Weak-strong uniqueness principle for compressible barotropic self-gravitating fluids

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Abstract

The aim of this work is to prove the weak–strong uniqueness principle for the compressible Navier–Stokes–Poisson system on an exterior domain, with an isentropic pressure of the type $p(\varrho) = a\varrho^\gamma$ and allowing the density to be close or equal to zero. In particular, the result will be first obtained for an adiabatic exponent $\gamma \in [9/5, 2]$ and afterwards, this range will be slightly enlarged via pressure estimates "up to the boundary", deduced relaying on boundedness of a proper singular integral operator.

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1 Introduction

In this paper we consider the compressible Navier–Stokes–Poisson system, characterized by the following equations:

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= \mathrm{div}_x \, \mathbb{S}(\nabla_x \mathbf{u}) + G\varrho \nabla_x \Phi, \\
\sigma \Delta_x \Phi &= \varrho + g.
\end{aligned}
\tag{1.1}
$$

The system will be studied $(0,T) \times \Omega$, where $T > 0$ can be chosen arbitrarily large and $\Omega \subseteq \mathbb{R}^3$ is a bounded or unbounded domain. Here, the unknown variables are the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the potential $\Phi = \Phi(t,x)$, while $p = p(\varrho)$ represents the barotropic pressure,

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$\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ the viscous stress tensor, which we suppose to be a linear function of the velocity gradient, G is a positive constant and $g = g(x)$ a given function; further details can be found in Section (2).

Depending on the choice of $\sigma = \pm 1$, system (1.1) models two different physical phenomena:

- for $\sigma = 1$, it describes the transportation of charged particles in electronic devices and $(\varrho, \mathbf{u}, \Phi)$ represent the density, velocity and electrostatic potential of the charge, respectively (see [1] for more details);

- for $\sigma = -1$, it describes the motion of a gaseous star and $(\varrho, \mathbf{u}, \Phi)$ represent the density, velocity and gravitational potential of the star, respectively.

In view of its importance in many real world problems, the Navier-Stokes-Poisson system (1.1) is a matter of great interest in mathematics and physics. Unfortunately, well-posedness of strong solutions was achieved only on a small time interval and for initial data satisfying some compatibility conditions, see for instance the work of Tan and Zhang [15]. On the other hand, something more can be said if we turn our attention to the class of weak solutions. For $\sigma = 1$, the existence of global-in-time weak solutions was established on a bounded domain Ω and for a barotropic pressure of the type $p(\varrho) = a\varrho^\gamma$ by Donatelli [4] with the adiabatic exponent $\gamma \geq 3$, and by Kobayashi and Suzuki [10] for $\gamma > \frac{3}{2}$, while on the whole space $\Omega = \mathbb{R}^3$ it was proved by Li, Matsumura and Zhang [11]. For $\sigma = -1$, the existence of global-in-time weak solutions was proved on an exterior domain Ω and with a barotropic pressure of the type $p(\varrho) = a\varrho^\gamma$, $\gamma > \frac{3}{2}$, by Ducomet and Feireisl [2]; later on, this result was improved for a non-monotone pressure by Ducomet, Feireisl, Petzeltová and Straškraba [3].

In this context, a bridge between the classes of strong and weak solutions can be constructed by means of an important analytical tool known as *weak-strong uniqueness principle*: a weak solution of problem (1.1) coincides with the strong one, emanating from the same initial data, as long as the latter exists. The rather standard procedure in order to prove it is to introduce a positive functional measuring the "distance" between the weak and strong solutions and to show that it vanishes for any time as a consequence of Gronwall's Lemma. The functional in question is known as *relative energy* since it can be seen as a generalization of the mechanical energy associated to the system. However, the choice of $\sigma \in \{1, -1\}$ in the third equation of the Navier-Stokes-Poisson system (1.1) plays a key role in making the whole problem easier or more difficult, respectively. Indeed, notice that multiplying the second equation of system (1.1) by $\mathbf{u}$, integrating over $(0, T) \times \Omega$ and imposing suitable boundary conditions for $\mathbf{u}$ and $\nabla_x \Phi$, we can recover the energy inequality associated to the system:

$$\left[\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + \frac{\sigma}{2}\ G\ |\nabla_x \Phi|^2 \right)(t, \cdot) \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \mathrm{d}t \leq 0,$$

where $P = P(\varrho)$ denotes the pressure potential; further details can be found in Section 2.2.

For $\sigma = 1$, it then makes sense to consider the relative energy functional as

$$\mathcal{E}\left(\varrho, \mathbf{u}, \Phi \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\Phi}\right) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) + \frac{1}{2} G\ |\nabla_x (\Phi - \widetilde{\Phi})|^2 \right) \mathrm{d}x, \tag{1.2}$$

where $(\varrho, \mathbf{u}, \Phi)$ and $(\widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\Phi})$ denote the weak and strong solutions of system (1.1), respectively. Indeed, the convexity of the pressure potential $P = P(\varrho)$ guarantees the non-negativity of $\mathcal{E} = \mathcal{E}(t)$ for any time $t \in [0,T]$. Moreover, if $\varrho, \widetilde{\varrho} > 0$, proving the weak-strong uniqueness principle is equivalent to showing that $\mathcal{E}(t) \equiv 0$ for any time $t \in [0,T]$; this is the strategy pursued by He and Tan [8] to prove the weak-strong uniqueness principle on a bounded domain Ω.

For $\sigma = -1$, however, the problem gets more complicated. First of all, the analogous of (1.2) would be

$$\mathcal{E}\left(\varrho, \mathbf{u}, \Phi \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\Phi}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) - \frac{1}{2}G\,|\nabla_x(\Phi - \widetilde{\Phi})|^2\right) \mathrm{d}x,$$

but with this choice we cannot guarantee the non-negativity of $\mathcal{E} = \mathcal{E}(t)$ for any time $t \in [0,T]$. Moreover, for $\sigma = -1$, the Navier-Stokes-Poisson system (1.1) describes the motion of a gaseous star and thus the optimal choice for Ω is to be exterior to a rigid object; however, working on an unbounded domain prevents us from using some useful tools such as the Sobolev-Poincaré inequality. A third difficulty is represented by the fact that the density is close to zero, at least in the far field, and therefore we loose the strict positivity of $\varrho, \widetilde{\varrho}$.

To handle these problems for $\sigma = -1$, first of all we will consider the relative energy to be a function of the density and velocity only, cf. Section 4. Indeed, it is well–known that the solution Φ of the Poisson equation

$$-\Delta_x \Phi = f$$

on the whole space $\mathbb{R}^3$ is uniquely determined by the corresponding known term f. Therefore, in our context it makes sense to write $\Phi = (-\Delta_x)^{-1}(\varrho + g)$, provided ϱ can be extended to be zero outside Ω, and, as a consequence of the Hörmander-Mikhlin Theorem, we will be able to recover some useful estimates for $\nabla_x \Phi$ depending on the density only, which will be fundamental in proving the weak–strong uniqueness principle, cf. Section 5. The problem of the vanishing strong solution $\widetilde{\varrho}$ can be handled following the same idea developed by Feireisl and Novotný in [5], considering first $\widetilde{\varrho} + \varepsilon$, with $\varepsilon > 0$, instead of $\widetilde{\varrho}$ in the relative energy functional to get a strictly positive quantity and passing to the limit $\varepsilon \to 0$. In particular, in [5] the authors were able to prove the weak–strong uniqueness principle for a general compressible viscous fluid on an exterior domain and with a barotropic pressure of the type $p(\varrho) = a\varrho^\gamma$ with $1 < \gamma \leq 2$. In our context, the presence of the gravitational potential forces the range for the adiabatic exponent to be

$$\frac{9}{5} \leq \gamma \leq 2,$$

where, in particular, the lower bound coincides with the critical exponent appearing in the book of Lions [13]. However, the result can be improved if we manage to get better regularity for the density. This will be achieved deducing pressure estimates “up to the boundary”, obtained adapting the work of Feireisl and Petzeltová in [6] for a Lipschitz exterior domain and exploiting, in particular, the boundedness of the singular operator $\nabla_x(-\Delta_x)^{-1}\nabla_x$, cf. Section 6.

