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KFP operators with coefficients measurable in time and Dini continuous in space

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Abstract. We consider degenerate Kolmogorov–Fokker–Planck operators

$$\begin{aligned} \mathcal{L}u &= \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u \\ &\equiv \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} u + Yu \end{aligned}$$

(with $(x,t) \in \mathbb{R}^{N+1}$ and $1 \leq m_0 \leq N$) such that the corresponding model operator having constant a_{ij} is hypoelliptic, translation invariant w.r.t. a Lie group operation in $\mathbb{R}^{N+1}$ and 2-homogeneous w.r.t. a family of nonisotropic dilations. The matrix $(a_{ij})_{i,j=1}^{m_0}$ is symmetric and uniformly positive on $\mathbb{R}^{m_0}$. The coefficients a_{ij} are bounded and *Dini continuous in space*, and only bounded measurable in time. This means that, setting

$$\begin{aligned} &\text{(i) } S_T = \mathbb{R}^N \times (-\infty, T), \\ &\text{(ii) } \omega_{f,S_T}(r) = \sup_{\substack{(x,t),(y,t)\in S_T \\ \|x-y\|\leq r}} |f(x,t) - f(y,t)| \\ &\text{(iii) } \|f\|_{\mathcal{D}(S_T)} = \int_0^1 \frac{\omega_{f,S_T}(r)}{r} dr + \|f\|_{L^\infty(S_T)} \end{aligned}$$

we require the finiteness of $\|a_{ij}\|_{\mathcal{D}(S_T)}$. We bound $\omega_{u_{x_i x_j},S_T}$, $\|u_{x_i x_j}\|_{L^\infty(S_T)}$ $(i,j = 1,2,...,m_0)$, ω_{Yu,S_T}, $\|Yu\|_{L^\infty(S_T)}$ in terms of $\omega_{\mathcal{L}u,S_T}$, $\|\mathcal{L}u\|_{L^\infty(S_T)}$ and $\|u\|_{L^\infty(S_T)}$, getting a control on the uniform continuity in space of $u_{x_i x_j}$, Yu if $\mathcal{L}u$ is bounded and Dini-continuous in space. Under the additional assumption that both the coefficients a_{ij} and $\mathcal{L}u$ are log-Dini continuous, meaning the finiteness of the quantity

$$\int_0^1 \frac{\omega_{f,S_T}(r)}{r} |\log r|\, dr,$$

we prove that $u_{x_i x_j}$ and Yu are Dini continuous; moreover, in this case, the derivatives $u_{x_i x_j}$ are locally uniformly continuous in space *and time*.

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1. Introduction and statement of the main result

In this paper, we will be concerned with *Kolmogorov–Fokker–Planck* (KFP, in short) operators of the form

$$\mathcal{L}u = \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u, \qquad (x,t) \in \mathbb{R}^{N+1}, \quad (1.1)$$

where $1 \le m_0 \le N$. The first-order part of the operator $\mathcal{L}$, also called *the drift term*, is a smooth vector field which will be denoted by Y; more explicitly,

$$Yu = \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u. \quad (1.2)$$

Points of $\mathbb{R}^{N+1}$ will be sometimes denoted by the compact notation

$$\xi = (x,t), \ \eta = (y,s).$$

Given $T \in \mathbb{R}$, we set

$$S_T := \mathbb{R}^N \times (-\infty, T).$$

We will make the following assumptions on $\mathcal{L}$:

(H1) $A_0(x,t) = (a_{ij}(x,t))_{i,j=1}^{m_0}$ is a symmetric, uniformly positive matrix on $\mathbb{R}^{m_0}$ of bounded measurable coefficients, defined in $\mathbb{R}^{N+1}$; more precisely, there exists a constant $\nu > 0$ such that

$$\begin{gathered} \nu|v|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t) v_i v_j \le \nu^{-1}|v|^2 \\ \text{for every } v \in \mathbb{R}^{m_0},\ x \in \mathbb{R}^N \text{ and a.e. } t \in \mathbb{R}. \end{gathered} \quad (1.3)$$

The coefficients will be also assumed to be Dini continuous w.r.t. x, uniformly w.r.t. t. This assumption will be specified later (see Definition 1.2 and assumption (H3)), since it requires some preliminaries.

(H2) The matrix $B = (b_{ij})_{i,j=1}^{N}$ satisfies the following condition: for m_0 and suitable positive integers $m_1, \ldots, m_k$ such that

$$m_0 \ge m_1 \ge \ldots \ge m_k \ge 1 \quad \text{and} \quad m_0 + m_1 + \ldots + m_k = N, \quad (1.4)$$

we have

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \ldots & \ldots & \ldots \\ \mathbb{O} & B_2 & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & B_k & \mathbb{O} \end{pmatrix} \quad (1.5)$$

where B_j is an $m_j \times m_{j-1}$ matrix of rank m_j (for $j = 1, 2, \ldots, k$).

To the best of our knowledge, the study of the KFP operators has a long history which dates back to the 1934 paper by Kolmogorov [10] on the Theory of Gases. In this paper, Kolmogorov introduced the operator

$$\mathcal{K} = \Delta_u + \langle u, \nabla_v \rangle - \partial_t, \quad \text{with } u, v \in \mathbb{R}^n \text{ and } t \in \mathbb{R},$$

which can be obtained from (1.1) by choosing

$$N = 2n, \quad m_0 = m_1 = n, \quad A_0 = \mathrm{Id}_n, \quad , \quad B = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathrm{Id}_n & \mathbb{O}_n \end{pmatrix}.$$

It should be noticed that, since $m_0 < N$, the operator $\mathcal{K}$ is *not parabolic*; however, Kolmogorov proved in [10] that $\mathcal{K}$ is C^∞-hypoelliptic in $\mathbb{R}^{2n}$ by constructing an explicit smooth fundamental solution. The (global) C^∞-hypoellipticity of the operator $\mathcal{K}$ is cited by Hörmander as one of the main 'inspiration' for his celebrated work [7] on the hypoellipticity of the *sums of squares of vector fields* (plus a drift), of which the KFP operators with constant coefficients $a_{i,j}$'s are a particular case.

After the seminal paper by Hörmander, the KFP operators *with constant coefficients* have been studied by many authors, and from several point of views; in particular, at the beginning of the '90s' Lanconelli and Polidoro [11] started the study of constant coefficients KFP operators from a *geometrical viewpoint*, showing that these operators possess a rich underlying *subelliptic geometric structure*. More precisely, they proved that the $m_0 + 1$ vector fields

$$X_1 = \partial_{x_1}, \ldots, X_{m_0} = \partial_{x_{m_0}}, \; Y = \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} - \partial_t$$

(on which the KFP operators (1.1) are modeled) satisfy the following properties:

(a) $X_1, \ldots, X_{m_0}, Y$ are left-invariant on the *Lie group* $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$, where the (non-commutative) composition law $\circ$ is defined as follows

$$(y, s) \circ (x, t) = (x + E(t)y, t + s)$$
$$(y, s)^{-1} = (-E(-s)y, -s),$$

and $E(t) = \exp(-tB)$ (which is defined for every $t \in \mathbb{R}$ since the matrix B is nilpotent). For a future reference, we explicitly notice that

$$(y, s)^{-1} \circ (x, t) = (x - E(t - s)y, t - s), \tag{1.6}$$

and that the Lebesgue measure is the Haar measure, which is also invariant with respect to the inversion.

(b) $X_1, \ldots, X_{m_0}$ are homogeneous of degree 1 and Y is homogeneous of degree 2 with respect to a nonisotropic family of *dilations* in $\mathbb{R}^{N+1}$, which are automorphisms of $\mathbb{G}$ and are defined by

$$D(\lambda)(x, t) \equiv (D_0(\lambda)(x), \lambda^2 t) = (\lambda^{q_1} x_1, \ldots, \lambda^{q_N} x_N, \lambda^2 t), \tag{1.7}$$

where the N-tuple $(q_1, \ldots, q_N)$ is given by

$$(q_1, \ldots, q_N) = (\underbrace{1, \ldots, 1}_{m_0}, \underbrace{3, \ldots, 3}_{m_1}, \ldots, \underbrace{2k+1, \ldots, 2k+1}_{m_k}).$$

The integer

$$Q = \sum_{i=1}^{N} q_i > N \tag{1.8}$$

is called the (spatial) *homogeneous dimension* of $\mathbb{R}^N$, while $Q+2$ is the homogeneous dimension of $\mathbb{R}^{N+1}$. We explicitly point out that the exponential matrix $E(t)$ satisfies the following homogeneity property

$$E(\lambda^2 t) = D_0(\lambda) E(t) D_0\Big(\frac{1}{\lambda}\Big), \tag{1.9}$$

for every $\lambda > 0$ and every $t \in \mathbb{R}$ (see [11, Rem. 2.1.]).

(c) $X_1, \ldots, X_{m_0}, Y$ satisfy the *Hörmander Rank Condition* in $\mathbb{R}^{N+1}$.

Through the years, many Authors have studied KFP operators with *variable* coefficients $a_{ij}(x,t)$, modeled on the above class of left invariant hypoelliptic operators. For instance, Schauder estimates on bounded domains have been investigated first by Manfredini [14], and later by Di Francesco–Polidoro in [6] under more general assumptions, assuming the coefficients a_{ij} Hölder continuous with respect to the intrinsic distance induced in $\mathbb{R}^{N+1}$ by the vector fields $\partial_{x_1}, \ldots \partial_{x_{m_0}}, Y$. With regards to Schauder estimates for KFP operators, the reader is referred also to the papers by Lunardi [13], Priola [17], Imbert–Mouhot [8], Wang–Zhang [18], and the references therein. Also, continuity estimates on $u_{x_i x_j}$ under a Dini continuity assumption on a_{ij} and $\mathcal{L}u$ have been proved by Polidoro, Rebucci, Stroffolini in [16].

Recent contributions from the field of stochastic differential equations (see e.g. [15]) suggest the importance of developing a theory allowing the coefficients a_{ij} to be rough in t (say, L^∞), and uniformly continuous (for instance, Hölder continuous) only w.r.t. the space variables. The Schauder estimates that one can reasonably expect under this mild assumption consist in controlling the Hölder seminorms w.r.t. x of the derivatives involved in the equations, uniformly in time. These estimates are sometimes called "partial Schauder estimates". Similar results can be expected when Hölder continuity is replaced by Dini continuity. Results of this kind (in the Hölder case) are well-known for uniformly parabolic operators (see [4,9], and more recent papers quoted in the references in [1]). Also, in the parabolic case, it is known that $u_{x_i x_j}$ satisfy a continuity estimate *in time*, under the same assumptions of continuity in space of a_{ij} and $\mathcal{L}u$. Partial Schauder estimates for $u_{x_i x_j}, Yu$, together with local Hölder continuity in the joint variables, have been recently proved by the first two of us in [1]. Partial Schauder estimates for degenerate KFP operators have been proved also in the recent paper [5] by Chaudru de Raynal, Honoré, Menozzi, with different techniques and without getting the Hölder control in time of second order derivatives. We also quote [12], by Lucertini, Pagliarani, Pascucci, dealing with the construction of a fundamental solution for KFP operators with coefficients Hölder continuous in space and L^∞ in time.

In this paper, we address the problem of proving uniform continuity estimates w.r.t. the space variables on $u_{x_i x_j}$, Yu, assuming a_{ij} and $\mathcal{L}u$ to be Dini continuous w.r.t. the space variables, uniformly in t. We prove an estimate of this kind, which, in turn, implies the (partial) Dini continuity of $u_{x_i x_j}$, Yu under the stronger assumption that a_{ij} and $\mathcal{L}u$ are log-Dini continuous w.r.t. the space variables, uniformly in t (for the precise statement, see Theorem 1.6). These results are consistent with those proved in [16] when a_{ij} and $\mathcal{L}u$ are Dini-continuous in the joint variables. Moreover, under the same stronger assumption of log-Dini continuity of a_{ij} and $\mathcal{L}u$, we prove a bound on the modulus of continuity in the joint variables for $u_{x_i x_j}$, analogously to what happens in the Hölder case. (For the exact statement, see Theorem 1.7).

Statement of the main result. In order to introduce the function spaces and the quantities which will be involved in the statements of our results, we need to introduce some metric notions. First of all, the system

$$\mathbf{X} = \{X_1, \ldots, X_{m_0}, Y\}$$

induces in a standard way a (weighted) control distance $d_{\mathbf{X}}$ in $\mathbb{R}^{N+1}$, which is left invariant w.r.t. the group operation $\circ$ and jointly 1-homogeneous with respect to $D(\lambda)$. As a consequence, the function $\rho_{\mathbf{X}}(\xi) := d_{\mathbf{X}}(\xi, 0)$ satisfies

(1) $\rho_{\mathbf{X}}(\xi^{-1}) = \rho_{\mathbf{X}}(\xi)$;
(2) $\rho_{\mathbf{X}}(\xi \circ \eta) \leq \rho_{\mathbf{X}}(\xi) + \rho_{\mathbf{X}}(\eta)$.

(For these and related basic notions on Hörmander vector fields, we refer to [2, Chaps. 1–3]). In addition, since $d_{\mathbf{X}}$ is a distance, we also have

(1)' $\rho_{\mathbf{X}}(\xi) \geq 0$ and $\rho_{\mathbf{X}}(\xi) = 0 \Leftrightarrow \xi = 0$;
(2)' $\rho_{\mathbf{X}}(D(\lambda)\xi) = \lambda \rho_{\mathbf{X}}(\xi)$,

and this means that $\rho_{\mathbf{X}}$ is a *homogeneous norm* in $\mathbb{R}^{N+1}$. We then notice that, owing to the explicit expression of $D(\lambda)$ in (1.7), the function

$$\rho(\xi) = \rho(x, t) := \|x\| + \sqrt{|t|} = \sum_{i=1}^{N} |x_i|^{1/q_i} + \sqrt{|t|} \tag{1.10}$$

is also a homogeneous norm in $\mathbb{R}^{N+1}$, and therefore, it is *globally equivalent* to the norm $\rho_{\mathbf{X}}$. As a consequence, the map

$$d(\xi, \eta) := \rho(\eta^{-1} \circ \xi) = \|x - E(t-s)y\| + \sqrt{|t-s|} \tag{1.11}$$

is a left-invariant, 1-homogeneous *quasi-distance* on $\mathbb{R}^{N+1}$. More precisely, there exists $\boldsymbol{\kappa} \geq 1$ such that

$$d(\xi, \eta) \leq \boldsymbol{\kappa}\big(d(\xi, \zeta) + d(\eta, \zeta)\big) \qquad \forall\, \xi, \eta, \zeta \in \mathbb{R}^{N+1}; \tag{1.12}$$

$$d(\xi, \eta) \leq \boldsymbol{\kappa}\, d(\eta, \xi) \qquad \forall\, \xi, \eta \in \mathbb{R}^{N+1}. \tag{1.13}$$

The quasi-distance d is *globally equivalent* to the control distance $d_{\mathbf{X}}$; hence, we will systematically use this quasi-distance d in place of $d_{\mathbf{X}}$. We refer the reader to Sect. 2 for several properties of d which shall be used in the paper.

Using the quasi-distance d, we now introduce the relevant spaces of functions to which our main result applies.

Definition 1.1. [Hölder continuous functions] Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set, and let $\alpha \in (0, 1)$. Given a function $f : \Omega \to \mathbb{R}$, we introduce the notation

$$|f|_{C^\alpha(\Omega)} = \sup\left\{\frac{|f(\xi) - f(\eta)|}{d(\xi,\eta)^\alpha} : \xi, \eta \in \Omega \text{ and } \xi \neq \eta\right\}.$$

Accordingly, we define the space $C^\alpha(\Omega)$ as follows:

$$C^\alpha(\Omega) := \{f \in C(\Omega) \cap L^\infty(\Omega) : |f|_{C^\alpha(\Omega)} < \infty\}.$$

Finally, on this space $C^\alpha(\Omega)$ we introduce the norm

$$\|f\|_{C^\alpha(\Omega)} := \|f\|_{L^\infty(\Omega)} + |f|_{C^\alpha(\Omega)}.$$

Definition 1.2. [Partially Dini and log-Dini continuity] Let Ω be an arbitrary open set in $\mathbb{R}^{N+1}$, and let $f \in L^\infty(\Omega)$. For every $r > 0$, we set

$$\omega_{f,\Omega}(r) = \sup_{\substack{(x,t),(y,t)\in\Omega \\ \|x-y\|\leq r}} |f(x,t) - f(y,t)|.$$

We then say that

(i) f is *partially Dini-continuous in* Ω, and we write $f \in \mathcal{D}(\Omega)$, if

$$\int_0^1 \frac{\omega_{f,\Omega}(r)}{r}\,dr < \infty; \tag{1.14}$$

(ii) f is *partially log-Dini continuous*, and we write $f \in \mathcal{D}_{\log}(\Omega)$, if

$$\int_0^1 \frac{\omega_{f,\Omega}(r)}{r}|\log r|\,dr < \infty. \tag{1.15}$$

If $f \in \mathcal{D}(\Omega)$, we define

$$|f|_{\mathcal{D}(\Omega)} = \int_0^1 \frac{\omega_{f,\Omega}(r)}{r}\,dr \quad \text{and} \quad \|f\|_{\mathcal{D}(\Omega)} = \|f\|_{L^\infty(\Omega)} + |f|_{\mathcal{D}(\Omega)}.$$

Remark 1.3. Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set, and let $f \in \mathcal{D}_{\log}(\Omega)$. We will see in Sect. 2 that the following functions are *well-defined moduli of continuity* (that is, non-decreasing functions on $(0, \infty)$ vanishing for $r \to 0^+$):

$$\mathcal{M}_{f,\Omega}(r) = \omega_{f,\Omega}(r) + \int_0^r \frac{\omega_{f,\Omega}(s)}{s}\,ds + r\int_r^\infty \frac{\omega_{f,\Omega}(s)}{s^2}\,ds; \tag{1.16}$$

$$\mathcal{N}_{f,\Omega}(r) = \mathcal{M}_{f,\Omega}(r) + \int_0^r \frac{\mathcal{M}_{f,\Omega}(s)}{s}\,ds + r\int_r^\infty \frac{\mathcal{M}_{f,\Omega}(s)}{s^2}\,ds. \tag{1.17}$$

Furthermore, given any $\mu > 0$, we will see that also the functions

$$\mathcal{U}^{\mu}_{f,\Omega}(r) = \int_{\mathbb{R}^N} e^{-\mu|z|^2}\Big(\int_0^{r\|z\|} \frac{\omega_{f,\Omega}(s)}{s}\,ds\Big)dz \tag{1.18}$$

$$\mathcal{V}^{\mu}_{f,\Omega}(r) = \int_{\mathbb{R}^N} e^{-\mu|z|^2}\Big(\int_0^{r\|z\|} \frac{\mathcal{M}_{f,\Omega}(s)}{s}\,ds\Big)dz \tag{1.19}$$

are well-defined on the interval $(0, \infty)$. The continuity estimates appearing in our main results, namely Theorems 1.6–1.7, will depend on these functions.

Definition 1.4. Given any number $T > 0$, we define $\mathcal{S}^0(S_T)$ as the space of all functions $u : \overline{S_T} \to \mathbb{R}$ satisfying the following properties:

(i) $u \in C(\overline{S_T}) \cap L^\infty(S_T)$;

(ii) for every $1 \le i, j \le m_0$, $\partial_{x_i} u$, $\partial^2_{x_i x_j} u \in L^\infty(S_T)$;

(iii) $Yu \in L^\infty(S_T)$

(in the above (ii)–(iii), the derivatives $\partial_{x_i} u$, $\partial^2_{x_i x_j} u$ and Yu are intended in the sense of distributions). For every fixed $\tau < T$, we also define

$$S^0(\tau, T) = \{f \in \mathcal{S}^0(S_T) : \mathrm{supp}(f) \subset \mathbb{R}^N \times (\tau, T)\}.$$

Finally, we define $\mathcal{S}^D(S_T)$ as the space of functions $u \in \mathcal{S}^0(S_T)$ such that

$$\partial_{x_i} u,\ \partial^2_{x_i x_j} u \in \mathcal{D}(S_T) \quad (\text{for } i, j = 1, 2, \dots, m_0) \quad \text{and} \quad Yu \in \mathcal{D}(S_T).$$

Remark 1.5. If $u \in \mathcal{S}^0(S_T)$ then u and $\partial_{x_1} u, \dots, \partial_{x_{m_0}} u$ belong to $C^\alpha(S_T)$ for every $\alpha \in (0, 1)$. A quantitative estimate on these Hölder norms is proved in Theorem 2.20 [1], under the assumption of Hölder continuity (w.r.t. x) of a_{ij} and $\mathcal{L}u$, while in our main result (Theorem 1.6) this will be proved under the assumption of partial Dini continuity of a_{ij} and $\mathcal{L}u$.

We are now ready to state the main results of this paper.

Theorem 1.6. (Global continuity estimates) *Let $\mathcal{L}$ be an operator as in* (1.1)*, and assume that* (H1), (H2) *are satisfied. In addition, we assume that*

(H3) *$a_{ij} \in \mathcal{D}(\mathbb{R}^{N+1})$ for every $1 \le i, j \le m_0$.*

Then, for every $1 \le i, j \le m_0$, every $T > 0$ and $\alpha \in (0, 1)$ there exists a constant $c > 0$, depending on T, α, the matrix B in (1.5)*, the number ν in* (1.3) *and on the number*

$$A = \textstyle\sum_{i,j=1}^{m_0} \|a_{ij}\|_{\mathcal{D}(\mathbb{R}^{N+1})} \tag{1.20}$$

such that the following estimates hold for every $u \in \mathcal{S}^D(S_T)$*:*

$$\text{(i)} \quad \sum_{h,k=1}^{m_0} \|\partial^2_{x_h x_k} u\|_{L^\infty(S_T)} + \|Yu\|_{L^\infty(S_T)} + \sum_{i=1}^{m_0} \|\partial_{x_i} u\|_{C^\alpha(S_T)} + \|u\|_{C^\alpha(S_T)} \\ \leq c\big\{\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big\}$$

$$\text{(ii)} \quad \sum_{h,k=1}^{m_0} \omega_{\partial^2_{x_h x_k} u, S_T}(r) + \omega_{Yu,S_T}(r) \\ \leq c\big\{\mathcal{M}_{\mathcal{L}u,S_T}(cr) + (\mathcal{M}_{a,S_T}(cr) + r^\alpha)(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)})\big\}.$$

Here, $\mathcal{M}_{a,S_T} = \sum_{i,j=1}^{m_0} \mathcal{M}_{a_{ij},S_T}$ *and* $\mathcal{M}_{\cdot,\, S_T}$ *is as in* (1.16).

