KOSZUL GORENSTEIN ALGEBRAS FROM COHEN–MACAULAY SIMPLICIAL COMPLEXES

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ABSTRACT. We associate with every pure flag simplicial complex Δ a standard graded Gorenstein F-algebra R_{Δ} whose homological features are largely dictated by the combinatorics and topology of Δ . As our main result, we prove that the residue field F has a k-step linear R_{Δ} -resolution if and only if Δ satisfies Serre's condition (S_k) over F, and that R_{Δ} is Koszul if and only if Δ is Cohen-Macaulay over F. Moreover, we show that R_{Δ} has a quadratic Gröbner basis if and only if Δ is shellable. We give two applications: first, we construct quadratic Gorenstein F-algebras which are Koszul if and only if the characteristic of F is not in any prescribed set of primes. Finally, we prove that whenever R_{Δ} is Koszul the coefficients of its γ -vector alternate in sign, settling in the negative an algebraic generalization of a conjecture by Charney and Davis.

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1. INTRODUCTION

It is a common line of thought in combinatorial commutative algebra to build correspondences between rings and objects of a more combinatorial nature. This can be insightful for both sides of the story. A prototypical example is Stanley–Reisner theory, where topological and combinatorial properties of simplicial complexes are studied via quotients of polynomial rings by squarefree monomial ideals. In this article we follow the same general principle and associate with any pure simplicial complex Δ a non-monomial *Gorenstein* standard graded \mathbb{F} -algebra R_{Δ} , where \mathbb{F} is a field. If the complex Δ is flag and satisfies some mild homological condition, the algebra R_{Δ} will moreover be quadratic.

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In combinatorics, Gorenstein algebras show up as Stanley–Reisner rings of homology spheres, as well as Ehrhart rings of certain lattice polytopes. While Gorenstein rings are very rich in structure and well-studied both in algebra and combinatorics, it is not always so easy to construct examples with prescribed features, e.g. with fixed *h*-vector. A classical and useful tool for this purpose is Nagata idealization, prominently featured in a recent article by Mastroeni, Schenck and Stillman [MSS21a].

A purely combinatorial way to construct a sphere from any simplicial complex Δ was introduced by Thomas Bier [Bie92] (see also [BPSZ05, Mur11, DFN19]): such a *Bier sphere* is PL and simultaneously contains both Δ and its Alexander dual. In this paper we will take a twist on this construction and crucially exploit a beautiful result by Terai and Yanagawa [Yan00, Corollary 3.7] relating Serre's (S_k) property on Δ to the linearity of the resolution of the Stanley–Reisner ideal of the Alexander dual of Δ .

Our recipe to build the ring R_{Δ} is as follows: said $I_{\Delta} \subseteq \mathbb{F}[x_1, \ldots, x_n]$ the Stanley–Reisner ideal of a pure simplicial complex Δ on n vertices, we consider a new complex Γ on 2nvertices whose Stanley–Reisner ideal is

$$I_{\Gamma} \coloneqq I_{\Delta} + (x_i y_i : i \in [n]) \subseteq \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_n].$$

Such a complex turns out to be a PL ball whose boundary is the Bier sphere associated with Δ , as already noted by Murai in [Mur11]. Taking inspiration from [MSS21a], we then define the ring R_{Δ} as the Nagata idealization of the canonical module of $\mathbb{F}[\Gamma]$.

The main contribution of the present paper is to relate the homological features of the links of Δ with the behaviour of the minimal free resolution of the residue field \mathbb{F} as an R_{Δ} -module, under the extra assumption that Δ is *flag* (i.e., its Stanley–Reisner ideal is quadratically generated). In order to do this, we build some technical results comparing the resolutions of certain monomial ideals over different rings: see Corollary 5.11 and the lemmas preceding it. Our first result is as follows:

Theorem A (Theorem 5.1). Let Δ be a pure flag simplicial complex. The minimal resolution of the residue field \mathbb{F} as an R_{Δ} -module is linear for k steps if and only if Δ satisfies Serre's condition (S_k) over \mathbb{F} .

Note that flagness is indeed necessary for the resolution of \mathbb{F} to have a chance to be linear: as the monomials generating I_{Δ} lie in a minimal generating set of the ideal defining R_{Δ} , the second step of the R_{Δ} -resolution of \mathbb{F} fails to be linear whenever Δ is not flag.

Serre's condition (S_k) is an algebraic property that, when formulated for the Stanley–Reisner ring of Δ , translates to a certain vanishing condition about the homology of its links. Reisner's criterion [Rei76] is precisely this translation when k = d, where $d = \dim \mathbb{F}[\Delta] = \dim(\Delta) + 1$: when Δ satisfies (S_d) , we say that Δ is Cohen–Macaulay over \mathbb{F} .

The Cohen–Macaulayness of Δ over \mathbb{F} translates to the R_{Δ} -resolution of \mathbb{F} being linear not just until the *d*-th step, but at *every* step. Standard graded algebras with this behaviour are called *Koszul*.

Theorem B (Corollary 5.3). Let Δ be a pure flag simplicial complex. The standard graded Gorenstein \mathbb{F} -algebra R_{Δ} is Koszul if and only if Δ is Cohen–Macaulay over \mathbb{F} .

We remark that this is not the first time the Koszul and Cohen–Macaulay properties have been brought together. Polo [Pol95] and Woodcock [Woo98] independently proved that the incidence algebra associated with a finite graded poset is a (noncommutative) Koszul algebra if and only if all the open intervals of the poset are Cohen–Macaulay, i.e. the order complex of every open interval is Cohen–Macaulay. In a similar vein, Peeva, Reiner and Sturmfels showed in [PRS98] that a graded pointed affine semigroup algebra is Koszul if and only if every open interval of the poset associated with the semigroup is Cohen–Macaulay. These results have later been unified by Reiner and Stamate in [RS10]. Another intriguing connection between Koszulness and Cohen–Macaulayness has been highlighted by Vallette in the context of operads [Val07]. Moreover, the study of the Koszul property for quadratic Gorenstein algebras has been the object of intensive research in the last few years [Mat18, MSS21a, MSS21b, MS20]. In particular, the results in [MSS21a, MSS21b, MS20] provide an almost fully complete characterization of the pairs (c, r) for which there exists a non-Koszul quadratic Gorenstein algebra of codimension c and Castelnuovo–Mumford regularity r: the only cases that have not been settled yet are (c, r) = (6, 3) and (c, r) = (7, 3). Unfortunately, the techniques developed in this paper do not give further information on these cases, see Remark 5.14.

We find a pleasing application of Theorem B in Proposition 5.15. Fix a finite set P of prime numbers. By considering certain simplicial manifolds Δ whose vanishing in homology depends on the characteristic of the field, we obtain standard graded quadratic Gorenstein \mathbb{F} -algebras R_{Δ} that are Koszul if and only if the characteristic of \mathbb{F} is not in P.

Theorem B implies that checking Koszulness for the algebras R_{Δ} is equivalent to testing if Δ is Cohen–Macaulay. For instance, it suffices to compute the (finite) minimal free resolution of $\mathbb{F}[\Delta]$ to certify linearity of the (infinite) R_{Δ} -resolution of \mathbb{F} . This is in stark contrast with the general behaviour of Koszul algebras. Indeed, working over a field \mathbb{F} of characteristic zero, Roos [Roo93] constructed a family $(A_{\alpha})_{\alpha\geq 2}$ of standard graded Artinian quadratic \mathbb{F} algebras with fixed Hilbert series and such that the (A_{α}) -resolution of \mathbb{F} is linear for α steps, but fails to be so at the $(\alpha + 1)$ -st. Applying idealization to Roos' construction, McCullough and Seceleanu recently showed in [MS20, Theorem 5.1] that checking Koszulness is just as hard for quadratic Gorenstein algebras.

Since certifying Koszulness for a quadratic algebra is difficult, algebraists often resort to studying stronger conditions like the existence of a quadratic Gröbner basis for the defining ideal. Our next result shows that having a quadratic Gröbner basis for the algebras R_{Δ} corresponds indeed to a property of Δ which is much stronger than Cohen–Macaulayness.

Theorem C (Theorem 6.3). Let Δ be a pure flag simplicial complex. The Gorenstein \mathbb{F} algebra R_{Δ} has a quadratic Gröbner basis if and only if Δ is a shellable complex.

As the ideal defining R_{Δ} has a characteristic-free generating set consisting of monomials and binomials of the form $\mathbf{m} - \mathbf{m'}$ (Proposition 4.3), the existence of a quadratic Gröbner basis for such an object is a combinatorial property not depending on the choice of the field. The same is true for shellability of Δ .

As a last topic, we investigate a numerical invariant related to the Hilbert series of R_{Δ} . Gorenstein algebras are known to have a symmetric *h*-polynomial with nonnegative coefficients. Gal [Gal05] proposed the study of a certain linear transformation of these coefficients, called the γ -vector, and conjectured the nonnegativity of its entries in the case of Stanley–Reisner rings of flag \mathbb{F} -homology spheres. The validity of one of the inequalities given by Gal's conjecture is known as Charney–Davis conjecture [CD95] and has important implications in metric geometry: see [For07] for an excellent survey on the topic.

The Stanley–Reisner ring of a flag \mathbb{F} -homology sphere happens to be both Koszul and Gorenstein: using this observation as a starting point, Reiner and Welker began a more algebraic investigation of the Charney–Davis conjecture in [RW05]. An explicit generalization of the Charney–Davis conjecture in this direction can be found in a survey by Peeva and Stillman:

Question ([PS09, Problem 10.3]). Let S/I be a Koszul Gorenstein algebra with *h*-vector $(h_0, h_1, \ldots, h_{2e})$. Is it true that $(-1)^e(h_0 - h_1 + h_2 - \ldots + h_{2e}) \ge 0$?

The answer turns out to be negative. This is a consequence of our last result:

Theorem D (Proposition 7.9). Let Δ be a pure (d-1)-dimensional simplicial complex. The vector $\gamma(R_{\Delta})$ is given by $\gamma_0(R_{\Delta}) = 1$, $\gamma_1(R_{\Delta}) = 2h_1(\Delta) + \sum_{k=2}^d h_k(\Delta)$, and

$$\gamma_i(R_{\Delta}) = (-1)^{i-1} \sum_{k=2i-1}^d \left(\binom{k-i}{i-1} + \binom{k-i-1}{i-2} \right) h_k(\Delta)$$

for $2 \le i \le \lfloor \frac{d+1}{2} \rfloor$. In particular, if $h(\Delta)$ has nonnegative entries, then $(-1)^{i-1}\gamma_i(R_{\Delta}) \ge 0$.

As the *h*-vector of a Cohen–Macaulay complex has nonnegative entries, Theorem D and Theorem B show that whenever the algebra R_{Δ} is Koszul, its γ -vector has entries which alternate in sign. In particular, when Δ is a (d-1)-dimensional Cohen–Macaulay flag complex with $d \equiv 3 \mod 4$, the Charney–Davis quantity $\gamma_{\lfloor \frac{d+1}{2} \rfloor}(R_{\Delta})$ is nonpositive. For an explicit example where $\gamma_{\lfloor \frac{d+1}{2} \rfloor}(R_{\Delta}) < 0$, see Example 7.5.

Finally, in Section 8 we highlight a connection to the literature, showing that a Gorenstein algebra defined via apolarity by Gondim and Zappalà [GZ18] is an Artinian reduction of R_{Δ} in characteristic zero. As a consequence, Theorem B applies to the algebras in [GZ18] as well. Moreover, we show that Theorem C also goes through the Artinian reduction, and thus provides a criterion to decide if the algebras studied by Gondim and Zappalà have a quadratic Gröbner basis. Finally, in Remark 8.5 we fix a previous characterization [GZ18, Theorem 3.5] of quadraticity for such algebras.

2. Preliminaries

2.1. Graded algebras and modules. Let us fix a field \mathbb{F} and consider a standard graded \mathbb{F} -algebra $R = \mathbb{F}[x_1, \ldots, x_n]/I$, for some homogeneous ideal *I*. The *Hilbert series* of *R* is a rational function of the form

$$\operatorname{Hilb}(R,t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^d},$$

with $\sum_i h_i \neq 0$. The numerator of Hilb(R, t) expressed as above is the *h*-polynomial of R, and the integer sequence $h(R) = (h_0, \ldots, h_s)$ is the *h*-vector of R. In general, the entries of the *h*-vector can be either positive or negative.

If M is a finitely generated graded module over R, we can consider its *minimal graded* free resolution. This is the unique (up to isomorphism of chain complexes) complex of free R-modules and degree zero maps

(2.1)
$$\cdots \xrightarrow{\partial_{i+1}} \bigoplus_{j} R(-j)^{\beta_{i,j}} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} \bigoplus_{j} R(-j)^{\beta_{1,j}} \xrightarrow{\partial_1} \bigoplus_{j} R(-j)^{\beta_{0,j}} \to 0$$

which is exact in all positions but the zeroth, where $\operatorname{coker}(\partial_1) \cong M$, and is such that every ∂_i can be represented as a matrix of zeros and homogeneous polynomials of positive degree. The numbers $\beta_{i,j}$ in (2.1) are known as the graded Betti numbers of M. In a more functorial fashion, they can be written as $\beta_{i,j} = \beta_{i,j}^R(M) = \dim_{\mathbb{F}} \operatorname{Tor}_i^R(M, \mathbb{F})_j$. This highlights a second way to compute graded Betti numbers: namely, thanks to the commutativity of Tor, one can tensor a minimal resolution of the R-module \mathbb{F} with M, and then take the homology of the resulting chain complex. We will use this standard observation in the next sections. The quantity $\sup\{j-i: \beta_{i,j}^R(M) \neq 0\}$ is known as the *Castelnuovo–Mumford regularity* of M, denoted by $\operatorname{reg}_R(M)$.

The standard tool to record the information about graded Betti numbers is the *Poincaré* series of M over R, defined as

(2.2)
$$P_{R}^{M}(s,t) = \sum_{i,j} \beta_{i,j}^{R}(M) s^{j} t^{i}.$$

We will omit the superscript when M is the residue field $\mathbb{F} = R/(x_1, \ldots, x_n)$.

If M is generated in a single degree, we say that the minimal resolution of M is *linear for* 0 *steps*; if moreover all the nonzero entries of the matrices ∂_i in (2.1) are linear forms for every $1 \le i \le k$, we say that the minimal resolution of M is *linear for* k *steps*. If the nonzero entries of all matrices in the resolution are linear forms, we say that M has a *linear resolution*.

By a celebrated theorem of Hilbert, minimal resolutions of modules over the polynomial ring $S = \mathbb{F}[x_1, \ldots, x_n]$ are finite: the length of the minimal free resolution of M as an S-module is known as the *projective dimension* of M and denoted by $pd_S(M)$. When M = R, it is known that $n - d \leq pd_S(R) \leq n$, where d is the Krull dimension of R. Algebras attaining the above lower bound are well studied:

Definition 2.1. A *d*-dimensional standard graded \mathbb{F} -algebra R = S/I is *Cohen-Macaulay* over \mathbb{F} if $pd_S(R) = n - d$.

Cohen–Macaulay algebras play an important role in commutative algebra, algebraic geometry and combinatorics [BH98, Sta96], and so does the special subclass consisting of *Gorenstein* algebras.

Definition 2.2. A *d*-dimensional standard graded \mathbb{F} -algebra R = S/I is *Gorenstein* over \mathbb{F} if it is Cohen–Macaulay and $\dim_{\mathbb{F}} \operatorname{Tor}_{n-d}^{S}(R, \mathbb{F}) = 1$.

We conclude this section by introducing an object that will play a crucial role in the rest of the paper.

Definition 2.3. Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a standard graded polynomial ring and let R = S/I be a *d*-dimensional standard graded Cohen-Macaulay \mathbb{F} -algebra. Then:

- the canonical module of S is the \mathbb{Z}^n -graded module $\omega_S \coloneqq S(-1, -1, \dots, -1)$ (hence, if we are only interested in the \mathbb{Z} -graded structure, $\omega_S \cong S(-n)$);
- the canonical module of R is the \mathbb{Z} -graded module $\omega_R \coloneqq \operatorname{Ext}_S^{n-d}(R, \omega_S)$. If I happens to be \mathbb{Z}^n -graded, then so is ω_R ;
- the *a*-invariant of R is $a(R) \coloneqq -\min\{j \in \mathbb{Z} : (\omega_R)_j \neq 0\};$
- the algebra R is *level* if ω_R is generated in a single \mathbb{Z} -degree.

The minimal free resolution of ω_R as an S-module can be obtained dualizing a minimal free resolution of R via the contravariant functor $\operatorname{Hom}_S(-,\omega_S)$.