The work is organized as follows. Section 2 will be devoted to the detailed description of the system we are going to study, deducing, in particular, the energy inequality associated to it. In Section 3, we provide the definition of a dissipative weak solution, cf. Definition 3.1, while in Section 4, we recover the relative energy inequality, cf. Lemma 4.1. Section 5 will be devoted to the proof of the weak–strong uniqueness principle, cf. Theorem 5.2. Finally, in Section 6.1, we are able to improve the result obtained in the previous section, cf. Corollary 6.3, by means of the pressure estimates "up to the boundary", cf. Theorem 6.1.

2 The system

We consider the Navier-Stokes-Poisson system, describing the motion of a gaseous star:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + G\varrho \nabla_x \Phi, \tag{2.2}$$

$$-\Delta_x \Phi = \varrho + g. \tag{2.3}$$

Here, the unknown variables are the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the gravitational potential $\Phi = \Phi(t,x)$ of the star. For simplicity, we assume an isentropic pressure $p = p(\varrho)$ of the type

$$p(\varrho) = a\varrho^\gamma$$

for a constant $a > 0$, with the adiabatic exponent

$$\gamma > 1,$$

while the viscous stress tensor is a linear function of the velocity gradient, more specifically it satisfies Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}(\mathrm{div}_x\, \mathbf{u})\mathbb{I} \right) + \lambda(\mathrm{div}_x\, \mathbf{u})\mathbb{I}, \tag{2.4}$$

with $\mu > 0$ and $\lambda \geq 0$. Finally, G is a positive constant and $g = g(x)$ is a given function, which for simplicity we suppose to satisfy

$$g \in L^1 \cap L^\infty(\mathbb{R}^3).$$

We will study the system on $(0,T) \times \Omega$, where the time $T > 0$ can be chosen arbitrarily large while $\Omega \subset \mathbb{R}^3$ is a Lipschitz exterior domain, on the boundary of which we impose

$$\mathbf{u}|_{\partial\Omega} = 0; \tag{2.5}$$

moreover, we fix the conditions at infinity as

$$\varrho \to 0, \quad \mathbf{u} \to 0 \quad \text{as } |x| \to \infty. \tag{2.6}$$

The system is formally closed prescribing the initial conditions for the density and momentum:

$$\varrho(0,\cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0,\cdot) = \mathbf{m}_0. \tag{2.7}$$

2.1 Poisson equation

Noticing that the Poisson equation (2.3) holds on the whole space $\mathbb{R}^3$, provided ϱ is extended to be zero outside Ω, we can write

$$\Phi(t,x) = [\Gamma * (\varrho + g)](t,x) = \int_{\mathbb{R}^3} [\varrho(t,y) + g(y)]\ \Gamma(|x-y|)\ \mathrm{d}y$$

where

$$\Gamma(|x|) = \frac{1}{4\pi|x|}$$

is the *fundamental solution* of the Laplace equation (2.3). Therefore, the gravitational potential Φ is uniquely determined by the corresponding density ϱ and therefore it is not necessary to consider it as a third variable.

2.2 Energy inequality

Multiplying equation (2.2) by $\mathbf{u}$ and noticing that each term of this product can be rewritten as

$$\begin{aligned}
\partial_t(\varrho\mathbf{u}) \cdot \mathbf{u} &= \frac{\partial}{\partial t}\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{u}|^2\partial_t\varrho, \\
\operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) \cdot \mathbf{u} &= \operatorname{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) + \frac{1}{2}|\mathbf{u}|^2 \operatorname{div}_x(\varrho\mathbf{u}), \\
\nabla_x p(\varrho) \cdot \mathbf{u} &= \operatorname{div}_x[p(\varrho)\mathbf{u}] - p(\varrho)\operatorname{div}_x \mathbf{u}, \\
\operatorname{div}_x \mathbb{S}(\nabla_x\mathbf{u}) \cdot \mathbf{u} &= \operatorname{div}_x[\mathbb{S}(\nabla_x\mathbf{u})\mathbf{u}] - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}, \\
\varrho\nabla_x\Phi \cdot \mathbf{u} &= \operatorname{div}_x(\varrho\Phi\mathbf{u}) - \Phi\operatorname{div}_x(\varrho\mathbf{u}),
\end{aligned} \tag{2.8}$$

where, in particular, from (2.1) and (2.3),

$$\begin{aligned}
-\Phi\operatorname{div}_x(\varrho\mathbf{u}) &= \Phi\partial_t\varrho = -\Phi\partial_t\Delta_x\Phi = -\Phi\operatorname{div}_x[\partial_t\nabla_x\Phi] \\
&= -\operatorname{div}_x[\Phi\ \partial_t\nabla_x\Phi] + \nabla_x\Phi\cdot\partial_t\nabla_x\Phi \\
&= -\operatorname{div}_x[\Phi\ \partial_t\nabla_x\Phi] + \frac{\partial}{\partial t}\left(\frac{1}{2}|\nabla_x\Phi|^2\right),
\end{aligned}$$

from the continuity equation (2.1), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\varrho|\mathbf{u}|^2 - G|\nabla_x\Phi|^2\right) + \operatorname{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + p(\varrho)\right)\mathbf{u}\right] - p(\varrho)\operatorname{div}_x\mathbf{u} \\
&\qquad = \operatorname{div}_x\left[\left(\mathbb{S}(\nabla_x\mathbf{u}) + \varrho\Phi\right)\mathbf{u}\right] - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \operatorname{div}_x[\Phi\ \partial_t\nabla_x\Phi].
\end{aligned}$$

Integrating over Ω, keeping in mind that $\mathbf{u}$ satisfies the boundary condition (2.5) and imposing that

$$\nabla_x\Phi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.9}$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\varrho|\mathbf{u}|^2 - G|\nabla_x\Phi|^2\right)\mathrm{d}x - \int_\Omega p(\varrho)\operatorname{div}_x\mathbf{u}\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x = 0. \tag{2.10}$$

Introducing the *pressure potential* $P = P(\varrho)$ as a solution of

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho), \tag{2.11}$$

from the continuity equation (2.1), we can write

$$-p(\varrho)\,\mathrm{div}_x\,\mathbf{u} = \partial_t P(\varrho) + \mathrm{div}_x[P(\varrho)\mathbf{u}].$$

We finally get the *energy inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) - \frac{1}{2}G|\nabla_x\Phi|^2\right)(t,\cdot)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x \leq 0.$$

Alternatively, we can leave the last term in (2.8) unchanged and get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x \leq G\int_\Omega \varrho\nabla_x\Phi\cdot\mathbf{u}\ \mathrm{d}x, \tag{2.12}$$

with

$$E(t) := \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(t,\cdot)\ \mathrm{d}x. \tag{2.13}$$

Remark 2.1. Hereafter, we will consider (2.12), (2.13) and therefore we don't need the boundary condition (2.9) for the gravitation potential.

3 Dissipative weak solution

We are now ready to give the definition of a *dissipative weak solution* to the compressible Navier-Stokes-Poisson system. Following the same definition presented in [3], a dissipative weak solution of problem (2.1)–(2.7) is a couple $[\varrho, \mathbf{u}]$ such that

1. equation (2.1) and its renormalized version hold in a distributional sense on the whole $(0,T)\times\mathbb{R}^3$, provided ϱ and $\mathbf{u}$ are extended to be zero outside Ω;

2. equation (2.2) holds in a distributional sense on $(0,T)\times\Omega$;

3. equation (2.3) is satisfied a.e. on $\mathbb{R}^3$ for any fixed $t \in (0,T)$, provided ϱ is extended to be zero outside Ω;

4. the integral version of the energy inequality (2.12) holds on $(0,T)$.

More precisely, we have the following definition.

