators with respect to the variables* z *and* $\bar{z}$ *respectively which are given by*

$$(2.23) \qquad\qquad I_\infty(f)(z, \bar{z}) = \sum_{n=0}^{\infty} \bar{z}^n I(f_n)(z), \quad z \in \mathbb{C}$$

and

$$(2.24) \qquad\qquad J_\infty(g)(z, \bar{z}) = \sum_{m=0}^{\infty} z^m J(g_m)(\bar{z}), \quad z \in \mathbb{C}.$$

We point out that the operator I is the integration operator considered in [9], while J is the integration with respect to the conjugate variable.

Remark 2.21. It holds that

$$(2.25) \qquad\qquad I_\infty(\bar{z}^n z^m)(z) = \frac{\bar{z}^n z^{m+1}}{m+1}, \quad z \in \mathbb{C}$$

and

$$(2.26) \qquad\qquad J_\infty(\bar{z}^n z^m)(z) = \frac{\bar{z}^{n+1} z^m}{n+1}, \quad z \in \mathbb{C}.$$

As a consequence, we have the following result:

Theorem 2.22. *The adjoints of* R_∞ *and* L_∞ *satisfy*

$$(2.27) \qquad\qquad\qquad\qquad R_\infty^* = I_\infty$$

and

$$(2.28) \qquad\qquad\qquad\qquad L_\infty^* = J_\infty.$$

Proof. Let $f, g \in \mathcal{SF}(\mathbb{C})$; we will prove that

$$(2.29) \qquad\qquad \langle I_\infty(f), g \rangle_{\mathcal{SF}(\mathbb{C})} = \langle f, R_\infty(g) \rangle_{\mathcal{SF}(\mathbb{C})}.$$

Indeed, we write $f(z) = \sum_{n=0}^{\infty} \bar{z}^n f_n(z)$ and $g(w) = \sum_{n=0}^{\infty} \bar{z}^n g_n(z)$ with $f_n, g_n \in \mathcal{F}(\mathbb{C})$ for any $n \geq 0$. We have

$$I_\infty(f)(z) = \sum_{n=0}^{\infty} \bar{z}^n I(f_n)(z)$$

and

$$R_\infty(g)(z) = \sum_{m=0}^{\infty} \bar{z}^m R_0(g_m)(z).$$

Then, we use the scalar product on $\mathcal{SF}(\mathbb{C})$ and apply the result on the classical backward shift operator, see [9]. Therefore, it follows that

$$
\begin{aligned}
\langle I_\infty(f), g \rangle_{\mathcal{SF}(\mathbb{C})} &= \sum_{k=0}^{\infty} k! \langle I(f_k), g_k \rangle_{\mathcal{F}(\mathbb{C})} \\
&= \sum_{k=0}^{\infty} k! \langle f_k, R_0(g_k) \rangle_{\mathcal{F}(\mathbb{C})} \\
&= \langle f, R_\infty(g) \rangle_{\mathcal{SF}(\mathbb{C})}.
\end{aligned}
$$

The second part of the statement can be proved in a similar way. $\qquad\square$

3. A KERNEL FUNCTION RELATING POLYANALYTIC FOCK SPACES OF FINITE AND INFINITE ORDER

In this section we study how the polyanalytic Fock spaces of finite and infinite order are related between them. We denote by $\mathcal{F}_n(\mathbb{C})$ the classical polyanalytic Fock space whose kernel is given by the formula (1.4). The relation between the kernels K and $(K_n)_{n \geq 1}$ is described in the next result.

Proposition 3.1 (kernel formula). *For any $z, w \in \mathbb{C}$ we set*

$$G(z, w) = e^{z\bar{w} - (|z|^2 + |w|^2)}.$$

Then, it holds that

$$(3.1) \qquad \sum_{n=1}^{\infty} \frac{K_n(z, w)}{2^{n+1}} = G(z, w) K(z, w), \quad \text{for any } z, w \in \mathbb{C}.$$

Proof. We note that the polyanalytic Fock kernels given by (1.4) can be written in terms of the generalized Laguerre polynomials as follows

$$(3.2) \qquad K_n(z, w) = e^{z\bar{w}} L_{n-1}^1(|z - w|^2), \quad \text{for any } z, w \in \mathbb{C}.$$

Then, taking the series (3.2) we obtain

$$(3.3) \qquad \sum_{n=1}^{\infty} \frac{K_n(z, w)}{2^{n-1}} = e^{z\bar{w}} \sum_{n=1}^{\infty} \frac{L_{n-1}^1(|z - w|^2)}{2^{n-1}} = e^{z\bar{w}} \sum_{n=0}^{\infty} \frac{L_n^1(|z - w|^2)}{2^n}.$$

Morever, we note that for any $a, \alpha > 0$ we have the following expansion, see [28, Example 2, pp 89]

$$(3.4) \qquad e^{-ax} = (a+1)^{-(\alpha+1)} \sum_{n=0}^{\infty} \left(\frac{a}{a+1} \right)^n L_n^\alpha(x), \quad x \geq 0.$$

In particular, inserting $\alpha = a = 1$ and $x = |z - w|^2$ in (3.4) we obtain

$$e^{-|z-w|^2} = \frac{1}{2^2} \sum_{n=0}^{\infty} \frac{L_n^1(|z - w|^2)}{2^n}, \quad z, w \in \mathbb{C}.$$

Hence, with some computations involving (3.3) we obtain that

$$\sum_{n=1}^{\infty} \frac{K_n(z, w)}{2^{n+1}} = G(z, w) K(z, w),$$

for any $z, w \in \mathbb{C}$ where $G(z, w) = e^{z\overline{w} - (|z|^2 + |w|^2)}$. $\qquad \square$

Remark 3.2. We observe that the classical creation and annihilation operators are adjoint of each others on the polyanalytic Fock space of infinite order $\mathcal{SF}(\mathbb{C})$, see Theorem 2.16.

4. A Segal-Bargmann type transform and related operators

In this section, we deal with a Segal-Bargmann type transform related to the polyanalytic Fock spaces of infinite order. We discuss also some related operators.

Let $(\psi_n(x))_{n \geq 0}$ denote the normalized Hermite functions and consider the Segal-Bargmann kernel $A(z, x)$ which is given by

$$(4.1) \qquad A(z, x) := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_n(x) = e^{-\frac{1}{2}(z^2 + x^2) + \sqrt{2}zx}, \quad \text{for any } (z, x) \in \mathbb{C} \times \mathbb{R}.$$

For any $z \in \mathbb{C}$ fixed we use also the notation $A_z(x) = A(z, x)$ for all $x \in \mathbb{R}$. The kernel (2.3) can be factorized as follows:

Theorem 4.1. *For any $(z, w) \in \mathbb{C}^2$, we have*

$$(4.2) \qquad K(z, w) = \langle A_z \otimes A_{\bar{z}}, A_w \otimes A_{\bar{w}} \rangle_{L^2(\mathbb{R}^2)}.$$

Proof. The proof is based on computations using Fubini's theorem combined with the following well-known fact

$$\langle A_z, A_w \rangle_{L^2(\mathbb{R})} = e^{z\bar{w}}.$$

Indeed, for $z, w \in \mathbb{C}$ we have the explicit computations

$$\langle A_z \otimes A_{\bar{z}}, A_w \otimes A_{\bar{w}} \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (A_z \otimes A_{\bar{z}})(x, y) \overline{(A_w \otimes A_{\bar{w}})(x, y)} dx dy$$

$$= \int_{\mathbb{R}^2} A_z(x) A_{\bar{z}}(y) \overline{A_w(x)} \; \overline{A_{\overline{w}}(y)} dx dy$$

$$= \left(\int_{\mathbb{R}} A_z(x) \overline{A_w(x)} dx \right) \left(\int_{\mathbb{R}} \overline{A_{\bar{z}}(y)} A_{\bar{w}}(y) dy \right)$$

$$= \langle A_z, A_w \rangle_{L^2(\mathbb{R})} \langle A_{\bar{w}}, A_{\bar{z}} \rangle_{L^2(\mathbb{R})}$$

$$= e^{z\bar{w}} e^{w\bar{z}}$$

$$= K(z, w).$$

$\qquad \square$

Definition 4.2. *For a given $\varphi \in L^2(\mathbb{R}^2)$, we define the so-called first Segal-Bargmann type transform by*

$$(4.3) \qquad T(\varphi)(z, \bar{z}) = \langle \varphi, \overline{A_z \otimes A_{\bar{z}}} \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} A_z(x) A_{\bar{z}}(y) \varphi(x, y) dx dy.$$

Then, as a consequence, we can write the kernel function $K(z, w)$ as a function in the range of the transform T thanks to the following

Proposition 4.3. *For a fixed $w \in \mathbb{C}$, we set $\varphi_w(t_1, t_2) = (A_{\overline{w}} \otimes A_w)(t_1, t_2)$, with $t_1, t_2 \in \mathbb{R}$. Then*

$$K(z, w) = T(\varphi_w)(z, \overline{z}), \qquad \text{for any } z \in \mathbb{C}.$$

Proof. This result can be obtained as a direct application of Theorem 4.1 and Definition 4.2 taking into account that

$$\overline{A_{\overline{w}} \otimes A_w} = A_w \otimes A_{\overline{w}}, \qquad w \in \mathbb{C}.$$

Indeed, for any $z, w \in \mathbb{C}$ we have

$$\begin{aligned}
T(\varphi_w)(z, \overline{z}) &= \langle \varphi_w, \overline{A_z \otimes A_{\bar{z}}} \rangle_{L^2(\mathbb{R}^2)} \\
&= \langle A_{\overline{w}} \otimes A_w, \overline{A_z \otimes A_{\bar{z}}} \rangle_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} (A_{\overline{w}} \otimes A_w)(t_1, t_2)(A_z \otimes A_{\bar{z}})(t_1, t_2) dt_1 dt_2 \\
&= \langle A_z \otimes A_{\bar{z}}, \overline{A_{\overline{w}} \otimes A_w} \rangle_{L^2(\mathbb{R}^2)} \\
&= \langle A_z \otimes A_{\bar{z}}, A_w \otimes A_{\overline{w}} \rangle_{L^2(\mathbb{R}^2)} \\
&= K(z, w).
\end{aligned}$$