In particular, the functions $\partial^2_{x_h x_k} u$, $(h, k = 1, ..., m_0)$, Yu *are partially Dini continuous if both the coefficients* a_{ij} *and the function* $\mathcal{L}u$ *are partially log-Dini continuous.*

It is worthwhile noting that the full Hölder norms of the lower order terms u, $\partial_{x_i} u$ $(i = 1, 2, ..., m_0)$ can be bounded assuming the partial Dini continuity of $\mathcal{L}u$ and the coefficients a_{ij}. In particular, any function in $\mathcal{S}^D(S_T)$ has this regularity property.

Theorem 1.7. (Continuity estimates in space-time for $\partial^2_{x_i x_j} u$) *Let* $\mathcal{L}$ *be an operator as in* (1.1)*, and assume that* (H1), (H2) *are satisfied. In addition, we assume that*

(H3)' $a_{ij} \in \mathcal{D}_{\log}(\mathbb{R}^{N+1})$ *for every* $1 \leq i, j \leq m_0$.

Then, for every $1 \leq i, j \leq m_0$*, every* $-\infty < \tau < T$*, every* $\alpha \in (0, 1)$ *and every* compact set $K \subseteq \mathbb{R}^N$ *there exists a constant* $c > 0$*, depending on* K, τ, T, α *and* ν*, such that*

$$|\partial^2_{x_i x_j} u(x_1, t_1) - \partial^2_{x_i x_j} u(x_2, t_2)| \\ \leq c\left\{\mathcal{N}_{\mathcal{L}u,S_T}(cr) + \mathcal{V}^\mu_{\mathcal{L}u,S_T}\left(c\sqrt{|t_1 - t_2|}\right)\right. \\ \left. + \left(\mathcal{N}_{a,S_T}(cr) + \mathcal{V}^\mu_{a,S_T}\left(c\sqrt{|t_1 - t_2|}\right) + r^\alpha\right)\left(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\right)\right\} \tag{1.21}$$

for any $u \in \mathcal{S}^D(S_T)$ *with* $\mathcal{L}u \in \mathcal{D}_{\log}(S_T)$ *and any* $(x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$*. In the above estimate, we have used the notation*

$$r = d(x_1, t_1), (x_2, t_2)) + |t_1 - t_2|^{1/q_N}$$

where $q_N \geq 3$ *is the largest exponent in the dilations* $D(\lambda)$*, see* (1.7)*; in addition,*

$$\mathcal{N}_{a,S_T} = \textstyle\sum_{i,j=1}^{m_0} \mathcal{N}_{a_{ij},S_T}, \qquad \mathcal{V}^\mu_{a,S_T} = \textstyle\sum_{i,j=1}^{m_0} \mathcal{V}^\mu_{a_{ij},S_T},$$

and $\mathcal{N}_{\cdot,\, S_T}$, $\mathcal{V}^\mu_{\cdot,\, S_T}$ *are as in* (1.17)–(1.19)*, respectively* (*and* $\mu > 0$ *is a constant only depending on* ν).

More explicit bounds on the functions $\mathcal{U}^\mu_{\cdots}(r)$, $\mathcal{V}^\mu_{\cdots\cdot}(r)$ appearing in (1.21) will be given in Proposition 2.13.

Remark 1.8. (Dependence of the constants) Throughout the paper, we will call 'structural constant' any constant $c > 0$ only depending on the matrix B and the ellipticity constant' ν. Notice that the matrix B encodes in particular the numbers N, Q, q_i, m_i, $\boldsymbol{\kappa}$, and the functions $d_{\mathbf{X}}$, $\rho_{\mathbf{X}}$, d, ρ. Any other dependence will be specified.

Structure of the paper. Let us now briefly explain the strategy we follow to prove our a priori estimates. As in the classical Schauder theory, the operator with variable coefficients is seen as a small local perturbation of the constant one obtained by freezing the coefficients a_{ij} at some point $(\overline{x}, \overline{t})$. In our context, since the coefficients are not continuous in t, we can only see our operator as a small local perturbation of the operator with coefficients only depending on t, obtained by freezing the $a_{ij}(\cdot, t)$ at some point $\overline{x}$. Therefore, our model operator is the one with bounded measurable coefficients $a_{ij}(t)$:

$$\mathcal{L}u = \sum_{i,j=1}^{m_0} a_{ij}(t)\partial^2_{x_i x_j} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u.$$

So, the starting point of our strategy is a careful study of the operator $\mathcal{L}$ with bounded measurable coefficients $a_{ij}(t)$. For this operator, an explicit fundamental solution has been computed and studied by Bramanti and Polidoro in [3]; more properties and sharp estimates for this fundamental solution have been established in [1]. In Sect. 2, after recalling some known facts about the metrics (Sect. 2.1) and establishing some preliminary results on the Dini-type function spaces (Sect. 2.2), in Sects. 2.3 and 2.4 we recall some results proved in [1,3] about the fundamental solution of the model operator with coefficients $a_{ij}(t)$ and some interpolation inequalities for Hölder norms.

In Sect. 3 we keep studying the model operator with coefficients only depending on t. We first establish representation formulas for u and $u_{x_i x_j}$ in terms of $\mathcal{L}u$, exploiting this fundamental solution, under the partial Dini-continuity assumption on $\mathcal{L}u$. Then, by singular integral techniques, we prove the desired a priori estimates for this model operator (see Theorems 3.4–3.5). In Sect. 4 we then study the operator with coefficients $a_{ij}(x, t)$. Here we apply the classical "Korn's trick" of freezing the coefficients of $\mathcal{L}$, in our case only w.r.t. x, writing representation formulas for $u_{x_i x_j}$ and then regard the original operator as a small local perturbation of the frozen one. This allows us to prove the desired a priori estimates for functions with small compact support (Sect. 4.1). Removing this restriction requires the use of cutoff functions and interpolation inequalities for the derivatives of intermediate order; this is accomplished in Sect. 4.2, completing the proof of our first main result, Theorem 1.6. Finally, in Sect. 5 we prove our second main result, Theorem 1.7, that is the bound of the modulus of continuity of $u_{x_i x_j}$ in the joint variables (x, t).

2. Preliminaries

We collect in this section several preliminary results which will be used in the rest of the paper. For basic facts and more details about Hörmander vector fields, the metric they induce, and homogeneous groups, we refer to [2].

2.1. Some metric properties

As already discussed in the Introduction, Lanconelli and Polidoro [11] proved that there is an 'intrinsic subelliptic geometry' associated with any KFP operator. More precisely, if $\mathcal{L}$ is as in (1.1) and if

$$\mathbf{X} = \{\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\},$$

assumption (H2) ensures that the weighted distance $\rho_{\mathbf{X}}$ induced by $\mathbf{X}$ is well-defined, left-invariant w.r.t. the group operation $\circ$ and $D(\lambda)$-homogeneous of degree 1. Even if it seems natural to investigate the regularity properties of $\mathcal{L}$ using this distance, the lack of an explicit expression makes better suited the *quasi-distance*

$$d((x,t),(y,s)) = \rho((y,s)^{-1} \circ (x,t)) = \|x - E(t-s)y\| + \sqrt{|t-s|}, \quad (2.1)$$

which is *globally equivalent to* $\rho_{\mathbf{X}}$ *and has an explicit form.* We now list here below some simple properties of d which shall be used in the sequel.

We begin by observing that, since $E(0) = \mathbb{I}$, from (2.1) we infer that

$$d((x,t),(y,t)) = \|x - y\| = \sum_{i=1}^{N} |x_i - y_i|^{1/q_i} \quad \forall\, x, y \in \mathbb{R}^N,\ t \in \mathbb{R}. \quad (2.2)$$

As a consequence, we derive that d *is symmetric and independent of* t *when applied to points of* $\mathbb{R}^{N+1}$ *with the same* t *-coordinate*. Unfortunately, an analogous property for points with the same x-coordinate *does not hold*. In fact, for every fixed $x \in \mathbb{R}^N$ and every $t, s \in \mathbb{R}$, again by (2.1), we have

$$d((x,t),(x,s)) = \|x - E(t-s)x\| + \sqrt{|t-s|}.$$

Now, since the geometry of a metric space is encoded in the 'shape' of the balls, in our context we are led to consider the d-*balls* associated with d. Recalling that d is a *quasi-distance* (in particular, d is not symmetric), we fix once and for all the following definition: given any $\xi \in \mathbb{R}^{N+1}$ and any $r > 0$, we define

$$B_r(\xi) := \{\eta \in \mathbb{R}^{N+1} : d(\eta, \xi) < r\}.$$

Using the translation-invariance and the homogeneity of d, it is not difficult to recognize that the following properties are satisfied:

$$\text{(i) } B_r(\xi) = \xi \circ B_r(0) = \xi \circ D_r(B_1(0)) \quad \forall\, \xi \in \mathbb{R}^{N+1},\ r > 0; \quad (2.3)$$

$$\text{(ii) } |B_r(\xi)| = |B_r(0)| = \omega_Q r^{Q+2}, \quad \text{where } \omega_Q := |B_1(0)| > 0. \quad (2.4)$$

On the other hand, since d satisfies (1.12)–(1.13) with a positive constant $\boldsymbol{\kappa}$ *possibly greater that* 1, we readily derive that

(iii) if $\eta \in B_r(\xi)$, then $\xi \in B_{\boldsymbol{\kappa} r}(\eta)$;

(iv) if $\eta_1, \eta_2 \in B_r(\xi)$, then $d(\eta_1, \eta_2) < 2\boldsymbol{\kappa} r$.

Finally, we state for a future reference some elementary lemmas concerning the quasi-distance d; for a proof of these results we refer to [1].

Lemma 2.1. *There exists a structural constant $\boldsymbol{\vartheta} > 0$ such that, if ξ_1, ξ_2 and η are points in $\mathbb{R}^{N+1}$ which satisfy $d(\xi_1, \eta) \geq 2\boldsymbol{\kappa}\, d(\xi_1, \xi_2)$, one has*

$$\boldsymbol{\vartheta}^{-1} d(\xi_2, \eta) \leq d(\xi_1, \eta) \leq \boldsymbol{\vartheta} d(\xi_2, \eta), \tag{2.5}$$

Here, $\boldsymbol{\kappa} > 0$ is the constant appearing in (1.12)–(1.13).

Lemma 2.2. *There exists a structural constant $c > 0$ such that*

$$\|E(t)x\| \leq c\rho(x,t) = c\big(\|x\| + \sqrt{|t|}\big) \qquad \forall\, x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{2.6}$$

Lemma 2.3. *Let $K \subseteq \mathbb{R}^N$ be a fixed compact set, and let $T > \tau > -\infty$. There exists a constant $c = c(K, T, \tau) > 0$ such that, for every $x \in K$ and $t, s \in [\tau, T]$,*

$$\|x - E(t-s)x\| \leq c\,|t-s|^{1/q_N} \tag{2.7}$$

$$\|(E(t) - E(s))x\| \leq c\,|t-s|^{1/q_N}. \tag{2.8}$$

Here $q_N \geq 3$ is the maximum exponent appearing in (1.7).

2.2. Function spaces

Let us now turn our attention to the notion of *partial Dini and log-Dini continuity*. In what follows, $\Omega \subseteq \mathbb{R}^{N+1}$ is an arbitrary open set.

We begin by recalling that, according to Definition 1.2, a function $f : \Omega \to \mathbb{R}$ belongs to the space $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}_{\log}(\Omega)$) if $f \in L^\infty(\Omega)$ and

$$\begin{gathered}\int_0^1 \frac{\omega_{f,\Omega}(r)}{r}\,dr < \infty \quad \Big(\text{resp.}\ \int_0^1 \frac{\omega_{f,\Omega}(r)}{r}|\log r|\,dr < \infty\Big),\\ \text{where } \omega_{f,\Omega}(r) = \sup_{\substack{(x,t),(y,t)\in\Omega\\ \|x-y\|\leq r}} |f(x,t) - f(y,t)|.\end{gathered} \tag{2.9}$$

We obviously have $\mathcal{D}_{\log}(\Omega) \subseteq \mathcal{D}(\Omega)$.

We also notice that, given any $f \in L^\infty(\Omega)$, by (2.2) we can write

$$\omega_{f,\Omega}(r) = \sup_{\substack{(x,t),(y,t)\in\Omega\\ d((x,t),(y,t))\leq r}} |f(x,t) - f(y,t)|.$$

Moreover, $\omega_{f,\Omega}$ is non-negative, non-decreasing and *globally bounded* on $(0, \infty)$; more precisely, we have the obvious estimate

$$0 \leq \omega_{f,\Omega}(r) \leq 2\|f\|_{L^\infty(\Omega)} \quad \forall\, r > 0. \tag{2.10}$$

Remark 2.4. Here we list some remarks on partially Dini and log-Dini continuous functions which easily follow from the definition.

1. If $f \in L^\infty(\Omega)$ and $0 < a < b$, by (2.10) we have
$$\int_a^b \frac{\omega_{f,\Omega}(r)}{r}\,dr \leq 2\|f\|_{L^\infty(\Omega)} \log(b/a) < \infty. \tag{2.11}$$
Thus, condition (2.9) is actually an integrability condition *near* 0.
2. If $f \in \mathcal{D}(\Omega)$ and $r > 0$, from (2.11) we infer that
$$\begin{aligned}\int_0^r \frac{\omega_{f,\Omega}(s)}{s}\,ds &= \int_0^1 \frac{\omega_{f,\Omega}(s)}{s}\,ds + \int_1^r \frac{\omega_{f,\Omega}(s)}{s}\,ds \\ &\leq |f|_{\mathcal{D}(\Omega)} + 2\|f\|_{L^\infty(\Omega)} \log(r)\,\mathbf{1}_{(1,\infty)}(r) \\ &\leq \big(1 + 2\log(r)\mathbf{1}_{(1,\infty)}(r)\big)\|f\|_{\mathcal{D}(\Omega)},\end{aligned} \tag{2.12}$$
where $|\cdot|_{\mathcal{D}(\Omega)}$ and $\|\cdot\|_{\mathcal{D}(\Omega)}$ are as in Definition 1.2.
3. If $f \in \mathcal{D}(\Omega)$ (so that $f \in L^\infty(\Omega)$ and condition (2.9) is satisfied), it is readily seen that $\omega_{f,\Omega}(r) \to 0$ as $r \to 0^+$; thus, $\omega_{f,\Omega}$ is a *continuity modulus* (i.e., it is non-negative, non-decreasing and it vanishes as $r \to 0^+$).

Next, we can turn to the functions $\mathcal{M}_{f,\Omega}$, $\mathcal{N}_{f,\Omega}$ introduced in Remark 1.3 (and appearing in Theorems 1.6–1.7).

In the following, we want to prove that when f is a function satisfying suitable continuity properties (reflecting in properties of $\omega_{f,\Omega}$), then the moduli $\mathcal{M}_{f,\Omega}$, $\mathcal{N}_{f,\Omega}$ have suitable properties. By the definition of $\mathcal{N}_{f,\Omega}$, this will involve some iterative argument. Now, while the function $\omega_{f,\Omega}(r)$ is globally bounded as soon as f is bounded, the same is not true for $\mathcal{M}_{f,\Omega}(r)$ (see (2.12)). In view of this fact, it is useful to introduce the following definition.

Definition 2.5. We will say that a function $\omega : \mathbb{R}^+ \equiv (0,\infty) \to \mathbb{R}$ is a *continuity modulus of exponent* $\alpha \in (0,1)$ if

(P1) ω is non-decreasing on $\mathbb{R}^+$, and $\omega(r) \to 0$ as $r \to 0^+$;

(P2) there exists $\omega_0 > 0$ such that
$$\omega(r) \leq \omega_0\, r^\alpha \quad \forall\, r \geq 1;$$

If, in addition, we have
$$[\omega] := \int_0^1 \frac{\omega(r)}{r}\,dr < \infty, \tag{2.13}$$
we will say that ω is a *Dini continuity modulus* (of exponent α).

Lemma 2.6. *Let $\alpha \in (0,1)$, and let $\omega : \mathbb{R}^+ \to \mathbb{R}$ be a Dini continuity modulus of exponent α. Then, the function $M(\omega)$ defined by*
$$M(\omega)(r) = \omega(r) + \int_0^r \frac{\omega(s)}{s}\,ds + r\int_r^\infty \frac{\omega(s)}{s^2}\,ds. \tag{2.14}$$

is a continuity modulus with exponent α. In particular, there exists a constant $c > 0$, only depending on α, such that

$$M(\omega)(r) \leq \omega_0' r^\alpha \text{ for all } r \geq 1, \quad \text{where } \omega_0' = c([\omega] + \omega_0), \tag{2.15}$$

If, in addition, ω satisfies the stronger integrability property

$$\int_0^1 \frac{\omega(r)}{r} |\log(r)| \, dr < \infty, \tag{2.16}$$

then $M(\omega)$ is a Dini continuity modulus. *In particular, we have*

$$[M(\omega)] = \int_0^1 \frac{M(\omega)(s)}{s} \, ds \leq c\Big(\int_0^1 \frac{\omega(s)}{s} (1 + |\log(s)|) ds + \omega_0 \Big), \tag{2.17}$$

for a constant $c > 0$ only depending on α.

Proof. To ease the readability, we split the proof into three steps.

STEP I: In this first step we prove that $M(\omega)$ is well-defined on $\mathbb{R}^+$. To this end, we observe that, by (P1) and (2.13), we have

$$\begin{aligned} \int_0^r \frac{\omega(s)}{s} \, ds &\leq \int_0^1 \frac{\omega(s)}{s} \, ds + \int_1^{\max\{1,r\}} \frac{\omega(s)}{s} \, ds \\ &\leq [\omega] + \omega(\max\{r, 1\}) r < \infty \quad \forall\, r > 0. \end{aligned}$$

Moreover, by exploiting property (P2) (and since $\alpha < 1$), we also have

$$\begin{aligned} \int_r^\infty \frac{\omega(s)}{s^2} \, ds &\leq \int_{\min\{r,1\}}^1 \frac{\omega(s)}{s^2} \, ds + \int_1^\infty \frac{\omega(s)}{s^2} \, ds \\ &\leq \frac{\omega(1)}{r} + \omega_0 \int_1^\infty \frac{1}{s^{2-\alpha}} \, ds < \infty. \end{aligned}$$

Gathering these facts, we then conclude that $M(\omega)(r) < \infty$ for all $r > 0$.

STEP II: Now we have shown that $M(\omega)$ is well-defined, we then turn to prove that such a function is a continuity modulus of exponent α, further satisfying estimate (2.17). To this end, we first observe that, owing to the properties of ω, the (well-defined) function

$$F(r) := \omega(r) + \int_0^r \frac{\omega(s)}{s} \, ds \quad (r > 0)$$

is clearly non-negative, non-decreasing and it vanishes as $r \to 0^+$. Moreover, by using property (P2) of ω we see that, for every $r \geq 1$,

$$\begin{aligned} F(r) &\leq \omega_0 r^\alpha + \int_0^1 \frac{\omega(s)}{s} \, ds + \int_1^r \frac{\omega(s)}{s} \, ds \\ &\leq \omega_0 r^\alpha + [\omega] + \omega_0 \int_1^r s^{\alpha-1} \, ds \leq \frac{1}{\alpha}([\omega] + 2\omega_0) r^\alpha, \end{aligned}$$

and thus, F is a continuity modulus of exponent α satisfying (2.15). In view of these facts, and taking into account (2.14), to prove that $M(\omega)$ is a continuity modulus, we consider the (well-defined) map

$$G(r) := r\int_r^\infty \frac{\omega(s)}{s^2}\,ds \qquad (r>0),$$

and we show that also G satisfies the following properties:

(a) G is non-negative, non-decreasing and it vanishes as $r\to 0^+$;

(b) there exists a constant $\hat{c}>0$, only depending on α such that

$$G(r)\le \hat{c}\,\omega_0\,r^\alpha \quad \forall\, r\ge 1.$$

Proof of (a). Clearly, $G(r)\ge 0$ for every $r>0$ (as ω is non-negative); moreover, by Lebesgue's Differentiation Theorem (and recalling that the function ω is non-decreasing on $(0,\infty)$), for a.e. $r>0$ we have

$$G'(r) = \int_r^\infty \frac{\omega(s)}{s^2}\,ds - \frac{\omega(r)}{r} \ge \omega(r)\int_r^\infty \frac{1}{s^2}\,ds - \frac{\omega(r)}{r} = 0,$$

and this proves that G is non-decreasing. Finally, we turn to prove that $G(r)$ vanishes as $r\to 0^+$. To this end, it is useful to distinguish two cases.