Definition 3.1. The pair of functions $(\varrho, \mathbf{u})$ is called *dissipative weak solution* of the Navier-Stokes-Poisson system (2.1)–(2.7) with initial conditions ϱ_0, $\mathbf{m}_0$ satisfying

$$\varrho_0 \in L^1 \cap L^\gamma(\Omega), \quad \varrho_0 \geq 0 \text{ a.e. in } \Omega, \quad \frac{|\mathbf{m}_0|^2}{\varrho_0} \in L^1(\Omega).$$

if the following holds:

(i) *regularity class*:

$$\varrho \in C_{\rm weak}([0,T]; L^1 \cap L^\gamma(\Omega)),$$
$$\varrho \mathbf{u} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)),$$
$$\mathbf{u} \in L^2(0,T; D_0^{1,2}(\Omega;\mathbb{R}^3)),$$

and ϱ is a non-negative function a.e. in $(0,T)\times\Omega$;

(ii) *weak formulation of the continuity equation*: for any $\tau \in (0,T)$, the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ {\rm d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi]\ {\rm d}x{\rm d}t, \tag{3.1}$$

holds for any $\varphi \in C_c^1([0,T]\times\mathbb{R}^3)$, with $\varrho(0,\cdot) = \varrho_0$, provided ϱ and $\mathbf{u}$ are extended to be zero outside Ω;

(iii) *weak formulation of the renormalized continuity equation*: for any $\tau \in (0,T)$ and any functions

$$B \in C[0,\infty) \cap C^1(0,\infty),\ \ b \in C[0,\infty) \text{ bounded on } [0,\infty),$$

$$B(0) = b(0) = 0 \quad \text{and} \quad b(z) = zB'(z) - B(z) \text{ for any } z > 0,$$

the integral identity

$$\left[\int_\Omega B(\varrho)\varphi(t,\cdot)\ {\rm d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [B(\varrho)\partial_t\varphi + B(\varrho)\mathbf{u}\cdot\nabla_x\varphi + b(\varrho)\,{\rm div}_x\,\mathbf{u}\varphi]\ {\rm d}x{\rm d}t, \tag{3.2}$$

holds for any $\varphi \in C_c^1([0,T]\times\mathbb{R}^3)$, provided ϱ and $\mathbf{u}$ are extended to be zero outside Ω;

(iv) *weak formulation of the balance of momentum*: for any $\tau \in (0,T)$, the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ {\rm d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi} + p(\varrho)\,{\rm div}_x\,\boldsymbol{\varphi}]\ {\rm d}x{\rm d}t,\\ &- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\ {\rm d}x{\rm d}t + G\int_0^\tau\int_\Omega \varrho\nabla_x\Phi\cdot\boldsymbol{\varphi}\ {\rm d}x{\rm d}t\end{aligned} \tag{3.3}$$

holds for any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\overline{\Omega};\mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$, with $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0$;

(v) *Poisson equation*: for any fixed $t \in [0,T]$, equation (2.3) is satisfied a.e. on $\mathbb{R}^3$, provided ϱ is extended to be zero outside Ω;

(vi) *energy inequality*: inequality

$$[E(t)]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\ {\rm d}x{\rm d}t \le G\int_0^\tau\int_\Omega \varrho\nabla_x\Phi\cdot\mathbf{u}\ {\rm d}x{\rm d}t, \tag{3.4}$$

holds for a.e. $\tau \in (0,T)$, with

$$E(t) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(t,\cdot)\ {\rm d}x.$$

Remark 3.2. Hereafter, we denote with $D_0^{1,p}(\Omega)$, $1 \le p < \infty$ the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{1,p}(\Omega)} = \|\nabla_x u\|_{L^p(\Omega)}.$$

Remark 3.3. Notice that conditions (i) of Definition 3.1 come naturally from the assumption that the total mechanical energy of the system is bounded at the initial time $t = 0$.

Remark 3.4. By a density argument, the test functions in the weak formulations (3.1)–(3.3) can be taken less regular as long as all the integrals remain well–defined.

4 Relative energy inequality

The aim of this section is to prove that any dissipative weak solution $[\varrho, \mathbf{u}]$ of the compressible Navier–Stokes–Poisson system (2.1)–(2.7) satisfies an extended version of the energy inequality, known as *relative energy inequality*, for $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ regular enough. More precisely, our goal is to prove the following result.

Lemma 4.1. *Let $[\varrho, \mathbf{u}]$ be a dissipative weak solution of the compressible Navier-Stokes-Poisson system* (2.1)–(2.7) *in the sense of Definition 3.1. Then for any pair of functions $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ such that*

$$\widetilde{\varrho} \in C^1([0,T] \times \overline{\Omega}),$$
$$\widetilde{\mathbf{u}} \in C_c^1([0,T] \times \overline{\Omega}; \mathbb{R}^3),$$

with $\widetilde{\varrho} > 0$, the following inequality holds:

$$\begin{aligned}
\left[\int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(t,\cdot)\, \mathrm{d}x\right]_{t=0}^{t=\tau} &+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\mathbf{u} - \widetilde{\mathbf{u}})\, \mathrm{d}x\mathrm{d}t \\
&\le - \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho})]\, \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x \widetilde{\mathbf{u}} \cdot (\mathbf{u} - \widetilde{\mathbf{u}})\, \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega [p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho})]\, \mathrm{div}_x \widetilde{\mathbf{u}}\, \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega p'(\widetilde{\varrho}) \left(1 - \frac{\varrho}{\widetilde{\varrho}}\right) [\partial_t \widetilde{\varrho} + \mathrm{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})]\, \mathrm{d}x\mathrm{d}t \\
&\quad + G \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x \Phi\, \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.1}$$

with

$$E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}) = \frac{1}{2}\varrho|\mathbf{u} - \widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}).$$

Remark 4.2. We define the *relative energy* for any $\tau \in [0,T]$ as

$$\mathcal{E}(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau) = \int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau, \cdot)\, \mathrm{d}x,$$

and consequently, we refer to relation (4.1) as *relative energy inequality*.

Proof. First of all, we can take $\boldsymbol{\varphi} = \widetilde{\mathbf{u}}$ in the weak formulation of the momentum equation (3.3) to obtain

$$\begin{aligned} \left[\int_\Omega \varrho\mathbf{u}\cdot\widetilde{\mathbf{u}}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\widetilde{\mathbf{u}} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\widetilde{\mathbf{u}} + p(\varrho)\operatorname{div}_x\widetilde{\mathbf{u}}]\ \mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t + G\int_0^\tau\int_\Omega \varrho\nabla_x\Phi\cdot\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.2}$$

$\varphi = \frac{1}{2}|\widetilde{\mathbf{u}}|^2, P'(\widetilde{\varrho})$ in the weak formulation of the continuity equation (3.1) to get

$$\left[\int_\Omega \frac{1}{2}\varrho|\widetilde{\mathbf{u}}|^2(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\widetilde{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}} + \varrho\mathbf{u}\cdot\nabla_x\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{u}}]\ \mathrm{d}x\mathrm{d}t, \tag{4.3}$$

$$\left[\int_\Omega \varrho P'(\widetilde{\varrho})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\partial_t P'(\widetilde{\varrho}) + \varrho\mathbf{u}\cdot\nabla_x P'(\widetilde{\varrho})]\ \mathrm{d}x\mathrm{d}t, \tag{4.4}$$

respectively.

Subtracting equations (4.2), (4.4) and summing equation (4.3) to the energy inequality (3.4), we get

$$\begin{aligned} &\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - P(\widetilde{\varrho})\right)\mathrm{d}x\right]_{t=0}^{t=\tau} \\ &\quad + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\ &\quad \leq -\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot[\partial_t\widetilde{\mathbf{u}} + \nabla_x\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho})]\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\nabla_x\widetilde{\mathbf{u}}\cdot(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau\int_\Omega [p(\varrho) - p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - p(\widetilde{\varrho})]\operatorname{div}_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\ &\quad + G\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\nabla_x\Phi\ \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau\int_\Omega \big[[p'(\widetilde{\varrho}) - \varrho P''(\widetilde{\varrho})]\partial_t\widetilde{\varrho} + p'(\widetilde{\varrho})(\widetilde{\varrho}-\varrho)\operatorname{div}_x\widetilde{\mathbf{u}} + [p'(\widetilde{\varrho}) - \varrho P''(\widetilde{\varrho})]\nabla_x\widetilde{\varrho}\cdot\widetilde{\mathbf{u}}\big]\ \mathrm{d}x\mathrm{d}t \end{aligned}$$

where we summed and subtracted quantities $\varrho\widetilde{\mathbf{u}}\cdot\nabla_x P'(\widetilde{\varrho})$, $p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) + p(\widetilde{\varrho})$ and $\partial_t p(\widetilde{\varrho})$.