2.2. Simplicial complexes. Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$, i.e., a collection of subsets of [n] closed under inclusion. Elements of Δ are called *faces* and inclusion-maximal faces are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set all facets of Δ . The dimension af a face equals its cardinality minus one, and the dimension of Δ is the maximal dimension of one of its faces; if all facets have the same dimension, we say that Δ is *pure*. We record the number of faces in each dimension in a vector $f(\Delta)$ called the *f*-vector of Δ . If Δ is (d-1)-dimensional, then $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_{-1} = 1$, unless $\Delta = \emptyset$.

If F_1, \ldots, F_r are subsets of [n], we will denote by $\langle F_1, \ldots, F_r \rangle$ the simplicial complex generated by F_1, \ldots, F_r , i.e. the smallest simplicial complex on [n] containing F_1, \ldots, F_r .

Fix now a field \mathbb{F} . We denote by $I_{\Delta} \subseteq \mathbb{F}[x_1, \ldots, x_n]$ the squarefree monomial ideal generated by the monomials supported on the complement of Δ in $2^{[n]}$, namely

$$I_{\Delta} \coloneqq \left(\mathbf{x}^F : F \notin \Delta \right),$$

where $\mathbf{x}^F = \prod_{i \in F} x_i$.

The ideal I_{Δ} is known as the *Stanley-Reisner ideal* of Δ , and the graded \mathbb{F} -algebra $\mathbb{F}[\Delta] = \mathbb{F}[\mathbf{x}]/I_{\Delta}$ is the associated *Stanley-Reisner ring*. There is a rich interplay between combinatorial properties of Δ and algebraic features of $\mathbb{F}[\Delta]$: for instance, the Krull dimension of $\mathbb{F}[\Delta]$ is the dimension of Δ plus one. We can actually do better than this, as we can read off the whole *f*-vector of Δ from the Hilbert series of $\mathbb{F}[\Delta]$ and vice versa: the *h*-vector $h(\Delta) = h(\mathbb{F}[\Delta]) = (h_0, \ldots, h_d)$ is an invertible integer linear transformation of $f(\Delta)$. More precisely:

(2.3)
$$h_i = \sum_{j=0}^{i} (-1)^{i-j} {\binom{d-j}{d-i}} f_{j-1}$$

(2.4)
$$f_{i-1} = \sum_{j=0}^{i} {\binom{d-j}{d-i}} h_j.$$

Local properties of a simplicial complex around a face $F \in \Delta$ are described by the *link* of F, defined as

$$lk_{\Delta}(F) := \{ G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta \}.$$

If Δ is a pure (d-1)-dimensional complex, then $lk_{\Delta}(F)$ is a (pure) (d-|F|-1)-dimensional simplicial complex. As a slogan, the simpler the homology of the links of Δ is, the nicer Δ is. To make this precise, we give the following definition after Murai and Terai [MT09] (even though it was already implicit in work by Terai and Yanagawa [Yan00]):

Definition 2.4. Let $r \ge 1$. A simplicial complex Δ satisfies the (combinatorial) Serre condition (S_r) with respect to a field \mathbb{F} if

$$H_i(\mathrm{lk}_{\Delta}(F);\mathbb{F})=0$$

for every $F \in \Delta$ and for every $i < \min\{r-1, \dim(\operatorname{lk}_{\Delta}(F))\}$. A (d-1)-dimensional simplicial complex satisfying property (S_d) over \mathbb{F} is called *Cohen–Macaulay* over \mathbb{F} .

Definition 2.4 deserves some comments. It is clear that every complex Δ must satisfy (S_1) and that, if Δ has (S_r) , then it has (S_i) for every $1 \leq i \leq r$. Moreover, Δ satisfies property (S_2) if and only if it is pure and the link of any face of codimension at least two is connected. In particular, property (S_2) does not depend on the field. For $r \geq 2$, condition (S_r) states that the link of any face of codimension at most r is allowed to have nonvanishing homology only in top homological degree, while lower-dimensional faces need to have vanishing homology up to homological degree r-2. The combinatorial Serre condition (S_r) for Δ (with respect to \mathbb{F}) is equivalent to the usual algebraic Serre condition (S_r) for the Stanley-Reisner ring $\mathbb{F}[\Delta]$. In particular, a (d-1)-dimensional complex Δ is (S_d) if and only if $\mathbb{F}[\Delta]$ satisfies the algebraic (S_d) condition, which in turn means that $\mathbb{F}[\Delta]$ is a Cohen-Macaulay ring. This fact is known as *Reisner's criterion* [Rei76]. The equivalence between the algebraic and combinatorial Serre conditions for simplicial complexes is known and proved for instance in [Ter07, end of Section 1], but we will sketch a proof here for the interested reader.

Proposition 2.5. Let $r \ge 1$ and let Δ be a simplicial complex on [n]. Then Δ satisfies the combinatorial (S_r) condition (with respect to \mathbb{F}) if and only if the Stanley–Reisner ring $\mathbb{F}[\Delta]$ satisfies the usual algebraic (S_r) condition, i.e.

(2.5) $\operatorname{depth} \mathbb{F}[\Delta]_{\mathfrak{p}} \ge \min\{r, \dim \mathbb{F}[\Delta]_{\mathfrak{p}}\} \qquad for \ every \ \mathfrak{p} \in \operatorname{Spec}(\mathbb{F}[\Delta]).$

Proof: For a Noetherian ring R, the algebraic (S_1) condition is known to be equivalent to the absence of embedded primes in R, which is always the case for the reduced ring $\mathbb{F}[\Delta]$. On the other hand, the combinatorial (S_1) condition is met by every simplicial complex. Let now $r \geq 2$, and denote by S the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$. Note that both the combinatorial and the algebraic (S_2) conditions imply that $\mathbb{F}[\Delta]$ is equidimensional, i.e. Δ is pure: for the algebraic statement, see [Har62, Remark 2.4.1]. By [DMV19, Proposition 2.11], (2.5) is then equivalent to

$$\dim \operatorname{Ext}_{S}^{n-j}(\mathbb{F}[\Delta], \omega_{S}) \leq j - r \qquad \text{for every } j < \dim \mathbb{F}[\Delta]$$

which, since $\operatorname{Ext}_{S}^{n-j}(\mathbb{F}[\Delta], \omega_{S})$ is a squarefree module [Yan00], is in turn equivalent to

$$\operatorname{Ext}_{S}^{n-j}(\mathbb{F}[\Delta], \omega_{S})_{F} = 0 \quad \text{for every } j < \dim(\Delta) + 1 \text{ and } F \in \{0, 1\}^{n} \text{ with } |F| > j - r.$$

It is a useful fact of Stanley–Reisner theory that $\operatorname{Ext}_{S}^{n-j}(\mathbb{F}[\Delta], \omega_{S})_{F} \cong H_{j-|F|-1}(\operatorname{lk}_{\Delta} F, \mathbb{F})$: see for instance [Yan00, Proposition 3.1]. To conclude it is then enough to set i = j - |F| - 1 and note that, since Δ is pure, dim $\operatorname{lk}_{\Delta} F = \dim \Delta - |F|$.

The Alexander dual of Δ is the simplicial complex Δ^* on the same vertex set as Δ and with Stanley–Reisner ideal

$$I_{\Delta^*} = (\mathbf{x}^{[n] \smallsetminus G} : G \in \Delta),$$

where $\mathbf{x}^{[n] \setminus G} = \prod_{i \in [n] \setminus G} x_i$. A minimal generating set for I_{Δ^*} is hence given by monomials supported on the complements of facets of Δ in [n].

Eagon and Reiner [ER98] related the Cohen–Macaulay property of Δ to the Castelnuovo– Mumford regularity of the Alexander dual ideal I_{Δ^*} . This result was extended to (S_r) conditions by Terai and Yanagawa [Yan00].

Theorem 2.6. [ER98, Theorem 3], [Yan00, Corollary 3.7] Let Δ be a (d-1)-dimensional simplicial complex on [n]. For every $2 \leq r \leq d$, the complex Δ satisfies property (S_r) if and only if I_{Δ^*} has a resolution as an $\mathbb{F}[x_1, \ldots, x_n]$ -module which is linear for r-1 steps. In particular, Δ is Cohen–Macaulay if and only if I_{Δ^*} has a linear $\mathbb{F}[x_1, \ldots, x_n]$ -resolution.

2.3. Nagata idealization. A theme of this paper will be the construction of Gorenstein algebras from "good enough" objects. The required technical tool is a well-known operation called *idealization*, made popular by Nagata. This subsection follows closely [MSS21a, Section 3], and we refer the reader to that for more information.

Definition 2.7. The *idealization* of an A-module M, denoted $A \ltimes M$, is the A-algebra with underlying module $A \oplus M$ and multiplication defined by $(a_1, m_1) \cdot (a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$.

This operation is called idealization because it turns every submodule N of M into an ideal $\{0\} \ltimes N$ of $A \ltimes M$. For the rest of the paper, the ring A will be a standard graded \mathbb{F} -algebra. If M is graded, then so is $A \ltimes M$, by setting $(A \ltimes M)_j = A_j \oplus M_j$; moreover, $A \ltimes M$ is standard graded if and only if M is generated in degree one [MSS21a, Remark 3.1]. Note that, if A and M are \mathbb{Z}^n -graded, then $A \ltimes M$ inherits the \mathbb{Z}^n -grading.

We record here for later use a lemma that relates the homological information of an idealization to that of its building blocks.

Lemma 2.8. [Gul72, Theorem 2] Let A be a standard graded \mathbb{F} -algebra and let M be a finitely generated graded A-module generated in degree one. Then

(2.6)
$$P_{A \ltimes M}(s,t) = \frac{P_A(s,t)}{1 - t P_A^M(s,t)},$$

where $P_A^M(s,t)$ is the Poincaré series of M as an A-module, as defined in (2.2).

Definition 2.9. A standard graded Cohen–Macaulay \mathbb{F} -algebra A is superlevel if it is level and its canonical module ω_A has a linear presentation as an A-module, i.e., there is an exact sequence

$$A(a-1)^b \stackrel{\varphi_1}{\to} A(a)^g \stackrel{\varphi_0}{\to} \omega_A \to 0,$$

with a = a(A) as in Definition 2.3.

We end this subsection by collecting several results proved in [MSS21a, Section 3]. Parts iv and v follow immediately by analyzing the graded structure of the idealization and using the formula for the Hilbert series of the canonical module [BH98, Corollary 4.4.6].

Theorem 2.10 ([MSS21a, Proposition 3.2, Lemma 3.3]). Let A = S/I be a standard graded and level \mathbb{F} -algebra and let $\widetilde{A} := A \ltimes \omega_A(-a(A) - 1)$. The following statements hold:

- *i.* \widetilde{A} is a standard graded and Gorenstein \mathbb{F} -algebra;
- ii. if ω_A is minimally generated by g elements, then

(2.7)
$$\widetilde{A} = \frac{S[z_1, \dots z_g]}{I + \mathcal{L} + (z_1, \dots, z_g)^2},$$

where

(2.8)
$$\mathcal{L} := \left(\sum_{i=1}^{g} f_i z_i : (f_1, \dots, f_g) \in \operatorname{Syz}_1^S(\omega_A)\right);$$

iii. if A is quadratic, then \widetilde{A} is quadratic if and only if A is superlevel. iv. $h_i(\widetilde{A}) = h_i(A) + h_{s-i+1}(A)$ for every 1 < i < s - 1. v. $\dim(\widetilde{A}) = \dim(A)$. 2.4. Koszul algebras. The last ingredient we need is a special class of standard graded algebras called *Koszul algebras*: for a survey, we direct the reader to [CDNR13].

Definition 2.11. A standard graded \mathbb{F} -algebra A is *Koszul* (over \mathbb{F}) if \mathbb{F} has a linear resolution as an A-module.

In the above definition, \mathbb{F} is identified with the quotient of A by its maximal homogeneous ideal. Observe that, by definition, A is Koszul if and only if $P_A(s,t) \in \mathbb{Z}[[st]]$.

Due to the fact that a minimal resolution of \mathbb{F} as an A-module is typically infinite, it is in general very hard to prove that a certain algebra A is Koszul. However, the following is well-known.

Proposition 2.12. Let $A = \mathbb{F}[x_1, \ldots, x_n]/I$ be a standard graded \mathbb{F} -algebra and assume without loss of generality that $I \subseteq (x_1, \ldots, x_n)^2$. Then:

- *if A is Koszul, then I is generated in degree 2;*
- if I has a Gröbner basis of quadrics, then A is Koszul.

Having a Gröbner basis of quadrics is actually a much stronger condition than Koszulness, and we will see in Section 6 how this translates combinatorially for the algebras studied in the present paper.

We close this section by a statement highlighting how Koszulness interacts with suitable idealizations.

Lemma 2.13. Let A be a Koszul \mathbb{F} -algebra and let M be a finitely generated graded A-module generated in degree one. Then the residue field \mathbb{F} has an $(A \ltimes M)$ -resolution which is linear for k steps if and only if the A-module M has a resolution which is linear for k-1 steps. In particular, $A \ltimes M$ is Koszul if and only if M has a linear A-resolution.

Proof: The statement follows from comparing the coefficients in (2.6). Since A is Koszul, one has that $P_A(s,t) \in \mathbb{Z}[[st]]$. Let $P_{A \ltimes M}(s,t) = \sum_{i,j} b_{i,j} s^j t^i$ and $P_A^M(s,t) = \sum_{i,j} c_{i,j} s^j t^i$. After multiplying both sides of (2.6) by $1 - t P_A^M(s,t)$, we analyze the coefficient of $s^j t^i$ for every i, j with i < j, obtaining that

(2.9)
$$b_{i,j} - \sum_{h,\ell} b_{h,\ell} c_{i-h-1,j-\ell} = 0$$

Note that $b_{0,0} = 1$ and $b_{0,j} = 0$ for every j > 0. We want to show that

$$b_{i,j} = 0$$
 for every $1 \le i \le k$, $j > i \Leftrightarrow c_{i-1,j} = 0$ for every $1 \le i \le k$, $j > i$.

To prove the left to right arrow it is enough to observe that, for every $1 \le i \le k$ and j > i, (2.9) implies that $0 = b_{i,j} \ge b_{0,0}c_{i-1,j} = c_{i-1,j} \ge 0$.

Conversely, let us now assume that $c_{i-1,j} = 0$ for every $1 \le i \le k$ and j > i, and assume by contradiction that there exists $1 \le m \le k$ such that $b_{m,q} > 0$ for some q > m. We can then pick this m to be the smallest possible with respect to this property. Because of (2.9), it holds that $b_{m,q} = \sum_{h,\ell} b_{h,\ell}c_{m-h-1,q-\ell}$. Since every h in the sum must be strictly smaller than m, it follows that $b_{h,\ell}$ vanishes whenever $h < \ell$. Hence, $0 < b_{m,q} = \sum_h b_{h,h}c_{m-h-1,q-h}$, and hence there must be at least one nonvanishing $c_{m-h-1,q-h}$, contradicting the hypothesis.

3. BIER BALLS AND SPHERES

In this section we recall a simple construction which associates with any simplicial complex on [n] (other than the full simplex) an (n-1)-dimensional PL ball which is a proper subcomplex of the boundary complex of the *n*-dimensional cross-polytope. Such complexes were notably studied in higher generality by Murai [Mur11].

Definition 3.1. Let Δ be a simplicial complex on n vertices, and let $I_{\Delta} \subseteq \mathbb{F}[\mathbf{x}]$ be its Stanley–Reisner ideal. We define a new complex Γ on 2n vertices by

$$I_{\Gamma} := I_{\Delta} + (x_i y_i : i \in [n]) \subseteq \mathbb{F}[\mathbf{x}, \mathbf{y}].$$

If Δ is not the (n-1)-dimensional simplex, we will call Γ the *Bier ball* associated with Δ : see Proposition 3.4 below.

Remark 3.2. Most of the extant literature does not really deal with Bier balls, but rather with the PL spheres that bound them: these are known as *Bier spheres* after Thomas Bier, who introduced them in [Bie92] (see also [BPSZ05, DFN19]).

Remark 3.3. For Δ flag, the construction in Definition 3.1 has already appeared in the literature several times, notably in [CV13, Proposition 4.1]. The earliest reference we are aware of is [Vil90, Proposition 2.2], which has a slightly different take on the subject. Indeed, if Δ is flag, the ideal I_{Δ} can be seen as the edge ideal of a graph G, and I_{Γ} becomes the edge ideal of the graph obtained from G by adding extra leaves (or "whiskers") to every vertex.

Proposition 3.4. Let Δ be a simplicial complex on [n] and let Γ be as in Definition 3.1. *Then:*

- i. Γ is a shellable (n-1)-dimensional simplicial complex;
- *ii.* $h(\Gamma) = f(\Delta)$;
- iii. if Δ is not the (n-1)-simplex, then Γ is a PL (n-1)-ball and its boundary complex $\partial \Gamma$ is the PL (n-2)-sphere known as the Bier sphere associated with Δ [Bie92];
- iv. the canonical module $\omega_{\mathbb{F}[\Gamma]}$ is isomorphic as an $\mathbb{F}[\Gamma]$ -module to the ideal of $\mathbb{F}[\Gamma]$ generated by the monomials $\mathbf{y}^{[n] \setminus F}$, where F ranges over all facets of Δ (here \mathbf{y}^{\varnothing} is equal to 1). In other words, the canonical module of $\mathbb{F}[\Gamma]$ is the image of the Alexander dual ideal of Δ (in the y-variables) under the projection $\mathbb{F}[\mathbf{x}, \mathbf{y}] \twoheadrightarrow \mathbb{F}[\Gamma]$.