$\square$

Remark 4.4. As a consequence of the previous result, we observe that for any fixed $w \in \mathbb{C}$, we have

$$T(\varphi_w)(z) = K(z, w) = K_w(z), \quad z \in \mathbb{C}.$$

Then, $T^* = T^{-1}$ since T is a unitary operator; moreover, for any $w \in \mathbb{C}$

$$T^{-1}(K_w)(t_1, t_2) = T^*(K_w)(t_1, t_2) = (A_{\overline{w}} \otimes A_w)(t_1, t_2), \qquad \text{for all } (t_1, t_2) \in \mathbb{R}^2.$$

As a first example, we consider the family of functions given by

$$\psi_{m,n}(x, y) := (\psi_m \otimes \psi_n)(x, y) = \psi_m(x) \psi_n(y), \text{ for any } m, n \geq 0.$$

We have

Proposition 4.5. *For every $z \in \mathbb{C}$ we have*

$$(4.4) \qquad T(\psi_{m,n})(z, \bar{z}) = \frac{z^m \overline{z^n}}{\sqrt{m! n!}} = \phi_{m,n}(z, \overline{z}), \quad m, n = 0, 1, \dots$$

and

$$(4.5) \qquad \Delta_z \phi_{p,q}(z, \overline{z}) = 4\sqrt{pq} \phi_{p-1, q-1}(z, \overline{z}), \quad p, q = 1, 2, \dots.$$

Proof. Recalling that $A_{\overline{z}}(y) = \overline{A_z(y)}$ for any $y \in \mathbb{R}$, using the Fubini's theorem we have:

$$
\begin{aligned}
T(\psi_{(m,n)})(z,\bar{z}) &= \int_{\mathbb{R}^2} (A_z \otimes A_{\bar{z}})(x,y)\psi_{m,n}(x,y)dxdy \\
&= \int_{\mathbb{R}^2} A_z(x)A_{\overline{z}}(y)\psi_m(x)\psi_n(y)dxdy \\
&= \left(\int_{\mathbb{R}} A_z(x)\psi_m(x)dx\right)\left(\int_{\mathbb{R}} \overline{A_z(y)}\psi_n(y)dy\right) \\
&= B(\psi_m)(z)\overline{B(\psi_n)(z)} \\
&= \frac{z^m\overline{z}^n}{\sqrt{m!n!}} \\
&= \phi_{m,n}(z,\overline{z}).
\end{aligned}
$$

Now, using the fact that $\Delta_z = 4\dfrac{\partial^2}{\partial z\partial\overline{z}}$ we get

$$
\Delta_z(\phi_{p,q})(z,\overline{z}) = 4\frac{pq}{\sqrt{p!q!}}z^{p-1}\overline{z}^{q-1} = 4\sqrt{pq}\phi_{p-1,q-1}(z,\overline{z}).
$$

$\square$

Example 4.6. Let

$$
f(z,\overline{z}) = T(\psi_{n,m})(z,\overline{z}), \text{ for any } z \in \mathbb{C}.
$$

Then $f \in \mathcal{SF}(\mathbb{C})$, moreover we have

$$
||f||_{\mathcal{SF}(\mathbb{C})} = ||T(\psi_{n,m})|| = 1 = ||\psi_{n,m}||_{L^2(\mathbb{R}^2)}.
$$

Theorem 4.7. *The first Segal-Bargmann type transform T defines an isometric isomorphism from $L^2(\mathbb{R}^2)$ onto $\mathcal{SF}(\mathbb{C})$.*

Proof. The normalized Hermite functions $(\psi_{m,n})_{m,n\geq 0}$ form an orthonormal basis of $L^2(\mathbb{R}^2)$, thus for any $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$, there exist unique coefficients $(\beta_{m,n})_{m,n\geq 0}$ in $\mathbb{C}$ such that

$$
\varphi(x,y) = \sum_{m,n=0}^{\infty} \psi_{m,n}(x,y)\beta_{m,n}, \quad \text{and} \quad ||\varphi||_{L^2(\mathbb{R}^2)}^2 = \sum_{m,n=0}^{\infty} |\beta_{m,n}|^2 < \infty.
$$

Therefore, inserting φ in the definition of the transform T and using some standard arguments we have

$$
\begin{aligned}
T(\varphi)(z,\bar{z}) &= \int_{\mathbb{R}^2} A_z(x)A_{\bar{z}}(y)\left(\sum_{m,n=0}^{\infty} \psi_{m,n}(x,y)\beta_{m,n}\right)dxdy \\
&= \sum_{m,n=0}^{\infty}\left(\int_{\mathbb{R}^2} A_z(x)A_{\overline{z}}(y)\psi_{m,m}(x,y)dxdy\right)\beta_{m,n} \\
&= \sum_{m,n=0}^{\infty} T(\psi_{m,n})(z,\overline{z})\beta_{m,n}.
\end{aligned}
$$

Applying the first part of Proposition 4.5 we obtain

$$T(\varphi)(z, \bar{z}) = \sum_{m,n=0}^{\infty} \phi_{m,n}(z, \overline{z})\beta_{m,n}$$

$$= \sum_{m,n=0}^{\infty} \frac{z^m \overline{z}^n}{\sqrt{m!n!}}\beta_{m,n}$$

$$= \sum_{n=0}^{\infty} \overline{z}^n \left(\sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!n!}}\beta_{m,n} \right).$$

Then, setting $f_n(z) = \dfrac{1}{\sqrt{n!}} \displaystyle\sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}}\beta_{m,n}$ for any $n \geq 0$ we have

$$T(\varphi)(z, \overline{z}) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z), \quad z \in \mathbb{C}.$$

We observe that f_n are entire functions. Moreover, it is immediate to see that

$$||f_n||_{\mathcal{F}(\mathbb{C})}^2 = \frac{1}{n!} \sum_{m=0}^{\infty} |\beta_{m,n}|^2 < \infty.$$

Hence, we deduce

$$||T(\varphi)||_{\mathcal{SF}(\mathbb{C})}^2 = \sum_{n=0}^{\infty} n!||f_n||_{\mathcal{F}(\mathbb{C})}^2$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\beta_{m,n}|^2$$

$$= ||\varphi||_{L^2(\mathbb{R})}^2.$$

This proves that the transform T defines an isometric operator from $L^2(\mathbb{R})$ into the polyanalytic Fock space of infinite order $\mathcal{SF}(\mathbb{C})$. On the other hand, we note that using Proposition 4.5 we have

$$T(\psi_{m,n}) = \frac{z^m \overline{z}^n}{\sqrt{m!n!}}, \quad m, n \geq 0.$$

This allows to justify that if $f \in \mathcal{SF}(\mathbb{C})$ there exists $g \in L^2(\mathbb{R})$ such that $f = T(g)$. In particular, this shows that T is also surjective which ends the proof. $\qquad\square$

Remark 4.8. We know that T defines an isometric and surjective transform from $L^2(\mathbb{R}^2)$ onto $\mathcal{SF}(\mathbb{C})$. This shows that T is invertible and the inverse of T exists. However, we do not know how to explicitly calculate this inverse because of the lack of a geometric description of the space $\mathcal{SF}(\mathbb{C})$.

We recall the position operators given by

$$X\varphi(x, y) = x\varphi(x, y)$$

and

$$Y\varphi(x, y) = y\varphi(x, y).$$

We denote by $\mathcal{D}(X) = \left\{ \varphi \in L^2(\mathbb{R}), X(\varphi) \in L^2(\mathbb{R}) \right\}$ the domain of the position operator X.

Proposition 4.9. *The following relations hold:*

$$T^{-1}\left(\frac{\partial}{\partial z} + M_z\right)T = \sqrt{2}X, \quad \text{on } \mathcal{D}(X)$$

and

$$T^{-1}\left(\frac{\partial}{\partial \bar{z}} + M_{\bar{z}}\right)T = \sqrt{2}Y, \quad \text{on } \mathcal{D}(Y).$$

Proof. We will make the calculations for the operator X and for Y it can be done in a similar way. As in the classical case, for $\varphi \in \mathcal{D}(X)$ we have

$$\frac{\partial}{\partial z}[T(\varphi)](z) = \int_{\mathbb{R}^2} \frac{\partial}{\partial z}(A_z(x))A_{\bar{z}}(y)\varphi(x,y)dxdy$$

.

It is easy to check that

$$\frac{\partial}{\partial z}(A(z,x)) = (-z + \sqrt{2}x)A(z,x).$$

Thus, we obtain

$$\frac{\partial}{\partial z}T(\varphi)(z) = \int_{\mathbb{R}^2}(-z + \sqrt{2}x)A_z(x)A_{\bar{z}}(y)\varphi(x,y)dxdy$$

$$= -z\int_{\mathbb{R}^2}A_z(x)A_{\bar{z}}(y)\varphi(x,y)dxdy + \sqrt{2}\int_{\mathbb{R}^2}A_z(x)A_{\bar{z}}(y)x\varphi(x,y)dxdy$$

$$= -M_zT(\varphi)(z) + \sqrt{2}T(X\varphi)(z).$$

Therefore, it follows that for any $\varphi \in \mathcal{D}(X)$ we have

$$\frac{\partial}{\partial z}T(\varphi) + M_zT(\varphi) = \sqrt{2}TX(\varphi).$$

Hence, we obtain

$$\left(\frac{\partial}{\partial z} + M_z\right)T = \sqrt{2}TX.$$

Finally, this leads to

$$T^{-1}\left(\frac{\partial}{\partial z} + M_z\right)T = \sqrt{2}X, \quad \text{on } \mathcal{D}(X).$$

$\square$

Proposition 4.10. *We have*

$$\frac{1}{2}T^{-1}\left(\frac{1}{4}\Delta_z + z\frac{\partial}{\partial \bar{z}} + \bar{z}\frac{\partial}{\partial z} + M_{|z|^2}\right)T = YX, \text{ on } \mathcal{D}(YX).$$

Proof. We observe that Proposition 4.9 yields

$$\frac{1}{2}T^{-1}\left(\frac{\partial}{\partial \bar{z}} + M_{\bar{z}}\right)\left(\frac{\partial}{\partial z} + M_z\right)T = YX.$$

Then, the result holds since we have

$$\left(\frac{\partial}{\partial \bar{z}} + M_{\bar{z}}\right)\left(\frac{\partial}{\partial z} + M_z\right) = \frac{1}{4}\Delta_z + z\frac{\partial}{\partial \bar{z}} + \bar{z}\frac{\partial}{\partial z} + M_{|z|^2}.$$

$\square$

Let $\mathcal{D}(M_z) = \{f \in \mathcal{SF}(\mathbb{C}), M_z(f) \in \mathcal{SF}(\mathbb{C})\}$ be the domain of the creation operator M_z.