- If $\int_0^\infty \frac{\omega(s)}{s^2}\,ds<\infty$, we immediately get

$$\lim_{r\to 0^+} G(r) = \lim_{r\to 0^+} r\int_r^\infty \frac{\omega(s)}{s^2}\,ds = 0.$$

- If, instead, $\int_0^\infty \frac{\omega(s)}{s^2}\,ds=\infty$, we observe that

$$\frac{\left(\int_r^\infty \frac{\omega(s)}{s^2}\,ds\right)'}{(1/r)'} = \omega(r) \quad \text{for every } r>0;$$

thus, since $\omega(r)$ vanishes as $r\to 0^+$ (see assumption (ii)), an immediate application of De L'Hôpital's Theorem gives

$$\lim_{r\to 0^+} G(r) = \lim_{r\to 0^+} \frac{\int_r^\infty \frac{\omega(s)}{s^2}\,ds}{1/r} = 0.$$

Proof of (b). By exploiting property (P2) of ω, we immediately get

$$G(r)\le \omega_0 r\int_r^\infty s^{\alpha-2}\,ds = \omega_0 r\Big[\frac{s^{\alpha-1}}{\alpha-1}\Big]_r^\infty = \frac{\omega_0}{1-\alpha}r^\alpha \quad \forall\, r\ge 1,$$

and this proves that G satisfies (b). Summing up, the function G is a continuity modulus of exponent α satisfying (2.15), and thus the same is true for $M(\omega)$.

STEP III: In this last step, we prove that $M(\omega)$ satisfies (2.17) (hence, $M(\omega)$ is a Dini continuity modulus of exponent α), provided ω satisfies the *stronger property* (2.16). To prove this fact, and since ω satisfies (2.16), we set

$$M_1(r) = \int_0^r \frac{\omega(s)}{s}\,ds, \qquad M_2(r) = r\int_r^\infty \frac{\omega(s)}{s^2} = G(r), \tag{2.18}$$

and we show that both $M_1,\ M_2$ satisfy property (iii), that is,

$$\int_0^1 \frac{M_i(r)}{r}\,dr < \infty \quad \forall\ i = 1, 2.$$

As regards M_1, by Fubini–Tonelli's Theorem we have

$$\begin{aligned}\int_0^1 \frac{M_1(r)}{r}dr &= \int_0^1 \frac{1}{r}\left(\int_0^r \frac{\omega(s)}{s}ds\right)dr = \int_0^1 \frac{\omega(s)}{s}\left(\int_s^1 \frac{dr}{r}\right)ds \\ &= \int_0^1 \frac{\omega(s)}{s}|\log s|\,ds < \infty,\end{aligned} \tag{2.19}$$

where we have used the fact that ω satisfies (2.16). As regards M_2, again by using Fubini–Tonelli's Theorem (and since ω satisfies (2.16)), we obtain

$$\begin{aligned}\int_0^1 \frac{M_2(r)}{r}\,dr &= \int_0^1 \Big(\int_r^\infty \frac{\omega(s)}{s^2}\,ds\Big)dr \\ &= \int_0^\infty \frac{\omega(s)}{s^2}\Big(\int_0^{\min\{s,1\}} dr\Big)ds \\ &= \int_0^1 \frac{\omega(s)}{s}\,ds + \int_1^\infty \frac{\omega(s)}{s^2}\,ds \\ &\le \int_0^1 \frac{\omega(s)}{s}\,ds + \frac{\omega_0}{1-\alpha} < \infty,\end{aligned} \tag{2.20}$$

where we have also used the fact that ω satisfies property (P2) (with $\alpha < 1$). Finally, by combining (2.19)–(2.20), we conclude that

$$\begin{aligned}[M(\omega)] &= \int_0^1 \frac{M(\omega)(s)}{s}\,ds \le [\omega] + \int_0^1 \frac{\omega(s)}{s}|\log(s)|\,ds + [\omega] + \frac{\omega_0}{1-\alpha} \\ &\le c\Big(\int_0^1 \frac{\omega(s)}{s}(1+|\log(s)|)ds + \omega_0\Big).\end{aligned}$$

This ends the proof. □

Remark 2.7. Let $\alpha \in (0, 1)$, and let $\omega : \mathbb{R}^+ \to \mathbb{R}$ be a Dini continuity modulus of exponent α. It is contained in the proof of Lemma 2.6 the following useful (thought not sharp) bound, which will be repeatedly used in the sequel:

$$\int_0^r \frac{\omega(s)}{s}\,ds \le c_\alpha([\omega] + \omega_0)(1 + r^\alpha) \quad \forall\ r > 0 \tag{2.21}$$

(where $c > 0$ is a constant only depending on α).

Thanks to Lemma 2.6, we readily obtain the following

Proposition 2.8. *Assume that* $f \in \mathcal{D}(\Omega)$. *Then, the function* $\mathcal{M}_{f,\Omega}(r)$ *defined in* (1.16) *is a* modulus of continuity of exponent α*, for every* $\alpha \in (0, 1)$. *In particular, given any* $\alpha \in (0, 1)$ *there exists a constant* $c > 0$ *only depending on* α *such that*

$$\mathcal{M}_{f,\Omega}(r) \leq c\|f\|_{\mathcal{D}(\Omega)} r^\alpha \quad \forall\, r \geq 1, \tag{2.22}$$

If, in addition, $f \in \mathcal{D}_{\log}(\Omega)$*, then the function* $\mathcal{M}_{f,\Omega}$ *is a* Dini continuity modulus*; in particular, given any* $\alpha \in (0, 1)$ *there exists* $c = c_\alpha > 0$ *such that*

$$\int_0^1 \frac{\mathcal{M}_{f,\Omega}(r)}{r}\, dr \leq c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}\Big) < \infty. \tag{2.23}$$

Finally, the function $\mathcal{N}_{f,\Omega}(r)$ *defined in* (1.17) *is a* modulus of continuity of exponent α*, for every* $\alpha \in (0, 1)$.

Proof. First of all, we observe that, if $f \in \mathcal{D}(\Omega)$, then $\omega_{f,\Omega}$ is a Dini continuity modulus of exponent α, for every $\alpha \in (0, 1)$. To be more precise, if $\alpha \in (0, 1)$ is arbitrarily chosen, by exploiting (2.10) we get

$$\omega_{f,\Omega}(r) \leq \omega_0\, r^\alpha \text{ for all } r \geq 1, \quad \text{with } \omega_0 = 2\|f\|_{\mathcal{D}(\Omega)}.$$

Thus, since $\mathcal{M}_{f,\Omega} = M(\omega_{f,\Omega})$ (where M is as in (2.14)), from Lemma 2.6 we infer that $\mathcal{M}_{f,\Omega}$ is a modulus of continuity of exponent α. In particular,

$$\mathcal{M}_{f,\Omega}(r) \leq \omega_0'\, r^\alpha \quad \forall\, r \geq 1,$$

where $\omega_0' > 0$ is a constant which, by (2.15), is of the form

$$\omega_0' = c([\omega_{f,\Omega}] + \omega_0) = c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}\, ds + \omega_0\Big) \leq c\|f\|_{\mathcal{D}(\Omega)},$$

and $c > 0$ is a constant only depending on α. This gives (2.22). If, in addition, $f \in \mathcal{D}_{\log}(\Omega)$, the function $\omega_{f,\Omega}$ also satisfies assumption (2.16) in the statement of Lemma 2.6; we then infer from this lemma that

$$\mathcal{M}_{f,\Omega} = M(\omega_{f,\Omega})$$

is a *Dini continuity modulus*. Moreover, by (2.17), we have

$$\begin{aligned}\int_0^1 \frac{\mathcal{M}_{f,\Omega}(r)}{r}\, dr = [\mathcal{M}_{f,\Omega}] &\leq c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \omega_0\Big)\\ &\leq c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}\Big),\end{aligned}$$

where $c > 0$ depends on the fixed α. Finally, we also have that

$$\mathcal{N}_{f,\Omega} = M(\mathcal{M}_{f,\Omega})$$

is a well-defined modulus of continuity, and the proof is complete. □

Remark 2.9. It should be noticed that, even if $f \in \mathcal{D}_{\log}(\Omega)$, the function $\mathcal{N}_{f,\Omega}$ may not be a Dini continuity modulus; namely, we cannot ensure that

$$\int_0^1 \frac{\mathcal{N}_{f,\Omega}(r)}{r}\,dr < \infty. \tag{2.24}$$

In fact, by arguing as in the proof of Proposition 2.8, we see that a sufficient condition for (2.24) to hold is the *log-Dini continuity of* $\mathcal{M}_{f,\Omega}$, i.e.,

$$\int_0^1 \frac{\mathcal{M}_{f,\Omega}(r)}{r}|\log(r)|\,dr < \infty.$$

This, in turn, is readily seen to be satisfied as soon as

$$\int_0^1 \frac{\omega_{f,\Omega}(r)}{r}\log^2(r)\,dr < \infty,$$

that is when f is $\log^2$-Dini continuous.

Now that we have fully established Proposition 2.8, we proceed by studying the two functions $\mathcal{U}^{\mu}_{f,\Omega}$, $\mathcal{V}^{\mu}_{f,\Omega}$ introduced in Remark 1.3.

Lemma 2.10. *Let* $\alpha \in (0,1)$*, and* $\omega : \mathbb{R}^+ \to \mathbb{R}$ *be a Dini continuity modulus of exponent* α*. For a given* $\mu > 0$*, we consider the function*

$$U^{\mu}(\omega)(r) := \int_{\mathbb{R}^N} e^{-\mu|z|^2}\Big(\int_0^{r\|z\|}\frac{\omega(s)}{s}\,ds\Big)dz \qquad (r>0). \tag{2.25}$$

Then, the following facts hold:

(i) *there exists a constant* $c > 0$*, only depending on* μ *and* α*, such that*

$$0 \le U^{\mu}(\omega)(r) \le c(1+r^{\alpha})(\omega_0 + [\omega]) \quad \forall\, r > 0;$$

(ii) $U^{\mu}(\omega)(r) \to 0$ *as* $r \to 0^+$.

Proof. (i) Since ω is a Dini continuity modulus of exponent α, we get

$$\begin{aligned} &e^{-\mu|z|^2}\Big(\int_0^{r\|z\|}\frac{\omega(s)}{s}\,ds\Big)\\ &\quad\le e^{-\mu|z|^2}\Big([\omega] + \omega_0\int_1^{\max\{r\|z\|,1\}} s^{\alpha-1}\,ds\Big)\\ &\quad\le c([\omega]+\omega_0)\cdot e^{-\mu|z|^2}(1+r^{\alpha}\|z\|^{\alpha}) \quad \forall\, z \in \mathbb{R}^N,\ r>0. \end{aligned} \tag{2.26}$$

From this, since $\mu > 0$ and $\|z\| = \sum_j |z_j|^{1/q_j}$, we obtain

$$\begin{aligned} 0 \le U^{\mu}(\omega)(r) &\le c([\omega]+\omega_0)\int_{\mathbb{R}^N} e^{-\mu|z|^2}(1+r^{\alpha}\|z\|^{\alpha})\,dz\\ &\le c([\omega]+\omega_0)\Big(\int_{\mathbb{R}^N} e^{-\mu|z|^2}\,dz + r^{\alpha}\int_{\mathbb{R}^N} e^{-\mu|z|^2}\|z\|^{\alpha}\,dz\Big)\\ &= c([\omega]+\omega_0)(c_{1,\mu}+c_{2,\mu}r^{\alpha}) \le c([\omega]+\omega_0)(1+r^{\alpha}), \end{aligned}$$

where $c>0$ is a constant only depending on μ and α.

(ii) We fist observe that, since $\int_0^1 \frac{\omega(s)}{s}\,ds<\infty$, we have

$$\lim_{r\to 0^+} e^{-\mu|z|^2}\Big(\int_0^{r\|z\|}\frac{\omega(s)}{s}\,ds\Big)=0 \quad \forall\, z\in\mathbb{R}^N.$$

On the other hand, for every $z\in\mathbb{R}^N$ and every $r\in(0,1)$, by (2.26) we have

$$\begin{aligned}0&\le e^{-\mu|z|^2}\Big(\int_0^{r\|z\|}\frac{\omega(s)}{s}\,ds\Big)\\ &\le c\,e^{-\mu|z|^2}(1+\|z\|)\in L^1(\mathbb{R}^N).\end{aligned}$$

Then, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{r\to 0^+} U^\mu(\omega)(r)=\lim_{r\to 0^+}\int_{\mathbb{R}^N} e^{-\mu|z|^2}\Big(\int_0^{r\|z\|}\frac{\omega(s)}{s}\,ds\Big)dz=0.$$

This ends the proof. □

The next proposition collects some explicit bounds for $U^\mu(\omega)$.

Proposition 2.11. *Let $\alpha\in(0,1)$, and let $\omega:\mathbb{R}^+\to\mathbb{R}$ be a Dini continuity modulus of exponent α. Then, the following facts hold.*

(i) *There exist constants $c,\kappa>0$, only depending on μ and N, such that*

$$U^\mu(\omega)(r)\le\begin{cases} c\Big(\int_0^{\sqrt{r}}\frac{\omega(s)}{s}\,ds+([\omega]+\omega_0)e^{-\frac{\kappa}{r}}\Big) & \text{if } 0<r<1,\\ cr^\alpha([\omega]+\omega_0) & \text{if } r\ge 1.\end{cases}$$

(ii) *Assume that there exists $\omega_0>0$ such that*

$$\omega(r)\le\omega_0 r^\alpha \text{for every } r>0 \tag{2.27}$$

(that is, ω satisfies the estimate in property (P2) for every $r>0$*, and not only for $r\ge 1$); then, we have the following estimate*

$$U^\mu(\omega)(r)\le c_{\mu,\alpha}\cdot\omega_0\, r^\alpha \quad \forall r>0.$$

Proof. (i) For a fixed $R>1$ (to be chosen later on), we write

$$U^\mu(\omega)(r)=\int_{\{\|z\|\le R\}}\{\cdots\}\,dz+\int_{\{\|z\|>R\}}\{\cdots\}\,dz\equiv A_R+B_R. \tag{2.28}$$

Then, we proceed by estimating the two integrals A_R, B_R separately, distinguishing two cases.

CASE I: $0 < r < 1$. We have:

$$\begin{aligned} A_R &\leq \int_{\{\|z\|\leq R\}} e^{-\mu|z|^2}\Big(\int_0^{rR}\frac{\omega(s)}{s}\,ds\Big)dz \\ &\leq \Big(\int_0^{rR}\frac{\omega(s)}{s}\,ds\Big)\cdot\int_{\mathbb{R}^N} e^{-\mu|z|^2}\,dz \\ &= c\int_0^{rR}\frac{\omega(s)}{s}\,ds, \end{aligned} \tag{2.29}$$

where $c > 0$ is a constant only depending on the fixed μ. Next, since in B_R we have $\|z\| > R > 1$ and we are assuming $0 < r < 1$, from (2.21) we get

$$\begin{aligned} B_R &\leq \int_{\{\|z\|>R\}} e^{-\mu|z|^2}\Big(\int_0^{\|z\|}\frac{\omega(s)}{s}\,ds\Big)dz \\ &\leq c([\omega]+\omega_0)\int_{\{\|z\|>R\}}(1+\|z\|^\alpha)e^{-\mu|z|^2}\,dz \\ &\leq c([\omega]+\omega_0)\int_{\{\|z\|>R\}}\|z\|^\alpha e^{-\mu|z|^2}\,dz. \end{aligned}$$

Then, by performing the change of variables $z = D_0(R)u$, we obtain

$$B_R \leq c([\omega]+\omega_0)\,R^{Q+\alpha}\int_{\{\|u\|>1\}} e^{-\mu|D_0(R)u|^2}\|u\|^\alpha\,du. \tag{2.30}$$

Now, since we are assuming $R > 1$, we have

$$|D_0(R)u|^2 = \sum_{j=1}^N R^{2q_j}u_j^2 \geq R^2|u|^2 \quad \forall\ u \in \mathbb{R}^N.$$

As a consequence, we obtain the following estimate

$$\begin{aligned} e^{-\mu|D_0(R)u|^2}\|u\|^\alpha &\leq e^{-\mu R^2|u|^2}\|u\|^\alpha \leq c\sum_{j=1}^N e^{-\mu R^2|u|^2}|u_j|^{\alpha/q_j} \\ &\leq c\sum_{j=1}^N e^{-\mu R^2|u|^2}|u|^{\alpha/q_j}. \end{aligned} \tag{2.31}$$

In view of (2.31), and since $\{\|u\| > 1\} \subseteq \{|u| > \delta\}$ for some constant $\delta > 0$ only depending on the dimension N, from (2.30) we finally get

$$\begin{aligned} B_R \quad &\leq c([\omega]+\omega_0)\,R^Q\textstyle\sum_{j=1}^N\int_{\{|u|>\delta\}} e^{-\frac{\mu R^2}{2}|u|^2}|u|^{\alpha/q_j}\,du \\ &= c([\omega]+\omega_0)\,R^Q\textstyle\sum_{j=1}^N\int_\delta^\infty e^{-\frac{\mu R^2}{2}\rho^2}\rho^{N+\frac{\alpha}{q_j}-1}\,d\rho \\ &\quad\text{(by the change of variables } \rho = s/R, \text{ and since } R > 1) \\ &\leq c([\omega]+\omega_0)\,R^Q\textstyle\sum_{j=1}^N\int_{\delta R}^\infty e^{-\frac{\mu s^2}{2}}s^{N+\frac{\alpha}{q_j}-1}\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq c([\omega]+\omega_0)\, R^Q \textstyle\int_{\delta R}^{\infty} e^{-\frac{\mu s^2}{4}}\, ds \\
&\leq c([\omega]+\omega_0)\, R^Q e^{-\frac{\mu \delta^2 R^2}{4}} \leq c([\omega]+\omega_0)\, e^{-\kappa R^2},
\end{aligned} \tag{2.32}
$$

where $c > 0$ is a suitable constant, possibly different from line to line but only depending on μ, N and α, and $\kappa = \mu\,\delta^2/8$. Gathering (2.29)–(2.32), and choosing $R = 1/\sqrt{r} > 1$, we then conclude that

$$
U^\mu(\omega)(r) \leq A_R + B_R \leq c\Big(\int_0^{\sqrt{r}} \frac{\omega(s)}{s}\, ds + ([\omega]+\omega_0) e^{-\frac{\kappa}{r}} \Big),
$$

where $c, \kappa > 0$ are constants only depending on μ and N.

Case II: $r \geq 1$. Since $rR \geq R > 1$, by combining (2.21) with estimate (2.29) (which actually holds for every $r > 0$) we obtain

$$
\begin{aligned}
A_R &\leq c \int_0^{rR} \frac{\omega(s)}{s}\, ds \leq c([\omega]+\omega_0)(1 + (rR)^\alpha) \\
&\leq c(rR)^\alpha ([\omega]+\omega_0).
\end{aligned} \tag{2.33}
$$

Next, since in B_R we have $r\|z\| > rR > 1$, again by (2.21) we get

$$
\int_0^{r\|z\|} \frac{\omega(s)}{s}\, ds \leq c([\omega]+\omega_0)\big(1 + (r\|z\|)^\alpha\big) \leq c(r\|z\|)^\alpha ([\omega]+\omega_0);
$$

from this, since $\mu > 0$ and $\|z\| = \sum_j |z_j|^{1/q_j}$, we obtain

$$
B_R \leq c r^\alpha ([\omega]+\omega_0) \int_{\mathbb{R}^N} e^{-\mu |z|^2} \|z\|^\alpha\, dz = c r^\alpha ([\omega]+\omega_0), \tag{2.34}
$$

where $c > 0$ is a constant depending on μ and α. Gathering (2.33)–(2.32), and choosing $R = 2$, we then conclude that

$$
U^\mu(\omega)(r) \leq A_R + B_R \leq c r^\alpha ([\omega]+\omega_0),
$$

where $c > 0$ is a constant only depending on μ and α.

(ii) If (2.27) holds, by definition of $U^\mu(\omega)$ we have

$$
\begin{aligned}
U^\mu(\omega)(r) &\leq \omega_0 \int_{\mathbb{R}^N} e^{-\mu|z|^2} \Big(\int_0^{r\|z\|} s^{\alpha-1}\, ds \Big) dz \\
&= \frac{\omega_0\, r^\alpha}{\alpha} \int_{\mathbb{R}^N} e^{-\mu|z|^2} \|z\|^\alpha\, dz \equiv c_{\mu,\alpha} \cdot \omega_0\, r^\alpha \quad \forall r > 0.
\end{aligned}
$$

This ends the proof. □

Thanks to Lemmas 2.10–2.11, we readily obtain the following results.

Lemma 2.12. *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an arbitrary open set, and let $f \in \mathcal{D}(\Omega)$. For a given $\mu > 0$, let $\mathcal{U}^\mu_{f,\Omega}$ be as in (1.18). Then, the following facts hold.*

(i) *for every* $\alpha \in (0, 1)$ *there exists a constant* $c > 0$ *only depending on* μ *and* α*, such that*

$$0 \le \mathcal{U}^{\mu}_{f,\Omega}(r) \le c(1 + r^{\alpha})\|f\|_{\mathcal{D}(\Omega)} \quad \forall\, r > 0;$$

(ii) $\mathcal{U}^{\mu}_{f,\Omega}(r) \to 0$ *as* $r \to 0^+$.

If, in addition, $f \in \mathcal{D}_{\log}(\Omega)$*, then the function* $\mathcal{V}^{\mu}_{f,\Omega}$ *defined in* (1.19) *satisfies the following properties, analogous to* (i)–(ii) *above:*

(i)' *for every* $\alpha \in (0, 1)$ *there exists a constant* $c > 0$ *only depending on* μ *and* α*, such that*

$$0 \le \mathcal{V}^{\mu}_{f,\Omega}(r) \le c(1 + r^{\alpha})\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}\Big),$$

(ii)' $\mathcal{V}^{\mu}_{f,\Omega}(r) \to 0$ *as* $r \to 0^+$.

