CHAPMAN UNIVERSITY | **INSTITUTE FOR QUANTUM STUDIES**

REGULAR PAPER

Check for updates

On superoscillations and supershifts in several variables

**Y. Aharonov · F. Colombo · A. N. Jordan ·
I. Sabadini · T. Shushi · D. C. Struppa ·
J. Tollaksen**

Abstract The aim of this paper is to study a class of superoscillatory functions in several variables, removing some restrictions on the functions that we introduced in a previous paper. Since the tools that we used with our approach are not common knowledge we will give detailed proof for the case of two variables. The results proved for superoscillatory functions in several variables can be further extended to supershifts in several variables.

Y. Aharonov · A. N. Jordan · J. Tollaksen
Schmid College of Science and Technology and Institute of Quantum Studies, Chapman University, Orange, CA 92866, USA
e-mail: aharonov@chapman.edu

A. N. Jordan
e-mail: jordan@chapman.edu

J. Tollaksen
e-mail: tollakse@chapman.edu

F. Colombo · I. Sabadini (✉)
Politecnico di Milano, Dipartimento di Matematica, Via E. Bonardi 9, 20133 Milan, Italy
e-mail: irene.sabadini@polimi.it

F. Colombo
e-mail: fabrizio.colombo@polimi.it

T. Shushi
Department of Business Administration, Guilford Glazer Faculty of Business and Management & Center for Quantum Information Science and Technology, Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: tomershu@bgu.ac.il

D. C. Struppa
The Donald Bren Presidential Chair in Mathematics, Chapman University, Orange, CA, USA
e-mail: struppa@chapman.edu

1 Introduction

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. Physical phenomena associated with superoscillatory functions have been known for a long time and in more recent years there has been a wide interest both from the physical and the mathematical point of view. In 1952, Toraldo di Francia observed the superoscillation phenomenon in antennas theory, see [43], and Y. Aharonov discovered it in the context of weak values in quantum mechanics, see [1]. An introduction to superoscillatory functions in one variable and some applications to Schrödinger evolution of superoscillatory initial data can be found in [9]. Superoscillatory functions in several variables have been rigorously defined and studied in [8]. The aim of this paper is to remove the restrictions that were used in [8] and to generalize the existing theory to the more general phenomenon of supershift.

Our results are directed to a double audience of physicists and mathematicians and since our tools, that consist of infinite order differential operators acting on spaces of entire holomorphic functions, are not widely known we consider first the case of two superoscillatory variables. In this case, we avoid heavy notations so that the reader can better follow the main points of the proofs.

The literature on superoscillations is quite large, and even without claiming completeness we have tried to mention some of the most relevant (and recent) results. Papers [3–10,15,19,27,33,40] deal with the issue of permanence of superoscillatory behavior when evolved under a suitable Schrödinger equation; papers [22–26,34–39,41,42] are mostly concerned with the physical nature of superoscillations, while papers [8,10,13,14,16–18,29–32] develop in depth the mathematical theory of superoscillations. Finally, we have cited [9] as a good reference for the state of the art in the mathematics of superoscillations until 2017, and [21], the *Roadmap on Superoscillations* from the Institute of Physics, where the most recent advances in superoscillations and their applications to technology are well explained by the leading experts in this field.

The prototypical superoscillating function is

$$F_n(x, a) = \left(\cos\left(\frac{x}{n}\right) + ia \sin\left(\frac{x}{n}\right) \right)^n = \sum_{j=0}^{n} C_j(n, a) e^{i(1-2j/n)x}, \quad x \in \mathbb{R}, \tag{1}$$

where $a > 1$ and the coefficients $C_j(n, a)$ can be calculated to be

$$C_j(n, a) = \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^j. \tag{2}$$

If we fix $x \in \mathbb{R}$ and we let n go to infinity, we obtain that

$$\lim_{n \to \infty} F_n(x, a) = e^{iax},$$

and the limit is uniform on the compact sets of the real line. The term superoscillations comes from the fact that in the Fourier representation of the function (1) the frequencies $1 - 2j/n$ are bounded by 1, but the limit function e^{iax} has a frequency a that can be arbitrarily larger than 1.