Proposition 4.11. *The following relations hold*

$$T\left(X - \frac{\partial}{\partial x}\right)T^{-1} = \sqrt{2}M_z, \quad on\ \mathcal{D}(X) \cap \mathcal{D}(\frac{\partial}{\partial x})$$

and

$$T\left(Y - \frac{\partial}{\partial y}\right)T^{-1} = \sqrt{2}M_{\bar{z}}, \quad on\ \mathcal{D}(Y) \cap \mathcal{D}(\frac{\partial}{\partial y}).$$

Proof. We will prove the first statement of this result; the second one follows with similar arguments. Indeed, let $\varphi \in \mathcal{D}(X) \cap \mathcal{D}(\frac{\partial}{\partial x})$ we have

$$T(X - \frac{\partial}{\partial x})\varphi(z) = T(X\varphi)(z) - T(\frac{\partial}{\partial x}\varphi)(z).$$

By definition we have

$$T(X\varphi)(z) = \int_{\mathbb{R}^2} A_z(x)A_{\bar{z}}(y)x\varphi(x,y)dxdy,$$

so that

$$T(\frac{\partial}{\partial x}\varphi)(z) = \int_{\mathbb{R}^2} A_z(x)A_{\bar{z}}(y)\frac{\partial}{\partial x}\varphi(x,y)dxdy = -\int_{\mathbb{R}^2} \frac{\partial}{\partial x}(A_z(x))A_{\bar{z}}(y)\varphi(x,y)dxdy,$$

since

$$\frac{\partial}{\partial x}(A_z(x)) = (-x + \sqrt{2}z)A_z(x),$$

we are lead to

$$T\left(X - \frac{\partial}{\partial x}\right)\varphi(z) = \sqrt{2}zT(\varphi)(z).$$

Hence, we obtain

$$T\left(X - \frac{\partial}{\partial x}\right)T^{-1} = \sqrt{2}M_z, \quad on\ \mathcal{D}(X) \cap \mathcal{D}(\frac{\partial}{\partial x}).$$

$\square$

As a consequence of the previous result we can prove the following

Corollary 4.12. *We have*

$$T\left(\frac{\partial^2}{\partial x\partial y} + XY - X\frac{\partial}{\partial y} - Y\frac{\partial}{\partial x}\right)T^{-1} = 2M_{|z|^2}.$$

Proof. Indeed, we just need to apply Proposition 4.11 combined with the relation

$$\left(X - \frac{\partial}{\partial x}\right)\left(Y - \frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x\partial y} + XY - X\frac{\partial}{\partial y} - Y\frac{\partial}{\partial x}.$$

$\square$

5. A Berezin transform and related operators

We now use the kernel function (2.3) to study a Berezin integral transform and develop further results on it.

Definition 5.1. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ and let $d\mu(w) = \frac{1}{\pi}e^{-|w|^2}dA(w)$ be the Gaussian measure. Then, we consider the following integral transform*

$$(5.1) \qquad \mathcal{B}(f)(z) = \int_{\mathbb{C}} e^{-|z|^2}K(z,w)f(w)d\mu(w), \quad z \in \mathbb{C},$$

when the integral exists.

Remark 5.2. We observe that
$$e^{-|z|^2}K(z,w)e^{-|w|^2} = e^{-|z-w|^2}, \quad z,w \in \mathbb{C}.$$

Thus, it turns out that the integral transform given by (5.1) coincides with the so-called Berezin transform considered in [33, p. 101]. However, since $\mathcal{B}$ can be expressed in terms of the kernel function $K(z,w)$ we can use various properties of this kernel in order to develop further results. A similar transform in the case of two complex variables was considered in [19]. It is important to note that the Berezin transform was first introduced by Berezin in [20] as a general concept of quantization.

We start first by observing that the Berezin transform $\mathcal{B}$ fixes all the complex monomials z^n, $n \in \mathbb{N}$ which form an orthogonal basis of the classical Fock $\mathcal{F}(\mathbb{C})$:

Proposition 5.3. *For $n = 0, 1, \dots$ it holds that*

$$(5.2) \qquad \mathcal{B}(z^n) = z^n, \quad \forall z \in \mathbb{C}.$$

Proof. We set $f_n(z) = z^n$ with $n = 0, 1, \dots$. Then, for every $\alpha \in \mathbb{C}$ we can write

$$(5.3) \qquad \sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{N}(f_n)(z)}{n!} = \frac{1}{\pi} e^{-|z|^2} \int_{\mathbb{C}} K(z,w) e^{\alpha w} e^{-|w|^2} dA(w).$$

Let us set $w = t_1 + it_2$ and let us replace $K(z,w)$ by its expression to obtain

$$\sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{N}(f_n)(z)}{n!} = \frac{1}{\pi} e^{-|z|^2} \int_{\mathbb{R}^2} e^{z(t_1-it_2)+(t_1+it_2)\overline{z}+\alpha(t_1+it_2)} e^{-(t_1^2+t_2^2)} dt_1 dt_2.$$

Therefore, thanks to Fubini's theorem we have

$$\sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{B}(f_n)(z)}{n!} = \frac{1}{\pi} e^{-|z|^2} \left(\int_{\mathbb{R}} e^{-t_1^2 + t_1(z+\overline{z}+\alpha)} dt_1 \right) \left(\int_{\mathbb{R}} e^{-t_2^2 + t_2 i(\overline{z}-z+\alpha)} dt_2 \right).$$

Now, we recall the classical Gaussian integral

$$(5.4) \qquad \int_{\mathbb{R}} e^{-at^2+bt} dt = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \qquad a > 0, b \in \mathbb{C}.$$

Thus, setting $b_1(z,\overline{z}) = z + \overline{z} + \alpha$ and $b_2(z,\overline{z}) = i(\overline{z} - z + \alpha)$ we obtain

$$\sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{B}(f_n)(z)}{n!} = \frac{1}{\pi} e^{-|z|^2} \pi e^{\frac{b_1^2(z,\overline{z})}{4}} e^{\frac{b_2^2(z,\overline{z})}{4}}.$$

Therefore, we have

$$\sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{B}(f_n)(z)}{n!} = e^{-|z|^2} e^{\frac{b_1^2(z,\overline{z})+b_2^2(z,\overline{z})}{4}}$$

We have $b_1^2(z,\overline{z}) = z^2+\overline{z}^2+2|z|^2+\alpha^2+2\alpha z+2\alpha\overline{z}$ and $b_2^2(z,\overline{z}) = -(\overline{z}^2+z^2-2|z|^2+\alpha^2+2\alpha\overline{z}-2\alpha z)$, which leads to

$$b_1^2(z,\overline{z}) + b_2^2(z,\overline{z}) = 4(|z|^2 + \alpha z).$$

Hence

$$\sum_{n=0}^{\infty} \alpha^n \frac{\mathcal{B}(f_n)(z)}{n!} = e^{\alpha z} = \sum_{n=0}^{\infty} \alpha^n \frac{z^n}{n!}, \quad \forall \alpha \in \mathbb{C}.$$

Finally, we identify the coefficients with respect to the variable α and get

$$\mathcal{B}(f_n)(z) = z^n, \quad n = 0, 1, \dots$$

$\square$

Remark 5.4. It is impotant to note that in [33] it was proved that a function $f \in L^\infty(\mathbb{C})$ is a fixed point for the Berezin transform $\mathcal{B}$ if and only if f is constant. Then, Proposition 5.3 shows that monomials are fixed by the Berezin transform. In particular, this suggests to consider a more general fixed point problem: to find when $\mathcal{B}(f) = f$ for f in some suitable function space.

Remark 5.5. We observe that

$$(5.5) \qquad \mathcal{B}(f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} e^{\overline{w}z + w\overline{z} - |z|^2} f(w) e^{-|w|^2} dA(w).$$

If $z = 0$, we have

$$\mathcal{B}(f)(0) = \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{-|w|^2} dA(w).$$

In particular, using the Cauchy-Schwarz inequality we obtain

$$|\mathcal{B}(f)(0)| \leq ||f||_{L^2(\mathbb{C},\mu)},$$

and by taking the function $f = 1$, we get

$$\mathcal{B}(1)(z) = e^{-|z|^2} \frac{1}{\pi} \int_{\mathbb{C}} K(z,w) e^{-|w|^2} dA(w).$$

Theorem 5.6. *For any $\alpha > 0$, let $d\mu_\alpha(w) = \frac{\alpha}{\pi} e^{-\alpha|w|^2} dA(w)$ be the weighted Gaussian measure. For any $\beta > 2$ the operator $\mathcal{B}$ is bounded from $L^2(\mathbb{C}, d\mu)$ into $L^2(\mathbb{C}, d\mu_\beta)$. In particular, for any $f \in L^2(\mathbb{C}, d\mu)$ it holds that*

$$(5.6) \qquad ||\mathcal{B}(f)||_{L^2(\mathbb{C},d\mu_\beta)} \leq \frac{1}{\sqrt{\beta - 2}} ||f||_{L^2(\mathbb{C},d\mu)}.$$

Proof. For $z \in \mathbb{C}$ fixed, we have

$$\mathcal{B}(f)(z) = e^{-|z|^2} \int_{\mathbb{C}} K(z,w) f(w) d\mu(w).$$

Thus, setting $K_z(w) = K(z,w)$ for any $w \in \mathbb{C}$ and noting that $f, K_z \in L^2(\mathbb{C}, d\mu)$ we obtain

$$|\mathcal{B}(f)(z)| \leq e^{-|z|^2} \int_{\mathbb{C}} |K(z,w)||f(w)| d\mu(w),$$

and using the Cauchy Schwarz inequality

$$|\mathcal{B}(f)(z)| \leq e^{-|z|^2} ||K_z||_{L^2(\mathbb{C},d\mu)} ||f||_{L^2(\mathbb{C},d\mu)}.$$