Proof. If $f \in \mathcal{D}(\Omega)$, both the properties (i) and (ii) of $\mathcal{U}^{\mu}_{f,\Omega}$ immediately follow from Lemma 2.10, taking into account that

$$\mathcal{U}^{\mu}_{f,\Omega} = U^{\mu}(\omega_{f,\Omega}) \quad \text{and} \quad [\omega_{f,\Omega}] + \omega_0 = \int_0^1 \frac{\omega_{f,\Omega}(s)}{s}\,ds + \omega_0 \le 3\|f\|_{\mathcal{D}(\Omega)},$$

see the the *incipit* of the proof of Proposition 2.8. If, in addition, $f \in \mathcal{D}_{\log}(\Omega)$, from Proposition 2.8 we know that $\mathcal{M}_{f,\Omega}$ is a *Dini continuity modulus* of exponent α, for every $\alpha \in (0, 1)$; more precisely,

(1) $\mathcal{M}_{f,\Omega} \le \omega_0'^{\alpha}$ for all $r \ge 1$, where $\omega_0' = c\|f\|_{\mathcal{D}(\Omega)}$;

(2) $[\mathcal{M}_{f,\Omega}] = \displaystyle\int_0^1 \frac{\mathcal{M}_{f,\Omega}(s)}{s}\,ds \le c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}\Big)$,

where $c > 0$ is a constant only depending on the fixed α. As a consequence, properties (i)'–(ii)' of $\mathcal{V}^{\mu}_{f,\Omega}$ follow again from Lemma 2.10, since

$$\mathcal{V}^{\mu}_{f,\Omega} = U^{\mu}(\mathcal{M}_{f,\Omega})$$

and since, by (2.22)–(2.23) in Proposition 2.8, we have

$$\begin{aligned} [\mathcal{M}_{f,\Omega}] + \omega_0' &\le \int_0^1 \frac{\mathcal{M}_{f,\Omega}(s)}{s}\,ds + c\|f\|_{\mathcal{D}(\Omega)} \\ &\le c\Big(\int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}\Big), \end{aligned} \tag{2.35}$$

where $c > 0$ is a constant only depending on α. This ends the proof. □

Proposition 2.13. *Let* $\Omega \subseteq \mathbb{R}^{N+1}$ *be an arbitrary open set, and let* $f \in \mathcal{D}(\Omega)$*. Moreover, let* $\mu > 0$ *and* $\alpha \in (0, 1)$ *be fixed. Then, we have*

$$\mathcal{U}^{\mu}_{f,\Omega}(r) \le \begin{cases} c\big(\int_0^{\sqrt{r}} \frac{\omega_{f,\Omega}(s)}{s}\,ds + \|f\|_{\mathcal{D}(\Omega)} e^{-\frac{\kappa}{r}}\big) & \text{if } 0 < r < 1, \\ c r^{\alpha}\|f\|_{\mathcal{D}(\Omega)} & \text{if } r \ge 1. \end{cases}$$

If in addition $f \in \mathcal{D}_{\log}(\Omega)$, we have

$$\mathcal{V}^{\mu}_{f,\Omega}(r) \leq \begin{cases} c\left(\int_0^{\sqrt{r}} \frac{\omega_{f,\Omega}(s)}{s}\,ds + \|f\|_{\mathcal{D}_{\log}(\Omega)} e^{-\frac{\kappa}{r}}\right) & \text{if } 0 < r < 1, \\ c r^{\alpha} \|f\|_{\mathcal{D}_{\log}(\Omega)} & \text{if } r \geq 1. \end{cases}$$

where $c, \kappa > 0$ only depends on μ, N and α, and

$$\|f\|_{\mathcal{D}_{\log}(\Omega)} := \int_0^1 \frac{\omega_{f,\Omega}(s)}{s}(1 + |\log(s)|)ds + \|f\|_{L^\infty(\Omega)}$$

Proof. This is an immediate consequence of Proposition 2.11, taking into account the following identities (see (2.10) and Proposition 2.8)

$$[\omega] + \omega_0 \leq c \cdot \begin{cases} \|f\|_{\mathcal{D}(\Omega)}, & \text{if } \omega = \omega_{f,\Omega}; \\ \|f\|_{\mathcal{D}_{\log}(\Omega)}, & \text{if } \omega = \mathcal{M}_{f,\Omega}. \end{cases}$$

This ends the proof. □

We conclude this part of the section with a couple of technical lemmas, which will be repeatedly used in the sequel.

Lemma 2.14. *Let $f \in \mathcal{D}(\Omega)$, and let $\xi \in \Omega$, $r > 0$ be such that $B = B_r(\xi) \Subset \Omega$. We assume that $f \equiv 0$ in $\Omega \setminus B$ (i.e., $\operatorname{supp}(f) \subseteq \overline{B}$). Then,*

$$\omega_{f,\Omega}(r) = \omega_{f,\overline{B}}(r) \quad \forall\, r > 0.$$

Proof. First of all, since $B \Subset \Omega$ we have $\omega_{f,\Omega} \geq \omega_{f,B}$ on $(0, \infty)$. To prove the reverse inequality, we fix $r > 0$ and we let $(x, t), (y, t) \in \Omega$ be such that

$$d((x, t), (y, t)) = \|x - y\| \leq r.$$

We then distinguish three cases.

(a) $(x, t), (y, t) \in B$. In this case, by definition of $\omega_{f,B}$ we have

$$|f(x, t) - f(y, t)| \leq \sup_{\substack{(z_1,s),(z_2,s)\in B \\ \|z_1-z_2\|\leq r}} |f(z_1, s) - f(z_2, s)| = \omega_{f,\overline{B}}(r). \tag{2.36}$$

(b) $(x, t), (y, t) \notin B$. In this case, since $f \equiv 0$ out of B, we have

$$|f(x, t) - f(y, t)| = 0 \leq \omega_{f,\overline{B}}(r). \tag{2.37}$$

(c) $(x, t) \in B$, $(y, t) \notin B$. In this last case, we consider the segment

$$\gamma(\tau) = (x + \tau(y - x), t) \qquad (\tau \in [0, 1])$$

and we first observe that, for every $0 \leq \tau \leq 1$, we have:

$$d((x, t), \gamma(\tau)) = \|\tau(y - x)\| = \sum_{j=1}^{N} \tau^{1/q_j} |x_j - y_j|^{1/q_j} \leq \tau^{1/q_N} \|x - y\| \leq r.$$

On the other hand, since $\gamma(0) \in B$ and $\gamma(1) \notin B$, there exists $\tau^* \in (0,1]$ such that $\gamma(\tau^*) \in \partial B$. Thus, since $f \equiv 0$ in $\Omega \setminus B \supset \partial B$, we have

$$
\begin{aligned}
|f(x,t) - f(y,t)| &= |f(x,t)| = |f(x,t) - f(\gamma(\tau^*))| \\
&\leq \sup_{\substack{(z_1,s),(z_2,s)\in\overline{B} \\ \|z_1-z_2\|\leq r}} |f(z_1,s) - f(z_2,s)| = \omega_{f,\overline{B}}(r).
\end{aligned} \tag{2.38}
$$

Gathering (2.36)-to-(2.38), we then conclude that

$$
\omega_{f,\Omega}(r) = \sup_{\substack{(x,t),(y,t)\in\Omega \\ d((x,t),(y,t))\leq r}} |f(x,t) - f(y,t)| \leq \omega_{f,\overline{B}}(r),
$$

and the proof is complete. □

Lemma 2.15. *There exists a structural constant $c > 0$ such that, for every $f \in \mathcal{D}(\Omega)$ every $r > 0$ and every $\gamma > 0$, the following estimates hold true:*

$$
\int_{\{\eta\in\mathbb{R}^{N+1}:\, d(\xi,\eta)>r\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+3}}\, d\eta \leq c \int_{2r}^{\infty} \frac{\omega_{f,\Omega}(\gamma s)}{s^2}\, ds \tag{2.39}
$$

$$
\int_{\{\eta\in\mathbb{R}^{N+1}:\, d(\xi,\eta)<r\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+2}}\, d\eta \leq c \int_{0}^{2r} \frac{\omega_{f,\Omega}(\gamma s)}{s}\, ds \tag{2.40}
$$

Here, $Q \geq 1$ is as in (1.8), and c is independent of both f and r.

Proof. The proof of both (2.39)–(2.40) is based on the fact that, since $f \in \mathcal{D}(\Omega)$, the function $\omega_{f,\Omega}$ is non-negative, non-decreasing and satisfies (2.9); moreover, we exploit the fact that $|B_r(\xi)| = \omega_Q\, r^{Q+2}$, see (2.4).

(I) Proof of (2.39). Taking into account (1.13), we have

$$
\begin{aligned}
&\int_{\{\eta\in\mathbb{R}^{N+1}:\, d(\xi,\eta)>r\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+3}}\, d\eta \\
&\quad= \sum_{k=0}^{\infty} \int_{\{\eta\in\mathbb{R}^{N+1}:\, 2^k r\leq d(\xi,\eta)<2^{k+1}r\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+3}}\, d\eta \\
&\quad\leq \sum_{k=0}^{\infty} \frac{\omega_{f,\Omega}(2^{k+1}\gamma r)}{(2^k r)^{Q+3}} \cdot \left|\{\eta \in \mathbb{R}^{N+1} :\, 2^k r \leq d(\xi,\eta) < 2^{k+1} r\}\right| \\
&\quad\leq \sum_{k=0}^{\infty} \frac{\omega_{f,\Omega}(2^{k+1}\gamma r)}{(2^k r)^{Q+3}} \cdot |B_{2^{k+1}\boldsymbol{\kappa} r}(\xi)| \\
&\quad= \omega_Q (2\boldsymbol{\kappa})^{Q+2} \sum_{k=0}^{\infty} \frac{\omega_{f,\Omega}(2^{k+1}\gamma r)}{2^k r}.
\end{aligned} \tag{2.41}
$$

On the other hand, since $\omega_{f,\Omega}$ is non-decreasing we have

$$
\begin{aligned}
\int_{2r}^{\infty} \frac{\omega_{f,\Omega}(\gamma s)}{s^2}\, ds &= \sum_{k=0}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \frac{\omega_{f,\Omega}(\gamma s)}{s^2}\, ds \\
&\geq \sum_{k=0}^{\infty} \frac{\omega_{f,\Omega}(2^{k+1}\gamma r)}{(2^{k+2}r)^2} \cdot 2^{k+1} r = \frac{1}{8} \sum_{k=0}^{\infty} \frac{\omega_{f,\Omega}(2^{k+1}\gamma r)}{2^k r}.
\end{aligned} \tag{2.42}
$$

By combining (2.41)–(2.42), we immediately get (2.39).

(II) Proof of (2.40). Using once again (1.13), we have

$$\begin{aligned}
&\int_{\{\eta\in\mathbb{R}^{N+1}:\, d(\xi,\eta)<r\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+2}}\,d\eta \\
&\quad= \sum_{k=0}^{\infty}\int_{\{\eta\in\mathbb{R}^{N+1}:\, r/2^{k+1}<d(\xi,\eta)\le r/2^k\}} \frac{\omega_{f,\Omega}(\gamma\, d(\xi,\eta))}{d(\xi,\eta)^{Q+2}}\,d\eta \\
&\quad\le \sum_{k=0}^{\infty}\frac{\omega_{f,\Omega}(\gamma r/2^k)}{(r/2^{k+1})^{Q+2}}\cdot\left|\{\eta\in\mathbb{R}^{N+1}\,:\, r/2^{k+1}<d(\xi,\eta)\le r/2^k\}\right| \qquad (2.43)\\
&\quad\le \sum_{k=0}^{\infty}\frac{\omega_{f,\Omega}(\gamma r/2^k)}{(r/2^{k+1})^{Q+2}}\cdot|B_{\boldsymbol{\kappa} r/2^k}(\xi)| \\
&\quad= \omega_Q(2\boldsymbol{\kappa})^{Q+2}\sum_{k=0}^{\infty}\omega_{f,\Omega}(\gamma\, r/2^k).
\end{aligned}$$

On the other hand, since $\omega_{f,\Omega}$ is non-decreasing we have

$$\begin{aligned}
\int_0^{2r}\frac{\omega_{f,\Omega}(s)}{s}\,ds &= \sum_{k=0}^{\infty}\int_{r/2^k}^{r/2^{k-1}}\frac{\omega_{f,\Omega}(s)}{s}\,ds \\
&\ge \sum_{k=0}^{\infty}\frac{\omega_{f,\Omega}(\gamma r/2^k)}{r/2^{k-1}}\cdot\frac{r}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty}\omega_{f,\Omega}(\gamma r/2^k).
\end{aligned} \qquad (2.44)$$

By combining (2.43)–(2.44), we immediately get (2.40). □

2.3. Fundamental solution and representation formulas for the operator with coefficients only depending on t

In this section we collect some results established in [1,3] concerning the KFP operators $\mathcal{L}$ with *coefficients* a_{ij} *only depending on* t, that is,

$$\mathcal{L}u = \sum_{i,j=1}^{m_0} a_{ij}(t)\partial^2_{x_i x_j}u + \sum_{k,j=1}^{N} b_{jk}x_k\partial_{x_j}u - \partial_t u. \qquad (2.45)$$

Throughout what follows, we tacitly understand that $\mathcal{L}$ satisfies the structural assumptions (H1)–(H2) stated in the Introduction.

We begin by stating a result proved in [3], which provides an explicit expression for the global fundamental solution (heat kernel) of $\mathcal{L}$.

Theorem 2.16. (Fundamental solution for operators as in (2.45)) *Let* $C(t,s)$ *be the* $N\times N$ *matrix defined as follows:*

$$C(t,s) = \int_s^t E(t-\sigma)\cdot\begin{pmatrix} A_0(\sigma) & 0\\ 0 & 0\end{pmatrix}\cdot E(t-\sigma)^T\,d\sigma \quad (\textit{with } t>s) \qquad (2.46)$$

(*we recall that* $E(\sigma) = \exp(-\sigma B)$*, see* (1.5)). *Then,* $C(t, s)$ *is* symmetric and positive definite *for every* $t > s$*. Moreover, if we define*

$$\Gamma(x, t; y, s) = \frac{1}{(4\pi)^{N/2}\sqrt{\det C(t, s)}} e^{-\frac{1}{4}\langle C(t,s)^{-1}(x-E(t-s)y),\, x-E(t-s)y\rangle} \cdot \mathbf{1}_{\{t>s\}} \tag{2.47}$$

(*where* $\mathbf{1}_A$ *denotes the indicator function of a set* A), *then* Γ *enjoys the following properties, so that* Γ *is the* fundamental solution *for* $\mathcal{L}$ *with pole at* (y, s).

1. *In the open set* $\mathcal{O} := \{(x, t; y, s) \in \mathbb{R}^{2N+2} : (x, t) \neq (y, s)\}$*, the function* Γ *is* jointly continuous *in* $(x, t; y, s)$ *and* C^∞ *with respect to* x, y*. Moreover, for every multi-indices* α, β *the functions*

$$\partial_x^\alpha \partial_y^\beta \Gamma = \frac{\partial^{\alpha+\beta}\Gamma}{\partial x^\alpha \partial y^\beta}$$

   are jointly continuous *in* $(x, t; y, s) \in \mathcal{O}$*. Finally,* Γ *and* $\partial_x^\alpha \partial_y^\beta \Gamma$ *are* Lipschitz continuous *with respect to* t, s *in any region* $\mathcal{R}$ *of the form*

$$\mathcal{R} = \{(x, t; y, s) \in \mathbb{R}^{2N+2} : H \leq s + \delta \leq t \leq K\},$$

   where $H, K \in \mathbb{R}$ *and* $\delta > 0$ *are arbitrarily fixed.*
2. *For every fixed* $y \in \mathbb{R}^N$ *and* $t > s$*, we have*

$$\lim_{|x|\to+\infty} \Gamma(x, t; y, s) = 0.$$

3. *For every fixed* $(y, s) \in \mathbb{R}^{N+1}$*, we have*

$$\mathcal{L}\Gamma(\cdot; y, s)(x, t) = 0 \qquad \text{for every } x \in \mathbb{R}^N \text{ and a.e. } t.$$

4. *For every* $x \in \mathbb{R}^N$ *and* $t > s$*, we have*

$$\int_{\mathbb{R}^N} \Gamma(x, t; y, s)\, dy = 1. \tag{2.48}$$

5. *For every* $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $s \in \mathbb{R}$*, the function*

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; y, s) f(y)\, dy$$

   is the unique solution to the Cauchy problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^N \times (s, \infty) \\ u(\cdot, s) = f \end{cases} \tag{2.49}$$

   In particular, $u(\cdot, s) \to f$ *uniformly in* $\mathbb{R}^N$ *as* $t \to s^+$.

Finally, the function $\Gamma^*(x,t;y,s) := \Gamma(y,s;x,t)$ *satisfies dual properties of* (2)–(4) *with respect to the* formal adjoint *of* $\mathcal{L}$*, that is,*

$$\mathcal{L}^* = \textstyle\sum_{i,j=1}^{m_0} a_{ij}(s)\partial_{y_i y_j} - \sum_{k,j=1}^{N} b_{jk} y_k \partial_{y_i} + \partial_s,$$

and thus Γ^* *is the fundamental solution of* $\mathcal{L}^*$.

The precise definition of *solution to the Cauchy problem* (2.49) requires some care, see [3, Definitions 1.2 and 1.3] for the details.

In the particular case when the coefficients a_{ij} of $\mathcal{L}$ are *constant*, the results of the previous theorem apply in a simpler form (see also [11]).

Theorem 2.17. (Fundamental solution for operators with constant coefficients) *Let* $\alpha > 0$ *be fixed, and let* $\mathcal{L}_\alpha$ *be the constant coefficient KFP operator*

$$\mathcal{L}_\alpha u = \alpha \sum_{i=1}^{m_0} \partial^2_{x_i x_i} u + \sum_{k,j=1}^{N} b_{jk} x_k \partial_{x_j} u - \partial_t u. \tag{2.50}$$

Moreover, let Γ_α *be the fundamental solution of* $\mathcal{L}_\alpha$*, whose existence is guaranteed by Theorem* 2.16*. Then, the following facts hold true:*

1. Γ_α *is a* kernel of convolution type*, that is,*

$$\begin{aligned} \Gamma_\alpha(x,t;y,s) &= \Gamma_\alpha\big(x - E(t-s)y, t-s; 0, 0\big) \\ &= \Gamma_\alpha\big((y,s)^{-1} \circ (x,t); 0, 0\big); \end{aligned} \tag{2.51}$$

2. *the matrix* $C(t,s)$ *in* (2.46) *takes the simpler form*

$$C(t,s) = C_0(t-s), \tag{2.52}$$

where $C_0(\tau)$ *is the* $N \times N$ *matrix defined as*

$$C_0(\tau) = \alpha \int_0^\tau E(t-\sigma) \cdot \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix} \cdot E(t-\sigma)^T \, d\sigma \qquad (\tau > 0).$$

Furthermore, one has the 'homogeneity property'

$$C_0(\tau) = D_0(\sqrt{\tau}) C_0(1) D_0(\sqrt{\tau}) \qquad \forall\, \tau > 0. \tag{2.53}$$

In particular, by combining (2.47) *with* (2.52)–(2.53)*, we can write*

$$\begin{aligned} \Gamma_\alpha(x,t;0,0) &= \frac{1}{(4\pi\alpha)^{N/2}\sqrt{\det C_0(t)}} e^{-\frac{1}{4\alpha}\langle C_0(t)^{-1}x, x\rangle} \\ &= \frac{1}{(4\pi\alpha)^{N/2} t^{Q/2} \sqrt{\det C_0(1)}} e^{-\frac{1}{4\alpha}\langle C_0(1)^{-1}\left(D_0\left(\frac{1}{\sqrt{t}}\right)x\right), D_0\left(\frac{1}{\sqrt{t}}\right)x\rangle}. \end{aligned} \tag{2.54}$$

Now we have recalled Theorems 2.16–2.17, we collect in the next theorem some *fine properties* of Γ and of its derivatives which will be extensively used in the sequel. In what follows, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$, we set

$$|\boldsymbol{\alpha}| := \sum_{i=1}^N \alpha_i \quad \text{and} \quad \omega(\boldsymbol{\alpha}) := \sum_{i=1}^N q_i \alpha_i,$$

where the q_i's are the exponents appearing in the dilation $D_0(\lambda)$, see (1.7).

Theorem 2.18. (See [1, Thm.s 3.5 and 3.9]) *Let Γ be as in Theorem 2.16, and let $\nu > 0$ be as in (1.3). Then, the following assertions hold:*

1. *there exists a structural constant $c_1 > 0$ and, for every pair of multi-indices $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in (\mathbb{N} \cup \{0\})^N$, there exists $c = c(\nu, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) > 0$, such that*

$$\begin{aligned} \left| D_x^{\boldsymbol{\alpha}_1} D_y^{\boldsymbol{\alpha}_2} \Gamma(x, t; y, s) \right| &\leq \frac{c}{(t-s)^{\omega(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)/2}} \, \Gamma_{c_1 \nu^{-1}}(x, t; y, s) \\ &\leq \frac{c}{d((x,t),(y,s))^{Q + \omega(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)}}, \end{aligned} \tag{2.55}$$

for every $(x, t), (y, s) \in \mathbb{R}^N$ with $t \neq s$. In particular, we have

$$\left| D_x^{\boldsymbol{\alpha}_1} D_y^{\boldsymbol{\alpha}_2} \Gamma(x, t; y, s) \right| \leq \frac{c}{d((x,t),(y,s))^{Q + \omega(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)}} \quad \forall\, (x,t) \neq (y,s).$$

2. *Let $\eta = (y, s) \in \mathbb{R}^{N+1}$ be fixed, and let $\boldsymbol{\alpha} \in (\mathbb{N} \cup \{0\})^N$ be a multi-index. Then, there exists a constant $c = c(\boldsymbol{\alpha}, \nu) > 0$ such that*

$$|D_x^{\boldsymbol{\alpha}} \Gamma(\xi_1, \eta) - D_x^{\boldsymbol{\alpha}} \Gamma(\xi_2, \eta)| \leq c \frac{d(\xi_1, \xi_2)}{d(\xi_1, \eta)^{Q + \omega(\boldsymbol{\alpha}) + 1}} \tag{2.56}$$

for every $\xi_1 = (x_1, t_1), \xi_2 = (x_2, t_2) \in \mathbb{R}^{N+1}$ such that

$$d(\xi_1, \eta) \geq 4\boldsymbol{\kappa} d(\xi_1, \xi_2) > 0.$$

We conclude this subsection by recalling a representation formula for functions $u \in \mathcal{S}^0(\tau, T)$.