We note that Proposition 3.4 was essentially already proved in [Mur11, Lemma 1.4, Theorem 1.14 and Theorem 3.6], where a more general construction associating a Bier ball with every multicomplex is studied. However, we will sketch our approach below for the reader's benefit. First of all, note that the facets of Γ are precisely the sets $F^{\sharp} = \{x_i \mid i \in F\} \cup \{y_j \mid j \notin F\}$, where F ranges over all faces of Δ . Order (partially) the facets of Γ so that $F^{\sharp} < G^{\sharp}$ when dim $(F) < \dim(G)$. It is left to the reader to show that any total order refining < gives a shelling order for Γ (for the definition of a shelling order, see Definition 6.1). Moreover, since the minimal new face we are adding at each step of the shelling has cardinality |F|, the claim about the *h*-vector of $\mathbb{F}[\Gamma]$ follows from [Sta96, Proposition III.2.3].

Now note that every face of Γ of codimension 1 is contained in at most two facets. Indeed, every codimension 1 face is of the form $F^{\sharp} \setminus \{v_i\}$, where $i \in [n]$ and v_i is either x_i or y_i . As $\{x_j, y_j\}$ is a nonface of Γ for every $j \in [n]$, one can extend the given codimension 1 face to a facet only by adding either y_i (which always yields a facet) or x_i (which might yield a facet, depending on Δ). By [Bjö95, Theorem 11.4], the shellable complex Γ must then be either a PL ball or a PL sphere, with the former case occurring when there exists a codimension 1 face contained in exactly one facet. This happens whenever Δ has at least a missing face, i.e., it is not the full (n-1)-simplex.

Finally, the last statement follows from the description of the canonical module of any homology ball (see for instance [BH98, Theorem 5.7.1]), which says that $\omega_{\mathbb{F}[\Gamma]}$ is the image of $I_{\partial\Gamma}$ under the projection $\mathbb{F}[\mathbf{x}, \mathbf{y}] \twoheadrightarrow \mathbb{F}[\Gamma]$. Recalling that $\partial\Delta$ consists of the codimension 1 faces of Γ contained in exactly one facet, one checks that $I_{\partial\Gamma} = I_{\Gamma} + (\mathbf{y}^{[n] \setminus F} | F \in \mathcal{F}(\Delta))$. This is precisely the definition of a Bier sphere, see for instance [DFN19, Corollary 5.3]. Note that if Δ is the (n-1)-simplex, then Γ is actually a sphere, $\mathbb{F}[\Gamma]$ is Gorenstein and $\omega_{\mathbb{F}[\Gamma]} \cong \mathbb{F}[\Gamma]$; since $\mathbf{y}^{\varnothing} = 1$, we are done also in this case.

The simplicial complex Γ is actually vertex decomposable [Mur11, Remark 1.8], a property which is stronger than shellability. Moreover, any Bier (n-1)-ball or (n-2)-sphere is naturally a subcomplex of the boundary complex \diamond_n of the *n*-cross-polytope, with $I_{\diamond_n} = (x_i y_i : i \in [n])$.

Example 3.5. Let n = 3 and Δ be the simplicial complex $\{\emptyset, x_1, x_2, x_3, x_1x_2\}$. Then $I_{\Gamma} = (x_1x_3, x_2x_3) + (x_1y_1, x_2y_2, x_3y_3)$, and Γ is the 2-dimensional pure shellable homology ball with shelling order

 $y_1y_2y_3 \prec \underline{x_1}y_2y_3 \prec y_1\underline{x_2}y_3 \prec y_1y_2\underline{x_3} \prec \underline{x_1x_2}y_3.$

Observe that $h(\Gamma) = f(\Delta) = (1,3,1)$. The complex $\partial \Gamma = \langle x_1x_2, x_2y_1, y_1x_3, y_2x_3, x_1y_2 \rangle$ is a 5-cycle.



FIGURE 1. The simplicial complex Δ as in Example 3.5 and the associated Bier ball Γ .

Remark 3.6. The Bier sphere $\partial \Gamma$ in Example 3.5 is flag, but this is usually not the case in general. In [HK12], the authors prove that there are only 8 possible complexes Δ which give rise to flag Bier spheres.

4. From Bier balls to Gorenstein Algebras: the ring R_{Δ}

For the rest of this article we will be concerned with idealizing the canonical module of the Stanley–Reisner ring of a Bier ball. We set some notation for the rest of the paper.

Notation 4.1. From now on, if not specified otherwise:

• Δ is a pure (d-1)-dimensional simplicial complex on n vertices, and its Stanley-Reisner ideal I_{Δ} lives in $\mathbb{F}[x_1, \ldots, x_n]$;

- S is the polynomial ring $\mathbb{F}[\mathbf{x}, \mathbf{y}] = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_n];$
- Γ is the (n-1)-dimensional Bier ball obtained from Δ as in Definition 3.1 (with a slight abuse of notation, we will use the name "Bier ball" also when Δ is the full simplex);
- R_{Δ} is the *n*-dimensional standard graded Gorenstein \mathbb{F} -algebra $\mathbb{F}[\overline{\Gamma}] = \mathbb{F}[\Gamma] \ltimes \omega_{\mathbb{F}[\Gamma]}(n d 1)$, as in Theorem 2.10 (note that, by Proposition 3.4.iv, $\mathbb{F}[\Gamma]$ is level and $a(\mathbb{F}[\Gamma]) = -(n d) = d n$);
- given two facets F_1 and F_2 of Δ , we will denote by b_{F_1,F_2} the binomial $\mathbf{y}^{F_1 \smallsetminus F_2} z_{F_1} \mathbf{y}^{F_2 \smallsetminus F_1} z_{F_2}$ inside $S[z_F \mid F \in \mathcal{F}(\Delta)]$. Notice that $b_{F_2,F_1} = -b_{F_1,F_2}$, and the case where F_1 and F_2 coincide yields the zero binomial.

Remark 4.2. The construction of R_{Δ} shows that if $f = (1, f_0, \ldots, f_{d-1})$ is the *f*-vector of a pure simplicial complex, then $(1, f_0 + f_{d-1}, f_1 + f_{d-2}, \ldots, f_{d-1} + f_0, 1)$ is the *h*-vector of a Gorenstein standard graded algebra. In particular, it is an *M*-sequence (see [Sta96]). We will see a stronger condition on Δ for this vector to be the *h*-vector of a quadratic – or even Koszul – Gorenstein algebra.

Proposition 4.3. Let Δ be a pure simplicial complex of dimension d-1. Then

$$R_{\Delta} = \frac{S[z_F \mid F \in \mathcal{F}(\Delta)]}{I_{\Gamma} + \mathcal{L} + (z_F \mid F \in \mathcal{F}(\Delta))^2},$$

where

$$\mathcal{L} = (x_i z_F \mid F \in \mathcal{F}(\Delta), \ i \notin F) + (b_{F_1, F_2} \mid F_1, F_2 \in \mathcal{F}(\Delta), F_1 \neq F_2).$$

In particular, the defining ideal of R_{Δ} is generated by

- \mathbf{x}^{N_i} , where N_i is a minimal nonface of Δ ,
- $x_i y_i$, where *i* ranges between 1 and *n*,
- $z_{F_1}z_{F_2}$, where F_1 and F_2 are (possibly coincident) facets of Δ ,
- $x_i z_F$, where F is a facet of Δ and $i \notin F$,
- b_{F_1,F_2} , where F_1 and F_2 are distinct facets of Δ and b_{F_1,F_2} is the binomial defined in Notation 4.1,

and such a presentation does not depend on the characteristic of the field \mathbb{F} .

Proof: Since the defining ideal of the claimed presentation of R_{Δ} involves only monomials and binomials of the form $\mathbf{m} - \mathbf{m'}$, such a presentation must hold in every characteristic, see e.g. [D'A17, Lemma A.1.(iii)].

By Theorem 2.10.ii, in order to find a presentation for R_{Δ} it is enough to compute the syzygies of $\omega_{\mathbb{F}[\Gamma]}$ as an *S*-module. (Since we are quotienting by I_{Γ} in (2.7), we are actually investigating the $\mathbb{F}[\Gamma]$ -syzygies of $\omega_{\mathbb{F}[\Gamma]}$, as $\operatorname{Syz}_1^{\mathbb{F}[\Gamma]}(\omega_{\mathbb{F}[\Gamma]}) \cong \operatorname{Syz}_1^S(\omega_{\mathbb{F}[\Gamma]})/I_{\Gamma}\operatorname{Syz}_1^S(\omega_{\mathbb{F}[\Gamma]})$.) By Proposition 3.4.iv and by the usual Alexander duality (see e.g. [MS05, Proposition 1.37]), the canonical module $\omega_{\mathbb{F}[\Gamma]}$ can be identified with the monomial ideal of $\mathbb{F}[\Gamma] = S/I_{\Gamma}$ generated by $\{\overline{\mathbf{y}^{F^c}}: F \text{ facet of } \Delta\}$. Here and until the end of the proof, we will use the shorthand F^c to denote $[n] \smallsetminus F$, the complement of F, while the overline denotes the class of \mathbf{y}^{F^c} in $\mathbb{F}[\Gamma]$. Hence, the purity of Δ implies that all such generators have the same degree n - d. Calling F_1, \ldots, F_r the facets of Δ , our task is to compute the kernel of the map of S-modules

$$S(-n+d)^{|\mathcal{F}(\Delta)|} \to S/I_{\Gamma}$$
$$\mathbf{1}_{F_i} \mapsto \overline{\mathbf{y}^{F_i^c}}$$

where each $\mathbf{1}_{F_i}$ has multidegree F_i^c . To do so, we follow [GP08, Remark 2.5.6]: let N_1, \ldots, N_s be the minimal nonfaces of Δ and let A be the 1-by-(r + n + s) matrix

$$A = (\mathbf{y}^{F_1^c} \dots \mathbf{y}^{F_r^c} \mid x_1 y_1 \dots x_n y_n \mid \mathbf{x}^{N_1} \dots \mathbf{x}^{N_s}).$$

Ignoring the degrees, the matrix A represents a map $S^{r+n+s} \to S$. The S-module obtained by projecting the syzygy module $\operatorname{Syz}^S(A)$ on its first r coordinates is precisely $\operatorname{Syz}_1^S(\omega_{\mathbb{F}[\Gamma]})$. Since all entries of A are monomials, Schreyer's theorem [Eis95, Theorem 15.10] implies that computing syzygies between such monomials is enough to seize $\operatorname{Syz}^S(A)$ and hence $\operatorname{Syz}_1^S(\omega_{\mathbb{F}[\Gamma]})$. Let a and b be two of the entries in A. If a and b both belong to I_{Γ} , the syzygy between them will live in the last n + s components of S^{r+n+s} and will hence not matter for our final aim. From now on, assume thus that $a = \mathbf{y}^{F_i^c}$ for some i.

- If $b \in I_{\Gamma}$ and gcd(a, b) = 1, then the syzygy between a and b will vanish after projecting and taking the quotient modulo I_{Γ} . In particular, this syzygy will not be involved in the presentation of R_{Δ} .
- If $b \in I_{\Gamma}$ and gcd(a,b) > 1, then it must be that $b = x_j y_j$ and y_j divides $\mathbf{y}^{F_i^c}$, i.e. $j \notin F_i$. This gives rise to the syzygy $x_j \mathbf{1}_{F_i}$.
- Finally, assume $b = \mathbf{y}^{F_j^c}$ for some $j \neq i$. Then, noting that $F_j^c \smallsetminus F_i^c = F_i \smallsetminus F_j$, we obtain the syzygy $\mathbf{y}^{F_i \smallsetminus F_j} \mathbf{1}_{F_i} \mathbf{y}^{F_j \smallsetminus F_i} \mathbf{1}_{F_j}$ and this ends the proof.

As an application, we determine when R_{Δ} is quadratic.

Proposition 4.4. Let Δ be a pure flag simplicial complex. Then R_{Δ} is quadratic if and only if Δ is (S_2) .

Proof: Proceeding as in the proof of Proposition 4.3, one sees that $\omega_{\mathbb{F}[\Gamma]}$ has first linear $\mathbb{F}[\Gamma]$ -syzygies if and only if the Alexander dual ideal of Δ has first linear $\mathbb{F}[\mathbf{y}]$ -syzygies. By Theorem 2.6, this happens precisely when Δ is (S_2) .

Proposition 4.4 will be vastly generalized in the next section: see Theorem 5.1.

Proposition 4.5. The collection \mathcal{U} of monomials and binomials listed in Proposition 4.3 is a universal Gröbner basis for the ideal defining R_{Δ} in any characteristic.

Proof: Proposition 4.3 says that the defining ideal of R_{Δ} admits a generating set \mathcal{G} consisting of monomials and binomials of the form $\mathbf{m} - \mathbf{m'}$. Any Gröbner basis obtained by applying Buchberger's algorithm to the generating set \mathcal{U} will have the same property, and will hence be characteristic-independent.

It is then enough to show that, for any term order <, any possible S-polynomial reduces to zero with respect to \mathcal{U} . The only interesting situation arises when we consider the Spolynomial of two distinct binomials b and b', as all the other generators are monomial. If the leading terms of b and b' do not share the same z-variable, then S(b,b') has degree two in the z-variables and hence reduces to zero with respect to \mathcal{U} . We can hence assume that $b = b_{F,G_1}$, $b' = b_{F,G_2}$, $\operatorname{in}_{<}(b_{F,G_1}) = \mathbf{y}^{F \smallsetminus G_1} z_F$ and $\operatorname{in}_{<}(b_{F,G_2}) = \mathbf{y}^{F \smallsetminus G_2} z_F$ (where F, G_1 and G_2 are three distinct facets of Δ). We then get that

$$S(b,b') = -\mathbf{y}^{(F \smallsetminus G_2) \smallsetminus (F \smallsetminus G_1)} \mathbf{y}^{G_1 \smallsetminus F} z_{G_1} + \mathbf{y}^{(F \smallsetminus G_1) \smallsetminus (F \smallsetminus G_2)} \mathbf{y}^{G_2 \smallsetminus F} z_{G_2}$$
$$= -\mathbf{y}^{(F \smallsetminus G_2) \cap G_1} \mathbf{y}^{G_1 \smallsetminus F} z_{G_1} + \mathbf{y}^{(F \smallsetminus G_1) \cap G_2} \mathbf{y}^{G_2 \smallsetminus F} z_{G_2}$$
$$= \mathbf{y}^{(G_1 \cap G_2) \smallsetminus F} b_{G_2, G_1}$$

and, since $b_{G_2,G_1} \in \mathcal{U}$, we are done.

Example 4.6. Let Δ be the pure flag simplicial complex with facet list {123, 234, 345}. The binomials arising from comparing facets of Δ are

$$b_{123,234} = y_1 z_{123} - y_4 z_{234},$$

$$b_{234,345} = y_2 z_{234} - y_5 z_{345},$$

$$b_{123,345} = y_1 y_2 z_{123} - y_4 y_5 z_{345}.$$

However, the binomial $b_{123,345}$ is redundant, since it can be written as $y_2b_{123,234} + y_4b_{234,345}$. Therefore, R_{Δ} can be presented as the quotient of $\mathbb{F}[x_1, \ldots, x_5, y_1, \ldots, y_5, z_{123}, z_{234}, z_{345}]$ by the ideal

$$(x_1x_4, x_1x_5, x_2x_5, x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, z_{123}^2, z_{234}^2, z_{345}^2, z_{123}z_{234}, z_{123}z_{345}, z_{234}z_{345}, x_4z_{123}, x_5z_{123}, x_1z_{234}, x_5z_{234}, x_1z_{345}, x_2z_{345}, y_4z_{234} - y_1z_{123}, y_5z_{345} - y_2z_{234}).$$

Note this agrees with Proposition 4.4: being Cohen-Macaulay, Δ is also (S_2) .

5. When is R_{Δ} Koszul?

We now wish to investigate the Koszul property of the \mathbb{F} -algebra R_{Δ} . The main result, whose proof is postponed to the end of the section, is the following:

Theorem 5.1. Let Δ be a flag (d-1)-dimensional simplicial complex and let $1 \leq k \leq d$. \mathbb{F} has a resolution as an R_{Δ} -module which is linear for k steps if and only if Δ satisfies property (S_k) . Moreover, if \mathbb{F} has an R_{Δ} -resolution which is linear for d steps, then it has a linear R_{Δ} -resolution.