Now, keeping in mind condition (2.11) and in particular the fact that $p'(\widetilde{\varrho}) = \widetilde{\varrho}P''(\widetilde{\varrho})$, we can rewrite the last line as follows

$$\begin{aligned} &\int_0^\tau\int_\Omega \big[[p'(\widetilde{\varrho}) - \varrho P''(\widetilde{\varrho})]\partial_t\widetilde{\varrho} + p'(\widetilde{\varrho})(\widetilde{\varrho}-\varrho)\operatorname{div}_x\widetilde{\mathbf{u}} + [p'(\widetilde{\varrho}) - \varrho P''(\widetilde{\varrho})]\nabla_x\widetilde{\varrho}\cdot\widetilde{\mathbf{u}}\big]\ \mathrm{d}x\mathrm{d}t \\ &= \int_0^\tau\int_\Omega p'(\widetilde{\varrho})\left(1 - \frac{\varrho}{\widetilde{\varrho}}\right)[\partial_t\widetilde{\varrho} + \widetilde{\varrho}\operatorname{div}_x\widetilde{\mathbf{u}} + \nabla_x\widetilde{\varrho}\cdot\widetilde{\mathbf{u}}]\ \mathrm{d}x\mathrm{d}t \\ &= \int_0^\tau\int_\Omega p'(\widetilde{\varrho})\left(1 - \frac{\varrho}{\widetilde{\varrho}}\right)[\partial_t\widetilde{\varrho} + \operatorname{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})]\ \mathrm{d}x\mathrm{d}t. \end{aligned}$$

We finally got (4.1). □

5 Weak-strong uniqueness

In this section our goal is to prove that a dissipative weak solution coincides with the strong one emanating from the same initial data, as long as the latter exists. The strategy consists in showing, through a standard Gronwall argument and relying on the relative energy inequality (4.1), that $\mathcal{E}(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau)$ vanishes for any $\tau \in [0,T]$, where $(\varrho, \mathbf{u})$ and $(\widetilde{\varrho}, \widetilde{\mathbf{u}})$ denote the weak and strong solutions, respectively. One difficulty is represented by the fact the $\widetilde{\varrho} = 0$, at least in the far field, and therefore we cannot simply plug it in (4.1). We can then follow the same idea developed by Feireisl and Novotný [5], working with the couple $(\widetilde{\varrho} + \varepsilon, \widetilde{\mathbf{u}})$, $\varepsilon > 0$, and performing the limit $\varepsilon \to 0$. While in [5] the result was proved for $1 < \gamma \leq 2$, the presence of the gravitational potential in this context reduces the interval to $\gamma_1 \leq \gamma \leq 2$, with $\gamma_1 = \frac{9}{5}$. In order to prove the main result of this section, we need the following lemma.

Lemma 5.1. *Let Φ be the solution of the Laplace equation*

$$-\Delta_x \Phi = f \quad \text{on } \mathbb{R}^3.$$

If $f \in L^p(\mathbb{R}^3)$, $1 < p < 3$, then

$$\|\nabla_x \Phi\|_{L^{p^*}(\mathbb{R}^3;\mathbb{R}^3)} \leq c\|f\|_{L^p(\mathbb{R}^3)} \tag{5.1}$$

with $c = c(p)$ a positive constant and p^* the Sobolev conjugate of p,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}.$$

Proof. Writing $\Phi = (-\Delta_x)^{-1}(f)$, from Sobolev's inequality we have that

$$\|\nabla_x \Phi\|_{L^{p^*}(\mathbb{R}^3;\mathbb{R}^3)} = \left\|\nabla_x(-\Delta_x)^{-1}(f)\right\|_{L^{p^*}(\mathbb{R}^3;\mathbb{R}^3)} \leq c\left\|\nabla_x(-\Delta_x)^{-1}\nabla_x(f)\right\|_{L^p(\mathbb{R}^3;\mathbb{R}^{3\times3})}.$$

Now, let $\mathcal{F}$,$\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. We can write

$$\nabla_x(-\Delta_x)^{-1}\nabla_x(f) = \left[\frac{\partial^2}{\partial x_j \partial x_k}\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\mathcal{F}(f)\right)\right]_{j,k=1}^3 = \left[\mathcal{F}^{-1}\left(-\frac{\xi_j\xi_k}{|\xi|^2}\mathcal{F}(f)\right)\right]_{j,k=1}^3.$$

It is easy to show that the multipliers $m_{jk}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}$, $j,k = 1,\dots,3$, satisfy the hypothesis of the Hörmander-Mikhlin Theorem (see [14], Chapter 4, Theorem 3). Hence the pseudo-differential operator $\nabla_x(-\Delta_x)^{-1}\nabla_x$ is a bounded linear operator on $L^p(\mathbb{R})$ for any $1 < p < \infty$ and therefore

$$\left\|\nabla_x(-\Delta_x)^{-1}\nabla_x\ (f)\right\|_{L^p(\mathbb{R}^3;\mathbb{R}^{3\times3})} \leq \|f\|_{L^p(\mathbb{R}^3)}, \tag{5.2}$$

implying (5.1). ☐

Theorem 5.2. *Let*

$$\frac{9}{5} \leq \gamma \leq 2 \tag{5.3}$$

and let $(\varrho, \mathbf{u})$ *be a dissipative weak solution of problem* (2.1)–(2.7) *in the sense of Definition 3.1. Let* $(\widetilde{\varrho}, \widetilde{\mathbf{u}})$ *be a strong solution of the same problem such that*

$$
\begin{aligned}
\widetilde{\mathbf{u}} &\in C([0,T]; D_0^{1,2} \cap D^{3,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; D^{4,2}(\Omega;\mathbb{R}^3)) \\
\partial_t \widetilde{\mathbf{u}} &\in L^\infty(0,T; D_0^{1,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; D^{2,2}(\Omega;\mathbb{R}^3)), \\
\widetilde{\varrho},\ p(\widetilde{\varrho}) &\in C([0,T]; W^{3,2}(\Omega)).
\end{aligned} \tag{5.4}
$$

Moreover, suppose that

$$
\varrho(0,\cdot) = \widetilde{\varrho}(0,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\cdot) = (\widetilde{\varrho}\widetilde{\mathbf{u}})(0,\cdot) = \mathbf{m}_0,
$$

where

$$
\varrho_0 \in C_c(\overline{\Omega}), \quad (\varrho_0)^{\gamma-1} \in W^{1,6} \cap W^{1,\infty}(\Omega;\mathbb{R}^3). \tag{5.5}
$$

Then

$$
\varrho \equiv \widetilde{\varrho}, \quad \mathbf{u} \equiv \widetilde{\mathbf{u}} \quad \textit{in } (0,T) \times \Omega. \tag{5.6}
$$

Remark 5.3. We point out that the regularity class (5.4) chosen for $(\widetilde{\varrho}, \widetilde{\mathbf{u}})$ is the one introduced by Huang, Li and Xin [9]. Moreover, the two conditions in (5.5) for the initial density ϱ_0 guarantee

$$
\widetilde{\varrho}(t,x) = 0 \quad \text{for any } t \in [0,T] \text{ and } |x| > R, \tag{5.7}
$$

$$
\sup_{t\in(0,T)} \|\nabla_x(\widetilde{\varrho}^{\ \gamma-1})\|_{L^6\cap L^\infty(\Omega;\mathbb{R}^3)} \lesssim 1, \tag{5.8}
$$

respectively, for $R > 0$ sufficiently large. Indeed, from Lemma 2.1 in [5] we deduce that if the velocity field $\widetilde{\mathbf{u}}$ is smooth enough, the regularity in (5.5) will propagate in time. Conditions (5.7), (5.8), on the other hand, are fundamental in providing a proper bound for the term

$$
\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G\nabla_x \widetilde{\Phi};
$$

see Sections 5.1 and 5.2 in [5] for further details.