Theorem 2.19. (See [1, Thm. 3.11 and Cor. 3.12]) *Let $T \in \mathbb{R}$ be fixed, and let $\tau < T$. Moreover, let $\mathcal{L}$ be as in (2.45), and let $u \in \mathcal{S}^0(\tau; T)$. Then,*

$$u(x, t) = -\int_{\mathbb{R}^N \times (\tau, t)} \Gamma(x, t; y, s) \mathcal{L}u(y, s)\, dy\, ds \quad \forall\, (x, t) \in S_T. \tag{2.57}$$

Furthermore, given any $1 \leq k \leq m_0$, the function $\partial_{x_k} u$ exists pointwise *on S_T in the classical sense, and for every $(x, t) \in S_T$ we have*

$$\partial_{x_k} u(x, t) = -\int_{\mathbb{R}^N \times (\tau, t)} \partial_{x_k} \Gamma(x, t; y, s) \mathcal{L}u(y, s)\, dy\, ds. \tag{2.58}$$

Starting from the representation formula (2.57), in [1] the Authors proved a representation formula for $\partial^2_{x_i x_j} u$ (when $u \in \mathcal{S}^0(\tau, T)$ and $1 \leq i, j \leq m_0$) under the assumption that $\mathcal{L}u$ is partially Hölder-continuous w.r.t. x, uniformly in t (see, precisely, [1, Cor. 3.12 and Thm. 3.14]). In Sect. 3, we will extend such formulas to all functions $u \in \mathcal{S}^0(\tau, T)$ with $\mathcal{L}u$ only belonging to $\mathcal{D}(S_T)$.

2.4. Interpolation inequalities

We conclude this preliminary section by stating some interpolation inequalities, established in [1], which will be exploited in the proof of Theorem 1.6.

Theorem 2.20. (See [1, Thm. 4.2]) *Let $T \in \mathbb{R}$ be arbitrarily fixed, and let $u \in \mathcal{S}^0(S_T)$. Then, for every $\alpha \in (0,1)$, $\xi \in S_T$ and $r > 0$ we have*

$$u,\ \partial_{x_k} u \in C^\alpha(B_r^T(\xi)) \quad \forall\ 1 \le k \le m_0,$$

where we set:

$$B_r^T(\xi) = B_r(\xi) \cap S_T.$$

Moreover, the following interpolation inequality *holds:* for every $\alpha \in (0,1)$ and every $r > 0$ there exist constants $c > 0$ and $\gamma > 1$ such that

$$\begin{aligned} &\sum_{h=1}^{m_0} \|\partial_{x_h} u\|_{C^\alpha(B_r^T(\xi))} + \|u\|_{C^\alpha(B_r^T(\xi))} \\ &\quad \le \varepsilon \Bigg\{ \sum_{h,k=1}^{m_0} \|\partial^2_{x_k x_h} u\|_{L^\infty(B_{4r}^T(\xi))} + \|Yu\|_{L^\infty(B_{4r}^T(\xi))} \Bigg\} \\ &\qquad + \frac{c}{\varepsilon^\gamma} \|u\|_{L^\infty(B_{4r}^T(\xi))}. \end{aligned} \tag{2.59}$$

and this estimate holds for every $\varepsilon \in (0,1)$, $\xi \in S_T$ and $u \in \mathcal{S}^0(S_T)$. *We stress that the constant c depends on r and α, but is independent of ε, ξ and u.*

Remark 2.21. We explicitly highlight, for a future reference, the following easy yet important fact: *if $\Omega \subseteq \mathbb{R}^{N+1}$ is an arbitrary open set and if $f \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then $f \in \mathcal{D}(\Omega)$ and we have the estimate*

$$\begin{aligned} \omega_{f,\Omega}(r) &= \sup_{\substack{(x,t),(y,t)\in\Omega \\ \|x-y\|\le r}} |f(x,t) - f(y,t)| \\ &\le |f|_{C^\alpha(\Omega)} \cdot \sup_{\substack{(x,t),(y,t)\in\Omega \\ \|x-y\|\le r}} d\big((x,t),(y,t)\big)^\alpha \le r^\alpha |f|_{C^\alpha(\Omega)}, \end{aligned}$$

where we have also used (2.1). As a consequence, we easily deduce

$$\mathcal{M}_{f,\Omega}(r) \le c r^\alpha |f|_{C^\alpha(\Omega)}.$$

3. Operators with coefficients only depending on t

In this section we establish some 'weaker' versions of Theorems 1.6–1.7 for KFP operators with coefficients a_{ij} only depending on t; we will use these results as a crucial tool to prove Theorems 1.6–1.7 in Sect. 4. Throughout what follows, we tacitly understand that $\mathcal{L}$ is as in (2.45), and that the structural assumptions (H1)–(H2) stated in the Introduction are satisfied (without the need of repeat it); moreover, Γ denotes the fundamental solution of $\mathcal{L}$, as in Theorem 2.16.

To begin with, we extend to all functions $u \in \mathcal{S}^0(\tau, T)$ with $\mathcal{L}u \in \mathcal{D}(S_T)$ the representation formula for $\partial^2_{x_i x_j} u$ (where $1 \le i, j \le m_0$) proved in [1, Thm. 3.14] under the more restrictive assumption that $\mathcal{L}u$ is *partially Hölder continuous w.r.t.* x. In this direction, a first key tool is the following proposition.

Proposition 3.1. *There exist structural constants* $c, \mu > 0$ *such that, for every fixed* $T \in \mathbb{R}$, *every* $f \in \mathcal{D}(S_T)$, $x \in \mathbb{R}^N$ *and* $\tau < t < T$, *one has*

$$\begin{aligned} &\int_{\mathbb{R}^N \times (\tau,t)} |\partial^2_{x_i x_j} \Gamma(x,t;y,s)| \cdot \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy\, ds \\ &\qquad \le c\, \mathcal{U}^{\mu}_{f,S_T}(\sqrt{t-\tau}), \end{aligned} \tag{3.1}$$

where $\mathcal{U}^{\mu}_{f,S_T}$ *is as in* (1.18). *In particular, we have*

$$\begin{aligned} &\int_{\mathbb{R}^N \times (t-\varepsilon,t)} |\partial^2_{x_i x_j} \Gamma(x,t;y,s)| \cdot \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy\, ds \to 0 \\ &\qquad \text{uniformly w.r.t. } (x,t) \in \mathbb{R}^{N+1} \text{ as } \varepsilon \to 0^+. \end{aligned} \tag{3.2}$$

Proof. The proof of this proposition is similar to that of [1, Prop. 3.13]; we sketch it here for the sake of completeness, but we omit the details. First of all, by combining estimate (2.55) with (2.51)–(2.54), we have

$$\begin{aligned} &\int_{\mathbb{R}^N \times (\tau,t)} |\partial^2_{x_i x_j} \Gamma(x,t;y,s)| \cdot \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy\, ds \\ &\quad \le c \int_{\mathbb{R}^N \times (\tau,t)} \frac{e^{-\mu \left|D_0\left(\frac{1}{\sqrt{t-s}}\right)(E(s-t)x-y)\right|^2}}{(t-s)^{Q/2+1}} \cdot \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy\, ds \\ &\quad \le c \int_\tau^t \frac{1}{(t-s)^{Q/2+1}} \cdot \mathcal{I}_{x,t}(s)\, ds = (\bigstar), \end{aligned}$$

where we have introduced the notation

$$\mathcal{I}_{x,t}(s) = \int_{\mathbb{R}^N} e^{-\mu \left|D_0\left(\frac{1}{\sqrt{t-s}}\right)(E(s-t)x-y)\right|^2} \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy,$$

and $c, \mu > 0$ are structural constants. We explicitly mention that, in the above estimate, we have also used (1.9) and the non-singularity of $E(1)$. Then, by performing in the integral $\mathcal{I}_{x,t}(s)$ the change of variables

$$y = E(s-t)x - D_0(\sqrt{t-s})z,$$

and by exploiting the 1-homogeneity of $\|\cdot\|$, we obtain

$$\begin{aligned}(\bigstar) &= c\int_\tau^t \frac{1}{t-s}\bigg(\int_{\mathbb{R}^N} e^{-\mu|z|^2}\omega_{f,S_T}(\sqrt{t-s}\|z\|)\,dz\bigg)ds \\ &= c\int_{\mathbb{R}^N} e^{-\mu|z|^2}\bigg(\int_\tau^t \frac{\omega_{f,S_T}(\sqrt{t-s}\|z\|)}{t-s}\,ds\bigg)dz \\ &= c\int_{\mathbb{R}^N} e^{-\mu|z|^2}\bigg(\int_0^{\sqrt{t-\tau}\|z\|} \frac{\omega_{f,S_T}(\sigma)}{\sigma}\,d\sigma\bigg)dz = c\,\mathcal{U}^\mu_{f,S_T}(\sqrt{t-\tau}).\end{aligned}$$

Since the constants $c, \mu > 0$ only depend on ν, this is exactly the desired (3.1). As regards (3.2), it is a direct consequence of (3.1) and Lemma 2.12. □

Thanks to Proposition 3.1, we can now prove the following theorem.

Theorem 3.2. *For $T > t > \tau > -\infty$, let $u \in \mathcal{S}^0(\tau; T)$ be such that $\mathcal{L}u \in \mathcal{D}(S_T)$. Then, for every $1 \le i, j \le m_0$ the second-order derivatives $\partial^2_{x_i x_j} u$* exist pointwise *on S_T in the classical sense, and for every $(x,t) \in S_T$ we have*

$$\partial^2_{x_i x_j} u(x,t) = \int_{\mathbb{R}^N\times(\tau,t)} \partial^2_{x_i x_j}\Gamma(x,t;y,s)\big[\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u(y,s)\big]\,dy\,ds. \tag{3.3}$$

Proof. The proof of this result is essentially analogous to that of [1, Thm. 3.14], but we use Proposition 3.1 in place of [1, Prop. 3.13]. For the sake of completeness we sketch the argument, but we refer to [1] for the details. First of all we observe that, owing to Proposition 3.1 (and taking into account the very definition of ω_{f,S_T} in Definition 1.2), we have

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^N\times(\tau,t)} |\partial^2_{x_i x_j}\Gamma(x,t;y,s)|\cdot|\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u(y,s)|\,dy\,ds\bigg| \\ &\quad\le \bigg|\int_{\mathbb{R}^N\times(\tau,t)} |\partial^2_{x_i x_j}\Gamma(x,t;y,s)|\cdot\omega_{\mathcal{L}u,S_T}(\|E(s-t)x-y\|)\,dy\,ds\bigg| \\ &\quad\le c\,\mathcal{U}^\mu_{\mathcal{L}u,S_T}(\sqrt{|t-\tau|}) \quad \forall\ (x,t)\in S_T\end{aligned}$$

(where $c, \mu > 0$ only depend on ν); hence, the function

$$g(x,t) := \int_{\mathbb{R}^N\times(\tau,t)} \partial^2_{x_i x_j}\Gamma(x,t;y,s)\big[\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u(y,s)\big]\,dy\,ds$$

is well-defined on S_T. We then turn to prove that $\partial^2_{x_i x_j} u = g$ pointwise in S_T by an approximation argument. To this end, we fix $0 < \varepsilon \ll 1$ and we define

$$v_\varepsilon(x,t) := -\int_{\mathbb{R}^N\times(\tau,t-\varepsilon)} \partial_{x_j}\Gamma(x,t;\cdot)\,\mathcal{L}u\,dy\,ds.$$

Now, using (2.58) and taking into account the regularity of Γ, we see that

(i) $v_\varepsilon \in C(S_T)$ and $v_\varepsilon \to \partial_{x_j} u$ pointwise in S_T as $\varepsilon \to 0^+$;
(ii) v_ε is continuously differentiable w.r.t. x_i on S_T, and

$$\partial_{x_i} v_\varepsilon(x,t) = \int_{\mathbb{R}^N \times (\tau, t-\varepsilon)} \partial^2_{x_i x_j} \Gamma(x,t;y,s) \times \\ \times \big[\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u(y,s)\big]\, dy\, ds \quad \forall\, (x,t) \in S_T.$$

We explicitly stress that, in computing $\partial_{x_i} v_\varepsilon$, we have also used (2.48). On the other hand, again by Proposition 3.1, we have

$$\begin{aligned}
&|\partial_{x_i} v_\varepsilon(x,t) - g(x,t)| \\
&\quad = \int_{\mathbb{R}^N \times (t-\varepsilon, t)} |\partial^2_{x_i x_j} \Gamma(x,t;\cdot)|\, |\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u|\, dy\, ds \\
&\quad \le \int_{\mathbb{R}^N \times (t-\varepsilon, t)} |\partial^2_{x_i x_j} \Gamma(x,t;\cdot)| \ \cdot \omega_{\mathcal{L}u, S_T}(\|E(s-t)x - y\|)\, dy\, ds \\
&\quad \le c\, \mathcal{U}^\mu_{\mathcal{L}u, S_T}(\sqrt{\varepsilon}) \qquad \text{for every } (x,t) \in S_T,
\end{aligned}$$

and this shows that $\partial_{x_i} v_\varepsilon \to g$ uniformly on S_T as $\varepsilon \to 0^+$. Gathering these facts, by standard results, we then conclude that there exists

$$\partial^2_{x_i x_j} u = \partial_{x_i}(\partial_{x_j} u) = g \quad \text{pointwise in } S_T.$$

This ends the proof. □

Now that we have established the representation formula (3.3), we can prove the announced weaker version of Theorem 1.6 for KFP operators with coefficients a_{ij} only depending on t. Actually, we will deduce this result from the following general theorem, which will be used as a key tool in the next section.

Theorem 3.3. [Singular integrals and Dini-continuous functions] *For every fixed $T \in \mathbb{R}$ and $-\infty < \tau < T$, let us introduce the function space*

$$\mathcal{D}(\tau; T) := \{f \in \mathcal{D}(S_T) : \ f(x,t) = 0 \, \textit{for every } t \le \tau\},$$

and define, on this space $\mathcal{D}(\tau; T)$, the linear operator

$$f \mapsto T_{ij} f(x,t) := \int_{\mathbb{R}^N \times (\tau, t)} \partial^2_{x_i x_j} \Gamma(x,t;y,s) \big[f(E(s-t)x,s) - f(y,s)\big]\, dy\, ds.$$

Then, there exist structural constants $c, \mu > 0$ such that

$$\|T_{ij} f\|_{L^\infty(S_T)} \le c\, \mathcal{U}^\mu_{f,S_T}(\sqrt{T-\tau}) \tag{3.4}$$

$$\omega_{T_{ij} f}(r) \le c\, \mathcal{M}_{f,S_T}(cr) \quad \forall\, r > 0. \tag{3.5}$$

Here, $\mathcal{M}_{f,S_T}$ is the function defined in (1.16).

Proof. The proof of this theorem is similar to that of [1, Thm. 3.17], where the Authors deal with the particular case of functions $f \in \mathcal{D}(S_T)$ such that

$$\omega_{f,S_T}(r) \leq c\, r^\alpha \quad \text{for some } \alpha \in (0,1)$$

(that is, f is partially Hölder continuous w.r.t. x, uniformly in t). We then limit ourselves to sketch the argument exploited in [1], but we highlight how the main estimates have to be modified in our more general context. Let $f \in \mathcal{D}(\tau; T)$ be arbitrarily fixed. Since $f(\cdot, t) \equiv 0$ for all $t \leq \tau$, we clearly have $T_{ij} f(x,t) = 0$ for all $x \in \mathbb{R}^N$ and $t \leq \tau$. Thus, it is readily seen that

$$\|T_{ij} f\|_{L^\infty(S_T)} = \|T_{ij} f\|_{L^\infty(\Omega)} \quad \text{and} \quad \omega_{T_{ij} f, S_T} \equiv \omega_{T_{ij} f, \Omega}, \tag{3.6}$$

where we have set $\Omega = \mathbb{R}^N \times (\tau, T)$. On account of (3.6), to prove (3.4)–(3.5) it then suffices to study the function $T_{ij} f(x,t)$ only for $(x,t) \in \Omega$. As regards (3.4) we observe that, by Proposition 3.1, we have

$$\begin{aligned}
|T_{ij} f(x,t)| &\leq \int_{\mathbb{R}^N \times (\tau,t)} |\partial^2_{x_i x_j} \Gamma(x,t;y,s)| \cdot |f(E(s-t)x, s) - f(y,s)|\, dy\, ds \\
&\leq \int_{\mathbb{R}^N \times (\tau,t)} |\partial^2_{x_i x_j} \Gamma(x,t;y,s)| \cdot \omega_{f,S_T}(\|E(s-t)x - y\|)\, dy\, ds \\
&\leq c\, \mathcal{U}^\mu_{f,S_T}(\sqrt{t-\tau}) \leq c\, \mathcal{U}^\mu_{f,S_T}(\sqrt{T-\tau}) \quad \forall\, (x,t) \in \Omega,
\end{aligned}$$

where $c, \mu > 0$ are structural constants. Hence,

$$\|T_{ij} f\|_{L^\infty(S_T)} \leq c\, \mathcal{U}^\mu_{f,S_T}(\sqrt{T-\tau}).$$

We then turn to prove (3.5). To begin with, we arbitrarily fix $r > 0$ and we let $\xi_1 = (x_1, t)$, $\xi_2 = (x_2, t) \in \Omega$ be such that $d(\xi_1, \xi_2) = \|x_1 - x_2\| \leq r$. Using the compact notation $\eta = (y,s)$, we write

$$\begin{aligned}
&T_{ij} f(x_1,t) - T_{ij} f(x_2,t) \\
&\quad = \int_{\mathbb{R}^N \times (\tau,t)} \Big\{ \partial^2_{x_i x_j} \Gamma(x_1,t;y,s) \big[f(E(s-t)x_1, s) - f(y,s) \big] \\
&\qquad - \partial^2_{x_i x_j} \Gamma(x_2,t;y,s) \big[f(E(s-t)x_2, s) - f(y,s) \big] \Big\}\, dy\, ds \\
&\quad = \int_{\{\eta:\, d(\xi_2,\eta) \geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \{\cdots\}\, dy\, ds \\
&\qquad + \int_{\{\eta:\, d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \{\cdots\}\, dy\, ds \\
&\quad =: \mathrm{A}_1 + \mathrm{A}_2,
\end{aligned} \tag{3.7}$$

where $\boldsymbol{\kappa} > 0$ is as in (1.12)–(1.13). We then turn to estimate A_1 and A_2.

- Estimate of A_1. To begin with, we write A_1 as follows:

$$\begin{aligned}\mathrm{A}_1 = &\int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \Big\{\big[f(E(s-t)x_1,s) - f(y,s)\big] \\ &\times \big[\partial^2_{x_ix_j}\Gamma(x_1,t;y,s) - \partial^2_{x_ix_j}\Gamma(x_2,t;y,s)\big]\Big\}\,dy\,ds \\ &+ \int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \Big\{\partial^2_{x_ix_j}\Gamma(x_2,t;y,s) \\ &\times \big[f(E(s-t)x_1,s) - f(E(s-t)x_2,s)\big]\Big\}\,dy\,ds \\ =:&\ \mathrm{A}_{11} + \mathrm{A}_{12}.\end{aligned}$$

- *Estimate of* A_{11}. By using the mean value inequalities (2.56) in Theorem 2.18, together with Lemma 2.1, (1.13) and the expression of d in (2.1), we have

$$\begin{aligned}|\mathrm{A}_{11}| &\leq c\int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{d(\xi_2,\xi_1)}{d(\xi_2,\eta)^{Q+3}}\cdot \omega_{f,S_T}(\|E(s-t)x_1-y\|)\,dyds \\ &\leq c\,d(\xi_2,\xi_1)\int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(d(\eta,\xi_1))}{d(\xi_2,\eta)^{Q+3}}\,dyds \\ &\leq c\,d(\xi_2,\xi_1)\int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(c\,d(\xi_2,\eta))}{d(\xi_2,\eta)^{Q+3}}\,dyds.\end{aligned}$$

From this, by applying Lemma 2.15 (and since $f\in\mathcal{D}(S_T)$), we obtain

$$|\mathrm{A}_{11}| \leq c\,d(\xi_2,\xi_1)\int_{8\boldsymbol{\kappa} d(\xi_2,\xi_1)}^{\infty}\frac{\omega_{f,S_T}(cs)}{s^2}\,ds \leq cr\int_{cr}^{\infty}\frac{\omega_{f,S_T}(s)}{s^2}\,ds, \tag{3.8}$$

where $c>0$ is a structural constant. We explicitly mention that, in the above estimate, we have also used the monotonicity of the map

$$r\mapsto \mathcal{M}_{2,f,S_T}(r) = r\int_r^{\infty}\frac{\omega_{f,S_T}(s)}{s^2}\,ds$$

proved in Lemma 2.6, jointly with the fact that

$$d(\xi_2,\xi_1) = \|x_1-x_2\| \leq r.$$

- *Estimate of* A_{12}. Arguing as in [1], by using Lemma 2.3 we have

$$\begin{aligned}|\mathrm{A}_{12}| &\leq \int_\tau^t \big|f(E(s-t)x_1,s) - f(E(s-t)x_2,s)\big|\cdot\mathcal{J}(s)\,ds \\ &\leq \int_\tau^t \omega_{f,S_T}(\|E(s-t)(x_1-x_2)\|)\cdot\mathcal{J}(s)\,ds \\ &\leq \int_\tau^t \omega_{f,S_T}\big(c(\|x_1-x_2\|+\sqrt{t-s})\big)\cdot\mathcal{J}(s)\,ds,\end{aligned} \tag{3.9}$$

where $c>0$ is a structural constant and

$$\mathcal{J}(s) := \left|\int_{\{y\in\mathbb{R}^N:\, d(\xi_2,(y,s))\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \partial^2_{x_ix_j}\Gamma(x_2,t;y,s)\,dy\right|.$$

From this, since ω_{f,S_T} shares with the map $r \mapsto r^\alpha$ the same monotonicity, we can proceed *exactly* as in the proof of [1, Thm. 3.17], obtaining

$$|A_{12}| \le c\,\omega_{f,S_T}(c\|x_1 - x_2\|) \le c\,\omega_{f,S_T}(cr). \tag{3.10}$$

for a suitable structural constant $c > 0$.