Remark 5.2. The homological behaviour of R_{Δ} is especially remarkable, since for a general standard graded quadratic \mathbb{F} -algebra Q the linearity of the Q-resolution of \mathbb{F} might fail at an arbitrarily high homological position [Roo93].

As a corollary of Theorem 5.1 we obtain the following:

Corollary 5.3. Let Δ be a flag simplicial complex. Then R_{Δ} is Koszul over \mathbb{F} if and only if Δ is Cohen–Macaulay over \mathbb{F} .

5.1. The generalized Koszul complex for a quadratic Stanley–Reisner ring. Let Σ be a flag simplicial complex and consider its Stanley-Reisner ring $\mathbb{F}[\Sigma] = \mathbb{F}[z_i \mid \{i\} \in \Sigma]/I_{\Sigma}$ which is a Koszul algebra. It was already known to Fröberg [Frö75] how to explicitly describe a minimal $\mathbb{F}[\Sigma]$ -resolution of the residue field \mathbb{F} : we will briefly describe such a *generalized* Koszul complex here, following the treatment in [MP15, Section 8].

The Koszul algebra $\mathbb{F}[\Sigma]$ admits a *Koszul dual algebra* $\mathbb{F}[\Sigma]^!$, which is obtained by quotienting the noncommutative polynomial ring $\mathbb{F}\langle Z_i | \{i\} \in \Sigma \rangle$ by the relations

(5.1)
$$Z_i^2 \text{ for every vertex } i \text{ of } \Sigma$$
$$Z_i Z_j + Z_j Z_i \text{ for every edge } \{i, j\} \text{ of } \Sigma.$$

In particular, given a noncommutative monomial in the Z-variables, two consecutive distinct variables v and v' can anticommute unless $vv' \in I_{\Sigma}$.

Notation 5.4. In what follows we will think of noncommutative monomials \mathbf{w} in the Zvariables as *words*, and will denote by $[\mathbf{w}]$ the equivalence class of \mathbf{w} with respect to the relations (5.1) (sometimes writing $[-\mathbf{w}]$ to denote $-[\mathbf{w}]$). We will write $\mathbf{w} \| v$ for the word obtained by adding the letter v at the end of the word w. Finally, for any nonzero [w], the support $supp([\mathbf{w}])$ will be the signless commutative monomial obtained by multiplying together the z-variables corresponding to the letters of any representative of $[\mathbf{w}]$.

Definition 5.5. The generalized Koszul complex $(\mathbb{GK}_{\bullet}(\mathbb{F}[\Sigma]), \partial)$ is the chain complex of free $\mathbb{F}[\Sigma]$ -modules given by the following data:

- GK_j(F[Σ]) = F[Σ] ⊗_F F[Σ][!]_j;
 if j > 0 and w = Z_{i1}Z_{i2}...Z_{ij} is a j-letter word, the differential ∂ is given by

(5.2)
$$\partial(1 \otimes [\mathbf{w}]) = \sum_{k \in \text{head}(\mathbf{w})} (-1)^{k-1} z_{i_k} \otimes [\mathbf{w} \setminus \{Z_{i_k}\}],$$

where $\mathbf{w} \setminus \{Z_{i_k}\}$ is the (j-1)-letter word obtained by erasing the letter Z_{i_k} from \mathbf{w} and head (\mathbf{w}) is the set of those indices k for which there exists a representative of $[\mathbf{w}]$ with Z_{i_k} as its first letter. One can check that such a differential is indeed well-defined.

Example 5.6. Let $\Sigma = \{12, 23, 1, 2, 3, \emptyset\}$.

Then an \mathbb{F} -basis for $\mathbb{F}[\Sigma]_2^!$ is $\{[Z_1Z_2], [Z_1Z_3], [Z_2Z_3], [Z_3Z_1]\}, \text{ whereas an } \mathbb{F}$ -basis for $\mathbb{F}[\Sigma]_3^!$ is $\{[Z_1Z_2Z_3], [Z_1Z_3Z_1], [Z_2Z_3Z_1], [Z_3Z_1Z_3]\}.$ Note that, for instance, $[Z_1^2] = 0$ and $[Z_2Z_3Z_1] = -[Z_3Z_2Z_1] = [Z_3Z_1Z_2]$, but $[Z_2Z_3Z_1] \neq -[Z_3Z_1Z_2]$

 $[Z_2Z_1Z_3]$. If $\mathbf{w} = Z_2Z_3Z_1$, we have that head $(\mathbf{w}) = \{1, 2\}$ and

$$\partial(1\otimes [Z_2Z_3Z_1]) = z_2\otimes [Z_3Z_1] - z_3\otimes [Z_2Z_1] = z_2\otimes [Z_3Z_1] + z_3\otimes [Z_1Z_2].$$

The matrix describing the map $\mathbb{GK}_3(\mathbb{F}[\Sigma]) \xrightarrow{\partial} \mathbb{GK}_2(\mathbb{F}[\Sigma])$ in the proposed basis is

	$1 \otimes [Z_1 Z_2 Z_3]$	$1 \otimes [Z_1 Z_3 Z_1]$	$1 \otimes [Z_2 Z_3 Z_1]$	$1 \otimes [Z_3 Z_1 Z_3]$	
$1 \otimes [Z_1 Z_2]$	(0	0	z_3	0)
$1 \otimes [Z_1 Z_3]$	$-z_{2}$	0	0	z_3	
$1 \otimes [Z_2 Z_3]$	z_1	0	0	0	ŀ
$1 \otimes [Z_3 Z_1]$	0	z_1	z_2	0	J
		15			

5.2. Homological properties of Stanley–Reisner rings of flag Bier balls. For the rest of this section we fix a pure flag simplicial complex Δ and the associated Bier ball Γ .

In the following we will be interested in \mathbb{N}^{2n} -graded objects; we will reserve the word "multidegree" to denote either a vector in \mathbb{N}^{2n} or the associated monomial in $x_1, \ldots, x_n, y_1, \ldots, y_n$. When working with multidegrees, we shall distinguish between two types of variables.

Definition 5.7. Let **m** be a multidegree. A variable $v|\mathbf{m}$ is *red* with respect to **m** (and Γ) if there exists $v'|\mathbf{m}$ such that $vv' \in I_{\Gamma}$. Observe that necessarily $v \neq v'$. A variable $v|\mathbf{m}$ which is not red is called *blue*. We say that a multidegree is blue if all the variables in its support are blue, and red otherwise.

Remark 5.8. By definition, blue multidegrees correspond bijectively to monomials of $\mathbb{F}[\Gamma]$. In particular, any \mathbb{Z}^{2n} -graded ideal of $\mathbb{F}[\Gamma]$ is generated by a collection of blue monomials.

We now establish two technical lemmas about the homological behaviour of \mathbb{Z}^{2n} -graded ideals in $\mathbb{F}[\mathbf{x}, \mathbf{y}]$ and $\mathbb{F}[\Gamma]$: these results will be crucial for the proof of Theorem 5.1.

Lemma 5.9 (Blue Lemma). Let **m** be a blue multidegree, let \mathcal{M} be a collection of blue monomials, and denote by $I^{\mathcal{M}}$ (respectively, $J^{\mathcal{M}}$) the \mathbb{Z}^{2n} -graded ideal of $\mathbb{F}[\mathbf{x}, \mathbf{y}]$ (respectively, $\mathbb{F}[\Gamma]$) generated by \mathcal{M} . Then:

- *i.* if **n** divides **m**, then **n** is also a blue multidegree;
- ii. the \mathbb{F} -vector spaces $I_{\mathbf{m}}^{\mathcal{M}}$ and $J_{\mathbf{m}}^{\mathcal{M}}$ are either both one-dimensional or both $\{0\}$;
- iii. $\mathbb{GK}_{\bullet}(\mathbb{F}[\Gamma])_{\mathbf{m}} = \mathbb{K}_{\bullet}(\mathbf{x}, \mathbf{y})_{\mathbf{m}}$, where $\mathbb{K}_{\bullet}(\mathbf{x}, \mathbf{y})$ is the usual Koszul complex on the variables \mathbf{x} and \mathbf{y} ; iv. $\beta_{i,\mathbf{m}}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) = \beta_{i,\mathbf{m}}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}})$ for every $i \in \mathbb{N}$.

Proof:

- i. This is a direct consequence of the definition of blue multidegree.
- ii. Both $J_{\mathbf{m}}^{\mathcal{M}}$ and $I_{\mathbf{m}}^{\mathcal{M}}$ can be at most one-dimensional; if they differ, then it must be that $\mathbf{m} \in I_{\Gamma}$, but this cannot happen precisely because \mathbf{m} is blue.
- iii. Fix $j \in \mathbb{N}$ and consider

$$\mathbb{GK}_{j}(\mathbb{F}[\Gamma])_{\mathbf{m}} = (\mathbb{F}[\Gamma] \otimes \mathbb{F}[\Gamma]_{j}^{!})_{\mathbf{m}} = \bigoplus_{\substack{\mathbf{n} \mid \mathbf{m} \\ |\mathbf{n}|=j}} \mathbb{F}[\Gamma]_{\frac{\mathbf{m}}{\mathbf{n}}} \otimes \mathbb{F}[\Gamma]_{\mathbf{n}}^{!}.$$

Pick a multidegree **n** dividing **m** and such that $|\mathbf{n}| = j$; by part i, **n** is blue. For any word **w** with support **n**, one has that $[\mathbf{w}] \neq 0$ if and only if **n** is squarefree: otherwise, since all letters can anticommute, we would be able to put next to each other two occurrences of the same letter, causing the whole class to vanish. We can hence assume that **n** is squarefree; in particular, since $x_h y_h \in I_{\Gamma}$ for every $1 \le h \le n$, we can write **n** as $z_{i_1}z_{i_2}\ldots z_{i_j}$, where $i_1 < i_2 < \ldots < i_j$ and each z_{i_k} is either x_{i_k} or y_{i_k} . All words w with support n belong to the same class (up to a global sign), since the order of the letters does not really matter in this case: we will denote such class by $z_{i_1} z_{i_2} \dots z_{i_j}$. Such a characterization proves that $\mathbb{GK}_j(\mathbb{F}[\Gamma])_m$ coincides with $\mathbb{K}_j(\mathbf{x}, \mathbf{y})_{\mathbf{m}}$. As for the differential, since head $(z_{i_1} z_{i_2} \dots z_{i_j}) = \{1, 2, \dots, j\}$, we get

$$\partial \left(\frac{\mathbf{m}}{\mathbf{n}} \otimes \mathbf{1}_{z_{i_1} z_{i_2} \dots z_{i_j}}\right) = \sum_{k=1}^{j} (-1)^{k-1} \frac{z_{i_k} \mathbf{m}}{\mathbf{n}} \otimes \mathbf{1}_{z_{i_1} z_{i_2} \dots \hat{z}_{i_k} \dots z_{i_j}},$$

i.e. the differential of the (degree \mathbf{m} part of the) usual Koszul complex on the variables \mathbf{x} and \mathbf{y} .

iv. It is enough to show that the chain complexes of \mathbb{F} -vector spaces $(J^{\mathcal{M}} \otimes \mathbb{GK}_{\bullet}(\mathbb{F}[\Gamma]))_{\mathbf{m}}$ and $(I^{\mathcal{M}} \otimes \mathbb{K}_{\bullet}(\mathbf{x}, \mathbf{y}))_{\mathbf{m}}$ coincide.

For every $j \in \mathbb{N}$ one has that

$$(J^{\mathcal{M}} \otimes \mathbb{GK}_j(\mathbb{F}[\Gamma]))_{\mathbf{m}} = \bigoplus_{\mathbf{n}|\mathbf{m}} J^{\mathcal{M}}_{\frac{\mathbf{m}}{\mathbf{n}}} \otimes (\mathbb{GK}_j(\mathbb{F}[\Gamma]))_{\mathbf{n}};$$

by part i, both **n** and $\frac{\mathbf{m}}{\mathbf{n}}$ are blue multidegrees. By parts ii and iii, one then has that $J_{\frac{\mathbf{m}}{\mathbf{n}}}^{\mathcal{M}} = I_{\frac{\mathbf{m}}{\mathbf{n}}}^{\mathcal{M}}$ and $\mathbb{GK}_{j}(\mathbb{F}[\Gamma])_{\mathbf{n}} = \mathbb{K}_{j}(\mathbf{x}, \mathbf{y})_{\mathbf{n}}$. Analyzing the differential maps as in the proof of part iii yields the claim.

Lemma 5.10 (Red Lemma). Let J be a \mathbb{Z}^{2n} -graded ideal of $\mathbb{F}[\Gamma]$, let \mathbf{m} be a red multidegree and assume that $\beta_{i,\mathbf{m}}(J) \neq 0$ for some i > 0. Then there exists a red variable v such that $\beta_{i-1,\underline{m}}(J) \neq 0$.

Proof: Let us start by considering a cycle z which has homological degree i, internal multidegree \mathbf{m} and is not a boundary. Since $\operatorname{Tor}_i(J, \mathbb{F})_{\mathbf{m}} = H_i(J \otimes \mathbb{GK}_{\bullet}(\mathbb{F}[\Gamma]))_{\mathbf{m}}$, such a cycle can be written as

(5.3)
$$z = \sum_{j \in \mathcal{C}} \lambda_j n_j \otimes [\mathbf{w}_j],$$

where for each j in the finite set C one has that $\lambda_j \in \mathbb{F} \setminus \{0\}$, n_j is a monomial in J, $[\mathbf{w}_j]$ is the nonzero class of a word \mathbf{w}_j with i letters, and $n_j \cdot \text{supp}([\mathbf{w}_j]) = \mathbf{m}$. Note that each word \mathbf{w}_j in (5.3) must contain at least one red letter v. If this were not the case for some \mathbf{w}_j , then any two red variables v and \bar{v} such that $v\bar{v} \in I_{\Gamma}$ would both divide n_j , causing the whole term to vanish.

Since blue letters anticommute with every other letter, there must be at least a red letter v_1 that appears as the last letter of a representative of some $[\mathbf{w}_j]$, up to sign. Let us fix such a v_1 . By possibly tweaking the sign of some λ_j , we can assume without loss of generality that

(5.4)
$$z = \sum_{j \in \mathcal{A}} \lambda_j n_j \otimes \left[\tilde{\mathbf{w}}_j \| v_1 \right] + \sum_{j \in \mathcal{B}} \lambda_j n_j \otimes \left[\mathbf{w}_j \right] =: z_{v_1} + z',$$

where each $\tilde{\mathbf{w}}_j$ has i-1 letters, $\mathcal{A} \sqcup \mathcal{B} = \mathcal{C}$, and the nonempty set \mathcal{A} contains all the indices for which v_1 can travel to the end of the associated word.

Claim: z_{v_1} is a cycle (and hence so is z').

To see this, note that all the basis elements appearing in $\partial(1 \otimes [\tilde{\mathbf{w}}_j || v_1])$ must end in v_1 , except possibly for those words $\tilde{\mathbf{w}}_j || v_1$ where v_1 is free to navigate to the front. However, this last instance can happen only if the support of $\tilde{\mathbf{w}}_j$ contains no variable \bar{v}_1 with $v_1 \bar{v}_1 \in I_{\Gamma}$; then, since \bar{v}_1 divides \mathbf{m} , it must be that \bar{v}_1 divides n_j . But then, when v_1 exits the word, it multiplies \bar{v}_1 and vanishes. Hence, ∂z_{v_1} can be written as a combination of words ending in v_1 . On the other hand, $\partial z'$ can contain no such words. But then, since $\partial z = 0$, it must be that $\partial z_{v_1} = \partial z' = 0$. Note that both of these new cycles have again homological degree *i* and internal multidegree \mathbf{m} . If z_{v_1} is not a boundary, we keep it and we stop. If instead z_{v_1} is a boundary, then z' cannot be so (otherwise the original z would also be a boundary) and we can iterate the same procedure as before, finding a new red variable $v_2 \neq v_1$ and writing $z' = z'_{v_2} + z''$. This process can be repeated only finitely many times and will yield the desired non-boundary cycle.

From now on we can hence assume without loss of generality that

(5.5)
$$z = \sum_{j \in \mathcal{A}} \lambda_j n_j \otimes \left[\tilde{\mathbf{w}}_j \| v \right].$$

Now let

$$\hat{z} \coloneqq \sum_{j \in \mathcal{A}} \lambda_j n_j \otimes [\tilde{\mathbf{w}}_j].$$

Reasoning as in the proof of the claim above, it follows that \hat{z} is a cycle; moreover, if there existed \hat{y} such that $\partial \hat{y} = \hat{z}$, then z would also be a boundary (appending v at the end of each word in \hat{y}). By definition, the cycle \hat{z} has homological degree i - 1 and internal multidegree $\frac{\mathbf{m}}{v}$, and this concludes the proof.