A fundamental problem is to determine how large the class of superoscillatory functions. Many of the works in the reference list, as we pointed out before, are devoted to the question of permanence of superoscillations when they are taken as initial values for a given Schrödinger equation, but as a byproduct they also offered a very powerful way to extend the class of superoscillatory functions. These extensions, nevertheless, are still very closely connected to the archetypical function defined earlier on. To address this issue, we have recently introduced, [11], a new method to generate superoscillatory functions for different configurations of points in the interval $[-1, 1]$. More precisely: let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, ..., n$ for $n \in \mathbb{N}$, and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$ for every $i \neq j$ the function

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, \, k \neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) e^{ixh_j(n)}, \quad x \in \mathbb{R}$$

is such that $\frac{\mathrm{d}^p}{\mathrm{d}x^p} f_n(0) = (ia)^p$ for $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and therefore when the sequence of holomorphic extensions of (f_n) converges in the space A_1 of functions of exponential type we have that

$$\lim_{n \to \infty} f_n(x) = e^{iax}, \quad x \in \mathbb{R}.$$

Two explicit examples are the following:

1. (I) Let $n \in \mathbb{N}$ and set $h_j(n) = 1 - \frac{2}{n} j$ where $j = 0, ..., n$; in this case we obtain the superoscillatory function

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, \ k \neq j}^{n} \frac{n}{2} \Big(\frac{1 - \frac{2}{n}k - a}{j - k} \Big) e^{i(1 - \frac{2}{n}j)x}, \quad x \in \mathbb{R}.$$

2. (II) Set $h_j(n) = 1 - \frac{2}{n^p} j$ where $j = 0, ..., n$, for a fixed $p \in \mathbb{N}$, then we have:

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, \ k \neq j}^{n} \frac{n^p}{2} \Big(\frac{1 - \frac{2}{n^p}k - a}{j - k} \Big) e^{i(1 - \frac{2}{n^p}j)x}, \quad x \in \mathbb{R}.$$

In both cases the sequences converge to e^{iax} for every $x \in \mathbb{R}$, and numerous other examples can be easily constructed explicitly.

In the paper [8] we described and studied superoscillations in several variables. The methods we used, however, required us to accept some constraints on the kind of superoscillations we could include. In this paper, on the other hand, we develop a new approach that allows us to remove those constraints and study a more general class of superoscillations, by showing how they can be constructed starting from superoscillatory functions in one variable. The main idea is to consider a superoscillating function

$$f_n(x) := \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$

for some coefficients $Z_j(n, a)$ (see Definition 2.1) and to assume that its holomorphic extension to the entire function $f_n(\xi)$ converges to $e^{ia\xi}$ in A_1, i.e. that there exists $C \geq 0$ such that

$$\lim_{n \to \infty} \sup_{\xi \in \mathbb{C}} \big| f_n(\xi) - f(\xi) \big| e^{-C|\xi|} = 0.$$

For $p_1, p_2, \ldots, p_d \in \mathbb{N}, d \in \mathbb{N}$, we then define

$$F_n(x_1, x_1, \ldots, x_d) = \sum_{j=0}^{n} Z_j(n, a) e^{ix_1(h_j(n))^{p_1}} e^{ix_2(h_j(n))^{p_2}} \ldots e^{ix_d(h_j(n))^{p_d}}$$

and we show that

$$\lim_{n \to \infty} F_n(x_1, x_1, \ldots, x_d) = e^{ix_1 a^{p_1}} e^{ix_2 a^{p_2}} \ldots e^{ix_d a^{p_d}},$$

so that, when $|a| > 1$, $F_n(x_1, x_1, \ldots, x_d)$ is superoscillating.

In some earlier work, we have also shown that superoscillating functions are a particular case of supershifts, and for this reason we now introduce, and study for the first time, the case of supershifts in $d \geq 2$ variables. The results that we have obtained and, even more, the techniques that we have used have convinced us of the existence of an intimate relation between global analyticity in $\mathbb{C}^d$ and superoscillations and the supershift property on the real space R^d. We have discussed this in detail this very subtle question with a colleague [44], and we plan to come back to it with a joint paper in the near future.

The paper is organized into four sections. After this introduction, Section 2 contains the preliminary material on superoscillations, the relevant function spaces and their topology, and we study the continuity of some infinite order differential operators on such spaces. Section 3 is the main part of the paper and contains the definition of superoscillating functions in several variables and some results proved in the specific case of two variables. Section 4 discusses the notion of supershift in several variables.

2 Superoscillations, function spaces and operators

In this section, we summarize the preliminary definitions and results to treat superoscillatory functions in several variables. We begin with the precise definition of superoscillatory functions in one variable.

Definition 2.1 We call *generalized Fourier sequence* a sequence of the form

$$f_n(x) := \sum_{j=0}^{n} Z_j(n, a) e^{i h_j(n) x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{3}$$

where $a \in \mathbb{R}$, $Z_j(n, a)$ and $h_j(n)$ are complex and real valued functions of the variables n, a and n, respectively. The sequence (3) is said to be *a superoscillating sequence* if $\sup_{j,n} |h_j(n)| \leq 1$ and there exists a compact subset of $\mathbb{R}$, which will be called *a superoscillation set*, on which $f_n(x)$ converges uniformly to $e^{i g(a) x}$, where g is a continuous real-valued function such that $|g(a)| > 1$.