Summing up, by combining (3.8) with (3.10), we conclude that

$$|\mathrm{A}_1| \le c\left\{\omega_{f,S_T}(cr) + cr\int_{cr}^{\infty}\frac{\omega_{f,S_T}(s)}{s^2}\,ds\right\}, \tag{3.11}$$

for a suitable structural constant $c > 0$.

- ESTIMATE OF A_2. First of all, we write

$$|\mathrm{A}_2| \le \mathrm{A}_{21} + \mathrm{A}_{22}, \tag{3.12}$$

where, for $k = 1, 2$, we have introduced the notation

$$\begin{aligned}\mathrm{A}_{2k} := \int_{\{\eta:\, d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} & |\partial^2_{x_i x_j}\Gamma(x_k, t; y, s)| \\ & \times \omega_{f,S_T}(\|E(s-t)x_k - y\|)\,dy\,ds.\end{aligned}$$

We then proceed by estimating the two integrals $\mathrm{A}_{21},\ \mathrm{A}_{22}$ separately.

- *Estimate of* A_{21}. By exploiting the estimates for $\partial^2_{x_i x_j}\Gamma$ in Theorem 2.18-(1), together with (1.12)–(1.13) and the expression of d, we have

$$\begin{aligned}\mathrm{A}_{21} &\le c\int_{\{\eta:\, d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(\|E(s-t)x_1 - y\|)}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds \\ &\le c\int_{\{\eta:\, d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(d(\eta,\xi_1))}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds \\ &\le c\int_{\{\eta:\, d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(\boldsymbol{\kappa} d(\xi_1,\eta))}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds \\ &\quad \big(\text{since } d(\xi_2,\eta) < 4\boldsymbol{\kappa} d(\xi_2,\xi_1) \Rightarrow d(\xi_1,\eta) < c\,d(\xi_2,\xi_1)\big) \\ &\le c\int_{\{\eta:\, d(\xi_1,\eta) < c d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(\boldsymbol{\kappa} d(\xi_1,\eta))}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds,\end{aligned}$$

where $c > 0$ is a suitable structural constant. From this, using once again Lemma 2.15, we obtain

$$\mathrm{A}_{21} \le c\int_0^{2cd(\xi_2,\xi_1)}\frac{\omega_{f,S_T}(\boldsymbol{\kappa} s)}{s}\,ds \le c\int_0^{cr}\frac{\omega_{f,S_T}(s)}{s}\,ds. \tag{3.13}$$

- *Estimate of* A_{22}. By proceeding exactly as in the estimate of A_{21} (but without the need of enlarging the domain of integration), we obtain

$$\begin{aligned}
\mathrm{A}_{22} &\leq c \int_{\{\eta:\, d(\xi_2,\eta)<4\kappa d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(\|E(s-t)x_2-y\|)}{d(\xi_2,\eta)^{Q+2}}\,dy\,ds \\
&\leq c \int_{\{\eta:\, d(\xi_2,\eta)<4\kappa d(\xi_2,\xi_1)\}} \frac{\omega_{f,S_T}(\kappa d(\xi_2,\eta))}{d(\xi_2,\eta)^{Q+2}}\,dy\,ds \\
&\leq c \int_0^{8\kappa d(\xi_2,\xi_1)} \frac{\omega_{f,S_T}(\kappa s)}{s}\,ds \leq c \int_0^{cr} \frac{\omega_{f,S_T}(s)}{s}\,ds.
\end{aligned} \tag{3.14}$$

Summing up, by combining (3.13)–(3.14) with (3.12), we conclude that

$$|\mathrm{A}_2| \leq c \int_0^{cr} \frac{\omega_{f,S_T}(s)}{s}\,ds, \tag{3.15}$$

where $c > 0$ is a suitable structural constant.

Now we have estimated A_1 and A_2, we are ready to complete the proof: in fact, gathering (3.11)–(3.15), and recalling (3.7), we conclude that

$$\begin{aligned}
|T_{ij}f(x_1,t) - T_{ij}f(x_2,t)| &\leq |\mathrm{A}_1| + |\mathrm{A}_2| \\
&\leq c\left\{\omega_{f,S_T}(cr) + \int_0^{cr} \frac{\omega_{f,S_T}(s)}{s}\,ds + cr\int_{cr}^{\infty} \frac{\omega_{f,S_T}(s)}{s^2}\,ds\right\} \\
&= c\,\mathcal{M}_{f,S_T}(cr) \quad \forall\ (x_1,t),\ (x_2,t) \in \Omega \text{ with } \|x_1-x_2\| \leq r,
\end{aligned}$$

from which we readily obtain the desired (3.5). □

Thanks to Theorem 3.3, we can finally prove the following theorem.

Theorem 3.4. [Moduli of continuity of derivatives] *Let $T > \tau > -\infty$. Then, there exist structural constants $c, \mu > 0$, such that*

$$\begin{aligned}
&\sum_{i,j=1}^{m_0} \|\partial^2_{x_ix_j}u\|_{L^\infty(S_T)} \leq c\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{T-\tau}), \\
&\omega_{\partial^2_{x_ix_j}u,S_T}(r) \leq c\,\mathcal{M}_{\mathcal{L}u,S_T}(cr) \quad \forall\ r > 0.
\end{aligned} \tag{3.16}$$

for every $u \in \mathcal{S}^0(\tau,T)$ with $\mathcal{L}u \in \mathcal{D}(S_T)$. Moreover, we have

$$\begin{aligned}
&\|Yu\|_{L^\infty(S_T)} \leq c\left(\|\mathcal{L}u\|_{L^\infty(S_T)} + \mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{T-\tau})\right) \\
&\omega_{Yu,S_T}(r) \leq c\,\mathcal{M}_{\mathcal{L}u,S_T}(cr) \quad \forall\ r > 0.
\end{aligned} \tag{3.17}$$

In particular, if $\mathcal{L}u \in \mathcal{D}_{\log}(S_T)$ we have $\partial^2_{x_ix_j}u$, $Yu \in \mathcal{D}(S_T)$.

Proof. Let $u \in \mathcal{S}^0(\tau;T)$ be such that $\mathcal{L}u \in \mathcal{D}(S_T)$. By applying the representation formula (3.3), we can write

$$\begin{aligned}
\partial^2_{x_ix_j}u(x,t) &= \int_{\mathbb{R}^N\times(\tau,t)} \partial^2_{x_ix_j}\Gamma(x,t;y,s)\cdot\left[\mathcal{L}u(E(s-t)x,s) - \mathcal{L}u(y,s)\right]dy\,ds \\
&= T_{ij}(\mathcal{L}u)(x,t) \qquad \text{for every } (x,t) \in S_T \text{ and } 1 \leq i,j \leq m_0,
\end{aligned}$$

where T_{ij} is as in Theorem 3.3. Then, from (3.4)–(3.5) we obtain (3.16). On the other hand, using the definition of $\mathcal{L}$, and recalling that the coefficients $a_{ij}(\cdot)$ are bounded and *independent of* x, from (3.16) we also get

$$\begin{aligned}\|Yu\|_{L^\infty(S_T)} &= \Big\| \mathcal{L}u - \sum_{i,j=1}^{m_0} a_{ij}\, \partial^2_{x_i x_j} u \Big\|_{L^\infty(S_T)} \\ &\leq c\big(\|\mathcal{L}u\|_{L^\infty(S_T)} + \mathcal{U}^\mu_{\mathcal{L}u,S_T}(\sqrt{T-\tau})\big);\end{aligned}$$

analogously, if A is as in (1.20), we obtain

$$\omega_{Yu,S_T}(r) \leq \omega_{\mathcal{L}u}(r) + A \sum_{i,j=1}^{m_0} \omega_{\partial^2_{x_i x_j} u}(r) \leq c\, \mathcal{M}_{\mathcal{L}u}(cr).$$

This is precisely (3.17). Finally, the Dini continuity of the functions $\partial^2_{x_i x_j} u$, Yu, under the additional assumption $\mathcal{L}u \in \mathcal{D}_{\log}(S_T)$, immediately follows from Proposition 2.8. □

We end this section with a weaker version of Theorem 1.7 for operators with coefficients only depending on t; we will use this result to prove Theorem 1.7.

Theorem 3.5. (Continuity estimates in space-time) *Let $\mathcal{L}$ be as in* (2.45)*, and let $T > \tau > -\infty$. Moreover, let $K \subseteq \mathbb{R}^N$ be a* compact set.

Then, there exist a structural constant $\mu > 0$ and a constant $c(K, \tau, T) > 0$ such that, for every $u \in \mathcal{S}^0(\tau; T)$ such that $\mathcal{L}u \in \mathcal{D}(S_T)$, one has

$$\begin{aligned}&|\partial^2_{x_i x_j} u(x_1, t_1) - \partial^2_{x_i x_j} u(x_2, t_2)| \\ &\quad \leq c\big\{\mathcal{M}_{\mathcal{L}u,S_T}\big(c(d((x_1, t_1), (x_2, t_2)) + |t_1 - t_2|^{1/q_N})\big) \\ &\qquad + \mathcal{U}^\mu_{\mathcal{L}u,S_T}(\sqrt{|t_2 - t_1|})\big\}\end{aligned} \tag{3.18}$$

for every $1 \leq i, j \leq m_0$ and $(x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$.

Here, $\mathcal{U}^\mu_{\mathcal{L}u,S_T}$ is as in (1.18) *and $\mathcal{M}_{\mathcal{L}u,S_T}$ is as in* (1.16)*; moreover, $q_N \geq 3$ is the largest exponent in the dilations $D_0(\lambda)$, see* (1.7).

In particular, from (3.18) *we deduce that the derivatives $\partial^2_{x_i x_j} u$ are locally uniformly continuous in the joint variables (x, t).*

Proof. Let $u \in \mathcal{S}^0(\tau; T)$ be such that $\mathcal{L}u \in \mathcal{D}(S_T)$. To prove (3.18) we first observe that, owing to Theorem 3.4, for every $(x_1, t), (x_2, t) \in S_T$ we have

$$\begin{aligned}|\partial^2_{x_i x_j} u(x_1, t) - \partial^2_{x_i x_j} u(x_2, t)| &\leq \omega_{\partial^2_{x_i x_j} u}(\|x_1 - x_2\|) \\ &\leq c\, \mathcal{M}_{\mathcal{L}u,S_T}(c\|x_1 - x_2\|)\end{aligned} \tag{3.19}$$

where $c > 0$ is a structural constant. As a consequence of (3.19), and taking into account Lemma 2.3 , to prove (3.18) it suffices to show that

$$\begin{aligned}&|\partial^2_{x_i x_j} u(x, t_1) - \partial^2_{x_i x_j} u(x, t_2)| \\ &\quad \leq c\,\big\{\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1 - t_2|^{1/q_N}) + \mathcal{U}^\mu_{\mathcal{L}u,S_T}(\sqrt{|t_1 - t_2|})\big\},\end{aligned} \tag{3.20}$$

for every $(x,t_1),(x,t_2)\in K\times[\tau,T]$, where $c>0$ is a constant independent of u (but possibly depending on K,τ,T), while $\mu>0$ is a structural constant. In fact, once (3.20) has been established, by (3.19)–(3.20) we get

$$\begin{aligned}
&|\partial^2_{x_ix_j}u(x_1,t_1)-\partial^2_{x_ix_j}u(x_2,t_2)|\\
&\quad\le |\partial^2_{x_ix_j}u(x_1,t_1)-\partial^2_{x_ix_j}u(x_2,t_1)|+|\partial^2_{x_ix_j}u(x_2,t_1)-\partial^2_{x_ix_j}u(x_2,t_2)|\\
&\quad\le c\left\{\mathcal{M}_{\mathcal{L}u,S_T}(c\|x_1-x_2\|)+\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N})\right.\\
&\qquad\left.+\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\}\\
&\qquad\text{(by the explicit expression of } d,\ \text{see (2.1))}\\
&\quad= c\left\{\mathcal{M}_{\mathcal{L}u,S_T}\big(cd((x_1,t_1),(x_2,t_1))\big)+\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N})\right.\\
&\qquad\left.+\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\}=:(\bigstar);
\end{aligned}$$

from this, using the quasi-triangle inequality (1.12) jointly with Lemma 2.3, and recalling that $\mathcal{M}_{\mathcal{L}u,S_T}$ is non-decreasing, see Proposition 2.8, we obtain

$$\begin{aligned}
(\bigstar)&\le c\left\{\mathcal{M}_{\mathcal{L}u,S_T}\big(c(d((x_1,t_1),(x_2,t_2))+d((x_2,t_1),(x_2,t_2)))\big)\right.\\
&\quad\left.+\,\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N})+\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\}\\
&= c\left\{\mathcal{M}_{\mathcal{L}u,S_T}\big(c(d((x_1,t_1),(x_2,t_2))+\|x_2-E(t_1-t_2)x_2\|+\sqrt{|t_1-t_2|})\big)\right.\\
&\quad\left.+\,\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N})+\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\}\\
&\quad\text{(since } |t_1-t_2|\le T-\tau \text{ and } q_N\ge 3)\\
&\le c\left\{\mathcal{M}_{\mathcal{L}u,S_T}\big(c(d((x_1,t_1),(x_2,t_2))+|t_1-t_2|^{1/q_N})\big)\right.\\
&\quad\left.+\,\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N})+\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\}\\
&\le c\left\{\mathcal{M}_{\mathcal{L}u,S_T}\big(c(d((x_1,t_1),(x_2,t_2))+|t_1-t_2|^{1/q_N})\big)\right.\\
&\quad\left.+\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\right\},
\end{aligned}$$

which is exactly (3.18). Hence, we turn to prove (3.20). This can be done adapting several computations already exploited in the proof of Theorem 3.4. We will point out just the relevant differences.

To begin with, we fix $\xi_1=(x,t_1)$, $\xi_2=(x,t_2)\in K\times[\tau,T]$ and we exploit the representation formula (3.3): assuming, to fix ideas, that $t_2\ge t_1$ (and using the compact notation $\eta=(y,s)$), we can write

$$\begin{aligned}
&\partial^2_{x_ix_j}u(x,t_1)-\partial^2_{x_ix_j}u(x,t_2)\\
&\quad=\int_{\mathbb{R}^N\times(\tau,t_1)}\Big\{\partial^2_{x_ix_j}\Gamma(x,t_1;y,s)\big[\mathcal{L}u(E(s-t_1)x,s)-\mathcal{L}u(y,s)\big]\\
&\qquad-\partial^2_{x_ix_j}\Gamma(x,t_2;y,s)\big[\mathcal{L}u(E(s-t_2)x,s)-\mathcal{L}u(y,s)\big]\Big\}\,dy\,ds\\
&\qquad-\int_{\mathbb{R}^N\times(t_1,t_2)}\partial^2_{x_ix_j}\Gamma(x,t_2;y,s)\big[\mathcal{L}u(E(s-t_2)x,s)-\mathcal{L}u(y,s)\big]\,dy\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \{\cdots\}\, dy\, ds \\
&\quad + \int_{\{\eta:\, d(\xi_2,\eta)< 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \{\cdots\}\, dy\, ds \\
&\quad - \int_{\mathbb{R}^N\times(t_1,t_2)} \{\cdots\}\, dy\, ds \\
&=: \mathrm{A}_1 + \mathrm{A}_2 - \mathrm{A}_3, \qquad (3.21)
\end{aligned}
$$

where $\boldsymbol{\kappa} > 0$ is as in (1.12)–(1.13) We now turn to estimate A_1, A_2 and A_3.

- ESTIMATE OF A_1. To begin with, we write A_1 as follows:

$$
\begin{aligned}
\mathrm{A}_1 = &\int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \Big\{\big[\mathcal{L}u(E(s-t_1)x,s) - \mathcal{L}u(y,s)\big]\times \\
&\quad \times \big[\partial^2_{x_i x_j}\Gamma(x,t_1;y,s) - \partial^2_{x_i x_j}\Gamma(x,t_2;y,s)\big]\Big\}\, dy\, ds \\
&\quad + \int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \Big\{\partial^2_{x_i x_j}\Gamma(x,t_2;y,s)\times \\
&\quad \times \big[\mathcal{L}u(E(s-t_1)x,s) - \mathcal{L}u(E(s-t_2)x,s)\big]\Big\}\, dy\, ds \\
&=: \mathrm{A}_{11} + \mathrm{A}_{12}.
\end{aligned}
$$

We then turn to estimate A_{11} and A_{12} separately.

- *Estimate of* A_{11}. First of all, by proceeding *exactly* as in the estimate of A_{11} in the proof of Theorem 3.4, we get the following estimate

$$
\begin{aligned}
|\mathrm{A}_{11}| &\leq c \int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{d(\xi_2,\xi_1)}{d(\xi_2,\eta)^{Q+3}} \cdot \omega_{\mathcal{L}u,S_T}(\|E(s-t_1)x-y\|)\, dyds \\
&\leq c\, d(\xi_2,\xi_1) \int_{\{\eta:\, d(\xi_2,\eta)\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{\mathcal{L}u,S_T}(c d(\xi_2,\eta))}{d(\xi_2,\eta)^{Q+3}}\, dyds \\
&\leq c d(\xi_2,\xi_1) \int_{cd(\xi_2,\xi_1)}^{\infty} \frac{\omega_{\mathcal{L}u,S_T}(s)}{s^2}\, ds,
\end{aligned}
$$

where $c > 0$ is a structural constant. On the other hand, by exploiting Lemma 2.3 (and since $t_1, t_2 \in [\tau, T]$), we have

$$
d(\xi_2,\xi_1) = \|x - E(t_2-t_1)x\| + \sqrt{|t_2-t_1|} \leq c\, |t_2-t_1|^{1/q_N}, \qquad (3.22)
$$

where $c > 0$ is a constant depending on K, τ, T. Hence, we obtain

$$
|\mathrm{A}_{11}| \leq c|t_1-t_2|^{1/q_N} \int_{c|t_1-t_2|^{1/q_N}}^{\infty} \frac{\omega_{f,S_T}(s)}{s^2}\, ds. \qquad (3.23)
$$

- *Estimate of* A_{12}. First of all, using once again Lemma 2.3 we get

$$\begin{aligned}
|\mathrm{A}_{12}| &\leq \int_\tau^{t_1} \left|\mathcal{L}u(E(s-t_1)x,s) - \mathcal{L}u(E(s-t_2)x,s)\right| \cdot \mathcal{J}(s)\,ds \\
&\leq \int_\tau^{t_1} \omega_{\mathcal{L}u,S_T}(\|(E(s-t_1)-E(s-t_2))x\|)\cdot \mathcal{J}(s)\,ds \\
&\quad \text{(since } |s-t_1|, |s-t_2| \leq T-\tau \text{ for all } \tau \leq s \leq t_1\text{)} \\
&\leq \omega_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N}) \int_\tau^{t_1} \mathcal{J}(s)\,ds =: (\bigstar)
\end{aligned}$$

where $c>0$ is a constant depending on K,τ,T and

$$\mathcal{J}(s) := \left|\int_{\{y\in\mathbb{R}^N:\, d((x,t_2),(y,s))\geq 4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \partial^2_{x_ix_j}\Gamma(x,t_2;y,s)\,dy\right|.$$

From this, using the cancellation property of $\mathcal{J}$ in [1, Thm. 3.16], we obtain

$$(\bigstar) \leq c\,\omega_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N}), \tag{3.24}$$

for a suitable constant $c>0$ depending on K,τ,T. By combining (3.23) with (3.24), we conclude that

$$\begin{aligned}
|\mathrm{A}_1| \leq c\Big\{&\omega_{\mathcal{L}u,S_T}(|t_1-t_2|^{1/q_N}) \\
&+ c|t_1-t_2|^{1/q_N}\int_{c|t_1-t_2|^{1/q_N}}^{\infty} \frac{\omega_{f,S_T}(s)}{s^2}\,ds\Big\},
\end{aligned} \tag{3.25}$$

for a suitable constant $c>0$ possibly depending on K,T,τ.

- ESTIMATE OF A_2. By proceeding exactly as in the estimate of A_2 in the proof of Theorem 3.4, and by taking into account (3.22), we obtain the estimate

$$\begin{aligned}
|\mathrm{A}_2| &\leq c\Big\{\int_{\{\eta:\, d(\xi_2,\eta)<4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{\mathcal{L}u,S_T}(\|E(s-t_1)x-y\|)}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds \\
&\quad + \int_{\{\eta:\, d(\xi_2,\eta)<4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{\mathcal{L}u,S_T}(\|E(s-t_2)x-y\|)}{d(\xi_2,\eta)^{Q+2}}\,dy\,ds\Big\} \\
&\leq c\Big\{\int_{\{\eta:\, d(\xi_1,\eta)<c d(\xi_2,\xi_1)\}} \frac{\omega_{\mathcal{L}u,S_T}(\boldsymbol{\kappa} d(\xi_1,\eta))}{d(\xi_1,\eta)^{Q+2}}\,dy\,ds \\
&\quad + \int_{\{\eta:\, d(\xi_2,\eta)<4\boldsymbol{\kappa} d(\xi_2,\xi_1)\}} \frac{\omega_{\mathcal{L}u,S_T}(\boldsymbol{\kappa} d(\xi_2,\eta))}{d(\xi_2,\eta)^{Q+2}}\,dy\,ds\Big\} \\
&\leq c\int_0^{c d(\xi_2,\xi_1)} \frac{\omega_{\mathcal{L}u,S_T}(s)}{s}\,ds \leq c\int_0^{c|t_1-t_2|^{1/q_N}} \frac{\omega_{\mathcal{L}u,S_T}(s)}{s}\,ds,
\end{aligned} \tag{3.26}$$

where $c>0$ is a suitable constant depending on K,T,τ.