Corollary 5.11. Let \mathcal{M} be a collection of blue monomials, and denote by $I^{\mathcal{M}}$ (respectively, $J^{\mathcal{M}}$) the \mathbb{Z}^{2n} -graded ideal of $\mathbb{F}[\mathbf{x}, \mathbf{y}]$ (respectively, $\mathbb{F}[\Gamma]$) generated by \mathcal{M} . Let k and h be two nonnegative integers. Then:

- i. for every i, j for which β^{F[Γ]}_{i,j}(J^M) ≠ 0 there exists 0 ≤ h ≤ i for which β^{F[**x**,**y**]}_{i-h,j-h}(I^M) ≠ 0;
 ii. if β^{F[**x**,**y**]}_{i,j}(I^M) = 0 for every 0 ≤ i ≤ k and j > i + h, then β^{F[Γ]}_{i,j}(J^M) = 0 for every 0 ≤ i ≤ k and j > i + h; in particular, if the F[**x**,**y**]-resolution of I^M is linear for k steps, so is the F[Γ]-resolution of J^M;
- *iii.* $\operatorname{reg}_{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) \leq \operatorname{reg}_{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}).$

If moreover all the monomials in \mathcal{M} contain only y-variables, then:

- iv. if $\beta_{i,j}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}) \neq 0$, then $\beta_{i,j}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) \neq 0$;
- $v. \ \beta_{i,j}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}) = 0 \text{ for every } 0 \le i \le k \text{ and } j > i + h \text{ if and only if } \beta_{i,j}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) = 0 \text{ for every } 0 \le i \le k \text{ and } j > i + h;$
- vi. $\operatorname{reg}_{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) = \operatorname{reg}_{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}})$. In particular, $J^{\mathcal{M}}$ has a linear $\mathbb{F}[\Gamma]$ -resolution if and only if $I^{\mathcal{M}}$ has a linear $\mathbb{F}[\mathbf{x},\mathbf{y}]$ -resolution.

Proof:

- i. Pick *i* and *j* such that $\beta_{i,j}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) \neq 0$, and choose a multidegree **m** such that $|\mathbf{m}| = j$ and $\beta_{i,\mathbf{m}}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) \neq 0$. If **m** is blue, Lemma 5.9.iv yields the claim with h = 0. If instead **m** is red, we know by Lemma 5.10 that we can decrease the homological index from *i* to i 1 while staying in the (j i)-th linear strand of the $\mathbb{F}[\Gamma]$ -resolution of $J^{\mathcal{M}}$. This can be repeated until we hit a blue multidegree, which happens at the latest when we reach the zeroth homological degree, since all generators of $J^{\mathcal{M}}$ are blue. Then applying Lemma 5.9.iv finishes the proof.
- ii.–iii. These statements follow from part i.
 - iv. Pick *i* and *j* such that $\beta_{i,j}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}) \neq 0$, and choose a multidegree **m** such that $|\mathbf{m}| = j$ and $\beta_{i,\mathbf{m}}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}) \neq 0$. Since no *x*-variables are involved in any minimal generator of $I^{\mathcal{M}}$, one has that $\beta_{i,\mathbf{m}}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}})$ can be nonzero only if **m** is a monomial in the *y*variables; however, any such **m** must be blue, as by construction the only variable

v with $vy_i \in I_{\Gamma}$ is $v = x_i$. Applying Lemma 5.9. iv then yields that $\beta_{i,\mathbf{m}}^{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) =$ $\beta_{i,\mathbf{m}}^{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}})$, and the claim follows. v.-vi. These statements follow from combining parts i and iv.

Remark 5.12. Under the hypotheses of Corollary 5.11.iv–vi, the Betti table of $J^{\mathcal{M}}$ is a "horizontal echo" of the (finite) Betti table of $I^{\mathcal{M}}$.

As an example, take $\mathbb{F}[\Gamma] = \mathbb{F}[\mathbf{x}, \mathbf{y}]/(x_1x_2, x_1x_3, x_1y_1, x_2y_2, x_3y_3)$ and $\mathcal{M} = \{y_1y_2, y_2^2y_3^2, y_3^4\}.$ Using Macaulay2 [GS21], we find out that the Betti tables of $I^{\mathcal{M}}$ and $J^{\mathcal{M}}$ look as follows:

	0	1	2		0	1	2	3	4	5	6	7
2	1		•	2	1	2	5	13	34	89	233	610
3				3								
4	2	1	•	4	2	4	10	26	68	178	466	1220
5		2	1	5		2	6	16	42	110	288	754

TABLE 1. The Betti table of $I^{\mathcal{M}}$ (left) and the beginning of the Betti table of $J^{\mathcal{M}}$ (right) from Remark 5.12.

Remark 5.13. Note that the statements of Corollary 5.11.iv-vi can indeed fail when the elements of \mathcal{M} are not in the *y*-variables only.

For instance, take $\mathbb{F}[\Gamma] = \mathbb{F}[\mathbf{x}, \mathbf{y}]/(x_1x_3, x_1y_1, x_2y_2, x_3y_3, x_4y_4)$ and $\mathcal{M} = \{x_1x_2, x_3x_4\}$. Since $I^{\mathcal{M}} = (x_1x_2, x_3x_4)$ is a complete intersection in $\mathbb{F}[\mathbf{x}, \mathbf{y}]$, one has that $\operatorname{reg}_{\mathbb{F}[\mathbf{x}, \mathbf{y}]}(I^{\mathcal{M}}) = 3$. However, since $J^{\mathcal{M}}$ has linear quotients with respect to the Koszul filtration consisting of all subsets of variables of $\mathbb{F}[\Gamma]$, then $J^{\mathcal{M}}$ has a 2-linear $\mathbb{F}[\Gamma]$ -resolution [CDNR13, Lemma 17]; hence, $2 = \operatorname{reg}_{\mathbb{F}[\Gamma]}(J^{\mathcal{M}}) < \operatorname{reg}_{\mathbb{F}[\mathbf{x},\mathbf{y}]}(I^{\mathcal{M}}) = 3.$

We are finally ready to prove Theorem 5.1.

Proof: (of Theorem 5.1) Let $\mathcal{M} := \{\mathbf{y}^{[n] \setminus F} \mid F \text{ facet of } \Delta\}$, and let $I^{\mathcal{M}} \subseteq \mathbb{F}[\mathbf{x}, \mathbf{y}], J^{\mathcal{M}} \subseteq \mathbb{F}[\Gamma]$ be as in Corollary 5.11. Note that $I^{\mathcal{M}}$ is the extension of the Alexander dual ideal of Δ to a polynomial ring containing also x-variables. By Theorem 2.6, the simplicial complex Δ is (S_k) if and only if the Alexander dual ideal of Δ has an $\mathbb{F}[\mathbf{y}]$ -resolution which is linear for k-1 steps. This is equivalent to stating that $I^{\mathcal{M}}$ has an $\mathbb{F}[\mathbf{x}, \mathbf{y}]$ -resolution which is linear for k-1 steps. By Corollary 5.11.v, this is in turn equivalent to stating that $J^{\mathcal{M}}$ has an $\mathbb{F}[\Gamma]$ -resolution which is linear for k-1 steps. By Proposition 3.4.iv, the ideal $J^{\mathcal{M}}$ coincides with the canonical module $\omega_{\mathbb{F}[\Gamma]}$. Finally, applying Lemma 2.13 with $A = \mathbb{F}[\Gamma]$, $M = \omega_{\mathbb{F}[\Gamma]}(-a(\mathbb{F}[\Gamma]) - 1)$ concludes the proof.

Remark 5.14. In [MSS21a] the authors consider the problem of characterizing the pairs of positive integers (c, r) for which there exists a non-Koszul quadratic Gorenstein algebra R with codimension c and regularity r. The results of [MSS21a], [MSS21b] and [MS20] leave only two cases open, namely (c,r) = (6,3) and (c,r) = (7,3). One might wonder if the results in this section provide examples of non-Koszul quadratic Gorenstein algebras with such invariants. However, this is not the case: if R_{Δ} has regularity 3, then Δ must be 1dimensional, and in this case Δ satisfies (S_2) if and only if it is Cohen–Macaulay (over any field).

5.3. An application: quadratic Gorenstein algebras which are not Koszul in prescribed characteristics. We conclude this section with an application of Corollary 5.3. Given a finite list of prime numbers $P = \{p_1, \ldots, p_m\}$, our result can be used to construct quadratic Gorenstein \mathbb{F} -algebras which are Koszul if and only if char(\mathbb{F}) $\notin P$. Moreover, the presentations of these algebras will be characteristic-free, see Proposition 4.3.

To do so, it suffices to exhibit a flag simplicial complex which is Cohen–Macaulay over \mathbb{F} if and only if the characteristic of \mathbb{F} is not in P. Note that such a complex will automatically be (S_2) (which corresponds to R_{Δ} having a quadratic presentation, see Proposition 4.4), since the (S_2) property is characteristic-free and the complex is Cohen–Macaulay in some characteristic.

We will focus on flag triangulations of 3-dimensional lens spaces. A 3-dimensional lens space is an orientable 3-manifold obtained as a quotient of the 3-sphere by certain rotations (see [Hat02, Example 2.43]). Such spaces are parametrized by two coprime integers $p, q \ge 1$, and $L(p_1, q_1) \cong L(p_2, q_2)$ if and only if $p_1 = p_2$ and $q_1 = \pm q_1^{\pm 1} \mod p$. The reduced integral homology groups of L(p,q) are as follows: $\tilde{H}_0(L(p,q)) = \tilde{H}_2(L(p,q)) = 0$, $\tilde{H}_3(L(p,q)) = \mathbb{Z}$ and $\tilde{H}_1(L(p,q)) = \mathbb{Z}_p$. As lens spaces are homology 3-manifolds, the link of any nonempty face in any triangulation Δ of L(p,q) is a homology sphere. Hence, by Reisner's criterion together with the universal coefficient theorem [Hat02, Section 3A], Δ is Cohen–Macaulay over a field \mathbb{F} if and only if $\tilde{H}_1(\Delta;\mathbb{F}) = \tilde{H}_1(\Delta;\mathbb{Z}) \otimes \mathbb{F} = \mathbb{F}/p\mathbb{F} = 0$. In particular, every triangulation of L(p,q) is Cohen–Macaulay over \mathbb{F} if and only if $\operatorname{char}(\mathbb{F}) \neq p$. Therefore if Δ is a flag triangulation of L(p,q), then R_{Δ} is a quadratic Gorenstein \mathbb{F} -algebra which is Koszul if and only if $\operatorname{char}(\mathbb{F}) \neq p$. We will generalize this fact by considering connected sums of lens spaces.

Proposition 5.15. Let $P = \{p_1, \ldots, p_m\}$ be a set of prime numbers. Let Δ be any flag triangulation of the connected sum $L(p_1, q_1) # \cdots # L(p_m, q_m)$, for some positive integers q_1, \ldots, q_m . Then R_{Δ} is a quadratic Gorenstein \mathbb{F} -algebra that is Koszul if and only if char(\mathbb{F}) $\notin P$.

Proof: As Δ is a triangulated 3-manifold, one has that $\hat{H}_i(\mathrm{lk}_{\Delta}(F);\mathbb{Z}) = 0$ for every $\emptyset \neq F \in \Delta$ and for every $i < \dim(F)$. Moreover, we can compute the homology of a connected sum of manifolds by a standard application of the Mayer–Vietoris sequence. This yields that $\tilde{H}_2(\Delta;\mathbb{Z}) = 0$ and $\tilde{H}_1(\Delta;\mathbb{Z}) = \bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z}$. Thus, we have that Δ is Cohen–Macaulay over a field \mathbb{F} if and only if $\tilde{H}_1(\Delta;\mathbb{Z}) \otimes \mathbb{F} = \bigoplus_{p \in P} \mathbb{F}/p\mathbb{F} = 0$, which in turn happens if and only if $\mathrm{char}(\mathbb{F}) \notin P$. We conclude by using Corollary 5.3.

not restrictive, as any triangulable space has a flag triangulation given, for example, by its barycentric subdivision [Sta96, III.4]. However, these simplicial complexes typically have a lot of vertices, and it is a challenging problem to find vertex-minimal flag triangulations of a given space.

Finally, we observe that Proposition 5.15 can be applied verbatim to flag triangulations of higher-dimensional lens spaces, yielding quadratic Gorenstein algebras with the same behaviour as in Proposition 5.15 but with an *h*-polynomial of higher degree.

Example 5.16. Let Δ be a flag triangulation of \mathbb{RP}^2 . In [BOW⁺20] the authors construct two such non-isomorphic triangulations with the same *f*-vector $f(\Delta) = (1, 11, 30, 20)$. We let Δ be the simplicial complex whose list of facets is reported in the left column of [BOW⁺20, Table 1]. The algebra R_{Δ} is then presented as T/J, with

$$T = \mathbb{F}[x_1, \dots, x_9, x_a, x_b, y_1, \dots, y_9, y_a, y_b, z_{145}, z_{126}, z_{156}, z_{237}, z_{347}, z_{267}, z_{148}, z_{478}, z_{129}, z_{189}, z_{23a}, z_{34a}, z_{45a}, z_{29a}, z_{56b}, z_{67b}, z_{78b}, z_{89b}, z_{5ab}, z_{9ab}]$$

and

- $$\begin{split} J &= \big(x_b z_{145}, x_a z_{145}, x_9 z_{145}, x_8 z_{145}, x_7 z_{145}, x_6 z_{145}, x_3 z_{145}, x_2 z_{145}, x_b z_{126}, x_a z_{126}, x_9 z_{126}, x_8 z_{126}, x_8 z_{126}, x_7 z_{126}, x_5 z_{126}, x_4 z_{126}, x_3 z_{126}, x_b z_{156}, x_a z_{156}, x_9 z_{156}, x_8 z_{156}, x_7 z_{156}, x_4 z_{156}, x_3 z_{156}, x_2 z_{156}, x_b z_{237}, x_a z_{237}, x_9 z_{237}, x_8 z_{237}, x_6 z_{237}, x_5 z_{237}, x_4 z_{237}, x_1 z_{237}, x_b z_{347}, x_a z_{347}, x_9 z_{347}, x_8 z_{347}, x_6 z_{347}, x_5 z_{347}, x_2 z_{347}, x_1 z_{347}, x_b z_{267}, x_a z_{267}, x_9 z_{267}, x_8 z_{267}, x_5 z_{267}, x_4 z_{267}, x_3 z_{267}, x_1 z_{267}, x_b z_{148}, x_a z_{148}, x_9 z_{148}, x_7 z_{148}, x_6 z_{148}, x_5 z_{148}, x_3 z_{148}, x_2 z_{148}, x_b z_{478}, x_a z_{478}, x_9 z_{478}, x_6 z_{478}, x_5 z_{478}, x_3 z_{478}, x_2 z_{478}, x_1 z_{478}, x_b z_{129}, x_a z_{129}, x_7 z_{129}, x_6 z_{129}, x_5 z_{129}, x_4 z_{129}, x_3 z_{129}, x_5 z_{23a}, x_5 z_{23a}, x_1 z_{248}, x_1 z_{478}, x_b z_{148}, x_9 z_{448}, x_9 z_{478}, x_6 z_{484}, x_9 z_{448}, x_9 z_{45a}, x_9 z_{45a}, x_8 z_{45a}, x_7 z_{45a}, x_6 z_{45a}, x_3 z_{45a}, x_9 z_{45a}, x_8 z_{45a}, x_7 z_{45a}, x_6 z_{45a}, x_9 z_{45a}, x_6 z_{45a}, x_7 z_{45a}, x_6 z_{45a}, x_3 z_{45a}, x_2 z_{47b}, x_1 z_{29a}, x_a z_{56b}, x_1 z_{56b}, x_2 z_{56b}, x_1 z_{56b}, x_3 z_{56b}, x_2 z_{56b}, x_1 z_{56b}, x_4 z_{78b}, x_3 z_{78b}, x_2 z_{78b}, x_1 z_{78b}, x_a z_{89b}, x_7 z_{89b}, x_6 z_{89b}, x_5 z_{89b}, x_4 z_{89b}, x_3 z_{89b}, x_2 z_{89b}, x_1 z_{89b}, x_3 z_{80b}, x_2 z_{89b}, x_1 z_{89b}, x_3 z_{80b}, x_2 z_{89b}, x_1 z_{89b}, x_4 z_{89b}, x_3 z_{89b}, x_2 z_{89b}, x_1 z_{89b}, x_4 z_{9ab}, x_3 z_{9ab}, x_2 z_{9ab}, x_1 z_{9ab}, x_2 z_$$
 - $+ (y_4 z_{145} y_6 z_{156}, y_2 z_{126} y_5 z_{156}, y_2 z_{237} y_4 z_{347}, y_1 z_{126} y_7 z_{267}, y_3 z_{237} y_6 z_{267}, y_5 z_{145} y_8 z_{148}, y_3 z_{347} y_8 z_{478}, y_1 z_{148} y_7 z_{478}, y_6 z_{126} y_9 z_{129}, y_4 z_{148} y_9 z_{189}, y_2 z_{129} y_8 z_{189}, y_a z_{23a} y_7 z_{237}, y_a z_{34a} y_7 z_{347}, y_2 z_{23a} y_4 z_{34a}, y_1 z_{145} y_a z_{45a}, y_3 z_{34a} y_5 z_{45a}, y_1 z_{129} y_a z_{29a}, y_3 z_{23a} y_9 z_{29a}, y_1 z_{156} y_b z_{56b}, y_2 z_{267} y_b z_{67b}, y_5 z_{56b} y_7 z_{67b}, y_4 z_{45a} y_b z_{5ab}, y_a z_{5ab} y_6 z_{56b}, y_4 z_{478} y_b z_{78b}, y_6 z_{67b} y_8 z_{78b}, y_1 z_{189} y_b z_{89b}, y_7 z_{78b} y_9 z_{89b}, y_2 z_{29a} y_b z_{9ab}, y_8 z_{89b} y_a z_{9ab}, y_5 z_{5ab} y_9 z_{9ab})$
 - $+ (x_2x_8, x_4x_6, x_3x_9, x_3x_6, x_7x_9, x_3x_8, x_6x_8, x_bx_2, x_4x_9, x_1x_a, x_5x_9, x_5x_7, x_6x_9, x_1x_7, x_ax_7, x_1x_b, x_ax_6, x_2x_5, x_ax_8, x_2x_4, x_bx_4, x_3x_5, x_5x_8, x_bx_3, x_1x_3)$
 - + $(x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_6y_6, x_7y_7, x_8y_8, x_9y_9, x_ay_a, x_by_b)$
 - $+ (z_{145}, z_{126}, z_{156}, z_{237}, z_{347}, z_{267}, z_{148}, z_{478}, z_{129}, z_{189}, z_{23a}, z_{34a}, z_{45a}, z_{29a}, z_{56b}, z_{67b}, z_{78b}, z_{89b}, z_{5ab}, z_{9ab})^2.$