The classical Fourier expansion is obviously not a superoscillating sequence since its frequencies are not, in general, bounded.

A simple, but important, example is

$$Y_n(x, a) = \sum_{j=0}^{n} C_j(n, a) e^{i(1 - 2j/n)^m x}, \quad \text{for} \quad x \in \mathbb{R} \text{ and } m \in \mathbb{N},$$

where, if $C_j(n, a)$ is defined as in (2), we have

$$\lim_{n \to \infty} Y_n(x, a) = e^{i a^m x}.$$

In the recent paper [11], we enlarged the class of superoscillating functions for coefficients and frequencies more general than $C_j(n, a)$ and $1 - 2j/n$, and we solved the following problem.

Problem 2.2 *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, ..., n$, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Determine the coefficients $X_j(n)$ of the sequence*

$$f_n(x) = \sum_{j=0}^{n} X_j(n) e^{i h_j(n) x}, \quad x \in \mathbb{R}$$

in such a way that

$$f_n^{(p)}(0) = (i a)^p, \quad \text{for} \quad p = 0, 1, ..., n.$$

Remark 2.3 The conditions $f_n^{(p)}(0) = (i a)^p$ mean that the functions $x \mapsto e^{i a x}$ and $x \mapsto f_n(x)$ have the same derivatives at the origin, for $p = 0, 1, ..., n$, and therefore the same Taylor polynomial of order n.

Theorem 2.4 (Solution of Problem 2.2) *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, ..., n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$, for every $i \neq j$, then the coefficients $X_j(n, a)$ are uniquely determined and given by*

$$X_j(n, a) = \prod_{k=0, \, k \neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right). \tag{4}$$

As a consequence, the sequence

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, \, k \neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) e^{i x h_j(n)}, \quad x \in \mathbb{R}$$

solves Problem 2.2. Moreover, when the holomorphic extensions of the functions f_n converge in A_1, we have

$$\lim_{n \to \infty} f_n(x) = e^{i a x}, \quad \text{for all } x \in \mathbb{R}.$$

Our mathematical tools to study superoscillatory functions in one or in several variables make use of infinite order differential operators. Such operators naturally act on holomorphic functions. This is the reason for which we consider the holomorphic extension to entire functions of the sequence $f_n(x)$ defined in (2.1) by replacing the real variable x by the complex variable ξ. For the sequences of entire functions we shall consider, a natural notion of convergence is the convergence in the space A_1 as in the following definition.

Definition 2.5 The space A_1 is the complex algebra of entire functions such that there exists $B > 0$ such that

$$\sup_{\xi \in \mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right) < +\infty. \tag{5}$$

The space A_1 has a rather complicated topology, see e.g. [20], since it is a linear space obtained via an inductive limit. For our purposes, it is enough to consider, for any fixed $B > 0$, the set $A_{1,B}$ of functions f satisfying (5), and to observe that

$$\|f\|_B := \sup_{\xi \in \mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right)$$

defines a norm on $A_{1,B}$, called the B-norm. One can prove that $A_{1,B}$ is a Banach space with respect to this norm.

Moreover, let f and a sequence $(f_n)_n$ belong to A_1; f_n converges to f in A_1 if and only if there exists B such that $f, f_n \in A_{1,B}$ and

$$\lim_{n \to \infty} \sup_{\xi \in \mathbb{C}} |f_n(\xi) - f(\xi)| e^{-C|\xi|} = 0$$

for some $C \geq 0$. With these notations and definitions we can make the notion of continuity explicit (see [18]):

A linear operator $\mathcal{U} : A_1 \to A_1$ is continuous if and only if for any $B > 0$ there exists $B' > 0$ and $C > 0$ such that

$$\mathcal{U}(A_{1,B}) \subset A_{1,B'} \quad \text{and} \quad \|\mathcal{U}(f)\|_{B'} \leq C\|f\|_B, \quad \text{for any } f \in A_{1,B}. \tag{6}$$

The following result, see Lemma 2.6 in [17], gives a characterization of the functions in A_1 in terms of the coefficients appearing in their Taylor series expansion.

Lemma 2.6 *The entire function*

$$f(\xi) = \sum_{j=0}^{\infty} f_j \xi^j$$

belongs to A_1 if and only if there exists $C_f > 0$ and $b > 0$ such that

$$|f_j| \leq C_f \frac{b^j}{\Gamma(j+1)}.$$

Remark 2.7 To say that $f \in A_1$ means that $f \in A_{1,B}$ for some $B > 0$. The computations in the proof of Lemma 2.6 in [17], show that $b = 2eB$, and that we can choose $C_f = \|f\|_B$.