- ESTIMATE OF A_3. Using the assumption $\mathcal{L}u \in \mathcal{D}(S_T)$, together with estimate (3.1) in Proposition 3.1, we immediately obtain

$$\begin{aligned}
|\mathrm{A}_3| &\leq \int_{\mathbb{R}^N\times(t_1,t_2)} |\partial^2_{x_ix_j}\Gamma(x,t_2;y,s)|\cdot \omega_{\mathcal{L}u,S_T}(\|E(s-t_2)x-y\|)\,dy\,ds \\
&\leq c\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|}),
\end{aligned} \tag{3.27}$$

where $c, \mu > 0$ ore structural constants.

Now we have estimated A_1, A_2 and A_3, we can complete the proof: in fact, gathering (3.25), (3.26) and (3.27), and recalling (3.21), we conclude that

$$\begin{aligned}
&|\partial^2_{x_ix_j}u(x,t_1) - \partial^2_{x_ix_j}u(x,t_2)| \le |\mathrm{A}_1| + |\mathrm{A}_2| + |\mathrm{A}_3| \\
&\quad\le c\left\{\omega_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N}) + c|t_1-t_2|^{1/q_N}\int_{c|t_1-t_2|^{1/q_N}}^{\infty}\frac{\omega_{\mathcal{L}u,S_T}(s)}{s^2}\,ds\right. \\
&\quad\quad\left. + \int_0^{c|t_1-t_2|^{1/q_N}}\frac{\omega_{\mathcal{L}u,S_T}(s)}{s}\,ds + \mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(|t_1-t_2|^{1/q_N})\right\} \\
&\quad= c\big\{\mathcal{M}_{\mathcal{L}u,S_T}(c|t_1-t_2|^{1/q_N}) + \mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(\sqrt{|t_1-t_2|})\big\},
\end{aligned}$$

which is exactly the desired (3.20). $\square$

4. Operators with coefficients depending on (x,t)

4.1. The basic estimate for functions with small support

We want to extend our results to operators with coefficients $a_{ij}(x,t)$ Dini continuous in x and bounded measurable in t. The first step is a local estimate for functions with small compact support.

Notation: Since in this section we will make *crucial use* of the interpolation inequality contained in Theorem 2.20 , we will adopt the following notation: given any $T > 0$, any $\overline{\xi} \in S_T$ and any $r > 0$, we set

$$B_r^T(\overline{\xi}) = B_r(\overline{\xi}) \cap S_T.$$

Theorem 4.1. *Let $\mathcal{L}$ be as in (1.1), satisfying assumptions* (H1), (H2), (H3) *stated in Sect. 1. Then, there exist constants $c, r_0 > 0$ depending on T, the matrix B in (1.5), the number ν in (1.3) and A in (1.20), respectively, such that*

$$\|\partial^2_{x_ix_j}u\|_{L^\infty(B_r^T(\overline{\xi}))} \le c\,\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(1) \tag{4.1}$$

$$\omega_{\partial^2_{x_ix_j}u,B_r^T(\overline{\xi})}(\rho) \le c\big(\mathcal{M}_{\mathcal{L}u,S_T}(c\rho) + \mathcal{M}_{a,S_T}(c\rho)\cdot\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(1)\big) \quad \forall\, \rho > 0, \tag{4.2}$$

and these estimates hold for every $\overline{\xi} \in S_T$, $0 < r \le r_0$, $1 \le i, j \le m_0$ and $u \in \mathcal{S}^D(S_T)$ with $\mathrm{supp}(u) \subseteq B_r(\overline{\xi}) \cap \overline{S_T}$. *Here,*

$$\mathcal{M}_{a,S_T} = \textstyle\sum_{i,j=1}^{m_0}\mathcal{M}_{a_{ij},S_T} \text{ and } \mathcal{M}_{\cdot,S_T} \text{ is as in (1.16).}$$

We stress that the constant c in (4.1)–(4.2) is independent of the ball $B_r(\overline{\xi})$. In particular, in view of Proposition 2.8, estimate (4.2) expresses Dini-continuity of $\partial^2_{x_ix_j}u$ provided that both $\mathcal{L}u$ and a_{ij} are log-Dini continuous, *while it just expresses uniform continuity if $\mathcal{L}u$ and a_{ij} are only Dini continuous.*

Proof. We follow and revise the proof of [1, Thm. 4.1]. To begin with, we arbitrarily fix $0 < r_0 \le 1/2$ (to be suitably chosen later on) and a point $\overline{\xi} = (\overline{x}, \overline{t}) \in S_T$. We then consider the operator $\mathcal{L}_{\overline{x}}$ with coefficients $a_{ij}(\overline{x}, t)$ (frozen in space, variable in time), and we let $\Gamma^{\overline{x}}$ be its fundamental solution. We now observe that, given any $u \in \mathcal{S}^D(S_T)$ with $\mathrm{supp}(u) \subseteq B_r(\overline{\xi}) \cap \overline{S_T}$ (for some $0 < r \le r_0$), we clearly have $u \in \mathcal{S}^0(\overline{t} - r, T)$ and $\mathcal{L}_{\overline{x}} u \in \mathcal{D}(S_T)$; thus, we can exploit the representation formula in Corollary 3.2, giving

$$\begin{aligned} &\partial^2_{x_i x_j} u(x,t) \\ &\quad = \int_{\overline{t}-r}^{t} \left(\int_{\mathbb{R}^N} \partial^2_{x_i x_j} \Gamma^{\overline{x}}(x,t;y,s) \left[\mathcal{L}_{\overline{x}} u(E(s-t)x, s) - \mathcal{L}_{\overline{x}} u(y,s) \right] dy \right) ds, \end{aligned}$$

for every $(x,t) \in B_r^T(\overline{\xi})$. From this, since we can write

$$\begin{aligned} \mathcal{L}_{\overline{x}} u &= \mathcal{L} u + (\mathcal{L}_{\overline{x}} - \mathcal{L}) u \\ &= \mathcal{L} u + \sum_{h,k=1}^{m_0} \big(a_{hk}(\overline{x}, t) - a_{hk}(x,t) \big) \partial^2_{x_h x_k} u, \end{aligned}$$

we obtain the following identity

$$\begin{aligned} \partial^2_{x_i x_j} u(x,t) &= \int_{\overline{t}-r}^{t} \left(\int_{\mathbb{R}^N} \partial^2_{x_i x_j} \Gamma^{\overline{x}}(x,t;y,s) \big\{ \mathcal{L} u(E(s-t)x,s) - \mathcal{L} u(y,s) \big\} dy \right) ds \\ &\qquad + \sum_{h,k=1}^{m_0} \int_{\overline{t}-r}^{t} \int_{\mathbb{R}^N} \partial^2_{x_i x_j} \Gamma^{\overline{x}}(x,t;y,s) \\ &\qquad \times \Big\{ \big(a_{hk}(\overline{x}, s) - a_{hk}(E(s-t)x, s) \big) \partial^2_{x_h x_k} u(E(s-t)x, s) \\ &\qquad - \big(a_{hk}(\overline{x}, s) - a_{hk}(y,s) \big) \partial^2_{x_h x_k} u(y,s) \Big\} dy ds \\ &= T_{ij}(\mathcal{L} u)(x,t) + \sum_{h,k=1}^{m_0} T_{ij}(f_{hk})(x,t), \end{aligned} \tag{4.3}$$

where $T_{ij}(\cdot)$ is as in Theorem 3.3, and

$$f_{hk}(y,s) = \big(a_{hk}(\overline{x}, s) - a_{hk}(y,s) \big) \partial^2_{x_h x_k} u(y,s) \in \mathcal{D}(\overline{t} - r, T).$$

To proceed further, we turn to estimate the L^∞-norm and the continuity modulus of $T_{ij}(\mathcal{L} u)$ and of $T_{ij}(f_{hk})$ (for $1 \le h, k \le m_0$) on $B_r^T(\overline{\xi})$.

(1) Estimate of the L^∞-norm. First of all we observe that, since we assuming $0 < r \le r_0 \le 1/2$, by Proposition 3.1 we get the following estimate

$$\begin{aligned} &\| T_{ij}(\mathcal{L} u) \|_{L^\infty(B_r^T(\overline{\xi}))} \\ &\quad \le \sup_{(x,t) \in B_r(\overline{\xi})} \int_{\mathbb{R}^N \times (\overline{t}-r, t)} \partial^2_{x_i x_j} \Gamma^{\overline{x}}(x,t;y,s) \omega_{\mathcal{L} u, S_T}(\| E(s-t)x - y \|) \, dy \, ds \\ &\quad \le c \sup_{(x,t) \in B_r(\overline{\xi})} \mathcal{U}^{\mu}_{\mathcal{L} u, S_T}(\sqrt{t - \overline{t} + r}) = c \, \mathcal{U}^{\mu}_{\mathcal{L} u, S_T}(\sqrt{2r}) \le c \, \mathcal{U}^{\mu}_{\mathcal{L} u, S_T}(1), \end{aligned} \tag{4.4}$$

where we have used the fact that $\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}$ is non-decreasing, see (1.18), and $c,\mu>0$ are structural constants. Analogously, since $0<r\le r_0$, we have

$$\begin{aligned}\|T_{ij}(f_{hk})\|_{L^\infty(B_r^T(\overline{\xi}))} &\le c\,\mathcal{U}^{\mu}_{f_{hk},S_T}(\sqrt{2r}) \le c\,\mathcal{U}^{\mu}_{f_{hk},S_T}(\sqrt{2r_0})\\ &= c\int_{\mathbb{R}^N} e^{-\mu|z|^2}\Big(\int_0^{\sqrt{2r_0}\|z\|}\frac{\omega_{f_{hk},S_T}(s)}{s}\,ds\Big)dz\end{aligned}\tag{4.5}$$

Now, by exploiting the *product structure* of f_{hk}, together with Lemma 2.14 (note that $f_{hk}=\partial^2_{x_hx_k}u=0$ on $S_T\setminus B_r(\overline{\xi})$) and (2.10), we can write

$$\begin{aligned}\omega_{f_{hk},S_T}(\rho) &= \omega_{f_{hk},\overline{B}_r(\overline{\xi})\cap S_T}(\rho)\\ &\le \sup_{(y,s)\in\overline{B}_r(\overline{\xi})\cap S_T}|a_{hk}(y,s)-a_{hk}(\overline{x},s)|\cdot\omega_{\partial^2_{x_hx_k}u,S_T}(\rho)\\ &\quad+\omega_{a_{hk},S_T}(\rho)\cdot\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\\ &\le 2\,\omega_{a_{hk},S_T}(\rho)\cdot\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}+\omega_{a_{hk},S_T}(\rho)\cdot\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\\ &\le 3\,\omega_{a,S_T}(\rho)\cdot\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\qquad(\text{for all }\rho>0),\end{aligned}\tag{4.6}$$

where $\omega_{a,S_T}(\cdot)=\sum_{h,k=1}^{m_0}\omega_{a_{hk},S_T}(\cdot)$. Thus, by combining (4.5)–(4.6) we get

$$\begin{aligned}&\|T_{ij}(f_{hk})\|_{L^\infty(B_r^T(\overline{\xi}))}\\ &\quad\le c\,\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\int_{\mathbb{R}^N}e^{-\mu|z|^2}\Big(\int_0^{\sqrt{2r_0}\|z\|}\frac{\omega_{a,S_T}(s)}{s}\,ds\Big)dz\\ &\quad\equiv c\,\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\,\mathcal{U}^{\mu}_{a,S_T}(\sqrt{2r_0})\qquad(\text{for all }1\le i,j\le m_0).\end{aligned}\tag{4.7}$$

Gathering (4.3), (4.4) and (4.7), we finally obtain

$$\begin{aligned}&\max_{1\le i,j\le m_0}\|\partial^2_{x_ix_j}u\|_{L^\infty(B_r^T(\overline{\xi}))}\\ &\quad\le c\Big(\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(1)+\mathcal{U}^{\mu}_{a,S_T}(\sqrt{2r_0})\sum_{h,k=1}^{m_0}\|\partial^2_{x_hx_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\Big)\\ &\quad\le c\Big(\mathcal{U}^{\mu}_{\mathcal{L}u,S_T}(1)+\max_{1\le i,j\le m_0}\|\partial^2_{x_ix_j}u\|_{L^\infty(B_r^T(\overline{\xi}))}\cdot\mathcal{U}^{\mu}_{a,S_T}(\sqrt{2r_0})\Big),\end{aligned}$$

where $c>0$ is a constant, possibly different from line to line. From this, if we choose $0<r_0\le 1/2$ so small that

$$c\,\mathcal{U}^{\mu}_{a,S_T}(\sqrt{2r_0})\le\frac{1}{2}\tag{4.8}$$

(recall that $\mathcal{U}^{\mu}_{a,S_T}(r)$ vanishes as $r\to 0^+$, see Lemma 2.12), we immediately derive the desired (4.1). We explicitly point out that the choice of r_0 (in such a way that (4.8) is satisfied) *only depends on the coefficients* a_{hk}.

(2) Estimate of the continuity modulus. First of all we observe that, by combining the representation formula (4.3) with Theorem 3.3, we get

$$\begin{aligned}\omega_{\partial^2_{x_i x_j}u, B_r^T(\overline{\xi})}(\rho) &\leq \omega_{T_{ij}(\mathcal{L}u), S_T}(\rho) + \sum_{h,k=1}^{m_0} \omega_{T_{ij}(f_{hk}), S_T}(\rho) \\ &\leq c\Big(\mathcal{M}_{\mathcal{L}u, S_T}(c\rho) + \sum_{h,k=1}^{m_0} \mathcal{M}_{f_{hk}, S_T}(c\rho)\Big),\end{aligned} \tag{4.9}$$

where $c > 0$ is a structural constant. On the other hand, using (4.6) (and taking into account the very definition of $\mathcal{M}_{f_{hk}, S_T}$, see (1.16)), we can write

$$\begin{aligned}\mathcal{M}_{f_{hk}, S_T}(\rho) &= \omega_{f_{hk}, S_T}(\rho) + \int_0^\rho \frac{\omega_{f_{hk}, S_T}(s)}{s}\,ds + \rho\int_\rho^\infty \frac{\omega_{f_{hk}, S_T}(s)}{s^2}\,ds \\ &\leq 3\,\|\partial^2_{x_h x_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\mathcal{M}_{a, S_T}(\rho) \qquad (\text{for all } \rho > 0).\end{aligned} \tag{4.10}$$

By combining (4.9)–(4.10) with (4.1) (which has been already proved), we then obtain the following estimate, provided that r_0 is small enough:

$$\begin{aligned}\omega_{\partial^2_{x_i x_j}u, B_r^T(\overline{\xi})}(\rho) &\leq c\Big(\mathcal{M}_{\mathcal{L}u, S_T}(c\rho) + \mathcal{M}_{a, S_T}(c\rho)\cdot \sum_{h,k=1}^{m_0} \|\partial^2_{x_h x_k}u\|_{L^\infty(B_r^T(\overline{\xi}))}\Big) \\ &\leq c\big(\mathcal{M}_{\mathcal{L}u, S_T}(c\rho) + \mathcal{M}_{a, S_T}(c\rho)\cdot \mathcal{U}^\mu_{\mathcal{L}u, S_T}(1)\big) \qquad (\text{for all } \rho > 0),\end{aligned}$$

This is precisely the desired (4.2), and the proof is complete. □

4.2. The continuity estimate in the general case

Given an arbitrary open set $\Omega \subseteq \mathbb{R}^{N+1}$ and a function $f : \Omega \to \mathbb{R}$, we recall that the (partial) continuity modulus $\omega_{f,\Omega}$ of f is defined as follows:

$$\omega_{f,\Omega}(r) = \sup_{\substack{(x,t),(y,t)\in\Omega \\ \|x-y\|\leq r}} |f(x,t) - f(y,t)| \qquad (r > 0).$$

In the following, we will get a control on ω_{f,S_T} starting with a uniform control on the moduli $\omega_{f,B_r(\overline{\xi}_i)}$ where $\{B_r(\overline{\xi}_i)\}_{i=1}^\infty$ is a covering of S_T.

This is possible in view of the following:

Proposition 4.2. *Let $r > 0$ be fixed, and let $\{B_r(\overline{\xi}_i)\}_{i=1}^\infty$ be a covering of S_T, that is, $S_T \subseteq \bigcup_i B_r(\overline{\xi}_i)$. Then, we have*

$$\omega_{f,S_T}(\rho) \leq \sup_i \omega_{f, B^T_{\theta r}(\overline{\xi}_i)}(\rho) \quad \textit{for every } 0 < \rho \leq r.$$

where $\theta \geq 1$ is a structural constant.

Proof. Let (x_1, t), $(x_2, t) \in S_T$ be two points satisfying $\|x_1 - x_2\| = s \leq r$, and let $i_1 \in \mathbb{N}$ be such that $(x_1, t) \in B_r^T(\overline{\xi}_{i_1})$. Using the quasi-triangle inequality of d, see (1.12), we infer that $(x_2, t) \in B_{\theta r}^T(\overline{\xi}_{i_1})$ for some structural constant $\theta \geq 1$ (actually, we have $\theta = \boldsymbol{\kappa}(1 + \boldsymbol{\kappa})$); as a consequence, we get

$$|f(x_1, t) - f(x_2, t)| \leq \omega_{f, B_{\theta r}^T(\overline{\xi}_{i_1})}(s) \leq \sup_i \omega_{f, B_{\theta r}^T(\overline{\xi}_i)}(s),$$

and this implies the assertion. □

Thanks to all the results established so far, we can now give the

Proof of Theorem 1.6. To begin with, we fix $r > 0$ so small that the *local continuity estimates* in Theorem 4.1 hold on balls of radius $2\theta r$ (where $\theta \geq 1$ is as in Proposition 4.2), and we let $\{B_r(\overline{\xi}_n)\}_{n \geq 1}$ be a covering of S_T. We then choose a function $\Phi \in C_0^\infty(B_{2\theta r}(0))$ satisfying $\Phi \equiv 1$ in $B_{\theta r}(0)$, and we define

$$\phi_n(\xi) = \Phi(\overline{\xi}_n^{-1} \circ \xi) \qquad (n \geq 1).$$

Note that, by (2.3), $\phi_n \in C_0^\infty(B_{2\theta r}(\overline{\xi}_n))$ and $\phi_n \equiv 1$ in $B_{\theta r}(\overline{\xi}_n)$; moreover, by the left-invariance of $\partial_{x_1}, \ldots, \partial_{x_{m_0}}, Y$ we see that the C^α-norms of $\phi_n, \partial_{x_k}\phi_n, \mathcal{L}(\phi_n)$ are *bounded independently of* n (for all $\alpha \in (0, 1)$). Throughout this proof, the constants involved may depend on r, which however is by now fixed.

We now arbitrarily fix $n \geq 1$ and we observe that, since $u_n = u\phi_n \in \mathcal{S}^D(S_T)$ and since $\mathrm{supp}(u_n) \subseteq B_{2\theta r}^T(\overline{\xi}_n)$, we can apply the estimates (4.1)–(4.2) in Theorem 4.1 to this function u_n: recalling that $\phi_n \equiv 1$ in $B_{\theta r}(\overline{\xi}_n)$, this gives

$$\|\partial^2_{x_i x_j} u\|_{L^\infty(B_{\theta r}^T(\overline{\xi}_n))} \leq \|\partial^2_{x_i x_j} u_n\|_{L^\infty(B_{2\theta r}^T(\overline{\xi}_n))} \leq c\,\mathcal{U}^\mu_{\mathcal{L}u_n, S_T}(1) \tag{4.11}$$

$$\begin{aligned} &\omega_{\partial^2_{x_i x_j} u, B_{\theta r}^T(\overline{\xi}_n)}(\rho) \leq \omega_{\partial^2_{x_i x_j} u_n, B_{2\theta r}^T(\overline{\xi}_n)}(\rho) \\ &\quad \leq c\big(\mathcal{M}_{\mathcal{L}u_n, S_T}(c\rho) + \mathcal{M}_{a, S_T}(c\rho) \cdot \mathcal{U}^\mu_{\mathcal{L}u_n, S_T}(1)\big) \quad \forall\, \rho > 0, \end{aligned} \tag{4.12}$$

where $c, \mu > 0$ are structural constants (and $1 \leq i, j \leq m_0$). On the other hand, since a direct computation shows that

$$\mathcal{L}u_n = (\mathcal{L}u)\phi_n + u(\mathcal{L}\phi_n) + 2\textstyle\sum_{h,k=1}^{m_0} a_{hk}\partial_{x_h} u \cdot \partial_{x_k}\phi_n,$$

we clearly have the following estimate

$$\begin{aligned} \omega_{\mathcal{L}u_n, S_T}(\rho) &\leq \omega_{(\mathcal{L}u)\phi_n, S_T}(\rho) + \omega_{u(\mathcal{L}\phi_n), S_T}(\rho) \\ &\quad + 2\textstyle\sum_{h,k=1}^{m_0} \omega_{a_{hk}\partial_{x_h} u \cdot \partial_{x_k}\phi_n, S_T}(\rho). \end{aligned} \tag{4.13}$$

In view of (4.11)–(4.12), and taking into account the above (4.13), to prove the theorem we then turn to estimate the three *continuity moduli*

$$(1)\ \omega_{(\mathcal{L}u)\phi_n, S_T}, \qquad (2)\ \omega_{u(\mathcal{L}\phi_n), S_T}, \qquad (3)\ \omega_{a_{hk}\partial_{x_h} u \cdot \partial_{x_k}\phi_n, S_T}.$$

To this end we will repeatedly use the following straightforward estimate, holding true for every open set $\Omega \subseteq \mathbb{R}^{N+1}$ and every $f, g \in \mathcal{D}(\Omega)$:

$$\omega_{fg,\Omega}(\rho) \le \|f\|_{L^\infty(\Omega)}\omega_{g,\Omega}(\rho) + \|g\|_{L^\infty(\Omega)}\omega_{f,\Omega}(\rho) \qquad \forall\ \rho > 0. \tag{4.14}$$

- ESTIMATE OF (1). On account of (4.14), and since $\phi_n \in C_0^\infty(\mathbb{R}^{N+1})$ (hence, in particular, $\phi_n \in C^\alpha(\mathbb{R}^{N+1})$ for every $\alpha \in (0,1)$), we immediately get

$$\begin{aligned}\omega_{(\mathcal{L}u)\phi_n,S_T}(\rho) &\le \|\mathcal{L}u\|_{L^\infty(S_T)}\omega_{\phi_n,S_T}(\rho) + \|\phi_n\|_{L^\infty(S_T)}\omega_{\mathcal{L}u,S_T}(\rho)\\ &\le c\big(\rho^\alpha\|\mathcal{L}u\|_{L^\infty(S_T)} + \omega_{\mathcal{L}u,S_T}(\rho)\big) \qquad \forall\ \rho > 0,\end{aligned} \tag{4.15}$$

where $c > 0$ is a constant only depending on Φ.