Moreover, developing some calculations using Gaussian integrals we get

$$||K_z||_{L^2(\mathbb{C},d\mu)} = e^{2|z|^2},$$

and so

$$|\mathcal{B}(f)(z)| \leq e^{|z|^2} ||f||_{L^2(\mathbb{C},d\mu)}.$$

As a consequence, for any $\beta > 2$ we have

$$||\mathcal{B}(f)||^2_{L^2(\mathbb{C},d\mu_\beta)} \leq ||f||^2_{L^2(\mathbb{C},d\mu)} \left(\frac{1}{\pi} \int_{\mathbb{C}} e^{-(\beta-2)|z|^2} dA(z) \right) = \frac{1}{\beta - 2} ||f||^2_{L^2(\mathbb{C},d\mu)}.$$

Hence, $\mathcal{B}$ is a bounded operator from $L^2(\mathbb{C}, d\mu)$ into $L^2(\mathbb{C}, d\mu_\beta)$ with $\beta > 2$. $\qquad \square$

Remark 5.7. In particular, for any $\beta > 2$ we have

$$||\mathcal{B}(f)||_{L^2(\mathbb{C},d\mu_\beta)} \leq ||\mathcal{B}(f)||_{\mathcal{SF}(\mathbb{C})}.$$

This explains somehow that

$$\mathcal{SF}(\mathbb{C}) \subset L^2(\mathbb{C}, d\mu_\beta), \quad \beta > 2.$$

Now, we can prove the following

Lemma 5.8. *For any $p, q = 0, 1, 2, \ldots$, we have*

$$\mathcal{B}(H_{p,q})(z, \overline{z}) = z^p \overline{z}^q.$$

We have also

$$\|\mathcal{B}(H_{p,q})\|_{\mathcal{SF}(\mathbb{C})} = \|H_{p,q}\|_{L^2(\mathbb{C}, d\mu)}.$$

Proof. For any $u, v \in \mathbb{C}$ we have (see [23, 26])

$$(5.7) \qquad \sum_{m,n=0}^{\infty} H_{m,n}(z, \overline{z}) \frac{u^m v^n}{m!n!} = e^{uz+v\overline{z}-uv}, \quad \forall z \in \mathbb{C}.$$

For $w \in \mathbb{C}$ we set

$$(5.8) \qquad \mathcal{R}(z, w) := \sum_{m,n=0}^{\infty} \frac{\overline{z}^m z^n}{m!n!} H_{m,n}(w, \overline{w}) = e^{\overline{z}w+z\overline{w}-|z|^2}, \quad z \in \mathbb{C}.$$

It is immediate that

$$\overline{\mathcal{R}(z, w)} = \mathcal{R}(z, w), \quad z, w \in \mathbb{C}.$$

From formula (5.1) we obtain

$$\mathcal{B}(H_{p,q})(z) = \int_{\mathbb{C}} e^{-|z|^2} K(z, w) H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \int_{\mathbb{C}} e^{-|z|^2+z\overline{w}+w\overline{z}} H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \int_{\mathbb{C}} \left(\sum_{m,n=0}^{\infty} \frac{\overline{z}^m z^n}{m!n!} H_{m,n}(w, \overline{w}) \right) H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \int_{\mathbb{C}} \mathcal{R}(z, w) H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \int_{\mathbb{C}} \overline{\mathcal{R}(z, w)} H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \sum_{m,n=0}^{\infty} \frac{z^m \overline{z}^n}{m!n!} \int_{\mathbb{C}} \overline{H_{m,n}(w, \overline{w})} H_{p,q}(w, \overline{w}) d\mu(w)$$

$$= \sum_{m,n=0}^{\infty} \frac{z^m \overline{z}^n}{m!n!} \langle H_{p,q}, H_{m,n} \rangle_{L^2(\mathbb{C}, d\mu)}.$$

However, we know that the complex Hermite polynomials $(H_{m,n})_{m,n \geq 0}$ form an orthonormal basis of $L^2(\mathbb{C}, d\mu)$ (see [26]) so that we have

$$\langle H_{p,q}, H_{m,n} \rangle_{L^2(\mathbb{C}, d\mu)} = p!q! \delta_{(p,q);(m,n)}.$$

In particular, this leads to

$$\mathcal{B}(H_{p,q})(z, \overline{z}) = z^p \overline{z}^q, \quad z \in \mathbb{C}.$$

As a consequence, it is clear that for any $p, q = 0, 1, 2, \ldots$ we have

$$\|\mathcal{B}(H_{p,q})\|_{\mathcal{SF}(\mathbb{C})} = \|z^p \overline{z}^q\|_{\mathcal{SF}(\mathbb{C})} = \sqrt{p!q!} = \|H_{p,q}\|_{L^2(\mathbb{C}, d\mu)}.$$

$\square$

Theorem 5.9. *The integral transform $\mathcal{B}$ is an isomorphism from $L^2(\mathbb{C}, d\mu)$ onto $\mathcal{SF}(\mathbb{C})$ and for any $f \in L^2(\mathbb{C}, d\mu)$ we have*

$$(5.9) \qquad ||\mathcal{B}(f)||_{\mathcal{SF}(\mathbb{C})} = ||f||_{L^2(\mathbb{C}, d\mu)}.$$

Proof. For any $f \in L^2(\mathbb{C}, d\mu)$ we can write the following decomposition using Hermite polynomials

$$f(z) = \sum_{p,q=0}^{\infty} H_{p,q}(z, \overline{z})\alpha_{p,q}, \quad z \in \mathbb{C},$$

and

$$||f||_{L^2(\mathbb{C}, d\mu)}^2 = \sum_{p,q=0}^{\infty} p!q!|\alpha_{p,q}|^2.$$

Thus, thanks to Lemma 5.8 we get

$$\mathcal{B}(f)(z) = \sum_{p,q=0}^{\infty} \mathcal{B}(H_{p,q})(z, \overline{z})\alpha_{p,q}$$
$$= \sum_{p,q=0}^{\infty} z^p \overline{z}^q \alpha_{p,q}.$$

Then, using also Proposition 2.13 we obtain

$$||\mathcal{B}(f)||_{\mathcal{SF}(\mathbb{C})}^2 = \sum_{p,q=0}^{\infty} p!q!|\alpha_{p,q}|^2$$
$$= ||f||_{L^2(\mu)}^2.$$

We observe that the surjectivity of the transform $\mathcal{B}$ is a direct consequence of Lemma 5.8. This allows to consider the integral transform $\mathcal{B} : L^2(\mathbb{C}, d\mu) \longrightarrow \mathcal{SF}(\mathbb{C})$ which defines an unitary operator.

$\square$

Proposition 5.10. *For any $f \in L^2(\mathbb{C}, d\mu)$, it holds that*

$$\frac{\partial}{\partial z}\mathcal{B}(f)(z) = -\overline{z}\mathcal{B}(f)(z) + \mathcal{B}(\overline{w}f)(z).$$

In a similar way, we have also

$$\frac{\partial}{\partial \overline{z}}\mathcal{B}(f)(z) = -z\mathcal{B}(f)(z) + \mathcal{B}(wf)(z).$$

Proof. We observe that thanks to the definition of $\mathcal{B}$ we have

$$\mathcal{B}(f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} e^{-|z|^2} K(z, w) f(w) e^{-|w|^2} dA(w), \quad z \in \mathbb{C}.$$

Thus, we have

$$(5.10) \qquad \frac{\partial}{\partial z}\mathcal{B}(f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial}{\partial z}(e^{-|z|^2} K(z, w)) f(w) e^{-|w|^2} dA(w), \quad z \in \mathbb{C}.$$

Applying the Leibniz rule and Proposition 2.5 we get

$$\frac{\partial}{\partial z}(e^{-|z|^2}K(z,w)) = e^{-|z|^2}\frac{\partial}{\partial z}K(z,w) + \frac{\partial}{\partial z}(e^{-|z|^2})K(z,w)$$
$$= e^{-|z|^2}\overline{w}K(z,w) - \overline{z}e^{-|z|^2}K(z,w)$$
$$= (\overline{w} - \overline{z})e^{-|z|^2}K(z,w).$$

Hence, we insert the previous computations in the formula (5.10) and this leads to

$$\frac{\partial}{\partial z}\mathcal{B}(f)(z) = \frac{1}{\pi}\int_{\mathbb{C}}e^{-|z|^2}(\overline{w} - \overline{z})K(z,w)f(w)e^{-|w|^2}dA(w)$$
$$= \mathcal{B}(\overline{w}f)(z) - \overline{z}\mathcal{B}(f)(z).$$

$\square$

As a consequence of the previous result we prove

Proposition 5.11. *It holds that*

(5.11)
$$\mathcal{B}^{-1}\left(\frac{\partial}{\partial z} + \overline{z}\right)\mathcal{B} = M_{\overline{w}},$$

and

(5.12)
$$\mathcal{B}^{-1}\left(\frac{\partial}{\partial \overline{z}} + z\right)\mathcal{B} = M_{w}.$$

Proof. We note that thanks to Proposition 5.10 we can write

$$\frac{\partial}{\partial z}\mathcal{B}(f)(z) + \overline{z}\mathcal{B}(f)(z) = \mathcal{B}(\overline{w}f)(z).$$

Then, using the fact that $\mathcal{B}$ is an isomorphism it is easy to see that

$$\mathcal{B}^{-1}\left(\frac{\partial}{\partial z} + \overline{z}\right)\mathcal{B} = M_{\overline{w}}.$$

In a similar way we can prove the second statement.