Lemma 2.6 has been proved in [17] and is a crucial fact in the proof of the following results. The reader must not be confused by the fact that the variables x, y appearing in the statement below are real, indeed they have the role of parameters (which can also be considered as complex numbers). We now define two infinite order differential operators that will be used to study superoscillatory functions and supershifts in two variables. We would like to stress, one more time, that the key ingredient in the theory that we have developed is the ability to characterize those operators that act continuously on the space A_1 or, more generally, on spaces of entire functions with growth conditions.

Proposition 2.8 *Let $x, y \in \mathbb{R}$ and $p, q \in \mathbb{N}$. Denote by D_ξ the derivation with respect to the complex variable ξ. We define the formal operator:*

$$\mathcal{U}(x, y, D_\xi) := \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} D_\xi^{pm-p\mu+q\mu}. \tag{7}$$

Then $\mathcal{U}(x, y, D_\xi) : A_1 \mapsto A_1$ is continuous for all $x, y \in \mathbb{R}$ and $p, q \in \mathbb{N}$.

Proof Let us consider

$$\mathcal{U}(x, y, D_\xi) f(\xi)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} D_\xi^{pm-p\mu+q\mu} f(\xi)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} D_\xi^{pm-p\mu+q\mu} \sum_{j=0}^{\infty} f_j \xi^j$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} \sum_{j=pm-p\mu+q\mu}^{\infty} f_j \frac{j!}{(j-(pm-p\mu+q\mu))!} \xi^{j-(pm-p\mu+q\mu)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} \sum_{k=0}^{\infty} f_{pm-p\mu+q\mu+k} \frac{(pm-p\mu+q\mu+k)!}{k!} \xi^k.$$

Taking the modulus we get

$$|\mathcal{U}(x, y, D_\xi) f(\xi)| \leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} |x|^{m-\mu} |y|^{\mu} \sum_{k=0}^{\infty} |f_{pm-p\mu+q\mu+k}| \frac{(pm-p\mu+q\mu+k)!}{k!} |\xi|^k,$$

and Lemma 2.6 gives the estimate on the coefficients $f_{pm-p\mu+q\mu+k}$

$$|f_{pm-p\mu+q\mu+k}| \leq C_f \frac{b^{pm-p\mu+q\mu+k}}{\Gamma(pm-p\mu+q\mu+k+1)}.$$

Using the estimate $(a+b)! \leq 2^{a+b} a! b!$ we also have

$$(pm-p\mu+q\mu+k)! \leq 2^{pm-p\mu+q\mu+k} (pm-p\mu+q\mu)! k!,$$

so we get

$$|\mathcal{U}(x, y, D_\xi) f(\xi)| \leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} |x|^{m-\mu} |y|^{\mu}$$

$$\times C_f \sum_{k=0}^{\infty} \frac{b^{pm-p\mu+q\mu+k}}{\Gamma(pm-p\mu+q\mu+k+1)} \frac{2^{pm-p\mu+q\mu+k} (pm-p\mu+q\mu)! k!}{k!} |\xi|^k.$$

We now use the Gamma function estimate

$$\frac{1}{\Gamma(a+b+2)} \leq \frac{1}{\Gamma(a+1)} \frac{1}{\Gamma(b+1)} \tag{8}$$

to separate the series, and we have

$$\frac{1}{\Gamma(pm-p\mu+q\mu-\frac{1}{2}+k-\frac{1}{2}+2)} \leq \frac{1}{\Gamma(pm-p\mu+q\mu+\frac{1}{2})} \frac{1}{\Gamma(k+\frac{1}{2})}$$

and so

$$|\mathcal{U}(x, y, D_\xi) f(\xi)| \leq C_f \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} |x|^{m-\mu} |y|^{\mu} (2b)^{pm-p\mu+q\mu} \times$$

$$\times \frac{(pm-p\mu+q\mu)!}{\Gamma(pm-p\mu+q\mu+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\frac{1}{2})} (2b|\xi|)^k,$$

and

$$|\mathcal{U}(x, y, D_\xi) f(\xi)| \leq C_f \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} [(2b)^p |x|]^{m-\mu} [(2b)^q |y|]^\mu \frac{(pm - p\mu + q\mu)!}{\Gamma(pm - p\mu + q\mu + \frac{1}{2})}$$

$$\times \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \frac{1}{2})} (2b|\xi|)^k.$$

Now observe that the series in k satisfies the estimate

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \frac{1}{2})} (2b|\xi|)^k \leq C e^{2b|\xi|}$$

where C is a positive constant, because of the properties of the Mittag-Leffler function, and the series