- ESTIMATE OF (2). Using once again (4.14), and taking into account that $u\mathcal{L}\phi_n$ is compactly supported in $B_{2\theta r}(\overline{\xi}_n)$, by Lemma 2.14 we can write

$$\begin{aligned}\omega_{u(\mathcal{L}\phi_n),S_T}(\rho) &= \omega_{u(\mathcal{L}\phi_n),\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\\ &\le \|u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))}\omega_{\mathcal{L}\phi_n,S_T}(\rho)\\ &\quad + \|\mathcal{L}\phi_n\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))}\omega_{u,\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\\ &\quad \text{(since } \mathcal{L}\phi_n \in C^\alpha(\mathbb{R}^{N+1}) \text{ for every } 0 < \alpha < 1)\\ &\le c\big(\rho^\alpha\|u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))} + \omega_{u,\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\big) = (\bigstar),\end{aligned}$$

where $c > 0$ is a constant only depending on Φ. On the other hand, since we know from Theorem 2.20 that $u \in C^\alpha(B \cap S_T)$ *for every ball* $B = B_R(\overline{\eta})$ (with $\overline{\eta} \in S_T$) and every $\alpha \in (0,1)$, we obtain

$$(\bigstar) \le c\,\rho^\alpha\|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} \qquad \forall\ \rho > 0. \tag{4.16}$$

- ESTIMATE OF (3). By repeatedly exploiting (4.14), and by taking into account the smoothness and support of ϕ_n, we derive the following estimate

$$\begin{aligned}&\omega_{a_{hk}\partial_{x_h}u\cdot\partial_{x_k}\phi_n,S_T}(\rho) = \omega_{a_{hk}\partial_{x_h}u\cdot\partial_{x_k}\phi_n,\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\\ &\quad\le \omega_{a_{hk},S_T}(\rho)\,\|\partial_{x_h}u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))}\|\partial_{x_k}\phi_n\|_{L^\infty(B_{2\theta r}(\overline{\xi}_n))}\\ &\qquad + \omega_{\partial_{x_h}u,\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\,\|a_{hk}\|_{L^\infty(\mathbb{R}^{N+1})}\|\partial_{x_k}\phi_n\|_{L^\infty(\mathbb{R}^{N+1})}\\ &\qquad + \omega_{\partial_{x_k}\phi_n,S_T}(\rho)\,\|a_{hk}\|_{L^\infty(\mathbb{R}^{N+1})}\|\partial_{x_h}u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))}\\ &\quad\le c\big(\omega_{a_{hk},S_T}(\rho)\,\|\partial_{x_h}u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))} + \omega_{\partial_{x_h}u,\overline{B}_{2\theta r}(\overline{\xi}_n)\cap S_T}(\rho)\\ &\qquad + \rho^\alpha\|\partial_{x_h}u\|_{L^\infty(B^T_{2\theta r}(\overline{\xi}_n))}\big) =: (\bigstar),\end{aligned}$$

where $c > 0$ is a constant only depending on Φ and on ν in (1.3). On the other hand, since we know from Theorem 2.20 that $\partial_{x_h}u \in C^\alpha(B \cap S_T)$ *for every ball* $B = B_R(\overline{\eta})$ (with $\overline{\eta} \in S_T$) and every $\alpha \in (0,1)$, we obtain

$$\begin{aligned}(\bigstar) &\le c\big(\omega_{a_{hk},S_T}(\rho)\,\|\partial_{x_h}u\|_{L^\infty(B_{2\theta r}(\overline{\xi}_n))}\\ &\quad + \rho^\alpha\|\partial_{x_h}u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\big) \qquad \forall\ \rho > 0.\end{aligned} \tag{4.17}$$

Gathering (4.15)-to-(4.17), from (4.13) we then get

$$
\begin{aligned}
\omega_{\mathcal{L}u_n,S_T}(\rho) &\leq c\Big(\rho^\alpha \|\mathcal{L}u\|_{L^\infty(S_T)} + \omega_{\mathcal{L}u,S_T}(\rho) + \rho^\alpha \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} \\
&\qquad + \sum_{h,k=1}^{m_0} \big(\omega_{a_{hk},S_T}(\rho)\, \|\partial_{x_h} u\|_{L^\infty(B_{2\theta r}(\overline{\xi}_n))} \\
&\qquad + \rho^\alpha \|\partial_{x_h} u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\big)\Big) \\
&\qquad \text{(setting, as usual, } \omega_{a,S_T} = \textstyle\sum_{h,k=1}^{m_0} \omega_{a_{hk},S_T}) \\
&\leq c\Big(\rho^\alpha \|\mathcal{L}u\|_{L^\infty(S_T)} + \omega_{\mathcal{L}u,S_T}(\rho) + \rho^\alpha \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} \\
&\qquad + \big(\omega_{a,S_T}(\rho) + \rho^\alpha\big) \sum_{h=1}^{m_0} \|\partial_{x_h} u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\Big),
\end{aligned}
\tag{4.18}
$$

and this estimate holds *for every* $\rho > 0$. With (4.18) at hand, we are now ready to establish assertions (i)–(ii) in the statement of the theorem.

- *Proof of* (i). First of all, by combining estimates (4.11)–(4.18) and by exploiting Lemma 2.12-(i), we derive the bound

$$
\begin{aligned}
\|\partial^2_{x_i x_j} u\|_{L^\infty(B^T_r(\overline{\xi}_n))} &\leq c\, \mathcal{U}^\mu_{\mathcal{L}u_n,S_T}(1) \\
&= c \int_{\mathbb{R}^N} e^{-\mu|z|^2} \Big(\int_0^{\|z\|} \frac{\omega_{\mathcal{L}u_n,S_T}(s)}{s}\, ds \Big) dz \\
&\leq c\Big(\|\mathcal{L}u\|_{L^\infty(S_T)} + \mathcal{U}^\mu_{\mathcal{L}u,S_T}(1) \\
&\qquad + \sum_{h=1}^{m_0} \|\partial_{x_h} u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} + \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\Big) \\
&\leq c\, \Big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \sum_{h=1}^{m_0} \|\partial_{x_h} u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} + \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\Big)
\end{aligned}
\tag{4.19}
$$

where $c > 0$ now depends on the chosen α and on the number A in (1.20). From this, by using the interpolation inequality (2.59) in Theorem 2.20, we obtain

$$
\begin{aligned}
&\|\partial^2_{x_i x_j} u\|_{L^\infty(B^T_r(\overline{\xi}_n))} \\
&\quad \leq c\Big\{ \|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \varepsilon\Big(\sum_{h,k=1}^{m_0} \|\partial^2_{x_k x_h} u\|_{L^\infty(S_T)} + \|Yu\|_{L^\infty(S_T)}\Big) \\
&\qquad + \frac{1}{\varepsilon^\gamma} \|u\|_{L^\infty(S_T)}\Big\},
\end{aligned}
\tag{4.20}
$$

and this estimate holds *for every* $\varepsilon \in (0,1)$. We then observe that, since $n \geq 1$ is arbitrarily fixed, by taking the sup over $\mathbb{N}$ in the previous inequality we get

$$\begin{aligned}\|\partial^2_{x_ix_j}u\|_{L^\infty(S_T)} \leq c\Big\{\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \varepsilon\Big(\sum_{h,k=1}^{m_0}\|\partial^2_{x_kx_h}u\|_{L^\infty(S_T)} + \|Yu\|_{L^\infty(S_T)}\Big) \\ + \frac{1}{\varepsilon^\gamma}\|u\|_{L^\infty(S_T)}\Big\};\end{aligned}$$

moreover, since $Yu = \mathcal{L}u - \sum_{h,k=1}^{m_0} a_{hk}\partial^2_{x_hx_k}u$, by exploiting assumption (H3) we can write (up to possibly change the constant c)

$$\|\partial^2_{x_ix_j}u\|_{L^\infty(S_T)} \leq c\Big\{\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \varepsilon\sum_{h,k=1}^{m_0}\|\partial^2_{x_kx_h}u\|_{L^\infty(S_T)} + \frac{1}{\varepsilon^\gamma}\|u\|_{L^\infty(S_T)}\Big\}.$$

Thus, if we choose $\varepsilon > 0$ so small that $c\,\varepsilon < 1/2$, we conclude that

$$\|\partial^2_{x_ix_j}u\|_{L^\infty(S_T)} \leq c\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big), \tag{4.21}$$

and this implies, again by the identity $Yu = \mathcal{L}u - \sum_{h,k=1}^{m_0} a_{hk}\partial^2_{x_hx_k}u$,

$$\begin{aligned}\|Yu\|_{L^\infty(S_T)} &\leq \|\mathcal{L}u\|_{L^\infty(S_T)} + \sum_{h,k=1}^{m_0}\|a_{hk}\|_{L^\infty(\mathbb{R}^{N+1})}\|\partial^2_{x_hx_k}u\|_{L^\infty(S_T)} \\ &\leq c\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big).\end{aligned} \tag{4.22}$$

In view of (4.21)–(4.22) and Theorem 2.20, assertion (i) is now established.

- *Proof of* (ii). First of all, by combining (4.12) with (4.18) (and by taking into account the very definition of $\mathcal{M}_{\cdot,S_T}$, see (1.16)), we get

$$\begin{aligned}\omega_{\partial^2_{x_ix_j}u,B^T_{\theta r}(\overline{\xi}_n)}(\rho) &\leq c\big(\mathcal{M}_{\mathcal{L}u_n,S_T}(c\rho) + \mathcal{M}_{a,S_T}(c\rho)\cdot\mathcal{U}^\mu_{\mathcal{L}u_n,S_T}(1)\big) \\ &\leq c\Big\{\rho^\alpha\|\mathcal{L}u\|_{L^\infty(S_T)} + \mathcal{M}_{\mathcal{L}u,S_T}(c\rho) \\ &\quad + \big(\mathcal{M}_{a,S_T}(c\rho) + \rho^\alpha\big)\Big(\sum_{h=1}^{m_0}\|\partial_{x_h}u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} + \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\Big) \\ &\quad + \mathcal{M}_{a,S_T}(c\rho)\cdot\mathcal{U}^\mu_{\mathcal{L}u_n,S_T}(1)\Big\} \\ &\quad \text{(by the same computation in (4.19))} \\ &\leq c\Big\{\mathcal{M}_{\mathcal{L}u,S_T}(c\rho) + \big(\mathcal{M}_{a,S_T}(c\rho) + \rho^\alpha\big)\|\mathcal{L}u\|_{\mathcal{D}(S_T)} \\ &\quad + \big(\mathcal{M}_{a,S_T}(c\rho) + \rho^\alpha\big)\Big(\sum_{h=1}^{m_0}\|\partial_{x_h}u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))} + \|u\|_{C^\alpha(B^T_{2\theta r}(\overline{\xi}_n))}\Big)\Big\},\end{aligned}$$

where $c > 0$ depends on $\alpha \in (0,1)$ and on the number A. From this, by exploiting the interpolation inequality (2.59) *with* $\varepsilon = 1$, jointly with the estimate in assertion (i) (which has been already established), we obtain

$$\begin{aligned}
&\omega_{\partial^2_{x_ix_j}u,B^T_{\theta r}(\overline{\xi}_n)}(\rho)\\
&\quad\leq c\Big\{\mathcal{M}_{\mathcal{L}u,S_T}(c\rho)+\big(\mathcal{M}_{a,S_T}(c\rho)+\rho^\alpha\big)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)}+\|u\|_{L^\infty(S_T)}\big)\\
&\qquad+\big(\mathcal{M}_{a,S_T}(c\rho)+\rho^\alpha\big)\Big(\sum_{h,k=1}^{m_0}\|\partial^2_{x_kx_h}u\|_{L^\infty(S_T)}+\|Yu\|_{L^\infty(S_T)}\Big)\Big\}\\
&\quad\leq c\Big\{\mathcal{M}_{\mathcal{L}u,S_T}(c\rho)+\big(\mathcal{M}_{a,S_T}(c\rho)+\rho^\alpha\big)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)}+\|u\|_{L^\infty(S_T)}\big)\Big\}.
\end{aligned}$$

We then observe that, since $n \geq 1$ is arbitrarily fixed, by taking the supremum over $\mathbb{N}$ in the above estimate and by using Proposition 4.2, we obtain

$$\begin{aligned}
&\omega_{\partial^2_{x_ix_j}u,S_T}(\rho)\leq\sup_{n\in\mathbb{N}}\omega_{\partial^2_{x_ix_j}u,B^T_{\theta r}(\overline{\xi}_n)}(\rho)\\
&\quad\leq c\Big\{\mathcal{M}_{\mathcal{L}u,S_T}(c\rho)+\big(\mathcal{M}_{a,S_T}(c\rho)+\rho^\alpha\big)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)}+\|u\|_{L^\infty(S_T)}\big)\Big\}
\end{aligned}$$

and this estimate holds for every $0 < \rho \leq r$ (note that $r > 0$ is fixed *once and for all*); this, together with the identity $Yu = \mathcal{L}u - \sum_{h,k} a_{hk}\partial^2_{x_hx_k}u$ and (4.14), immediately implies an analogous bound for the modulus

$$\omega_{Yu,S_T}(\rho)\quad(\text{for } 0<\rho\leq r).$$

Finally, when $\rho \geq r$ estimate (ii) is an immediate consequence of (i). $\square$

5. Time continuity of $\partial^2_{x_ix_j}u$

Now we have established Theorem 1.6, we are finally ready to give the

Proof. Let K, T, τ, α be as in the statement of the theorem, and let $\psi(t) \in C_0^\infty(\mathbb{R})$ be a cut-off function such that

$$\text{(i)}\ \ 0\leq\psi\leq 1 \text{ on } \mathbb{R},\quad \text{(ii)}\ \ \psi\equiv 1 \text{ on } [\tau,T],\quad \text{(iii)}\ \ \psi(t)=0 \text{ for } t\leq\tau-1.$$

We then fix a point $\overline{\xi} = (\overline{x}, \overline{t}) \in S_T$ and, for a given function $u \in \mathcal{S}^D(S_T)$ with $\mathcal{L}u \in \mathcal{D}_{\log}(S_T)$, we apply the continuity estimate (3.18) in Theorem 3.5 to the function $v := u\psi \in \mathcal{S}^0(\tau-1, T)$ (see property (iii) of ψ): this gives

$$\begin{aligned}
&|\partial^2_{x_ix_j}u(x_1,t_1)-\partial^2_{x_ix_j}u(x_2,t_2)|=|\partial^2_{x_ix_j}v(x_1,t_1)-\partial^2_{x_ix_j}v(x_2,t_2)|\\
&\quad\leq c\big\{\mathcal{M}_{\mathcal{L}_{\overline{x}}v,S_T}\big(c(d((x_1,t_1),(x_2,t_2))+|t_1-t_2|^{1/q_N})\big)\\
&\qquad+\mathcal{U}^\mu_{\mathcal{L}_{\overline{x}}v,S_T}(\sqrt{|t_2-t_1|})\big\},
\end{aligned}\tag{5.1}$$

for every couple of points $(x_1, t_1), (x_2, t_2) \in K \times [\tau, T]$. Owing to (5.1), and taking into account the definitions of $\mathcal{M}_{\cdot,S_T}$ and of $\mathcal{U}^\mu_{\cdot,S_T}$, to complete the proof we then turn to estimate the continuity modulus

$$\omega_{\mathcal{L}_{\overline{x}}v,S_T}(r)\quad(\text{for } r>0).$$

First of all, since ψ is independent of x, we get $\mathcal{L}_{\overline{x}}v = \psi(\mathcal{L}_{\overline{x}}u) - u\,\partial_t\psi$; moreover,

$$\begin{aligned}\mathcal{L}_{\overline{x}}u &= \mathcal{L}u + (\mathcal{L}_{\overline{x}} - \mathcal{L})u \\ &= \mathcal{L}u + \sum_{h,k=1}^{m_0} \big(a_{hk}(\overline{x},t) - a_{hk}(x,t)\big)\partial^2_{x_h x_k}u.\end{aligned}$$

In view of these facts, and since ψ is *constant w.r.t. x*, by repeatedly exploiting (4.14), together with estimates (i)–(ii) in Theorem 1.6, we then obtain

$$\begin{aligned}\omega_{\mathcal{L}_{\overline{x}}v,S_T}(\rho) &\leq \omega_{\psi(\mathcal{L}_{\overline{x}}u),S_T}(\rho) + \omega_{u\partial_t\psi,S_T}(\rho) \leq c\left(\omega_{\mathcal{L}_{\overline{x}}u,S_T}(\rho) + \omega_{u,S_T}(\rho)\right) \\ &\leq c\left\{\omega_{\mathcal{L}u,S_T}(\rho) + 2A\sum_{h,k=1}^{m_0}\omega_{\partial^2_{x_hx_k}u.S_T}(\rho)\right. \\ &\quad\left. + \,\omega_{a,S_T}(\rho)\sum_{h,k=1}^{m_0}\|\partial^2_{x_hx_k}u\|_{L^\infty(S_T)} + \omega_{u,S_T}(\rho)\right\} \\ &\leq c\,\big\{\omega_{\mathcal{L}u,S_T}(\rho) \\ &\quad + \big(\mathcal{M}_{\mathcal{L}u,S_T}(c\rho) + \big(\mathcal{M}_{a,S_T}(c\rho) + \rho^\alpha\big)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big)\big) \\ &\quad + \omega_{a,S_T}(\rho)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big) + \omega_{u,S_T}(\rho)\big\} \\ &\quad \text{(since, by definition, } \omega_{a\,S_T} \leq \mathcal{M}_{a,S_T}) \\ &\leq c\,\big\{\omega_{\mathcal{L}u,S_T}(\rho) + \omega_{u,S_T}(\rho) + \mathcal{M}_{\mathcal{L}u,S_T}(c\rho) \\ &\quad + \big(\big(\mathcal{M}_{a,S_T}(c\rho) + \rho^\alpha\big)\big(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\big)\big)\big\} \qquad (5.2)\end{aligned}$$

where, as usual, $\omega_{a,S_T} = \sum_{h,k=1}^{m_0}\omega_{a_{hk},S_T}$. With estimate (5.2) at hand, we can easily complete the proof of the theorem: indeed, by combining (5.1)–(5.2) and by taking into account the definitions of the functions involved, we get

$$\begin{aligned}&|\partial^2_{x_ix_j}u(x_1,t_1) - \partial^2_{x_ix_j}u(x_2,t_2)| \\ &\quad\leq c\big\{\mathcal{M}_{\mathcal{L}u,S_T}(cr) + \mathcal{M}_{u,S_T}(cr) + \mathcal{N}_{\mathcal{L}u,S_T}(cr) \\ &\qquad + (\mathcal{N}_{a,S_T}(cr) + r^\alpha)(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)})) \\ &\qquad + \mathcal{U}^\mu_{\mathcal{L}u,S_T}(c\sqrt{|t_1-t_2|}) + \mathcal{U}^\mu_{u,S_T}(c\sqrt{|t_1-t_2|}) + \mathcal{V}^\mu_{\mathcal{L}u,S_T}(c\sqrt{|t_1-t_2|}) \\ &\qquad + (\mathcal{V}^\mu_{a,S_T}(c\sqrt{|t_1-t_2|}) + |t_1-t_2|^{\alpha/2})(\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)})\big\}, \qquad (5.3)\end{aligned}$$

where we have set

$$r := d\big((x_1,t_1),(x_2,t_2)\big) + |t_1-t_2|^{1/q_N}.$$

Next we note that, on the one hand we have $\omega_{\mathcal{L}u,S_T} \leq \mathcal{M}_{\mathcal{L}u,S_T} \leq \mathcal{N}_{\mathcal{L}u,S_T}$, which implies that $\mathcal{U}^\mu_{\mathcal{L}u,S_T} \leq \mathcal{V}^\mu_{\mathcal{L}u,S_T}$. On the other hand, since by Theorem 1.6 we know that

$$\|u\|_{C^\alpha(S_T)} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)}\right\},$$

by Remark 2.21 and Proposition 2.11, we can write

$$\mathcal{M}_{u,S_T}(cr) + \mathcal{U}^{\mu}_{u,S_T}(c\sqrt{|t_1 - t_2|}) \leq cr^{\alpha} \left\{ \|\mathcal{L}u\|_{\mathcal{D}(S_T)} + \|u\|_{L^\infty(S_T)} \right\}.$$

Using these facts in (5.3) we obtain the desired (1.21). □

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