$\square$

As a consequence, we can prove the following

Corollary 5.12. *It holds that*

(5.13)
$$\mathcal{B}^{-1}\left(\frac{1}{4}\Delta_z + \frac{\partial}{\partial \overline{z}}\overline{z} + z\frac{\partial}{\partial z} + |z|^2\right)\mathcal{B} = M_{|w|^2}.$$

Proof. We observe that using the two expressions proved in Proposition 5.11 we obtain that

$$\mathcal{B}^{-1}\left(\frac{\partial}{\partial z} + \overline{z}\right)\left(\frac{\partial}{\partial \overline{z}} + z\right)\mathcal{B} = M_{\overline{w}}M_{w} = M_{|w|^2}.$$

On the other hand, we have

$$\left(\frac{\partial}{\partial z} + \overline{z}\right)\left(\frac{\partial}{\partial \overline{z}} + z\right) = \left(\frac{1}{4}\Delta_z + \frac{\partial}{\partial \overline{z}}\overline{z} + z\frac{\partial}{\partial z} + |z|^2\right),$$

and this ends the proof.

$\square$

In the next result, we will calculate $\dfrac{\partial^n}{\partial \overline{z}^n}\mathcal{B}(f)(z)$ and $\dfrac{\partial^n}{\partial z^n}\mathcal{B}(f)(z)$:

Proposition 5.13. *For any* $n = 0, 1, 2, \ldots$, *it holds that*

$$(5.14) \qquad \frac{\partial^n}{\partial z^n} \mathcal{B}(f)(z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n-k} f)(z) = \mathcal{B}((\overline{w} - \overline{z})^n f)(z),$$

and

$$(5.15) \qquad \frac{\partial^n}{\partial \overline{z}^n} \mathcal{B}(f)(z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k \mathcal{B}(w^{n-k} f)(z) = \mathcal{B}((w - z)^n f)(z).$$

Proof. The proof is done by induction. For $n = 0$ the relations are true. We assume that (5.14) holds for some n and we prove that we have

$$(5.16) \qquad \frac{\partial^{n+1}}{\partial z^{n+1}} \mathcal{B}(f)(z) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n+1-k} f)(z).$$

Indeed, using Proposition 5.10 we have

$$\frac{\partial}{\partial z} \mathcal{B}(f)(z) = -\overline{z} \mathcal{B}(f)(z) + \mathcal{B}(\overline{w} f)(z).$$

Thus, applying the operator $\dfrac{\partial^n}{\partial z^n}$ we get

$$\frac{\partial^{n+1}}{\partial z^{n+1}} \mathcal{B}(f)(z) = -\overline{z} \frac{\partial^n}{\partial z^n} \mathcal{B}(f)(z) + \frac{\partial^n}{\partial z^n} \mathcal{B}(\overline{w} f)(z).$$

Applying the induction hypothesis to the functions f and $g = \overline{w} f$ we obtain the chain of equalities

$$\frac{\partial^{n+1}}{\partial z^{n+1}} \mathcal{B}(f)(z) = -\overline{z} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n-k} f)(z) \right) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n+1-k} f)(z)$$

$$= \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \overline{z}^{k+1} \mathcal{B}(\overline{w}^{n-k} f)(z) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n+1-k} f)(z)$$

$$= \sum_{h=1}^{n+1} (-1)^h \binom{n}{h-1} \overline{z}^h \mathcal{B}(\overline{w}^{n+1-h} f)(z) + \sum_{h=0}^{n} (-1)^h \binom{n}{h} \overline{z}^h \mathcal{B}(\overline{w}^{n+1-h} f)(z)$$

$$= \binom{n}{0} \mathcal{B}(\overline{w}^{n+1} f)(z) + \sum_{h=1}^{n} (-1)^h \left(\binom{n}{h} + \binom{n}{h-1} \right) \overline{z}^h \mathcal{B}(\overline{w}^{n+1-h} f)(z)$$

$$+ (-1)^{n+1} \binom{n}{n} \overline{z}^{n+1} \mathcal{B}(f)(z).$$

By Pascal identity we have

$$\binom{n}{h} + \binom{n}{h-1} = \binom{n+1}{h}, \qquad \text{for any } h \geq 1,$$

so that

$$\frac{\partial^{n+1}}{\partial z^{n+1}} \mathcal{B}(f)(z) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \overline{z}^k \mathcal{B}(\overline{w}^{n+1-k} f)(z),$$

and this ends the proof. The second statement follows using similar arguments. $\qquad \square$

Theorem 5.14. *For any $f \in L^2(\mathbb{C}, d\mu)$, it holds that*

(5.17)
$$\mathcal{B}(\partial_{\overline{w}}f)(z) = \mathcal{B}(wf)(z) - z\mathcal{B}(f)(z).$$

In a similar way, we have

(5.18)
$$\mathcal{B}(\partial_w f)(z) = \mathcal{B}(\overline{w}f)(z) - \overline{z}\mathcal{B}(f)(z).$$

Proof. This result can be proved using some computations that are based on Proposition 7.2 of [31]. Indeed, setting $d\mu(z) := \frac{1}{\pi}e^{-|z|^2}dA(z)$ we get

$$\int_{\mathbb{C}} u(w)\overline{(\partial_{\overline{w}}v(w))}d\mu(w) = \int_{\mathbb{C}} (-\partial_w + \overline{w})\,(u(w))\overline{v(w)}d\mu(w).$$

In particular, this means that on $L^2(\mathbb{C}, d\mu)$ we have

$$(\partial_w)^* = -\partial_w + \overline{w}.$$

Then, using this fact we deduce the following computations

$$
\begin{aligned}
M_z\mathcal{B}(f)(z) &= z\mathcal{B}(f)(z) \\
&= e^{-|z|^2}\int_{\mathbb{C}} zK(z,w)f(w)d\mu(w) \\
&= e^{-|z|^2}\int_{\mathbb{C}} \frac{\partial}{\partial\overline{w}}(K(z,w))f(w)d\mu(w) \\
&= e^{-|z|^2}\langle \partial_{\overline{w}}K_z, \overline{f}\rangle_\mu \\
&= e^{-|z|^2}\langle K_z, \partial_{\overline{w}}^*(\overline{f})\rangle_\mu \\
&= e^{-|z|^2}\langle K_z, (-\partial_w + \overline{w})(\overline{f})\rangle_\mu \\
&= e^{-|z|^2}\int_{\mathbb{C}} K(z,w)\overline{(-\partial_w + \overline{w})(\overline{f})(w)}d\mu(w) \\
&= e^{-|z|^2}\int_{\mathbb{C}} K(z,w)(-\partial_{\overline{w}}f + wf)d\mu(w) \\
&= \mathcal{B}(-\partial_{\overline{w}}f + wf)(z) \\
&= \mathcal{B}(wf)(z) - \mathcal{B}(\partial_{\overline{w}}f)(z),
\end{aligned}
$$

where we used $\overline{\partial_w f} = \partial_{\overline{w}}\overline{f}$, which leads to

$$\overline{(-\partial_w + \overline{w})(\overline{f})} = (-\partial_{\overline{w}}f + wf).$$

This ends the proof. $\qquad\square$

As a consequence of the previous result, we prove that the operators $w - \partial_{\overline{w}}, \overline{w} - \partial_w$ are similar to M_z and $M_{\overline{z}}$ respectively:

Proposition 5.15. *It holds that*

(5.19)
$$\mathcal{B}\left(w - \partial_{\overline{w}}\right)\mathcal{B}^{-1} = M_z,$$

and

(5.20)
$$\mathcal{B}\left(\overline{w} - \partial_w\right)\mathcal{B}^{-1} = M_{\overline{z}}.$$

Proof. By Theorem 5.14 we have

$$\mathcal{B}(\partial_{\overline{w}}f)(z) = \mathcal{B}(wf)(z) - z\mathcal{B}(f)(z),$$

from which we deduce

$$\mathcal{B}((w - \partial_{\overline{w}})f)(z) = z\mathcal{B}(f)(z).$$

Hence, using the fact that the integral transform $\mathcal{B}$ is a unitary operator from $L^2(\mathbb{C}, d\mu)$ onto $\mathcal{SF}(\mathbb{C})$ we multiply the operator $\mathcal{B}^{-1}$ on the right and obtain

$$\mathcal{B}\left(w - \partial_{\overline{w}}\right)\mathcal{B}^{-1} = M_z.$$

The second part of the statement can be proved similarly. $\qquad\square$

Proposition 5.16. *We have*

(5.21) $$\partial_z \mathcal{B}(f) = \mathcal{B}(\partial_w f)$$

and

(5.22) $$\partial_{\overline{z}} N(f) = \mathcal{B}(\partial_{\overline{w}} f).$$

Proof. Applying Proposition 5.10 and Theorem 5.14 we have

$$\partial_z \mathcal{B}(f)(z) = -\overline{z}\mathcal{B}(f)(z) + \mathcal{B}(\overline{w}f)(z)$$

and

$$\mathcal{B}(\partial_w f)(z) = \mathcal{B}(\overline{w}f)(z) - \overline{z}\mathcal{B}(f)(z).$$

Therefore, from the two previous relations we obtain

$$\partial_z \mathcal{B}(f)(z) = \mathcal{B}(\partial_w f)(z).$$

The second statement can be proved in a similar way and this will end the proof. $\qquad\square$

In the table below, we summarize different operators that are equivalent to each others thanks to the Berezin transform $\mathcal{B}$.

TABLE 1. Equivalent operators using the transform $\mathcal{B}$

$\mathcal{SF}(\mathbb{C})$	$L^2(\mathbb{C}, d\mu)$
$\frac{\partial}{\partial z} + \overline{z}$	$\overline{w}$
$\frac{\partial}{\partial \overline{z}} + z$	w
$\frac{\partial^n}{\partial z^n}$	$\frac{\partial^n}{\partial w^n}$
$\frac{\partial^n}{\partial \overline{z}^n}$	$\frac{\partial^n}{\partial \overline{w}^n}$
$\frac{1}{4}\Delta_z + \frac{\partial}{\partial \overline{z}}\overline{z} + z\frac{\partial}{\partial z} + |z|^2$	$|w|^2$
$M_{|z|^2}$	$\frac{1}{4}\Delta_w - \frac{\partial}{\partial \overline{w}}\overline{w} - w\frac{\partial}{\partial w} + |w|^2$
z	$w - \frac{\partial}{\partial \overline{w}}$
$\overline{z}$	$\overline{w} - \frac{\partial}{\partial w}$

Inspired by [34] we consider the following integral operator on the Fock space $\mathcal{F}(\mathbb{C})$

Definition 5.17. *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ and denote by $d\mu(w) = \frac{1}{\pi}e^{|w|^2}dA(w)$ the normalized Gaussian measure. Then, we define the following integral transform when it exists*

$$(5.23) \qquad \mathcal{S}_\varphi(f)(z) := \int_\mathbb{C} e^{z\overline{w}} f(w)\varphi(w - z)d\mu(w), \quad f \in \mathcal{F}(\mathbb{C}).$$

Remark 5.18. If $\varphi = 1$, it is clear by the reproducing kernel property for the Fock space that in this case

$$\mathcal{S}_\varphi(f)(z) := \int_\mathbb{C} e^{z\overline{w}} f(w)d\mu(w) = f(z), \quad \forall f \in \mathcal{F}(\mathbb{C}).$$

Example 5.19. For every $a \in \mathbb{C}$ set $\varphi_a(w) = e^{\overline{a}w}$. Then, we have

$$\mathcal{S}_{\varphi_z}(f)(z) = \int_\mathbb{C} e^{z\overline{w}} f(w)\varphi_z(w - z)d\mu(w) = \mathcal{B}(f)(z).$$

It turns out that the transform $\mathcal{B}$ is a particular case of the general integral operator $\mathcal{S}_\varphi$.