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} [(2b)^p |x|]^{m-\mu} [(2b)^q |y|]^\mu \frac{(pm - p\mu + q\mu)!}{\Gamma(pm - p\mu + q\mu + \frac{1}{2})} \qquad (9)$$

is convergent to a number denoted by $C_{x,y,p,q} > 0$. In fact, using the Stirling's formula for the Gamma function, we have

$$m! \sim \sqrt{2\pi m}\, e^{-m} m^m, \quad \text{for} \quad m \to \infty$$

and then we deduce

$$\frac{\Gamma(m+1)}{\Gamma(m+1/2)} \sim \frac{\sqrt{2\pi m}\, e^{-m} m^m}{\sqrt{2\pi (m-1/2)}\, e^{-(m-1/2)} (m-1/2)^{(m-1/2)}} \sim \sqrt{m - 1/2}, \quad \text{for} \quad m \to \infty \qquad (10)$$

and so

$$\frac{(pm - p\mu + q\mu)!}{\Gamma(pm - p\mu + q\mu + \frac{1}{2})} \sim \sqrt{pm - p\mu + q\mu - 1/2}, \quad \text{for} \quad pm - p\mu + q\mu \to \infty.$$

Now observe that the series (9) has positive coefficients and so it converges if and only if the series

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} [(2b)^p |x|]^{m-\mu} [(2b)^q |y|]^\mu \sqrt{pm - p\mu + q\mu - 1/2}$$

converges. From the estimate

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} [(2b)^p |x|]^{m-\mu} [(2b)^q |y|]^\mu \sqrt{pm - p\mu + q\mu - 1/2}$$

$$\leq \sqrt{p+q} \sum_{m=0}^{\infty} \frac{\sqrt{m}}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} [(2b)^p |x|]^{m-\mu} [(2b)^q |y|]^\mu$$

$$= \sqrt{p+q} \sum_{m=0}^{\infty} \frac{\sqrt{m}}{m!} [(2b)^p |x| + (2b)^q |y|]^m$$

we have that the series (9) converges for all $x, y \in \mathbb{R}$ and $p, q \in \mathbb{N}$. So we finally have

$$|\mathcal{U}(x, y, D_\xi) f(\xi)| \leq C_f\, C_{x,y,p,q}\, C\, e^{2b|\xi|}, \quad x, y \in \mathbb{R}, \ \xi \in \mathbb{C}. \qquad (11)$$

The estimate (11) shows that $\mathcal{U}(x, y, D_\xi) f \in A_1$, in fact

$$|\mathcal{U}(x, y, D_\xi) f(\xi)|\, e^{-2b|\xi|}, \leq C_f\, C_{x,y,p,q}\, C \quad x, y \in \mathbb{R}, \ \xi \in \mathbb{C},$$

moreover, it also shows that its $2b$-norm satisfies

$$\|\mathcal{U}(x, y, D_\xi) f\|_{2b} \leq C_f\, C_{x,y,p,q}\, C = C_{x,y,p,q}\, C \|f\|_B$$

where $b = 2eB$. Thus the conditions in (6) hold and the continuity of the operator $\mathcal{U}(x, y, D_\xi)$ follows. $\qquad \square$

Remark 2.9 Proposition 2.8 can be stated and proved also for $d > 2$ variables.

Proposition 2.10 *Let* $(g_{1,m})$ *and* $(g_{2,m})$ *be two sequences of complex numbers such that*

$$\lim_{m \to \infty} |g_{\ell,m}|^{1/m} = 0, \quad for \ \ell = 1, 2. \tag{12}$$

We define the formal operator

$$\mathcal{V}(x, y, D_\xi) := \sum_{m_1=0}^{\infty} g_{1,m_1} \sum_{m_2=0}^{\infty} g_{2,m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} D_\xi^{m_1+m_2}, \quad x, y \in \mathbb{R}, \ \xi \in \mathbb{C}. \tag{13}$$

Then $\mathcal{V}(x, y, D_\xi) : A_1 \mapsto A_1$ *is continuous.*

Proof We apply the operator $\mathcal{V}(x, y, D_\xi)$ to a function f in A_1 and we have

$$\begin{aligned}
\mathcal{V}(x, y, D_\xi) f(\xi) &= \sum_{m_1=0}^{\infty} g_{1,m_1} \sum_{m_2=0}^{\infty} g_{2,m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} D_\xi^{m_1+m_2} f(\xi) \\
&= \sum_{m_1=0}^{\infty} g_{1,m_1} \sum_{m_2=0}^{\infty} g_{2,m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} D_\xi^{m_1+m_2} \sum_{j=0}^{\infty} f_j \xi^j \\
&= \sum_{m_1=0}^{\infty} g_{1,m_1} \sum_{m_2=0}^{\infty} g_{2,m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} \sum_{j=m_1+m_2}^{\infty} f_j \frac{j!}{(j-(m_1+m_2))!} \xi^{j-(m_1+m_2)} \\
&= \sum_{m_1=0}^{\infty} g_{1,m_1} \sum_{m_2=0}^{\infty} g_{2,m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} \sum_{k=0}^{\infty} f_{m_1+m_2+k} \frac{(m_1+m_2+k)!}{k!} \xi^k.
\end{aligned}$$