Proof. As we cannot plug $\widetilde{\varrho}$ in (4.1) since condition $\inf \widetilde{\varrho} > 0$ is necessary, we can plug in $\widetilde{\varrho} + \varepsilon$ to get

$$
\begin{aligned}
\int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho} + \varepsilon, \widetilde{\mathbf{u}})(\tau,\cdot)\ \mathrm{d}x &+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\mathbf{u} - \widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\
&\leq \int_\Omega \big(P(\varrho_0) + \varepsilon P'(\varrho_0 + \varepsilon) - P(\varrho_0 + \varepsilon)\big)\ \mathrm{d}x \\
&+ \int_0^\tau \int_\Omega \varrho(\widetilde{\mathbf{u}} - \mathbf{u}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho} + \varepsilon)]\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \varrho(\widetilde{\mathbf{u}} - \mathbf{u}) \cdot \nabla_x \widetilde{\mathbf{u}} \cdot (\widetilde{\mathbf{u}} - \mathbf{u})\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega [p(\varrho) - p'(\widetilde{\varrho} + \varepsilon)(\varrho - \widetilde{\varrho} - \varepsilon) - p(\widetilde{\varrho} + \varepsilon)]\ \mathrm{div}_x\, \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\
&+ G \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x \Phi\ \mathrm{d}x\mathrm{d}t \\
&+ \varepsilon \int_0^\tau \int_\Omega p'(\widetilde{\varrho} + \varepsilon) \left(1 - \frac{\varrho}{\widetilde{\varrho} + \varepsilon}\right) \mathrm{div}_x\, \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Repeating the same passages done in [5] and performing the limit $\varepsilon \to 0$ we obtain

$$\begin{aligned}
\int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau, \cdot) \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x (\mathbf{u} - \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t& \\
\leq \int_0^\tau \int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t& \\
+ \int_0^\tau \int_\Omega \varrho (\widetilde{\mathbf{u}} - \mathbf{u}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G \nabla_x \widetilde{\Phi}] \, \mathrm{d}x\mathrm{d}t& \\
+ G \int_0^\tau \int_\Omega \varrho (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x (\Phi - \widetilde{\Phi}) \, \mathrm{d}x\mathrm{d}t&
\end{aligned}$$

As the couple $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ is a strong solution of our problem, it satisfies

$$\widetilde{\varrho} \, [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G \nabla_x \widetilde{\Phi}] = \mathrm{div}_x \, \mathbb{S}(\nabla_x \widetilde{\mathbf{u}})$$

and hence, we can add on both sides of the previous inequality the quantity

$$\int_0^\tau \int_\Omega \widetilde{\varrho} \, (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G \nabla_x \widetilde{\Phi}] \, \mathrm{d}x\mathrm{d}t = - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \widetilde{\mathbf{u}}) : \nabla_x (\mathbf{u} - \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t$$

to get

$$\begin{aligned}
\int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau, \cdot) \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \widetilde{\mathbf{u}})) : \nabla_x (\mathbf{u} - \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t& \\
\leq \int_0^\tau \int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t& \\
+ \int_0^\tau \int_\Omega (\varrho - \widetilde{\varrho})(\widetilde{\mathbf{u}} - \mathbf{u}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G \, \nabla_x \widetilde{\Phi}] \, \mathrm{d}x\mathrm{d}t& \\
+ G \int_0^\tau \int_\Omega \varrho (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x (\Phi - \widetilde{\Phi}) \, \mathrm{d}x\mathrm{d}t&
\end{aligned}$$

From (2.4) and keeping in mind that $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ vanish on the boundary of Ω, we have

$$\begin{aligned}
&\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \widetilde{\mathbf{u}})) : \nabla_x (\mathbf{u} - \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \left[\mu |\nabla_x (\mathbf{u} - \widetilde{\mathbf{u}})|^2 + \left(\frac{1}{3} \mu + \lambda \right) |\,\mathrm{div}_x (\mathbf{u} - \widetilde{\mathbf{u}})|^2 \right] \mathrm{d}x\mathrm{d}t \\
&\geq \mu \int_0^\tau \int_\Omega |\nabla_x (\mathbf{u} - \widetilde{\mathbf{u}})|^2 \, \mathrm{d}x\mathrm{d}t
\end{aligned}$$

and thus finally we can infer

$$\begin{aligned}
\int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(\tau, \cdot) \, \mathrm{d}x + \int_0^\tau \int_\Omega |\nabla_x (\mathbf{u} - \widetilde{\mathbf{u}})|^2 \, \mathrm{d}x\mathrm{d}t& \\
\lesssim \int_0^\tau \int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}) \, \mathrm{d}x\mathrm{d}t& \\
+ \int_0^\tau \int_\Omega (\varrho - \widetilde{\varrho})(\widetilde{\mathbf{u}} - \mathbf{u}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G \, \nabla_x \widetilde{\Phi}] \, \mathrm{d}x\mathrm{d}t& \\
+ G \int_0^\tau \int_\Omega \varrho (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x (\Phi - \widetilde{\Phi}) \, \mathrm{d}x\mathrm{d}t.&
\end{aligned} \tag{5.9}$$

Since the hypothesis of Theorem 5.2 in [5] are satisfied, we can repeat the same passages to obtain

$$\int_\Omega (\varrho - \widetilde{\varrho})(\widetilde{\mathbf{u}} - \mathbf{u}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - G\, \nabla_x \widetilde{\Phi}](t,\cdot)\ \mathrm{d}x \lesssim \int_\Omega E(\varrho, \mathbf{u} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}})(t,\cdot)\ \mathrm{d}x,$$

as soon as

$$1 < \gamma \leq 2. \tag{5.10}$$

It remains to control the last term in (5.9). First of all, notice that $-\Delta_x(\Phi - \widetilde{\Phi}) = \varrho - \widetilde{\varrho}$ and thus we get

$$\begin{aligned}
&\int_\Omega \varrho\ (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x (\Phi - \widetilde{\Phi})\ \mathrm{d}x \\
&\qquad = \int_\Omega \varrho\ (\mathbf{u} - \widetilde{\mathbf{u}}) \cdot \nabla_x (-\Delta_x)^{-1} (\varrho - \widetilde{\varrho})\ \mathrm{d}x \\
&\qquad \leq \frac{1}{2} \int_\Omega \varrho |\mathbf{u} - \widetilde{\mathbf{u}}|^2\ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho |\nabla_x (-\Delta_x)^{-1} (\varrho - \widetilde{\varrho})|^2\ \mathrm{d}x,
\end{aligned}$$

where the first term of the right-hand side of the inequality can be controlled by the relative energy and therefore it remains to estimate the second term. Fix $\bar{\varrho} > 0$ so that $\widetilde{\varrho} \leq \frac{1}{2}\bar{\varrho}$; then, we obtain the following inequality

$$\begin{aligned}
&\int_\Omega \varrho |\nabla_x (-\Delta_x)^{-1} (\varrho - \widetilde{\varrho})|^2\ \mathrm{d}x \\
&\qquad \leq \int_\Omega \varrho |\nabla_x (-\Delta_x)^{-1} \big(\mathbb{1}_{\varrho \geq \bar{\varrho}} (\varrho - \widetilde{\varrho})\big)|^2\ \mathrm{d}x + \int_\Omega \varrho |\nabla_x (-\Delta_x)^{-1} \big(\mathbb{1}_{\varrho \leq \bar{\varrho}} (\varrho - \widetilde{\varrho})\big)|^2\ \mathrm{d}x,
\end{aligned}$$