6. The polyanalytic Hardy space of infinite order and Gleason problem

Let us denote by $\mathbb{D}$ the unit disk and by $\mathcal{H}^2(\mathbb{D})$ the classical Hardy space. In this section, we will prove the following main result:

Theorem 6.1. *The reproducing kernel Hilbert space with reproducing kernel* $\dfrac{1}{(1 - z\overline{w})(1 - \overline{z}w)}$ *is, up to a multiplicative positive factor, the only reproducing kernel Hilbert space of polyanalytic functions of infinite order, regular at the origin, and for which*

$$(6.1) \qquad\qquad R_\infty^* \;=\; M_z$$
$$(6.2) \qquad\qquad L_\infty^* \;=\; M_{\overline{z}}.$$

To prove this theorem we need a couple of preliminary results, including a sequential characterization of the space $\mathcal{H}(\mathsf{K})$.

Definition 6.2. *The polyanalytic Hardy space of infinite order $\mathcal{SH}(\mathbb{D})$ is the space of functions of the form*

$$(6.3) \qquad\qquad f(z) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z),$$

satisfying

    i) $f_n \in \mathcal{H}^2(\mathbb{D})$ for any $n \geq 0$;

    ii) $\|f\|_{\mathcal{SH}(\mathbb{D})}^2 = \displaystyle\sum_{n=0}^{\infty} \|f_n\|_{\mathcal{H}^2(\mathbb{D})}^2 < \infty.$

Then, we consider the scalar product on $\mathcal{SH}(\mathbb{D})$ given by

$$(6.4) \qquad\qquad \langle f, g\rangle_{\mathcal{SH}(\mathbb{D})} := \sum_{k=0}^{\infty} \langle f_k, g_k\rangle_{\mathcal{H}^2(\mathbb{D})},$$

for any $f = \displaystyle\sum_{k=0}^{\infty} \overline{z}^k f_k$ and $g = \displaystyle\sum_{k=0}^{\infty} \overline{z}^k g_k$ with $f_k, g_k \in \mathcal{H}^2(\mathbb{D})$ for every $k \geq 0$.

Proposition 6.3. *A function $f : \mathbb{D} \longrightarrow \mathbb{C}$ belongs to $\mathcal{SH}(\mathbb{D})$ if and only if f is of the form*

$$f(z) = \sum_{(m,n)\in\mathbb{N}^2} z^m \overline{z}^n \alpha_{m,n},$$

with $(\alpha_{m,n}) \subset \mathbb{C}$ and such that

$$(6.5) \qquad \qquad ||f||^2_{\mathcal{SH}(\mathbb{D})} = \sum_{(m,n)\in\mathbb{N}^2} |\alpha_{m,n}|^2 < \infty.$$

Moreover, if for any $(m,n) \in \mathbb{N}^2$ we set $v_{m,n}(z,\overline{z}) = z^m \overline{z}^n$. Then, the family of functions $\{v_{m,n}\}_{m,n\geq 0}$ form an orthonormal basis of $\mathcal{SH}(\mathbb{D})$.

Proof. This proof follows the arguments used to prove Proposition 2.13.   □

Lemma 6.4. *We have*

$$\mathcal{H}(\mathsf{K}) = \mathcal{SH}(\mathbb{D}).$$

Moreover, it holds that

$$(6.6) \qquad \qquad \mathsf{K}(z,w) = \sum_{m,n=0}^{\infty} v_{m,n}(z,\overline{z})\overline{v_{m,n}(w,\overline{w})},$$

for every $z, w \in \mathbb{D}$.

Proof. We note that $(v_{m,n})_{m,n\geq 0}$ is an orthonormal basis of the space $\mathcal{SH}(\mathbb{D})$. Thus, the associated reproducing kernel is given by the following series which converges uniformly on each compact so that

$$\sum_{m,n=0}^{\infty} v_{m,n}(z,\overline{z})\overline{v_{m,n}(w,\overline{w})} < \infty, \text{ for any } z, w \in \mathbb{D}.$$

More precisely, for any $(z,w) \in \mathbb{D}^2$ we have the equalities

$$\begin{aligned}
\sum_{m,n=0}^{\infty} v_{m,n}(z,\overline{z})\overline{v_{m,n}(w,\overline{w})} &= \sum_{m,n=0}^{\infty} z^m \overline{z}^n \overline{w}^m w^n \\
&= \left(\sum_{m=0}^{\infty} z^m \overline{w}^m\right)\left(\sum_{n=0}^{\infty} w^n \overline{z}^n\right) \\
&= \frac{1}{(1-z\overline{w})}\frac{1}{(1-\overline{z}w)} \\
&= \frac{1}{(1-z\overline{w})(1-\overline{z}w)} \\
&= \mathsf{K}(z,w).
\end{aligned}$$

□

Lemma 6.5. *It holds that*

$$(6.7) \qquad \qquad \langle R_{\infty}(f), g \rangle_{\mathcal{SH}(\mathbb{D})} = \langle f, M_z g \rangle_{\mathcal{SH}(\mathbb{D})},$$

and

$$(6.8) \qquad \qquad \langle L_{\infty}(f), g \rangle_{\mathcal{SH}(\mathbb{D})} = \langle f, M_{\overline{z}} g \rangle_{\mathcal{SH}(\mathbb{D})}.$$

Proof. Let $f = \sum_{k=0}^{\infty} \overline{z}^k f_k$ and $g = \sum_{k=0}^{\infty} \overline{z}^k g_k$ in $\mathcal{SH}(\mathbb{D})$. Since we have

$$M_z(g) = \sum_{k=0}^{\infty} \overline{z}^k M_z(g_k),$$

it follows that

$$\langle R_\infty(f), g\rangle_{\mathcal{SH}(\mathbb{D})} = \sum_{k=0}^\infty \langle R_0(f_k), g_k\rangle_{\mathcal{H}^2(\mathbb{D})}$$

$$= \sum_{k=0}^\infty \langle f_k, (R_0)^* g_k\rangle_{\mathcal{H}^2(\mathbb{D})}$$

$$= \sum_{k=0}^\infty \langle f_k, M_z(g_k)\rangle_{\mathcal{H}^2(\mathbb{D})}$$

$$= \sum_{k=0}^\infty \langle f_k, z g_k\rangle_{\mathcal{H}^2(\mathbb{D})}$$

$$= \langle f, M_z(g)\rangle_{\mathcal{SH}(\mathbb{D})}.$$

In a similar way, we can prove the second part of the statement. $\qquad\square$

Example 6.6. On the bidisc $\mathbb{D}^2$ we consider the inner function defined by

$$j(z_1, z_2) = \frac{z_1 + z_2 + 2z_1 z_2}{z_1 + z_2 + 2}, \quad \forall (z_1, z_2) \in \mathbb{D}^2.$$

Then, $j(z_1, z_2)$ is a contractive multiplier of the Hardy space $\mathcal{H}^2(\mathbb{D}^2)$ and hence (dimension 2) is in the Schur-Agler class, see [14, 15]. Moreover, if we set

$$\rho_w(z) = 1 - z\overline{w},$$

we can consider the kernel function

(6.9) $\qquad \mathsf{K}_j((z_1, z_2); (w_1, w_2)) := \dfrac{1 - j(z_1, z_2)\overline{j(w_1, w_2)}}{\rho_{w_1}(z_1)\rho_{w_2}(z_2)}, \quad \forall (z_1, z_2); (w_1, w_2) \in \mathbb{D}^2.$

We note that

$$\mathsf{K}_j((z_1, z_2); (w_1, w_2)) = \frac{2}{(z_1 + z_2 + 2)(\overline{w_1} + \overline{w_2} + 2)}$$
$$\cdot \left(\frac{(z_1 + 1)(\overline{w_1} + 1)}{(1 - z_1\overline{w_1})} + \frac{(z_2 + 1)(\overline{w_2} + 1)}{(1 - z_2\overline{w_2})} \right).$$

Then, by taking $z_1 = z, z_2 = \overline{z}$ and $w_1 = w, w_2 = \overline{w}$ we have

(6.10) $\qquad\qquad j(z, \overline{z}) = \dfrac{z + \overline{z} + 2|z|^2}{z + \overline{z} + 2} = \dfrac{\mathrm{Re}(z) + |z|^2}{1 + \mathrm{Re}(z)}, \quad \forall z \in \mathbb{D},$

and we can write

$$j(z, \overline{z}) = \frac{P(z, \overline{z})}{Q(z, \overline{z})},$$

where both the polynomials P and Q are polyanalytic of order 2. We observe that $Q(z, \overline{z}) = 0$ if and only if $\mathrm{Re}(z) = -1$, so, $Q(z, \overline{z}) \neq 0$ for every $z \in \mathbb{D}$. On the other hand, we note that $j(z, \overline{z}) = 1$ on the boundary $\partial\mathbb{D}$. We have

(6.11) $\qquad\qquad \mathsf{K}_j(z, w) = \dfrac{1 - j(z, \overline{z})\overline{j(w, \overline{w})}}{\rho_w(z)\rho_{\overline{w}}(\overline{z})}, \quad \forall (z, w) \in \mathbb{D}^2,$

and, as a consequence,

$$\mathsf{K}_j((z,\overline{z});(w,\overline{w})) = \frac{2}{(z + \overline{z} + 2)(\overline{w} + w + 2)}$$
$$\cdot \left(\frac{(z + 1)(\overline{w} + 1)}{(1 - z\overline{w})} + \frac{(\overline{z} + 1)(w + 1)}{(1 - \overline{z}w)} \right).$$

It is important to note also that the function j given by (6.10) is polyrational in the sense of [12, pp 175].