We take the modulus

$$|\mathcal{V}(x, y, D_\xi) f(\xi)| \leq \sum_{m_1=0}^{\infty} |g_{1,m_1}| \sum_{m_2=0}^{\infty} |g_{2,m_2}| |x|^{m_1} |y|^{m_2} \sum_{k=0}^{\infty} |f_{m_1+m_2+k}| \frac{(m_1+m_2+k)!}{k!} |\xi|^k$$

and we use the estimate in Lemma 2.6:

$$|f_{m_1+m_2+k}| \leq C_f \frac{b^{m_1+m_2+k}}{\Gamma(m_1+m_2+k+1)}$$

to get

$$\begin{aligned}
|\mathcal{V}(x, y, D_\xi) f(\xi)| \leq &\sum_{m_1=0}^{\infty} |g_{1,m_1}| \sum_{m_2=0}^{\infty} |g_{2,m_2}| |x|^{m_1} |y|^{m_2} \\
&\times C_f \sum_{k=0}^{\infty} \frac{b^{m_1+m_2+k}}{\Gamma(m_1+m_2+k+1)} \frac{(m_1+m_2+k)!}{k!} |\xi|^k.
\end{aligned}$$

With the estimates

$$(m_1+m_2+k)! \leq 2^{m_1+m_2+k} (m_1+m_2)! k!$$

and

$$\frac{1}{\Gamma(m_1+m_2-\frac{1}{2}+k-\frac{1}{2}+2)} \leq \frac{1}{\Gamma(m_1+m_2+\frac{1}{2})} \frac{1}{\Gamma(k+\frac{1}{2})}$$

we separate the series

$$|\mathcal{V}(x, y, D_\xi)f(\xi)| \leq \sum_{m_1=0}^{\infty} |g_{1,m_1}| \sum_{m_2=0}^{\infty} |g_{2,m_2}||x|^{m_1}|y|^{m_2} \times$$

$$\times \sum_{k=0}^{\infty} C_f b^{m_1+m_2+k} \frac{1}{\Gamma(m_1+m_2+\frac{1}{2})} \frac{1}{\Gamma(k+\frac{1}{2})} \frac{2^{m_1+m_2+k}(m_1+m_2)!k!}{k!} |\xi|^k.$$

Finally, we get

$$|\mathcal{V}(x, y, D_\xi)f(\xi)| \leq C_f \sum_{m_1=0}^{\infty} |g_{1,m_1}| \sum_{m_2=0}^{\infty} |g_{2,m_2}|(2b|x|)^{m_1}(2b|y|)^{m_2} \frac{(m_1+m_2)!}{\Gamma(m_1+m_2+\frac{1}{2})}$$

$$\times \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\frac{1}{2})} (2b|\xi|)^k.$$

Using (10) we have

$$\frac{(m_1+m_2)!}{\Gamma(m_1+m_2+\frac{1}{2})} \sim \sqrt{m_1+m_2-1/2}, \quad \text{for} \quad m_1+m_2 \to \infty,$$

and $\sqrt{m_1+m_2-1/2} \leq m_1 m_2$, since $m_1 \geq 2$ and $m_2 \geq 2$. Thus the series

$$\sum_{m_1 \geq 2}^{\infty} m_1|g_{1,m_1}|(2b|x|)^{m_1}$$

$$\sum_{m_2 \geq 2}^{\infty} m_2|g_{2,m_2}|(2b|y|)^{m_2}$$

converge to C_x, C_y respectively, for $x, y \in \mathbb{R}$. So we have

$$|\mathcal{V}(x, y, D_\xi)f(\xi)| \leq C_f C_x C_y (2b|\xi|)e^{2b|\xi|} \leq C_{x,y} e^{4b|\xi|},$$

from which, recalling that $C_f = \|f\|_B$ and $b = 2eB$, we deduce

$$\|\mathcal{V}(x, y, D_\xi)f\|_{4b} \leq C_{x,y}\|f\|_B.$$

Thus we have that the conditions in (6) are satisfied and the statement follows.   □

3 Superoscillating functions in several variables

We recall some preliminary definitions related to superoscillatory functions in several variables, then, for the sake of simplicity, we limit our study to the case of two variables and then we discuss how our results can be extended to the general case of $d > 2$ variables, see [28].

Definition 3.1 (*Generalized Fourier sequence in several variables*) For $d \in \mathbb{N}$ such that $d \geq 2$, we assume that $(x_1, ..., x_d) \in \mathbb{R}^d$. Let $(h_{j,\ell}(n))$, $j = 0, ..., n$ for $n \in \mathbb{N}_0$, be real-valued sequences for $\ell = 1, ..., d$. We call *generalized Fourier sequence in several variables* a sequence of the form

$$F_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} c_j(n)e^{ix_1 h_{j,1}(n)} e^{ix_2 h_{j,2}(n)} \ldots e^{ix_d h_{j,d}(n)}, \tag{14}$$

where $(c_j(n))_{j,n}$, $j = 0, \ldots, n$, for $n \in \mathbb{N}_0$ is a complex-valued sequence.