On one hand, from Hölder's inequality we get

$$\int_\Omega \varrho |\nabla_x (-\Delta_x)^{-1} \big(\mathbb{1}_{\varrho \geq \bar{\varrho}} (\varrho - \widetilde{\varrho})\big)|^2\ \mathrm{d}x \leq \|\varrho\|_{L^q(\Omega)} \left\| \nabla_x (-\Delta_x)^{-1} \big(\mathbb{1}_{\varrho \geq \bar{\varrho}} (\varrho - \widetilde{\varrho})\big) \right\|^2_{L^{\gamma^*}(\Omega;\mathbb{R}^3)},$$

with γ^* is the Sobolev conjugate of γ and

$$q = \frac{3\gamma}{5\gamma - 6}. \tag{5.11}$$

As $\varrho \in C_{\mathrm{weak}}([0,T]; L^1 \cap L^\gamma(\Omega))$, in order to guarantee that $\varrho(t,\cdot) \in L^q$, one should check that

$$1 \leq q = \frac{3\gamma}{5\gamma - 6} \leq \gamma, \tag{5.12}$$

which provides the restriction

$$\frac{9}{5} \leq \gamma \leq 3. \tag{5.13}$$

The combination of (5.10) and (5.13) justifies hypothesis (5.3). We can now apply Lemma 5.1 and, in particular, from (5.1) we get

$$\left\| \nabla_x (-\Delta_x)^{-1} \big(\mathbb{1}_{\varrho \geq \bar{\varrho}} (\varrho - \widetilde{\varrho})\big) \right\|^2_{L^{\gamma^*}(\Omega;\mathbb{R}^3)} \lesssim \|\mathbb{1}_{\varrho \geq \bar{\varrho}} (\varrho - \widetilde{\varrho})\|^2_{L^\gamma(\Omega)}$$

where

$$\|\mathbb{1}_{\varrho\geq\bar\varrho}(\varrho-\widetilde\varrho)\|^2_{L^\gamma(\Omega)}=\|\mathbb{1}_{\varrho\geq\bar\varrho}(\varrho-\widetilde\varrho)\|^{2-\gamma}_{L^\gamma(\Omega)}\|\mathbb{1}_{\varrho\geq\bar\varrho}(\varrho-\widetilde\varrho)\|^{\gamma}_{L^\gamma(\Omega)}\lesssim c(\bar\varrho)\|\mathbb{1}_{\varrho\geq\bar\varrho}(\varrho-\widetilde\varrho)\|^{\gamma}_{L^\gamma(\Omega)},$$

with $2-\gamma\geq 0$ from (5.3). Hence, we may conclude that

$$\int_\Omega \varrho|\nabla_x(-\Delta_x)^{-1}\big(\mathbb{1}_{\varrho\geq\bar\varrho}(\varrho-\widetilde\varrho)\big)|^2\ (t,\cdot)\ \mathrm{d}x\lesssim\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})(t,\cdot)\ \mathrm{d}x.$$

Similarly, on the other side, from Hölder's inequality and (5.1), we obtain

$$\begin{aligned}&\int_\Omega \varrho\ |\nabla_x(-\Delta_x)^{-1}\big(\mathbb{1}_{\varrho\leq\bar\varrho}(\varrho-\widetilde\varrho)\big)|^2\ \mathrm{d}x\\ &\qquad\leq\|\varrho\|_{L^{\frac32}(\Omega)}\left\|\nabla_x(-\Delta_x)^{-1}\big(\mathbb{1}_{\varrho\leq\bar\varrho}(\varrho-\widetilde\varrho)\big)\right\|^2_{L^6(\Omega;\mathbb{R}^3)}\\ &\qquad\lesssim\|\varrho\|_{L^{\frac32}(\Omega)}\|\mathbb{1}_{\varrho\leq\bar\varrho}(\varrho-\widetilde\varrho)\|^2_{L^2(\Omega)}\end{aligned}$$

From the fact that the pressure potential P is strictly convex on the interval $[0,\bar\varrho]$ we have

$$\|\mathbb{1}_{\varrho\leq\bar\varrho}(\varrho-\widetilde\varrho)\|^2_{L^2(\Omega)}\lesssim\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})(t,\cdot)\ \mathrm{d}x.$$

Getting back to (5.9), we finally obtain

$$\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})(\tau,\cdot)\ \mathrm{d}x+\int_0^\tau\int_\Omega|\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})|^2\ \mathrm{d}x\mathrm{d}t\lesssim\int_0^\tau\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t,$$

and, as a consequence of Gronwall Lemma,

$$\begin{aligned}\mathcal{E}(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})(\tau)=\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde\varrho,\widetilde{\mathbf{u}})(\tau,\cdot)\ \mathrm{d}x=0\quad\text{for any }\tau\in[0,T],\\ \int_0^T\int_\Omega|\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})|^2\ \mathrm{d}x\mathrm{d}t=0,\end{aligned}$$

which, in particular, implies (5.6). □

6 Pressure estimate up to the boundary

The range of γ in (5.3) can be enlarged finding some proper pressure estimates "up to the boundary". The idea is to adapt the procedure performed by Feireisl and Petzeltová in [6] for a bounded domain Ω in the context of an exterior domain. More precisely, our goal is to prove the following result.

Theorem 6.1. *Let $\Omega\subset\mathbb{R}^3$ be a Lipschitz exterior domain. Then, for any dissipative weak solution $(\varrho,\mathbf{u})$ of the Navier-Stokes-Poisson system* (2.1)–(2.7) *in the sense of Definition 3.1 there exists a positive constant K such that*

$$\int_0^T\int_\Omega\varrho^{\gamma+\omega}\ \mathrm{d}x\mathrm{d}t\leq K \tag{6.1}$$

for any

$$0<\omega\leq\frac23\gamma-1. \tag{6.2}$$

Proof. First of all, fix $R > \mathrm{diam}(\Omega^c)$, and define $\varphi \in C^1(\mathbb{R}^3)$ such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in \Omega^c, \\ 1 & \text{if } x \in B_R^c, \end{cases}$$

where $A^c := \mathbb{R}^3 \setminus A$ denotes the complementary of a set A while B_R denotes the ball of radius R and center the origin. Let us now consider the function

$$\boldsymbol{\varphi}(t,x) = \psi(t)\varphi(x)\ \nabla_x(-\Delta_x)^{-1}\big[b\big(\varrho(t,\cdot)\big)\big](x),$$

where $\psi \in C_c^1(0,T)$ and

$$b \in C^1(\mathbb{R}), \quad b(0) = 0, \quad b(z) = z^\omega \text{ for } z \geq 1.$$

Keeping in mind Remark 3.4, we can now use $\boldsymbol{\varphi}$ as test function in (3.3); indeed, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ p(\varrho)\ b(\varrho)\ \mathrm{d}x\mathrm{d}t = \sum_{k=1}^{9} I_k$$

with

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi'\ \varphi\ \varrho\mathbf{u}\cdot\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_2 &= +\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ \varrho\mathbf{u}\cdot\big(\nabla_x(-\Delta_x)^{-1}\mathrm{div}_x\big)\big[\big(b(\varrho)\mathbf{u}\big)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_3 &= +\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ \varrho\mathbf{u}\cdot\nabla_x(-\Delta_x)^{-1}\big[\big(b(\varrho)-\varrho b'(\varrho)\big)\,\mathrm{div}_x\,\mathbf{u}\big]\ \mathrm{d}x\mathrm{d}t, \\
I_4 &= -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ [(\varrho\mathbf{u}\otimes\mathbf{u})\nabla_x\varphi]\cdot\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_5 &= -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ (\varrho\mathbf{u}\otimes\mathbf{u}) : \big(\nabla_x(-\Delta_x)^{-1}\nabla_x\big)\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_6 &= -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ p(\varrho)\ \nabla_x\varphi\cdot\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_7 &= +\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ [\mathbb{S}(\nabla_x\mathbf{u})\nabla_x\varphi]\cdot\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_8 &= +\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ \mathbb{S}(\nabla_x\mathbf{u}) : \big(\nabla_x(-\Delta_x)^{-1}\nabla_x\big)\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t, \\
I_9 &= -G\int_{\mathbb{R}}\int_{\mathbb{R}^3} \psi\ \varphi\ \varrho\nabla_x\Phi\cdot\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)\big]\ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