The Hilbert series of the Gorenstein algebra R_{Δ} is equal to

$$\frac{1+31t+60t^2+31t^3+t^4}{(1-t)^{11}}$$

We can verify Corollary 5.3 on this example. As any triangulation of the projective plane is Cohen–Macaulay over \mathbb{F} if and only if the characteristic of the field is not equal to 2, we

expect to observe a different behaviour for the resolution of \mathbb{F} as an R_{Δ} -module in the cases $\mathbb{F} = \mathbb{Z}_2$ and $\mathbb{F} = \mathbb{Z}_3$. Indeed, when $\mathbb{F} = \mathbb{Z}_2$, we find a nonlinear syzygy in homological degree 3, as shown in Table 2. The numbers in Table 2 have been computed via Macaulay2 [GS21], using the commands use(T/J); betti res(ideal gens(T/J), LengthLimit => 3).

	0	1	2	3		0	1	2	3
0	1	42	1297	37883	 0	1	42	1297	37883
1				1	1				

TABLE 2. The beginning of the Betti tables of \mathbb{F} as an R_{Δ} -module in the cases $\mathbb{F} = \mathbb{Z}_2$ (left) and $\mathbb{F} = \mathbb{Z}_3$ (right). Here Δ is the flag triangulation of \mathbb{RP}^2 from Example 5.16.

6. Quadratic Gröbner bases and shelling orders

In this section we add a further connection between the combinatorial features of Δ and the algebraic properties of R_{Δ} . We begin by recalling the notion of shellability.

Definition 6.1. A pure (d-1)-dimensional simplicial complex Δ is *shellable* if there exists a linear order $F_1, \ldots, F_{|\mathcal{F}(\Delta)|}$ of its facets such that $\Delta_{i-1} \cap \langle F_i \rangle$ is pure and (d-2)-dimensional for every $i = 2, \ldots, |\mathcal{F}(\Delta)|$, where $\Delta_i = \langle F_1, \ldots, F_i \rangle$. The ordering $F_1, \ldots, F_{|\mathcal{F}(\Delta)|}$ is called a *shelling order*, and we will often refer to the step $\Delta_{i-1} \to \Delta_i$ as a *shelling*.

This property poses severe restrictions on the topology of Δ , as shellable simplicial complexes are homotopy equivalent to a wedge of some (possibly zero) (d-1)-spheres. Boundary complexes of simplicial polytopes of any dimension are shellable, while there exist nonshellable simplicial spheres and balls already in dimension 3. Moreover, all links of a shellable simplicial complex are again shellable. Shellability has also implications on a more algebraic level, like the following.

Proposition 6.2. Shellable simplicial complexes are Cohen–Macaulay over any field.

The converse is far from being true and fails already in dimension 2: it is well known that any triangulation of the so-called *dunce hat* is Cohen–Macaulay over any field, but not shellable. However, the set of f-vectors of Cohen–Macaulay complexes and that of shellable ones coincide in any dimension [Sta77, Theorem 6].

The goal of the previous sections was to show a connection between the Cohen–Macaulayness of a flag simplicial complex Δ and the Koszulness of R_{Δ} . The next result shows that we can tighten both conditions and still obtain a correspondence.

Theorem 6.3. Let Δ be a pure flag simplicial complex. Then Δ is shellable if and only if R_{Δ} has a quadratic Gröbner basis.

Recall that if a standard graded algebra has quadratic Gröbner basis, then it is Koszul. Before embarking on the proof of Theorem 6.3 we observe the following fact.

Lemma 6.4. Any term order < on $\mathbb{F}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ induces a total order < on the facets of Δ , defined as the reflexive closure of the following rule:

$$F_{\ell} \prec F_k \Leftrightarrow \operatorname{in}_{<}(b_{F_k,F_{\ell}}) = \mathbf{y}^{F_k \smallsetminus F_{\ell}} z_{F_k}.$$

Proof: By definition, for any two distinct facets F_k and F_ℓ of Δ at most one of $F_\ell < F_k$ and $F_k < F_\ell$ holds. To conclude that < is a partial order, we need to prove transitivity. Let F_j , F_k , F_ℓ be facets such that $F_j > F_k$, $F_k > F_\ell$. We claim that $\ln_<(b_{F_j,F_\ell}) = \mathbf{y}^{F_j \setminus F_\ell} z_{F_j}$. Indeed, multiplying b_{F_j,F_k} by $\mathbf{y}^{F_j \cap F_k}$ yields that $\mathbf{y}^{F_j} z_{F_j} > \mathbf{y}^{F_k} z_{F_k}$; analogously, one has that $\mathbf{y}^{F_k} z_{F_k} > \mathbf{y}^{F_\ell} z_{F_\ell}$, and hence $\mathbf{y}^{F_j} z_{F_j} > \mathbf{y}^{F_\ell} z_{F_\ell}$. Since $\mathbf{y}^{F_j} z_{F_j} - \mathbf{y}^{F_\ell} z_{F_\ell} = \mathbf{y}^{F_j \cap F_\ell} b_{F_j,F_\ell}$, the claim follows.

Definition 6.5. If < is a term order on $\mathbb{F}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, we will denote by < be the unique total order on $\mathcal{F}(\Delta)$ induced by < as in Lemma 6.4. Conversely, if < is any total order on $\mathcal{F}(\Delta)$, we will say that a term order < on $\mathbb{F}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is *compatible* with < if $\operatorname{in}_{\langle b_{F_k, F_\ell} \rangle} = \mathbf{y}^{F_k \setminus F_\ell} z_{F_k}$ whenever $F_\ell < F_k$.

Lemma 6.6. Fix a term order < on $\mathbb{F}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, let < be the associated total order on $\mathcal{F}(\Delta)$ and let F_1 and F_2 be facets of Δ with $F_1 > F_2$. Then the binomial b_{F_1,F_2} can be reduced via the binomial b_{G_1,G_2} if and only if $\{G_1, G_2\} = \{F_1, H\}$ for some facet H such that $H < F_1$ and $F_1 \cap F_2 \subseteq H$. If this is the case, the binomial b_{F_1,F_2} reduces to $\mathbf{y}^{F_2 \cap (H \setminus F_1)} b_{H,F_2}$.

Proof: Since $F_1 > F_2$, the leading term of b_{F_1,F_2} is $\mathbf{y}^{F_1 \smallsetminus F_2} z_{F_1}$. We can operate a reduction via b_{G_1,G_2} precisely when the leading term of b_{G_1,G_2} divides $\mathbf{y}^{F_1 \smallsetminus F_2} z_{F_1}$. In order for this to happen, b_{G_1,G_2} must be (up to sign) $b_{F_1,H}$, where H is a facet of Δ . For z_{F_1} to appear in the leading term of $b_{F_1,H}$, it must be that $F_1 > H$. Moreover, we need $\mathbf{y}^{F_1 \smallsetminus H}$ to divide $\mathbf{y}^{F_1 \land F_2}$, i.e. $F_1 \smallsetminus H \subseteq F_1 \smallsetminus F_2$, which happens precisely when $F_1 \cap F_2 \subseteq H$ (note that this in turn equivalent to $F_2 \smallsetminus H \subseteq F_2 \smallsetminus F_1$). One checks that all these necessary conditions are indeed sufficient to get the desired reduction. When such conditions are met, one has that

$$\begin{split} b_{F_1,F_2} &= \mathbf{y}^{F_1 \smallsetminus F_2} z_{F_1} - \mathbf{y}^{F_2 \smallsetminus F_1} z_{F_2} \\ &= \mathbf{y}^{F_1 \cap (H \smallsetminus F_2)} \mathbf{y}^{F_1 \smallsetminus H} z_{F_1} - \mathbf{y}^{F_2 \cap (H \smallsetminus F_1)} \mathbf{y}^{F_2 \smallsetminus H} z_{F_2} \\ &= \mathbf{y}^{F_1 \cap (H \smallsetminus F_2)} (\mathbf{y}^{F_1 \lor H} z_{F_1} - \mathbf{y}^{H \smallsetminus F_1} z_H) + \mathbf{y}^{F_1 \cap (H \smallsetminus F_2)} \mathbf{y}^{H \smallsetminus F_1} z_H - \mathbf{y}^{F_2 \cap (H \smallsetminus F_1)} \mathbf{y}^{F_2 \lor H} z_{F_2} \\ &= \mathbf{y}^{F_1 \cap (H \smallsetminus F_2)} b_{F_1,H} + \mathbf{y}^{F_2 \cap (H \smallsetminus F_1)} \mathbf{y}^{H \smallsetminus F_2} z_H - \mathbf{y}^{F_2 \cap (H \smallsetminus F_1)} \mathbf{y}^{F_2 \lor H} z_{F_2} \\ &= \mathbf{y}^{F_1 \cap (H \smallsetminus F_2)} b_{F_1,H} + \mathbf{y}^{F_2 \cap (H \smallsetminus F_1)} b_{H,F_2}. \end{split}$$

For the rest of the section we let

 $\mathcal{Q} \coloneqq \{ b_{F_i, F_j} : \dim(F_i \cap F_j) = d - 2 \}.$

Observe that the condition $\dim(F_i \cap F_j) = d-2$ implies that all binomials in \mathcal{Q} are quadratic. Conversely, for every binomial b_{F_i,F_j} with $\deg(b_{F_i,F_j}) = 2$ we have that $\dim(F_i \cap F_j) = d-2$. We now establish a technical lemma that will be crucial in the proof of Theorem 6.3.

Lemma 6.7. Fix a term order < on $\mathbb{F}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ and let < be the associated total order on $\mathcal{F}(\Delta)$. Let $b_{F,F_1}, b_{F,F_2} \in \mathcal{Q}$ be such that $\operatorname{in}(b_{F,F_1}) = y_{F \setminus F_1} z_F$ and $\operatorname{in}(b_{F,F_2}) = y_{F \setminus F_2} z_F$. Then



FIGURE 2. An illustration of Lemma 6.7.

 $S(b_{F,F_1}, b_{F,F_2})$ reduces to zero modulo Q if and only if there exists a sequence of facets $F_1 = F^{(1)}, \ldots, F^{(p)} = F_2$ such that

i. $F_1 \cap F_2 \subset F^{(i)}$ for every $1 \le i \le p$;

ii. there exists $1 \le c \le p$ such that

$$F^{(1)} \succ \dots \succ F^{(c)} \prec F^{(c+1)} \prec \dots \prec F^{(p)}$$

with $\dim(F^{(i)} \cap F^{(i+1)}) = d - 2$, for every $1 \le i \le p - 1$.

Proof: The facets F_1 and F_2 must meet in codimension one or two, and the proof of Proposition 4.5 yields that $S(b_{F,F_1}, b_{F,F_2}) = \mathbf{y}^{(F_1 \cap F_2) \setminus F} b_{F_2,F_1}$.

If F_1 and F_2 meet in codimension one, then $b_{F_2,F_1} \in \mathcal{Q}$. In particular, $S(b_{F,F_1}, b_{F,F_2})$ reduces to zero modulo \mathcal{Q} , and the sequence of facets consisting of F_1 and F_2 always satisfies the hypotheses.

From now on we will hence assume F_1 and F_2 meet in codimension two. Then $F_1 \cap F_2 \subseteq F$ and thus $S(b_{F,F_1}, b_{F,F_2}) = b_{F_2,F_1}$. Assume without loss of generality that $F_1 > F_2$.

Only if: We assume that $S(b_{F,F_1}, b_{F,F_2}) = b_{F_2,F_1}$ reduces to zero modulo Q and prove the existence of a sequence of facets satisfying the hypotheses. By hypothesis, $b_{F_2,F_1} \notin$ \mathcal{Q} becomes zero after applying finitely many reductions via binomials in \mathcal{Q} . Let (b_1,\ldots,b_{p-1}) (with $p \ge 3$) be the shortest possible sequence of binomials used to reduce b_{F_2,F_1} to zero. By Lemma 6.6, b_{F_2,F_1} can be reduced via a binomial in \mathcal{Q} if and only if such a binomial is (up to sign) b_{F_1,H_1} , where H_1 is a facet meeting F_1 in codimension one and such that $F_1 > H_1$ and $F_1 \cap F_2 \subseteq H_1$. Moreover, such reduction produces a y-multiple of b_{F_2,H_1} ; hence, if p = 3, then F_2 and H_1 must meet in codimension one and (F_1, H_1, F_2) is the desired chain. If instead p > 3, then F_2 and H_1 must meet in codimension two (since $F_1 \cap F_2 \subseteq H_1 \cap F_2$). Applying Lemma 6.6 to (the given y-multiple of) b_{F_2,H_1} , we find another facet H_2 meeting max{ H_1, F_2 } in codimension one and such that $\max\{H_1, F_2\} > H_2$ and $H_1 \cap F_2 \subseteq H_2$. Moreover, one also has that $F_1 \cap F_2 \subseteq H_1 \cap F_2 \subseteq H_2$, as requested. The reduction step produces a y-multiple of $b_{\min\{H_1,F_2\},H_2}$; reasoning as before, if p = 4 then $\min\{H_1,F_2\}$ and H_2 must meet in codimension one and we are done, otherwise we continue the process until we reach the end of the reducing sequence.

If: For the reverse implication, assume that we are given a sequence of facets $F^{(1)} > \cdots > F^{(c)} < F^{(c+1)} < \cdots < F^{(p)}$ satisfying the hypotheses. Then one gets the desired sequence of binomials by the following algorithm:

```
\begin{split} i \leftarrow 1, \ j \leftarrow 1, \ k \leftarrow p, \ b_1, \dots, b_{p-1} \leftarrow 0 \\ \text{while} \ i < p-1 \ \text{and} \ \dim(F^{(j)} \cap F^{(k)}) = d-3 \\ \quad \text{if} \ \max\{F^{(j)}, F^{(k)}\} = F^{(j)} \\ \quad b_i \leftarrow b_{F^{(j)}, F^{(j+1)}}, \ j \leftarrow j+1, \ i \leftarrow i+1 \\ \text{else} \\ \quad b_i \leftarrow b_{F^{(k-1)}, F^{(k)}}, \ k \leftarrow k-1, \ i \leftarrow i+1 \\ \text{end} \ \text{if} \\ \text{end while} \\ b_i \leftarrow b_{F^{(j)}, F^{(k)}} \\ \text{return} \ b_1, \dots, b_i \end{split}
```

Note that we are invoking Lemma 6.6 at every iteration inside the while cycle. This can be done because, whenever $F^{(j)}$ and $F^{(k)}$ meet in codimension two, then $F^{(j)} \cap F^{(k)} = F_1 \cap F_2$, and $F_1 \cap F_2$ is contained both in $F^{(j+1)}$ and $F^{(k-1)}$ by hypothesis. Moreover, when the condition $\dim(F^{(j)} \cap F^{(k)}) = d - 3$ is not met, then it must be that $F^{(j)}$ and $F^{(k)}$ meet in codimension one, and thus we can stop the process after a final reduction via $b_{F^{(j)},F^{(k)}}$.

We are now ready to prove the main result in this section. *Proof of Theorem* 6.3 We prove the two implications separately.