Definition 3.2 (*Superoscillating sequence*) A generalized Fourier sequence in several variables $F_n(x_1, \ldots, x_d)$, with $d \in \mathbb{N}$ such that $d \geq 2$, is said to be *a superoscillating sequence* if

$$\sup_{j=0,\ldots,n,\ n\in\mathbb{N}_0} |h_{j,\ell}(n)| \leq 1, \quad \text{for } \ell = 1, \ldots, d,$$

and there exists a compact subset of $\mathbb{R}^d$, which will be called *a superoscillation set*, on which $F_n(x_1, \ldots, x_d)$ converges uniformly to $e^{ix_1 g_1} e^{ix_2 g_2} \ldots e^{ix_d g_d}$, where $|g_\ell| > 1$ for $\ell = 1, \ldots, d$.

Remark 3.3 An important example of a generalized Fourier sequence in several variables is the sequence

$$Y_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} C_j(n, a) e^{ix_1(1-2j/n)^{q_1}} \ldots e^{ix_m(1-2j/n)^{q_d}}.$$

where $C_j(n, a)$ are given by (2) and $q_\ell \in \mathbb{N}$, for $\ell = 1, \ldots, d$.

Remark 3.4 In the paper [8], we studied the function theory of superoscillatory functions in several variables under the additional hypothesis that there exist $r_\ell \in \mathbb{N}$, such that

$$p = r_1 q_1 + \ldots + r_d q_d. \tag{15}$$

In that case, we proved that for $p, q_\ell \in \mathbb{N}$, $\ell = 1, \ldots, d$ the function

$$F_n(x, y_1, \ldots, y_d) = \sum_{j=0}^{n} C_j(n, a) e^{ix(1-2j/n)^p} e^{iy_1(1-2j/n)^{q_1}} \ldots e^{iy_d(1-2j/n)^{q_d}}$$

is superoscillating when $|a| > 1$.

In this paper, we work in a more general framework and we are able to remove the restriction (15) on the coefficients p, q_ℓ for $\ell = 1, \ldots, d$ and to show that general superoscillating functions as in (3) can be used to define superoscillatory functions in several variables.

We start by proving the following:

Theorem 3.5 (The case of two variables) *Let*

$$f_n(x) := \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{16}$$

be a superoscillating function as (3) and assume that its holomorphic extension to the entire function $f_n(\xi)$ converges to $e^{ia\xi}$ in the space A_1. For p and $q \in \mathbb{N}$ we define

$$F_n(x, y) = \sum_{j=0}^{n} Z_j(n, a) e^{ix(h_j(n))^p} e^{iy(h_j(n))^q}.$$

Then, we have

$$\lim_{n\to\infty} F_n(x, y) = e^{ixa^p} e^{iya^q},$$

and, in particular, $F_n(x, y)$ is superoscillating when $|a| > 1$.

Proof We write the chain of equalities

$$F_n(x, y) = \sum_{j=0}^{n} Z_j(n, a) e^{ix(h_j(n))^p} e^{iy(h_j(n))^q}$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m=0}^{\infty} \frac{1}{m!} \Big[ix(h_j(n))^p + iy(h_j(n))^q \Big]^m$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} (h_j(n))^{pm-p\mu} (h_j(n))^{q\mu}$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} (h_j(n))^{pm-p\mu+q\mu}.$$

Now observe that using the auxiliary complex variable ξ we have

$$\lambda^{\ell} = \frac{1}{i^{\ell}} D_{\xi}^{\ell} e^{i\xi\lambda} \Big|_{\xi=0} \quad \text{for} \quad \lambda \in \mathbb{C}, \quad \ell \in \mathbb{N}, \tag{17}$$

where D_{ξ} is the derivative with respect to ξ and $|_{\xi=0}$ denotes the restriction to $\xi = 0$. So we can write

$$(h_j(n))^{pm-p\mu+q\mu} = \frac{1}{i^{pm-p\mu+q\mu}} D_{\xi}^{pm-p\mu+q\mu} e^{i\xi h_j(n)} \Big|_{\xi=0} \tag{18}$$

and defining the infinite order differential operator

$$\mathcal{U}(x, y, D_{\xi}) := \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} D_{\xi}^{pm-p\mu+q\mu} \tag{19}$$

we get

$$F_n(x, y) = \mathcal{U}(x, y, D_{\xi}) \sum_{k=0}^{n} Z_k(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}.$$