(i) Denoting with q^* the Sobolev conjugate of q, given by

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{3},$$

the combination of Sobolev's inequality with the boundedness of the operator $\nabla_x(-\Delta_x)^{-1}\nabla_x$ form $L^p(\mathbb{R}^3)$ onto $L^p(\mathbb{R}^3;\mathbb{R}^{3\times3})$ for any $1<p<\infty$, cf. (5.2), provides

$$\begin{aligned}\left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_1^*}(\mathbb{R}^3;\mathbb{R}^3)} &\lesssim \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_1}(\mathbb{R}^3;\mathbb{R}^{3\times3})}\\ &\lesssim \|b(\varrho)(t,\cdot)\|_{L^{q_1}(\mathbb{R}^3)} = \|\varrho(t,\cdot)\|^{\omega}_{L^{q_1\omega}(\Omega)}.\end{aligned}$$

Therefore, from Hölder's inequality we get

$$\begin{aligned}|I_1| &\le c(\psi')\int_0^T \|\sqrt{\varrho}(t,\cdot)\|_{L^{2p}(\Omega)}\|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)}\left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_1^*}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t\\ &\le c(\psi')\int_0^T \|\varrho(t,\cdot)\|^{\frac{1}{2}}_{L^p(\Omega)}\|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)}\|\varrho(t,\cdot)\|^{\omega}_{L^{q_1\omega}(\Omega)}\,\mathrm{d}t,\end{aligned}\tag{6.3}$$

with

$$\frac{1}{q_1}=\frac{5}{6}-\frac{1}{2p}.$$

As $\varrho\in C_{\mathrm{weak}}([0,T];L^1\cap L^\gamma(\Omega))$ and $1\le q_1^*<\infty$, we must have

$$\frac{6\gamma}{5\gamma-3}\le q_1<3.$$

(ii) Similarly, form (5.2) and Hölder's inequality we have

$$\begin{aligned}\left\|\left(\nabla_x(-\Delta_x)^{-1}\operatorname{div}_x\right)\left[\big(b(\varrho)\mathbf{u}\big)(t,\cdot)\right]\right\|_{L^{q_2}(\mathbb{R}^3;\mathbb{R}^3)} &\lesssim \left\|\big(b(\varrho)\mathbf{u}\big)(t,\cdot)\right\|_{L^{q_2}(\mathbb{R}^3;\mathbb{R}^3)}\\ &\lesssim \|\mathbf{u}(t,\cdot)\|_{L^6(\Omega;\mathbb{R}^3)}\|b(\varrho)(t,\cdot)\|_{L^{\frac{6q_2}{6-q_2}}(\mathbb{R}^3)}\\ &\lesssim \|\nabla_x\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\|\varrho(t,\cdot)\|^{\omega}_{L^{\frac{6q_2\omega}{6-q_2}}(\Omega)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|I_2| &\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)}\|\mathbf{u}(t,\cdot)\|_{L^6(\Omega;\mathbb{R}^3)}\left\|\left(\nabla_x(-\Delta_x)^{-1}\operatorname{div}_x\right)\left[\big(b(\varrho)\mathbf{u}\big)(t,\cdot)\right]\right\|_{L^{q_2}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t\\ &\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)}\|\nabla_x\mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\|\varrho(t,\cdot)\|^{\omega}_{L^{\frac{6q_2\omega}{6-q_2}}(\Omega)}\,\mathrm{d}t,\end{aligned}\tag{6.4}$$

with

$$\frac{1}{q_2}=\frac{5}{6}-\frac{1}{p}$$

satisfying

$$\frac{6\gamma}{5\gamma-6}\le q_2<\infty.$$

(iii) Proceeding as in (i), we have

$$
\begin{aligned}
&\left\|\nabla_x(-\Delta_x)^{-1}\left[\left(b(\varrho)-\varrho b'(\varrho)\right)(t,\cdot)\operatorname{div}_x \mathbf{u}(t,\cdot)\right]\right\|_{L^{q_3^*}(\mathbb{R}^3;\mathbb{R}^3)}\\
&\qquad\lesssim \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[\left(b(\varrho)-\varrho b'(\varrho)\right)(t,\cdot)\operatorname{div}_x \mathbf{u}(t,\cdot)\right]\right\|_{L^{q_3}(\mathbb{R}^3;\mathbb{R}^{3\times 3})}\\
&\qquad\lesssim \left\|\left[b(\varrho)\operatorname{div}_x \mathbf{u}\right](t,\cdot)\right\|_{L^{q_3}(\mathbb{R}^3)}\\
&\qquad\le \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,\|b(\varrho)(t,\cdot)\|_{L^{\frac{2q_3}{2-q_3}}(\mathbb{R}^3)}\\
&\qquad= \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,\|\varrho(t,\cdot)\|^{\omega}_{L^{\frac{2q_3\omega}{2-q_3}}(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_3| &\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)}\|\mathbf{u}(t,\cdot)\|_{L^6(\Omega;\mathbb{R}^3)}\left\|\nabla_x(-\Delta_x)^{-1}\left[\left(b(\varrho)-\varrho b'(\varrho)\right)\operatorname{div}_x \mathbf{u}(t,\cdot)\right]\right\|_{L^{q_3^*}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t\\
&\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)}\,\|\nabla_x \mathbf{u}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,\|\varrho(t,\cdot)\|^{\omega}_{L^{\frac{2q_3\omega}{2-q_3}}(\Omega)}\,\mathrm{d}t,
\end{aligned}
\tag{6.5}
$$

with

$$
\frac{1}{q_3}=\frac{7}{6}-\frac{1}{p}
$$

satisfying

$$
\frac{6\gamma}{7\gamma-6}\le q_3<\infty.
$$

(iv) From the fact that

$$
\begin{aligned}
\varrho\mathbf{u}&\in L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega;\mathbb{R}^3)),\\
\mathbf{u}&\in L^2(0,T;L^6(\Omega;\mathbb{R}^3)),
\end{aligned}
$$

we can deduce

$$
\varrho\mathbf{u}\otimes\mathbf{u}\in L^1(0,T;L^q(\Omega;\mathbb{R}^{3\times 3}))\quad\text{with } q=\frac{3\gamma}{3+\gamma}.
$$

Therefore, proceeding as in (i) we have

$$
\begin{aligned}
|I_4| &\le c(\nabla_x\varphi)\int_0^T \|(\varrho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\|_{L^{\frac{3\gamma}{3+\gamma}}(\Omega;\mathbb{R}^{3\times 3})}\left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_4^*}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t\\
&\le c(\nabla_x\varphi)\int_0^T \|(\varrho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\|_{L^{\frac{3\gamma}{3+\gamma}}(\Omega;\mathbb{R}^{3\times 3})}\,\|\varrho(t,\cdot)\|^{\omega}_{L^{q_4\omega}(\Omega)}\,\mathrm{d}t,
\end{aligned}
\tag{6.6}
$$

with

$$
\frac{1}{q_4}=1-\frac{1}{\gamma}.
$$

(v) Similarly, we have

$$
\begin{aligned}
|I_5| &\lesssim \int_0^T \|(\varrho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\|_{L^{\frac{3\gamma}{3+\gamma}}(\Omega;\mathbb{R}^{3\times3})} \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_4^*}(\mathbb{R}^3;\mathbb{R}^{3\times3})}\,\mathrm{d}t \\
&\lesssim \int_0^T \|(\varrho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\|_{L^{\frac{3\gamma}{3+\gamma}}(\Omega;\mathbb{R}^{3\times3})} \|\varrho(t,\cdot)\|^{\omega}_{L^{q_4^*\omega}(\Omega;\mathbb{R}^{3\times3})}\,\mathrm{d}t.
\end{aligned}
\tag{6.7}
$$