Remark 6.7. According to [5, 16] we observe that

(6.12) $$j(z_1, z_2) = C(I_2 - ZA)^{-1}ZB,$$

and using the unitary matrix M given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

we have

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and $D = 0$ where we set $Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$.

Inspired by [8] we can prove the following result concerning the backward shift operators considered in Definition 2.18 in the case of the polyanalytic Hardy space of infinite order $\mathcal{SH}(\mathbb{D})$. Indeed, we have

Theorem 6.8. *A function $f \in \mathcal{SH}(\mathbb{D})$ is a common eigenfunction for the backward shift operators R_∞ and L_∞ with corresponding eigenvalues given respectively by $\lambda_1, \lambda_2 \in \mathbb{D}$ if and only if*

(6.13) $$f(z, \overline{z}) = \frac{f(0,0)}{(1 - \lambda_1 z)(1 - \lambda_2 \overline{z})}, \quad \forall z \in \mathbb{D}.$$

Proof. It is easy to check that if f is of the form (6.13), then we have $R_\infty(f) = \lambda_1 f$ and $L_\infty(f) = \lambda_2 f$. For the converse, we assume that f is an eigenfunction for R_∞ and L_∞ with corresponding eigenvalues given by λ_1 and λ_2. By writing

$$f(z, \overline{z}) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z),$$

we observe that $R_\infty(f) = \lambda_1 f$ if and only if

$$\sum_{n=0}^{\infty} \overline{z}^n R_0(f_n)(z) = \sum_{n=0}^{\infty} \overline{z}^n (\lambda_1 f_n)(z).$$

In particular, this shows that

$$R_0(f_n)(z) = \lambda_1 f_n(z), \quad \text{for any } n = 0, 1, 2, \ldots.$$

Using the classical result on the backward shift operator R_0 for any $n = 0, 1, 2, \ldots$ we get

$$f_n(z, \overline{z}) = \frac{f_n(0)}{1 - \lambda_1 z}, \quad z \in \mathbb{D}.$$

Now, we insert the expression of f_n in the f decomposition which leads to the following calculations

$$f(z) = \sum_{n=0}^{\infty} \overline{z}^n f_n(z)$$

$$= \sum_{n=0}^{\infty} \overline{z}^n \frac{f_n(0)}{1 - \lambda_1 z}$$

$$= \left(\sum_{m=0}^{\infty} z^m \lambda_1^m \right) \left(\sum_{n=0}^{\infty} \overline{z}^n f_n(0) \right)$$

$$= \sum_{m=0}^{\infty} z^m g_m(\overline{z})$$

where we set $g_m(\overline{z}) = \lambda_1^m \left(\sum_{n=0}^{\infty} \overline{z}^n f_n(0) \right)$ for any $z \in \mathbb{D}$. On the other hand, we note that by definition

$$L_\infty(f) = \sum_{m=0}^{\infty} z^m L_0(g_m),$$

with $L_0(g_m)(\overline{z}) = \dfrac{g_m(\overline{z}) - g_m(0)}{\overline{z}}$. Then, following a similar reasoning as we did in the case of R_∞, using the fact that $L_\infty(f) = \lambda_2 f$ we deduce tha $L_0(g_m)(\overline{z}) = \lambda_2 g_m(\overline{z})$. Thus, in particular this allows to write

$$g_m(\overline{z}) = \frac{g_m(0)}{(1 - \lambda_2 \overline{z})}, \quad z \in \mathbb{D}.$$

So, now we insert g_m in the expression of f and get

$$f(z) = \sum_{m=0}^{\infty} z^m \frac{g_m(0)}{1 - \lambda_2 \overline{z}}.$$

We note that $g_m(0) = \lambda_1^m f_0(0) = \lambda_1^m f(0)$. Then, we obtain

$$f(z) = \frac{f(0)}{1 - \lambda_2 \overline{z}} \sum_{m=0}^{\infty} \lambda_1^m z^m, \quad z \in \mathbb{D}.$$

Hence, we conclude that

$$f(z, \overline{z}) = \frac{f(0,0)}{(1 - \lambda_1 z)(1 - \lambda_2 \overline{z})}, \quad \text{for any } z \in \mathbb{D}.$$

$\square$

Lemma 6.9. *For any $f \in \mathcal{C}^1$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(tx, ty) = z \partial f + \overline{z} \overline{\partial} f$$

Proof. It follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} f(tx, ty) = x \frac{\partial f}{\partial x}(tx, ty) + y \frac{\partial f}{\partial y}(tx, ty)$$

$$= \frac{1}{2}(x + iy) \left(\frac{\partial f}{\partial x}(tx, ty) - -i \frac{\partial f}{\partial y}(tx, ty) \right) + \frac{1}{2}(x - iy) \left(\frac{\partial f}{\partial x}(tx, ty) + i \frac{\partial f}{\partial y}(tx, ty) \right)$$

$$= z \partial f + \overline{z} \overline{\partial} f.$$

$\square$

Let

$$(6.14) \qquad f(z, \overline{z}) = z f_1(z, \overline{z}) + \overline{z} f_2(z, \overline{z})$$

where f_1 and f_2 are required to be in the same space as f. Then

$$(6.15) \qquad (A_0 f)(z, \overline{z}) = \int_0^1 \frac{\partial}{\partial z} f(tz, t\overline{z}) dt$$

$$(6.16) \qquad (B_0 f)(z, \overline{z}) = \int_0^1 \frac{\partial}{\partial \overline{z}} f(tz, t\overline{z}) dt$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(tz, t\overline{z}) = z \frac{\partial}{\partial z} f(tz, t\overline{z}) dt + \overline{z} \frac{\partial}{\partial \overline{z}} f(tz, t\overline{z}) dt$$

and so

$$(6.17) \qquad f(z, \overline{z}) - f(0,0) = z A_0 f(z, \overline{z}) + \overline{z} B_0 f(z, \overline{z})$$

Lemma 6.10. *Let $\mathfrak{M}$ be a finite dimensional space in which (6.17) holds. Any $f \in \mathfrak{M}$ which is a common eigenfunction of A_0 and B_0 can be written in the form*

$$f(z, \overline{z}) = \frac{f(0,0)}{1 - az - b\overline{z}}.$$

Proof. Since A_0 and B_0 commute they can be simultaneously triangularized. Let

$$A_0 f = af \quad \text{and} \quad B_0 f = bf$$

We have

$$f(z, \overline{z}) = f(0,0) + (az + b\overline{z}) f$$

and so

$$f(z, \overline{z}) = \frac{f(0,0)}{1 - az - b\overline{z}}$$

$\square$

Lemma 6.11. *The following equalities hold:*

$$(6.18) \qquad A_0(z^n \overline{z}^m) = \frac{n}{n+m} z^{n-1} \overline{z}^m$$

$$(6.19) \qquad B_0(z^n \overline{z}^m) = \frac{m}{m+n} z^n \overline{z}^{m-1}.$$

Proof. It follows from

$$A_0(z^n \overline{z}^m) = \int_0^1 (n z^{n-1} \overline{z}^m)(tz, t\overline{z}) dt$$

$$= n z^{n-1} \overline{z}^m \int_0^1 n t^{n+m-1} dt$$

$$= \frac{n}{n+m} z^{n-1} \overline{z}^m,$$

and similarly for B_0. $\square$

Remark 6.12. We remark that A_0 is not R_0, in general; it will reduce to R_0 when f is analytic. We note that both the operators R_∞ and A_0 extend the classical backward shift operator R_0 on the Hardy space, but these two operators are different.

7. The Drury-Arveson space case

Let us consider the kernel function given by

$$(7.1) \qquad k(z, w) = \frac{1}{1 - (z\overline{w} + \overline{z}w)}.$$

We denote by $\mathfrak{H}(k)$ the associated reproducing kernel Hilbert space. Setting $z = x + iy$ and $w = t + iu$, we have

$$(7.2) \qquad \frac{1}{1 - (z\overline{w} + \overline{z}w)} = \frac{1}{1 - 2(xt + yu)}$$

We note that (7.1) is a complete Nevanlinna-Pick kernel, meaning that

$$\frac{1}{k(z, w)} = 1 - 2xt - 2yu$$

has one positive square in $B(0, 1/\sqrt{2})$. Such kernels were introduced by Agler, see [4] and also the paper of Quiggin [29].