Only if: Let $F_1 \prec \cdots \prec F_{|\mathcal{F}(\Delta)|}$ be a shelling order for Δ . Then the collection

$$C_{\Delta} = \mathcal{Q} \cup \{x_i y_i : i = 1, \dots, n\} \cup \{\mathbf{x}^{N_i} : N_i \text{ minimal nonface of } \Delta\} \cup \{z_{F_i} z_{F_j} : F_i, F_j \text{ facets of } \Delta\} \cup \{x_i z_{F_j} : F_j \text{ facet of } \Delta, i \notin F_j\}$$

is a Gröbner basis for the ideal defining R_{Δ} with respect to any term order compatible with \prec .

Since Δ is flag, all monomials in the collection above have degree 2. Moreover, shellable complexes are (S_2) , and hence the algebra R_{Δ} is quadratic by Proposition 4.4. In particular, the defining ideal of R_{Δ} is generated by C_{Δ} , as this is the set of degree 2 elements of the generating set described in Proposition 4.3. It is left to the reader to check that all the *S*polynomials obtained by comparing a monomial and a binomial in C_{Δ} reduce to zero. Hence, to prove the claim we only need to show that the *S*-polynomials of the form $S(b_1, b_2)$, with $b_1, b_2 \in Q$, reduce to zero. Observe that with the term order we fixed, the leading term of each binomial is determined by the *z*-variables alone. We begin by noting that if two binomials b_1 and b_2 are such that $in(b_1) = y_i z_F$ and $in(b_2) = y_j z_G$, with $F \neq G$, then $S(b_1, b_2)$ reduces to zero, even if i = j. This follows from the fact that both monomials of $S(b_1, b_2)$ contain a product of two *z*-variables. Since the collection C_{Δ} contains all monomials of degree 2 in the *z*-variables, these *S*-polynomials reduce to zero. Consider now $S(b_{F_k,F_\ell}, b_{F_k,F_m})$, with $k > \ell$ and k > m. By Lemma 6.7, the polynomial $S(b_{F_k,F_\ell}, b_{F_k,F_m}) = \mathbf{y}^{(F_\ell \cap F_m) \setminus F_k} b_{F_m,F_\ell}$ reduces to zero modulo the collection C_Δ if and only if it reduces to zero modulo the collection C_{Δ_k} , with $\Delta_k = \langle F_1, \ldots, F_k \rangle$.

Let $H = F_{\ell} \cap F_m \in \Delta_{k-1}$ and note that $|H| \ge d-2$.

If |H| = d - 1, then $b_{F_{\ell},F_m} \in \mathcal{Q}$, which implies that $S(b_{F_k,F_{\ell}}, b_{F_k,F_m})$ is a multiple of a binomial in \mathcal{Q} . In particular, $S(b_{F_k,F_{\ell}}, b_{F_k,F_m})$ reduces to zero.

Assume then that |H| = d - 2. Since Δ_{k-1} is shellable, the 1-dimensional simplicial complex $lk_{\Delta_{k-1}}(H)$ is shellable, with a shelling order induced by the one of Δ_{k-1} . Since shellable simplicial complexes of dimension greater than zero are connected, there exists a path in $lk_{\Delta_{k-1}}(H)$ connecting the two edges $F_{\ell} \times H$ and $F_m \times H$. The edges of this path correspond to facets of Δ_{k-1} via taking their union with H: therefore, we can identify the path with a sequence of facets of Δ_{k-1} which all contain H. We can describe the support of such a path via a binary string of length $|\mathcal{F}(\Delta_{k-1})|$ whose *i*-th character is 1 if $F_i \in P$ and 0 if $F_i \notin P$. We consider the colexicographic order on these strings, i.e.,

$$(s_1, \ldots, s_{|\mathcal{F}(\Delta_{k-1})|}) <_{\text{colex}} (t_1, \ldots, t_{|\mathcal{F}(\Delta_{k-1})|}) \iff s_i < t_i \text{ with } i = \max\{j : s_j \neq t_j\}.$$

It is important to observe that the path between F_{ℓ} and F_m might not be unique. We choose the one $P: F_{\ell} = F_{i_1} \rightarrow F_{i_2} \cdots \rightarrow F_{i_p} = F_m$ whose support is minimal with respect to the colexicographic order on $2^{\mathcal{F}(\Delta_{k-1})}$. We claim there exists $1 \leq c \leq p$ such that $z_{F_{i_1}} > z_{F_{i_2}} > \cdots > z_{F_{i_c}} < z_{F_{i_{c+1}}} < \cdots < z_{F_{i_p}}$.

Assume by contradiction there exist $z_{F_{i_r}} < z_{F_{i_s}} > z_{F_{i_t}}$, with r < s < t. The minimality of P implies that one step before the facet $F_{i_s} \\ > H$ of $lk_{\Delta_{k-1}}(H)$ is added via a shelling, there is no path joining the edges corresponding to F_{i_r} and F_{i_t} , which contradicts the connectedness of shellable complexes. Indeed, if such a path existed, we would be able to replace $\{F_{i_{r+1}}, \ldots, F_{i_s}, \ldots, F_{i_{t-1}}\}$ in the support of P with a subset containing only facets which are older than F_{i_s} : this would lead to a set colexicographically smaller than the support of P, which is impossible. By Lemma 6.7, $S(b_{F_k,F_\ell}, b_{F_k,F_m})$ reduces to zero modulo C_{Δ} .

If: If R_{Δ} has a quadratic Gröbner basis with respect to <, then the total order

 \prec on $\mathcal{F}(\Delta)$ induced by \lt is a shelling order for Δ .

Assume by contradiction that the total order induced by \langle is not a shelling order, and let $F_1 < \cdots < F_{|\mathcal{F}(\Delta)|}$ be the facets in this order. This implies that there exists $2 \leq i \leq |\mathcal{F}(\Delta)|$ such that $\Delta_{i-1} \cap \langle F_i \rangle$ is not pure and (d-2)-dimensional. In particular, there is a facet G of $\Delta_{i-1} \cap \langle F_i \rangle$ with dim $(G) \leq d-3$. We observe that $lk_{\Delta_i}(G)$ is not connected in codimension 1. Indeed, G is not properly contained in any face of $\Delta_{i-1} \cap \langle F_i \rangle$, so no face in $lk_{\Delta_i}(G)$ contains vertices of $\Delta_{i-1} \cap \langle F_i \rangle$. In particular, the facet $F_i \setminus G$ does not intersect any other facet of $lk_{\Delta_i}(G)$. However, since R_{Δ} has a quadratic Gröbner basis then it is Koszul, and hence by Corollary 5.3 Δ is Cohen–Macaulay. In particular, $lk_{\Delta}(G)$ is Cohen–Macaulay, and hence it is connected in codimension 1. This implies that $i < |\mathcal{F}(\Delta)|$. Let k be the smallest integer in $\{i+1,\ldots,|\mathcal{F}(\Delta)|\}$ such that $lk_{\Delta_k}(G)$ is connected in codimension 1. Then there exist $1 \leq r, s \leq k-1$ such that

- $G \subset F_r, G \subset F_s;$
- $(F_k \setminus G) \cap (F_r \setminus G)$ and $(F_k \setminus G) \cap (F_s \setminus G)$ are faces of codimension 1 of $lk_{\Delta_k}(G)$;
- $(F_r \smallsetminus G)$ and $(F_s \smallsetminus G)$ are not connected in codimension 1 in $lk_{\Delta_{k-1}}(G)$.

Hence dim $(F_k \cap F_r)$ = dim $(F_k \cap F_s)$ = d-2, which implies that $b_{F_k,F_r}, b_{F_k,F_s} \in \mathcal{Q}$.

We claim that $S(b_{F_k,F_r}, b_{F_k,F_s})$ does not reduce to zero modulo the set Q. By Lemma 6.7, the reduction to zero of this polynomial implies the existence of a sequence of facets $F_r = F^{(1)} \dots, F^{(p)} = F_s$ in Δ_{k-1} such that

- $F^{(t)} \leq \max\{F_r, F_s\}$ for every $1 \leq t \leq p$;
- $G \subset F_r \cap F_s \subset F^{(t)}$ for every $1 \le t \le p$;
- dim $(F^{(t)} \cap F^{(t+1)}) = d 2$ for every $1 \le t \le p 1$.

The first condition implies that $F^{(t)} \in \Delta_{k-1}$, the second yields that $F^{(t)} \smallsetminus G$ is a facet of $\operatorname{lk}_{\Delta_{k-1}}(G)$, while the third implies the existence of a path of facets of $\operatorname{lk}_{\Delta_{k-1}}(G)$ connecting $F_r \smallsetminus G$ and $F_s \smallsetminus G$. However, such a sequence does not exist, as $F_r \smallsetminus G$ and $F_s \smallsetminus G$ are not connected in codimension 1 in $\operatorname{lk}_{\Delta_{k-1}}(G)$. This proves that if Δ is not shellable, then the set C_{Δ} is not a Gröbner basis. To conclude that R_{Δ} has no quadratic Gröbner basis we use the fact that set \mathcal{U} given in Proposition 4.3 is a universal Gröbner basis for R_{Δ} by Proposition 4.5, and hence every quadratic reduced Gröbner basis must be a subset of the set of quadratic polynomials in \mathcal{U} , which is precisely C_{Δ} .

Example 6.8. Let $\Delta = \langle 123, 234, 345 \rangle$ as in Example 4.6. Up to sign, the only two binomials in Q are

$$f = y_4 z_{234} - y_1 z_{123}$$
 and $g = y_5 z_{345} - y_2 z_{234}$.

If we consider a term order compatible with the shelling order 123 < 234 < 345, then we have $in(f) = y_4 z_{234}$ and $in(g) = y_5 z_{345}$. Since the two leading terms are coprime in the z-variables, S(f,g) reduces to zero modulo the minimal generating set of $(z_{123}, z_{234}, z_{345})^2$. On the other hand, if we consider a term order compatible with 123 < 345 < 234, which is not a shelling order, we obtain $in(f) = y_4 z_{234}$ and $in(g) = y_2 z_{234}$. In this case we see that none of the monomials in $S(f,g) = y_4 y_5 z_{345} - y_1 y_2 z_{123}$ is divisible by any of the leading terms of the elements of Q. Hence, S(f,g) does not reduce to zero.

7. The γ -vector of R_{Δ}

The aim of this section is to study the γ -vector of the algebra R_{Δ} in terms of combinatorial properties of the simplicial complex Δ . In general, the γ -vector is often the "right" invariant to consider when we want to highlight the information encoded in a palindromic vector, like for instance the *h*-vector of a simplicial sphere. See [Brä06], [Gal05], and [Ath18] for more information on this topic.

In particular, recall that the coefficients of the *h*-polynomial of a standard graded Gorenstein algebra form a palindromic sequence. More precisely, if the degree of the *h*-polynomial (which coincides with the Castelnuovo–Mumford regularity since the algebra is Cohen–Macaulay) equals *s*, we have that $h_i = h_{s-i}$ for every *i*. The vector space of univariate palindromic polynomials of degree *s* has dimension $\lfloor \frac{s}{2} \rfloor + 1$, and Gal proposed the use of the basis $\{t^i(t + 1)^{s-2i}\}_{i=0}^{\lfloor \frac{s}{2} \rfloor}$ [Gal05, Section 2].

Definition 7.1. The γ -vector associated with the integer vector $h = (h_0, \ldots, h_s)$ with $h_i = h_{s-i}$ is the integer vector $\gamma = (\gamma_0, \ldots, \gamma_{\lfloor \frac{s}{2} \rfloor})$ defined by the identity

(7.1)
$$\sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \gamma_i t^i (t+1)^{s-2i} = \sum_{i=0}^s h_i t^i.$$

We denote by $\gamma(A)$ the γ -vector associated with the *h*-vector of a standard graded Gorenstein algebra A.

Each γ_i can be expressed as an integer linear combination of the h_i : for instance, one has that $\gamma_0 = h_0$, $\gamma_1 = h_1 - sh_0$ and $\gamma_2 = h_2 - (s-2)h_1 + \frac{s(s-3)}{2}h_0$. In general, the following recursive formula holds:

(7.2)
$$\gamma_i = h_i - \sum_{j=0}^{i-1} {s-2j \choose i-j} \gamma_j.$$

For the rest of the section we will focus on the γ -vector of the standard graded Gorenstein algebra R_{Δ} , with Δ a pure simplicial complex. If Δ is (d-1)-dimensional, then the *h*polynomial of R_{Δ} has degree d + 1, and we will then replace *s* with d + 1 in Definition 7.1. Perhaps surprisingly, we will show that when R_{Δ} is Koszul over \mathbb{F} – i.e., by Corollary 5.3, when Δ is Cohen–Macaulay over \mathbb{F} – many of the γ_i are nonpositive.

We begin with a result on the last entry of the γ -vector. Even though we will prove a more general result in Corollary 7.10, it is instructive to consider this special case separately, as its proof is rather elementary.

Lemma 7.2. For any (d-1)-dimensional Cohen–Macaulay simplicial complex Δ , with $d \ge 3$ odd, we have that

$$\gamma_{\frac{d+1}{2}}(R_{\Delta}) = (-1)^{\frac{d-1}{2}} 2\widetilde{\chi}(\Delta) = (-1)^{\frac{d-1}{2}} 2\dim_{\mathbb{F}} \widetilde{H}_{d-1}(\Delta;\mathbb{F}).$$

In particular, if $d \equiv 3 \mod 4$, then $\gamma_{\frac{d+1}{2}}(R_{\Delta})$ is nonpositive; moreover, for any integer $c \ge 0$ and every $d \equiv 3 \mod 4$, there exists a Cohen–Macaulay (even shellable) (d-1)-complex Δ such that $\gamma_{\frac{d+1}{2}}(R_{\Delta}) = -2c$. Such a complex can be chosen to be flag.

Proof: Let $f(\Delta) = (1, f_0, \dots, f_{d-1})$ be the *f*-vector of Δ and recall that, as noted in Remark 4.2, $h(R_{\Delta}) = (1, f_0 + f_{d-1}, f_1 + f_{d-2}, \dots, f_0 + f_{d-1}, 1)$. Evaluating (7.1) at t = -1 yields

$$(-1)^{\frac{d+1}{2}}\gamma_{\frac{d+1}{2}}(R_{\Delta}) = \sum_{i=0}^{d+1} (-1)^{i} h_{i}(R_{\Delta}) = 2 + \sum_{i=1}^{d} (-1)^{i} (f_{i-1} + f_{d-i}) = -2\widetilde{\chi}(\Delta).$$

It is a standard fact from algebraic topology that the reduced Euler characteristic can be written as the alternating sum of the dimensions of the reduced homology groups of Δ . Since Δ is Cohen–Macaulay, one has that $\widetilde{H}_j(\Delta; \mathbb{F}) = 0$ for every $0 \leq j < d-1$, and thus $\gamma_{\underline{d}\pm 1}(R_{\Delta}) = (-1)^{\underline{d}-1} 2 \dim_{\mathbb{F}} \widetilde{H}_{d-1}(\Delta; \mathbb{F}).$

To prove the last claim it is enough to exhibit a (d-1)-dimensional Cohen–Macaulay flag complex with $\tilde{\chi}(\Delta) = c$. For instance, we can glue together c copies of the boundary complex of the d-dimensional cross-polytope so that they all share the same single facet. It is elementary to check that this simplicial complex has the desired properties, and that it is indeed even shellable.

In their collection [PS09] of problems on syzygies and Hilbert functions, Peeva and Stillman suggest that it might make sense to consider a Charney–Davis-like conjecture for (not necessarily monomial) Koszul Gorenstein algebras:

Question 7.3 ([PS09, Problem 10.3]). Let S/I be a Koszul Gorenstein algebra with h-vector $(h_0, h_1, \ldots, h_{2e})$. Is it true that $(-1)^e(h_0 - h_1 + h_2 - \ldots + h_{2e}) \ge 0$?

Notice that $(-1)^e(h_0-h_1+h_2-\ldots+h_{2e}) = \gamma_e(S/I)$. Lemma 7.2 and Corollary 5.3 immediately yield a negative answer to Question 7.3:

Corollary 7.4. Let $d \equiv 3 \mod 4$, $c \in \mathbb{Z}_+$. Then there exists a Koszul Gorenstein algebra with h-vector $(h_0, h_1, \ldots, h_{d+1})$ and such that $(-1)^{\frac{d+1}{2}}(h_0 - h_1 + h_2 - \ldots + h_{2e}) = -2c < 0$. Such an algebra is of the form R_{Δ} , where Δ is a flag Cohen–Macaulay (d-1)-dimensional complex with $\dim_{\mathbb{F}} \widetilde{H}_{d-1}(\Delta; \mathbb{F}) = c$.