(vi) Noticing that

$$
\operatorname{supp}(\nabla_x\varphi)\subset\Omega_R := \Omega\cap B_R,
$$

and using the Sobolev embedding

$$
W^{1,q_5}(\Omega_R)\hookrightarrow L^\infty(\Omega_R)\quad\text{with } q_5>3
$$

to deduce

$$
\begin{aligned}
\left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^\infty(\Omega_R;\mathbb{R}^3)} &\lesssim \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_5}(\Omega_R;\mathbb{R}^{3\times3})} \\
&\le \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[b(\varrho)(t,\cdot)\right]\right\|_{L^{q_5}(\mathbb{R}^3;\mathbb{R}^{3\times3})} \\
&\le \|\varrho(t,\cdot)\|^{\omega}_{L^{q_5\omega}(\Omega)},
\end{aligned}
$$

from Hölder's inequality we have

$$
\begin{aligned}
|I_6| &\le c(\nabla_x\varphi)\int_0^T \|p(\varrho)(t,\cdot)\|_{L^1(\Omega)} \left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^\infty(\Omega_R;\mathbb{R}^3)}\,\mathrm{d}t \\
&\le c(\nabla_x\varphi)\int_0^T \|p(\varrho)(t,\cdot)\|_{L^1(\Omega)} \|\varrho(t,\cdot)\|^{\omega}_{L^{q_5\omega}(\Omega)}\,\mathrm{d}t.
\end{aligned}
\tag{6.8}
$$

(vii) From Hölder's inequality we have

$$
\begin{aligned}
|I_7| &\le c(\nabla_x\varphi)\int_0^T \|\mathbb{S}(\nabla_x\mathbf{u})(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \left\|\nabla_x(-\Delta_x)^{-1}\left[b(\varrho)(t,\cdot)\right]\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t \\
&\le c(\nabla_x\varphi)\int_0^T \|\mathbb{S}(\nabla_x\mathbf{u})(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \|\varrho(t,\cdot)\|^{\omega}_{L^{\frac{6}{5}\omega}(\Omega)}\,\mathrm{d}t.
\end{aligned}
\tag{6.9}
$$

(viii) Similarly,

$$
\begin{aligned}
|I_8| &\lesssim \int_0^T \|\mathbb{S}(\nabla_x\mathbf{u})(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \left\|\left(\nabla_x(-\Delta_x)^{-1}\nabla_x\right)\left[b(\varrho)(t,\cdot)\right]\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}\,\mathrm{d}t \\
&\le \int_0^T \|\mathbb{S}(\nabla_x\mathbf{u})(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \|\varrho(t,\cdot)\|^{\omega}_{L^{2\omega}(\Omega)}\,\mathrm{d}t.
\end{aligned}
\tag{6.10}
$$

(ix) From Sobolev's inequality and the boundedness of the operator $\nabla_x(-\Delta_x)^{-1}\nabla_x$ from $L^p(\Omega)$ onto $L^p(\Omega;\mathbb{R}^{3\times3})$ for any $1<p<\infty$, cf. (5.2), we have

$$
\begin{aligned}
\|\nabla_x(-\Delta_x)^{-1}(\varrho+g)(t,\cdot)\|_{L^{\gamma^*}(\Omega;\mathbb{R}^3)} &\lesssim \|\nabla_x(-\Delta_x)^{-1}\nabla_x(\varrho+g)(t,\cdot)\|_{L^{\gamma}(\Omega;\mathbb{R}^{3\times3})} \\
&\lesssim \|(\varrho+g)(t,\cdot)\|_{L^{\gamma}(\Omega)},
\end{aligned}
$$

Therefore, from Hölder's inequality we get

$$\begin{aligned}|I_9| &\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)} \|\nabla_x(-\Delta_x)^{-1}(\varrho+g)(t,\cdot)\|_{L^{\gamma^*}(\Omega;\mathbb{R}^3)} \left\|\nabla_x(-\Delta_x)^{-1}\big[b(\varrho)(t,\cdot)\big]\right\|_{L^{q_5^*}(\mathbb{R}^3;\mathbb{R}^3)} \,\mathrm{d}t \\ &\lesssim \int_0^T \|\varrho(t,\cdot)\|_{L^p(\Omega)} \|(\varrho+g)(t,\cdot)\|_{L^\gamma(\Omega)} \|\varrho(t,\cdot)\|^\omega_{L^{q_5\omega}(\Omega)} \,\mathrm{d}t,\end{aligned} \tag{6.11}$$

with

$$\frac{1}{q_5} = \frac{5}{3} - \frac{1}{\gamma} - \frac{1}{p},$$

satisfying

$$\frac{3\gamma}{5\gamma-6} \le q_5 \le \frac{3\gamma}{2\gamma-3}$$

At this point, in order to guarantee boundedness of the integrals (6.3)–(6.11) we must require $\alpha \le \gamma$ in every norm of the type

$$\|\varrho(t,\cdot)\|_{L^\alpha(\Omega)},$$

and thus

- from (6.3), we recover: $0 < \omega \le \frac{5}{6}\gamma - \frac{1}{2}$;
- from (6.4), (6.5), (6.7), we recover: $0 < \omega \le \frac{2}{3}\gamma - 1$;
- from (6.6), we recover: $0 < \omega \le \gamma - 1$;
- from (6.8), (6.9), (6.10), we recover: $0 < \omega \le \min\left\{\frac{\gamma}{3}, \frac{\gamma}{2}, \frac{5}{6}\gamma\right\} = \frac{\gamma}{3}$;
- from (6.11), we recover: $0 < \omega \le \frac{5}{3}\gamma - 2$.

Since

$$\frac{2}{3}\gamma - 1 \le \min\left\{\frac{5}{6}\gamma - \frac{1}{2},\ \gamma - 1,\ \frac{\gamma}{3},\ \frac{5}{3}\gamma - 2\right\},$$

whenever $1 \le \gamma \le 3$, it is enough to consider (6.2). Summing the estimates (6.3)–(6.11), we get

$$\int_0^T \int_\Omega \psi\ \varphi\ \varrho^{\gamma+\omega}\ \mathrm{d}x\mathrm{d}t \lesssim 1,$$

and therefore, it is enough to let $\psi,\ \varphi \to 1$ to obtain (6.1). □

Remark 6.2. Instead of the singular operator $\nabla_x(-\Delta_x)^{-1}$, to prove Theorem 6.1 we could have used the Bogovskii operator $\mathfrak{B}$, which can be interpreted as the inverse of div_x. More precisely, if we consider the equation

$$\mathrm{div}_x\, \mathbf{v} = f$$

it has been proved that it admits a solution operator $\mathfrak{B} : f \mapsto \mathbf{v}$, bounded from $L^p(\Omega)$ onto $D_0^{1,p}(\Omega;\mathbb{R}^3)$ for any $1 < p < \infty$ and any locally Lipschitz exterior domain $\Omega \subset \mathbb{R}^3$, see Galdi [7], Theorem

III.3.6. However, in order to get an analogous of estimate (6.4), we would have needed an additional requirement that if $f = \mathrm{div}_x \mathbf{g}$ for some $\mathbf{g} \in L^r(\Omega; \mathbb{R}^3)$, with $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$, then

$$\|\mathfrak{B}[f]\|_{L^r(\Omega;\mathbb{R}^3)} \leq c(p, r, \Omega) \|\mathbf{g}\|_{L^r(\Omega;\mathbb{R}^3)};$$

this result is known to be true for bounded domains (see, for instance, [7], Theorem III.3.4) but not for unbounded domains, to the best of the author's knowledge.

Theorem 6.1 in particular implies that

$$\varrho \in L^{\gamma+\omega}((0,T) \times \Omega) \quad \text{with} \quad \omega = \frac{2}{3}\gamma - 1, \tag{6.12}$$

and therefore the range of γ in (5.3) can be slightly improved. More precisely, we have the following last result.

Corollary 6.3. *There exists $\gamma^* < \frac{7}{4}$ such that Theorem 5.2 holds with condition* (5.3) *replaced by*

$$\gamma^* \leq \gamma \leq 2.$$

Proof. Due to condition (6.12), we only have to replace (5.12) in the proof of Theorem (5.2) with

$$1 \leq \frac{3\gamma}{5\gamma - 6} \leq \gamma + \omega = \frac{5}{3}\gamma - 1,$$

which provides

$$\gamma^* \leq \gamma \leq 3, \quad \text{with } \gamma^* = \frac{27}{25} + \frac{3\sqrt{31}}{25} < \frac{7}{4}. \tag{6.13}$$

Unifying conditions (6.13) and (5.10), we get the claim. □

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