Lemma 7.1. *The function* (7.1) *is positive definite in* $|z| < 1/\sqrt{2}$ *and the functions*

$$\frac{\partial k}{\partial t} \quad and \quad \frac{\partial k}{\partial u}$$

belong to $\mathfrak{H}(k)$. *Furthermore, it holds that*

$$(7.3) \qquad \langle f, \frac{\partial f}{\partial t} \rangle_{\mathfrak{H}(k)} = \frac{\partial f}{\partial x}$$

$$(7.4) \qquad \langle f, \frac{\partial f}{\partial u} \rangle_{\mathfrak{H}(k)} = \frac{\partial f}{\partial y}.$$

Proof. For a fixed choice of (t, u) and for $h \in \mathbb{R}$ small enough we set

$$f_h(x, y, t, u) = \frac{k(x, y, t, u + h) - k(x, y, t, u)}{h}.$$

Then $f_h \in \mathfrak{H}(k)$ and

$$\|f_h\|_{\mathcal{H}(k)}^2 = \frac{k(t, u + t, t, u + h) + k(t, u, t, u) - 2k(t, u + h, t, u)}{h^2}$$

uniformly bounded in h for h small. Thus f_h has a weakly convergent subsequence, with limit say $g_{t,u}$. Since weak convergence implies pointwise convergence we have

$$
\begin{aligned}
g_{t,u}(x, y) &= \langle g_{t,u}, k(\cdot, \cdot, x, y) \rangle_{\mathfrak{H}(k)} \\
&= \lim_{h \to 0} \langle f_h, k(\cdot, \cdot, x, y) \rangle_{\mathfrak{H}(k)} \\
&= \lim_{h \to 0} \left\langle \frac{k(\cdot, \cdot, t, u + h) - k(\cdot, \cdot, t, u)}{h}, k(\cdot, \cdot, x, y) \right\rangle_{\mathfrak{H}(k)} \\
&= \lim_{h \to 0} \frac{k(x, y, t, u + h) - k(x, y, t, u)}{h} \\
&= \frac{\partial f}{\partial u}(x, y, t, u).
\end{aligned}
$$

Furthermore, for $f \in \mathfrak{H}(k)$, we have:

$$
\begin{aligned}
\langle f, g_{t,u}\rangle_{\mathfrak{H}(k)} = &= \lim_{h \to 0} \langle f, f_h\rangle_{\mathfrak{H}(k)} \\
&= \lim_{h \to 0} \left\langle f(\cdot, \cdot), \frac{k(\cdot, \cdot, t, u + h) - k(\cdot, \cdot, t, u)}{h} \right\rangle_{\mathfrak{H}(k)} \\
&= \lim_{h \to 0} \frac{f(t, u + h) - f(t, u)}{h} \\
&= \frac{\partial f}{\partial y}(t, u)
\end{aligned}
$$

The other claims are proved in the same way. $\qquad\square$

Iterating the above result we get:

Corollary 7.2. *Let k be as in (7.1), then for $n, m = 0, 1, 2, \ldots$*

$$
(7.5) \qquad \frac{\partial^{n+m} k}{\partial t^n \partial u^m}(\cdot, \cdot, t, u) \in \mathfrak{H}(k)
$$

and

$$
(7.6) \qquad \langle f(\cdot, \cdot), \frac{\partial^{n+m} k}{\partial t^n \partial u^m}(\cdot, \cdot, t, u)\rangle_{\mathfrak{H}(k)} = \frac{\partial^{n+m} f}{\partial x^n \partial y^m}(t, u).
$$

Proof. This a direct consequence of Lemma 7.1. $\qquad\square$

Corollary 7.3. *For for $n, m = 0, 1, 2, \ldots$ we have*

$$
x^n y^m \in \mathfrak{H}(k).
$$

Proof. It suffices to set $t = u = 0$ in the previous corollary. $\qquad\square$

We now give a characterization of the space $\mathfrak{H}(k)$.

Proposition 7.4. *The space $\mathfrak{H}(k)$ consists of the functions of the form*

$$
(7.7) \qquad f(z, \overline{z}) = \sum_{a,b=0}^{\infty} c_{a,b} z^a \overline{z}^b
$$

with norm

$$
(7.8) \qquad \|f\|^2 = \sum_{a,b=0}^{\infty} |c_{a,b}|^2 \frac{a! b!}{(a+b)!}
$$

Proof. It suffices to observe that we have

$$
(7.9) \qquad k(z, w) = \sum_{a,b \in \mathbb{N}_0} \frac{(a+b)!}{a! b!} z^a \overline{z}^b w^b \overline{w}^a.
$$

$\qquad\square$

Proposition 7.5. *The operators M_z and $M_{\overline{z}}$ are bounded in $\mathfrak{H}(k)$, with $\|M_z\| \leq 1$ and $\|M_{\overline{z}}\| \leq 1$. Their adjoints are given by*

$$
(7.10) \qquad\qquad M_z^* = A_0
$$
$$
(7.11) \qquad\qquad M_{\overline{z}}^* = B_0.
$$

Proof. The first claim follows from

$$\frac{1 - z\overline{w}}{1 - 2\operatorname{Re} z\overline{w}} = 1 + \frac{\overline{z}w}{1 - 2\operatorname{Re} z\overline{w}} \geq 0$$

and

$$\frac{1 - \overline{z}w}{1 - 2\operatorname{Re} z\overline{w}} = 1 + \frac{z\overline{w}}{1 - 2\operatorname{Re} z\overline{w}} \geq 0$$

The second claim follows from

$$\langle M_z(z^n\overline{z}^m), z^u\overline{z}^v \rangle = \langle z^{n+1}\overline{z}^m, z^u\overline{z}^v \rangle$$

and similarly for $M_{\overline{z}}$. $\qquad\square$

Following [10, Corollary 2.4, p. 7] we introduce

$$(7.12) \qquad\qquad (A_a f)(z) \;=\; \frac{z}{1 - 2\operatorname{Re} z\overline{a}} f(z)$$

$$(7.13) \qquad\qquad (B_a f)(z) \;=\; \frac{\overline{z}}{1 - 2\operatorname{Re} \overline{z}a} f(z)$$

with $a \in B(0, 1/\sqrt{2})$.

Proposition 7.6. *Let $a \in B(0, 1/\sqrt{2})$. The operators A_a and B_a are bounded and it holds that*

$$(7.14) \qquad f(z) - f(a) = (z - a)(A_a^* f)(z) + (\overline{z} - \overline{a})(B_a^* f)(z), \quad f \in \mathfrak{H}(k).$$

Proof. We have

$$(7.15) \qquad\qquad (A_a^* k_b)(z) \;=\; \frac{\overline{b}}{1 - 2\operatorname{Re}(b\overline{a})} k_b(z)$$

$$(7.16) \qquad\qquad (B_a^* k_b)(z) \;=\; \frac{b}{1 - 2\operatorname{Re}(b\overline{a})} k_b(z)$$

$$
\begin{aligned}
(z - a)(A_a^* k_b)(z) + (\overline{z} - \overline{a})(B_a^* k_b)(z) &= \frac{(z - a)\overline{b} + (\overline{z} - \overline{a})b}{(1 - 2\operatorname{Re} b\overline{a})(1 - 2\operatorname{Re} z\overline{b})} \\
&= \frac{z\overline{b} + \overline{z}b - a\overline{b} - \overline{a}b}{(1 - 2\operatorname{Re} b\overline{a})(1 - 2\operatorname{Re} z\overline{b})} \\
&= k_b(z) - k_b(a)
\end{aligned}
$$

$\qquad\square$

Example 7.7. Let us consider the coefficients c_n such that

$$(7.17) \qquad\qquad 1 - \sqrt{1 - t} = \sum_{n=1}^{\infty} c_n t^n, \quad t < 1.$$

Then, the function

$$f_m(z, \overline{z}) = z + \sum_{n=1}^{m} c_n \overline{z}^{2m}$$

is bounded by one in modulus in $|z| < \frac{1}{\sqrt{2}}$, but is not a Schur multiplier.

Indeed, recalling that the $c_n > 0$ and satisfy $\sum_{n=1}^{\infty} c_n = 1$ we have

$$\left| z + \sum_{n=1}^{m} c_n \overline{z}^{2n} \right| \leq |z| + \sum_{n=1}^{m} c_n |z|^{2n}$$

$$\leq |z| + \sum_{n=1}^{\infty} c_n |z|^{2n}$$

$$= |z| + 1 - \sqrt{1 - |z|^2}$$

$$\leq \frac{1}{\sqrt{2}} + 1 - \sqrt{\frac{1}{2}}$$

$$= 1.$$

But $\|f_m\|^2 = 1 + \sum_{n=1}^{m} c_n \|\overline{z}^{2n}\|^2 > 1$.
For $a \in B(0, 1/\sqrt{2})$ we set

$$(7.18) \qquad b_a(z) = \frac{(1 - 2|a|^2)\,(z - a \quad \overline{z} - \overline{a})}{1 - \operatorname{Re} z\overline{a}} \sqrt{I_2 - \begin{pmatrix} \overline{a} \\ a \end{pmatrix} (a \quad \overline{a})}$$

We note that (with $c_1, c_2, \ldots$ as in (7.17))

$$\sqrt{I_2 - \begin{pmatrix} \overline{a} \\ a \end{pmatrix} (a \quad \overline{a})} = I_2 - \sum_{n=1}^{\infty} c_n \left(\begin{pmatrix} \overline{a} \\ a \end{pmatrix} (a \quad \overline{a}) \right)^n$$

$$= I_2 - \sum_{n=1}^{\infty} c_n \left(\begin{pmatrix} \overline{a} \\ a \end{pmatrix} (a \quad \overline{a}) \right) \left((\overline{a} \quad a) \begin{pmatrix} a \\ \overline{a} \end{pmatrix} \right)^{n-1}$$

$$= I_2 - \frac{(\overline{a} \quad a) \begin{pmatrix} a \\ \overline{a} \end{pmatrix}}{2|a|^2} \sum_{n=1}^{\infty} c_n (2|a|^2)^n$$

$$I_2 - \frac{(\overline{a} \quad a) \begin{pmatrix} a \\ \overline{a} \end{pmatrix}}{2|a|^2} \left(1 - \sqrt{1 - 2|a|^2} \right).$$

Theorem 7.8. *The function f belongs to $\mathfrak{H}(k)$ and $f(a) = 0$ if and only if*

$$f(z) = b_a(z) g(z),$$

with $g \in \mathfrak{H}(k)^2$.

Proof. We follow [10]. One direction is trivial while the converse is a direct consequence of (7.14) with $g(z) = \begin{pmatrix} A_a^* f \\ B_a^* f \end{pmatrix}$ since $f(a) = 0$. $\qquad \square$

More generally, as in Proposition 4.5 and Section 5 of [10] we have

Theorem 7.9. *Let $z_1, \ldots, z_N \in B(0, 1/\sqrt{2})$ and $w_1, \ldots, w_N \in \mathbb{C}$. There exists a Schur multiplier s such that*

$$(7.19) \qquad s(z_n) = w_n, \quad n = 1, \ldots, N$$

if and only if the $N \times N$ matrix with (n, m) entry

$$(7.20) \qquad \frac{1 - w_n \overline{w_m}}{1 - 2\operatorname{Re} z_n \overline{z_m}}$$

is non negative.

Proof. This holds thanks to the fact that the kernel $k(z,w) = \frac{1}{1-2\operatorname{Re}z\overline{w}}$ is a complete Nevanlinna-Pick kernel, and so the Nevanlinna-Pick interpolation problem is solved. $\qquad\square$

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