Example 7.5. Let Δ be the boundary of the 3-dimensional cross-polytope (or octahedron). Since $f(\Delta) = (1, 6, 12, 8)$, one has that $h(R_{\Delta}) = (1, 14, 24, 14, 1)$ and hence $\gamma_2(R_{\Delta}) = 1 - 14 + 24 - 14 + 1 = -2 < 0$. Labeling the antipodal pairs of vertices of Δ by $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ yields that R_{Δ} is presented as the quotient of the polynomial ring in $2f_0(\Delta) + f_2(\Delta) = 20$ variables

 $\mathbb{F}[x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6, z_{135}, z_{136}, z_{145}, z_{146}, z_{235}, z_{236}, z_{245}, z_{246}]$

by the ideal with 81 quadratic generators

 $(x_1x_2, x_3x_4, x_5x_6, x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_6y_6)$

- + $(z_{135}, z_{136}, z_{145}, z_{146}, z_{235}, z_{236}, z_{245}, z_{246})^2$
- $+ (x_{2}z_{135}, x_{4}z_{135}, x_{6}z_{135}, x_{2}z_{136}, x_{4}z_{136}, x_{5}z_{136}, x_{2}z_{145}, x_{3}z_{145}, x_{6}z_{145}, x_{2}z_{146}, x_{3}z_{146}, x_{5}z_{146}, x_{1}z_{235}, x_{4}z_{235}, x_{6}z_{235}, x_{1}z_{236}, x_{4}z_{236}, x_{5}z_{236}, x_{1}z_{245}, x_{3}z_{245}, x_{6}z_{245}, x_{1}z_{246}, x_{3}z_{246}, x_{5}z_{246})$
- $+ (y_5 z_{135} y_6 z_{136}, y_3 z_{135} y_4 z_{145}, y_1 z_{135} y_2 z_{235}, y_3 z_{136} y_4 z_{146}, y_1 z_{136} y_2 z_{236}, y_5 z_{145} y_6 z_{146}, y_1 z_{145} y_2 z_{245}, y_1 z_{146} y_2 z_{246}, y_5 z_{235} y_6 z_{236}, y_3 z_{235} y_4 z_{245}, y_3 z_{236} y_4 z_{246}, y_5 z_{245} y_6 z_{246}).$

Being the boundary complex of a simplicial polytope, the simplicial complex Δ is shellable. Therefore, by Theorem 6.3 the 81 generators listed above are a Gröbner basis with respect to any term order compatible with any shelling order. We remark that, by Propositions 8.1 and 8.3, one can get an analogous Artinian example with 14 variables by going modulo the regular sequence of linear forms $y_1 - x_1, \ldots, y_6 - x_6$.

Remark 7.6. Example 7.5 also provides a Koszul Gorenstein algebra which is not PF (in the sense of [RW05]) and has Castelnuovo–Mumford regularity 4. By [RW05, Corollary 4.14], all Koszul Gorenstein algebras of regularity at most 3 are PF and hence have the Charney–Davis (CD) property.

Example 7.7. Let Δ be the flag triangulation of \mathbb{RP}^2 described in Example 5.16. We have observed that $h(R_{\Delta}) = (1, 31, 60, 31, 1)$, and hence $\gamma_2(R_{\Delta}) = 0$. This is in line with Lemma 7.2, as $\dim_{\mathbb{F}} \widetilde{H}_2(\mathbb{RP}^2; \mathbb{F}) = 0$ for every field \mathbb{F} over which Δ is Cohen–Macaulay, i.e., every field of characteristic different than 2.

For the rest of the section we stipulate that a binomial coefficient $\binom{n}{k}$ is equal to zero whenever k < 0 or n < k. Moreover, whenever $r \ge 2i \ge 0$, we set

$$\ell_{r,i} \coloneqq \begin{cases} 2 & \text{if } (r,i) = (0,0) \\ \binom{r-i}{i} + \binom{r-i-1}{i-1} = \frac{r}{r-i} \binom{r-i}{i} & \text{otherwise.} \end{cases}$$

The positive integers $\ell_{r,i}$ appear as coefficients of Lucas polynomials¹ in sequence A034807 of the On-Line Encyclopedia of Integer Sequences [Inc21]. We begin with a technical lemma.

Lemma 7.8. The following identities hold for any $n, m, r \ge 0$:

$$1 + t^{r} = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^{i} \ell_{r,i} t^{i} (1+t)^{r-2i}$$

ii.

i.

$$\binom{n}{m} + \binom{n}{m-r} = \sum_{i=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^i \ell_{r,i} \binom{n+r-2i}{m-i}.$$

Proof: Statement i is a specialization of a classical identity due to Girard and Waring [Gou99, Identity 1] which can be derived from the Newton formulas relating power sums and elementary symmetric polynomials.

Expanding $(1+t)^{r-2i}$ inside identity *i* yields that

$$1 + t^{r} = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{j=0}^{r-2i} (-1)^{i} \ell_{r,i} \binom{r-2i}{j} t^{i+j}.$$

This is now an identity inside the free \mathbb{Z} -module with basis $\{1, t, t^2, \ldots, t^r\}$. We can now specialize this by substituting each t^i with the binomial coefficient $\binom{n}{m-i}$, obtaining

$$\binom{n}{m} + \binom{n}{m-r} = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^i \ell_{r,i} \left(\sum_{j=0}^{r-2i} \binom{r-2i}{j} \binom{n}{m-i-j} \right).$$

Applying Vandermonde's identity for binomial coefficients yields identity *ii*.

Proposition 7.9. Let Δ be a (d-1)-dimensional pure simplicial complex. Then

$$\gamma_i(R_{\Delta}) = (-1)^{i-1} \sum_{k=2i-1}^d \ell_{k-1,i-1} h_k(\Delta),$$

for every $i \ge 1$. In particular, each $\gamma_i(R_{\Delta})$ is $(-1)^{i-1}$ times a linear combination of the $h_k(\Delta)$ with positive integer coefficients.

Proof: Throughout this proof we will set $\gamma_j \coloneqq \gamma_j(R_\Delta)$, $h_j \coloneqq h_j(\Delta)$ and $f_j \coloneqq f_j(\Delta)$. We prove the desired formula for γ_i by induction on $i \ge 1$.

For the base case i = 1 it suffices to observe that $\gamma_1 = h_1(R_{\Delta}) - d - 1 = f_0 + f_{d-1} - d - 1 = 2h_1 + \sum_{k=2}^d h_k$, where we used (2.4) to express f_0 and f_{d-1} as functions of the h_j .

Let us now fix i > 1 and assume that the claim holds for every $1 \le j < i$. (7.2) gives us a way to express γ_i as a function of the γ_j with j < i; moreover, using again (2.4), we can explicitly write each $h_i(R_{\Delta}) = f_{i-1} + f_{d-i}$ as a combination of the $h_i = h_i(\Delta)$. Putting together these two

¹There appear to be at least two different families of polynomials known as Lucas polynomials, but in both cases the nonzero coefficients are the integers $\ell_{r,i}$.

facts with the inductive hypothesis on the γ_j with j < i and the observation that $\gamma_0 = h_0 = 1$, we can write

$$\begin{split} \gamma_{i} &= h_{i}(R_{\Delta}) - \sum_{j=0}^{i-1} \binom{d+1-2j}{i-j} \gamma_{j} \\ &= f_{i-1} + f_{d-i} - \binom{d+1}{i} \gamma_{0} - \sum_{j=1}^{i-1} \binom{d+1-2j}{i-j} \gamma_{j} \\ &= \sum_{k=0}^{i} \binom{d-k}{d-i} h_{k} + \sum_{k=0}^{d-i+1} \binom{d-k}{i-1} h_{k} - \binom{d+1}{i} h_{0} - \sum_{j=0}^{i-2} \binom{d-1-2j}{i-j-1} \gamma_{j+1} \\ &= \sum_{k=0}^{i} \binom{d-k}{d-i} h_{k} + \sum_{k=0}^{d-i+1} \binom{d-k}{i-1} h_{k} - \binom{d+1}{i} h_{0} - \sum_{j=0}^{i-2} \binom{d-1-2j}{i-j-1} (-1)^{j} \sum_{s=2j+1}^{d} \ell_{s-1,j} h_{s}. \end{split}$$

We have thus obtained an expression for γ_i as a linear combination of the h_j . We will denote by $[\gamma_i]_k$ the coefficient of h_k in this expression and will analyze such coefficients separately. One has immediately that $[\gamma_i]_0 = \binom{d}{i} + \binom{d}{i-1} - \binom{d+1}{i} = 0$. When $0 < k \le d$, we obtain

(7.3)
$$[\gamma_i]_k = \binom{d-k}{d-i} + \binom{d-k}{i-1} - \sum_{j=0}^{\min\{i-2,\lfloor\frac{k-1}{2}\rfloor\}} \binom{d-1-2j}{i-j-1} (-1)^j \ell_{k-1,j}$$

We wish to rewrite $\binom{d-k}{d-i} + \binom{d-k}{i-1}$ in the above expression. Noting that $\binom{d-k}{d-i} = \binom{d-k}{i-1-(k-1)}$, we are in the position to apply Lemma 7.8.ii with n = d - k, m = i - 1 and r = k - 1, obtaining that

$$\binom{d-k}{d-i} + \binom{d-k}{i-1} = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{d-1-2j}{i-j-1} (-1)^j \ell_{k-1,j}.$$

Hence, (7.3) immediately yields that $[\gamma_i]_k = 0$ whenever $\min\{i-2, \lfloor \frac{k-1}{2} \rfloor\} = \lfloor \frac{k-1}{2} \rfloor$, which happens when $k \leq 2i-2$.

Now assume that $2i - 1 \le k \le d$. Then $\min\{i - 2, \lfloor \frac{k-1}{2} \rfloor\} = i - 2$, and so (7.3) becomes

$$[\gamma_i]_k = \sum_{j=i-1}^{\lfloor \frac{k-1}{2} \rfloor} {\binom{d-1-2j}{i-j-1}} (-1)^j \ell_{k-1,j} = (-1)^{i-1} \ell_{k-1,i-1}$$

with the last equality holding since the binomial coefficient $\binom{d-1-2j}{i-j-1}$ equals 1 when j = i - 1 and vanishes otherwise.

Corollary 7.10. Let Δ be a (d-1)-dimensional simplicial complex for which $h_i(\Delta) \geq 0$ holds for every $0 \leq i \leq d$. Then

$$(-1)^{i-1}\gamma_i(R_\Delta) \ge 0,$$

for every $1 \le i \le \lfloor \frac{d+1}{2} \rfloor$.

In particular, the algebra R_{Δ} has γ -numbers which alternate in sign whenever Δ is Cohen-Macaulay. However, the hypothesis in Corollary 7.10 is satisfied by larger families of simplicial complexes, such as partitionable complexes.

Remark 7.11. We would like to stress that the formula in Proposition 7.9 is of purely combinatorial nature, and it holds for more general integer vectors than f- and h-vectors of simplicial complexes. Let $\mathbf{a} = (a_0, a_1, \ldots, a_{d+1})$ be any sequence of integers with $a_i = a_{d+1-i}$, and let $\mathbf{b} = (b_0, b_1, \ldots, b_d)$ be a sequence which satisfies $b_0 = a_0$ and $b_i + b_{d+1-i} = a_i$, for every $0 < i \leq d$. Proposition 7.9 allows to express the γ -vector of \mathbf{a} as a linear combination of the entries of a vector \mathbf{c} , obtained from \mathbf{b} via the linear transformation (2.3).

8. An Artinian reduction of R_{Δ} and a connection to work of Gondim and Zappalà

After the first version of this paper was completed, we noticed a connection between the algebras R_{Δ} defined here and the Artinian Gorenstein algebras A_{Δ} introduced by Gondim and Zappalà in their paper [GZ18]: more specifically, A_{Δ} is an Artinian reduction of R_{Δ} in characteristic zero. The aim of this final section is to clarify this connection.

For this section, let the characteristic of the field \mathbb{F} be zero². Under this assumption, it is known that every graded Artinian Gorenstein algebra corresponds to a single homogeneous polynomial f via Macaulay's inverse system (see e.g. [IK99, Section 0.2]). The key idea of the paper by Gondim and Zappalà is the following: given a pure simplicial complex Δ with n vertices, consider the polynomial ring $\mathbb{F}[x_i, z_F | i \in \{1, \ldots, n\}, F$ facet of Δ] and let

$$f_{\Delta} \coloneqq \sum_{F \text{ facet of } \Delta} z_F \cdot \prod_{i \in F} x_i.$$

Call A_{Δ} the Artinian Gorenstein algebra corresponding to f_{Δ} via Macaulay's inverse system. After unifying the notation (as the U and X-variables in [GZ18] are respectively our x- and z-variables), an analysis of the presentation of A_{Δ} exhibited in [GZ18, Theorem 3.2.5] reveals that $A_{\Delta} = R_{\Delta}/(y_i - x_i \mid i \in \{1, ..., n\})$. Our first observation is then that $y_1 - x_1, ..., y_n - x_n$ is a regular sequence of linear forms:

Proposition 8.1. Let Δ be a pure simplicial complex and let char(\mathbb{F}) = 0. Then $A_{\Delta} = R_{\Delta}/(y_i - x_i \mid i \in \{1, ..., n\})$, and $y_1 - x_1, ..., y_n - x_n$ is an R_{Δ} -regular sequence of linear forms.

Proof: Since R_{Δ} is Cohen–Macaulay, it is enough to prove that dim $R_{\Delta} = n + \dim R_{\Delta}/(y_i - x_i \mid i \in \{1, \ldots, n\})$, see e.g. the graded version of [BH98, Theorem 2.1.2.c]. Since $R_{\Delta}/(y_i - x_i \mid i \in \{1, \ldots, n\})$ is Artinian and the Krull dimension of R_{Δ} equals n (check Notation 4.1), the claim follows.

Corollary 8.2. Let Δ be a pure flag simplicial complex and let char(\mathbb{F}) = 0. Then the following conditions are equivalent:

- i. A_{Δ} is Koszul;
- ii. R_{Δ} is Koszul;
- iii. Δ is Cohen–Macaulay over \mathbb{F} .

Proof: Conditions ii and iii are equivalent by Corollary 5.3. If B is a standard graded \mathbb{F} -algebra and $\ell \in B_1$ is a regular element, one has that B is Koszul if and only if $B/\ell B$ is

 $^{^{2}}$ This is only needed in order to use Macaulay's inverse system, but does not really play a role in our observations: see Remark 8.4.

[BF85, Theorem 4.e.iv]; hence, the equivalence of conditions i and ii descends directly from Proposition 8.1.

If B is a standard graded \mathbb{F} -algebra, $\ell \in B_1$ is a regular element and $B/\ell B$ has a quadratic Gröbner basis, then so does B [Con00, Lemma 4]; the reverse implication does *not* hold in general. In our case, however, the proof of the "only if" implication of Theorem 6.3 goes through for A_{Δ} as well: the only difference lies in the presence of some new nontrivial S-pairs, namely the ones coming from a nonface-monomial \mathbf{x}^N and a binomial $\mathbf{x}^{F_1 \setminus F_2} z_{F_1} - \mathbf{x}^{F_2 \setminus F_1} z_{F_2}$. It can be checked that such S-pairs reduce to zero. Putting these observations together, we get the following result:

Proposition 8.3. Let Δ be a pure flag simplicial complex and let char(\mathbb{F}) = 0. Then the following conditions are equivalent:

i. A_Δ has a quadratic Gröbner basis;
ii. R_Δ has a quadratic Gröbner basis;
iii. Δ is shellable.

Remark 8.4. The running hypothesis of this section about characteristic zero is needed just to fit the original definition of A_{Δ} by Gondim and Zappalà, where Macaulay's inverse system is used. However, the proof that $y_1 - x_1, \ldots, y_n - x_n$ is an R_{Δ} -regular sequence of linear forms is characteristic-free, and so are the proofs of Corollary 8.2 and Proposition 8.3, if we substitute " A_{Δ} " by " $R_{\Delta}/(y_i - x_i | i \in \{1, \ldots, n\})$ " in the statements. In particular, the results in Section 5.3 and Section 7 can be adapted to fit the Artinian setting.

Remark 8.5. Because of Proposition 8.1, the algebra A_{Δ} has a quadratic defining ideal precisely when R_{Δ} does; in particular, the characterizations of quadraticity in [GZ18, Theorem 3.5] and in our Proposition 4.4 should coincide. However, Proposition 4.4 requires Δ to be flag and (S_2) , while [GZ18, Theorem 3.5] only asks for Δ to be flag and strongly connected, which is weaker.

We claim that the complex Δ with facet list {123, 235, 245, 457, 567, 167} provides a counterexample to [GZ18, Theorem 3.5]: indeed, Δ is strongly connected but not (S_2), and the cubical generator $x_2x_3z_{123} - x_6x_7z_{167}$ is needed in the presentation of A_{Δ} , as can be checked for instance with Macaulay2 using the InverseSystems package [EB].

Note that the examples constructed via Turán complexes in [GZ18] are still valid: since Turán complexes are flag and Cohen–Macaulay, by Corollary 8.2 the associated algebras are Koszul, and hence quadratically presented.

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