In Proposition 2.8 we have proved that the operator

$$\mathcal{U}(x, y, D_{\xi}) : A_1 \mapsto A_1$$

is continuous; therefore, we can take the limit inside $\mathcal{U}(x, y, D_{\xi})$ and we have:

$$\lim_{n \to \infty} \mathcal{U}(x, y, D_{\xi}) \sum_{j=0}^{n} Z_j(n, a) e^{i\xi h_j(n)} = \mathcal{U}(x, y, D_{\xi}) \lim_{n \to \infty} \sum_{j=0}^{n} Z_j(n, a) e^{i\xi h_j(n)}$$

$$= \mathcal{U}(x, y, D_{\xi}) e^{i\xi a}.$$

Since the limit function is continuous (it is in A_1), we can take the restriction to $\xi = 0$

$$\lim_{n \to \infty} F_n(x, y) = \mathcal{U}(x, y, D_{\xi}) e^{i\xi a} \Big|_{\xi=0}.$$

The explicit computation of the term $\mathcal{U}(x, y, D_{\xi}) e^{i\xi a}$ gives

$$\mathcal{U}(x, y, D_{\xi}) e^{i\xi a} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} \frac{1}{i^{pm-p\mu+q\mu}} D_{\xi}^{pm-p\mu+q\mu} e^{i\xi a}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu} (iy)^{\mu} a^{pm-p\mu+q\mu} e^{i\xi a},$$

so we finally get

$$
\begin{aligned}
\lim_{n\to\infty} F_n(x, y) &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu}(iy)^{\mu} a^{pm-p\mu+q\mu} e^{i\xi a}\Big|_{\xi=0} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ix)^{m-\mu}(iy)^{\mu} (a)^{pm-p\mu+q\mu} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu=0}^{m} \binom{m}{\mu} (ixa^p)^{m-\mu}(iya^q)^{\mu} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} (ixa^p + iya^q)^m
\end{aligned}
$$

from which we get the statement.   □

Remark 3.6 From the inspection of the proof we observe a few facts.

1. (I) The space of the entire functions on which the infinite order differential operator $\mathcal{U}(x, y, D_\xi)$ acts is the space A_1 in one complex variable.
2. (II) In our strategy, the two variables (x, y) of the superoscillating function $F_n(x, y)$ appear as parameters of the operator $\mathcal{U}(x, y, D_\xi)$.
3. (III) In the case of $d \geq 2$ variables $(x_1, x_1, \ldots, x_d)$ the variables become the coefficients of the infinite order differential operator $\mathcal{U}(x_1, x_2, \ldots, x_d, D_\xi)$, defined in (21), that still acts on the space A_1.

We now state the case of $d \geq 2$ variables, without giving all the details of the proof.

Theorem 3.7 (The general case of $d \geq 2$ variables) *Let*

$$
f_n(x) := \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{20}
$$

be superoscillating functions as in (6) *and assume that their entire extensions to the functions $f_n(\xi)$ converge to $e^{ia\xi}$ in A_1. Let $p_1, p_2, \ldots, p_d$ with $p_\ell \in \mathbb{N}$, $\ell = 1, 2, \ldots, d$ and $d \in \mathbb{N}$, for $d \geq 2$. Define*

$$
F_n(x_1, x_1, \ldots, x_d) = \sum_{j=0}^{n} Z_j(n, a) e^{ix_1(h_j(n))^{p_1}} e^{ix_2(h_j(n))^{p_2}} \ldots e^{ix_d(h_j(n))^{p_d}}.
$$

Then

$$
\lim_{n\to\infty} F_n(x_1, x_1, \ldots, x_d) = e^{ix_1 a^{p_1}} e^{ix_2 a^{p_2}} \ldots e^{ix_d a^{p_d}},
$$

and in particular $F_n(x_1, x_1, \ldots, x_d)$ is superoscillating when $|a| > 1$.

Proof To generalize the case of two variables we recall the multidimensional version of the Newton binomial expansion. Given variables $y_1, y_2, \ldots, y_d$ we have

$$
(y_1 + y_2 + \ldots + y_d)^m = \sum_{\mu_1+\mu_2+\ldots+\mu_d=m} \binom{m}{\mu_1, \mu_2, \ldots, \mu_d} y_1^{\mu_1} y_2^{\mu_2} \ldots y_d^{\mu_d}
$$

where

$$
\binom{m}{\mu_1, \mu_2, \ldots, \mu_d} := \frac{m!}{\mu_1! \mu_2! \ldots \mu_d!}.
$$

We write the chain of equalities

$$F_n(x_1, x_1, \ldots, x_d) = \sum_{j=0}^{n} Z_j(n, a) e^{ix_1(h_j(n))^{p_1} + ix_2(h_j(n))^{p_2} + \ldots + ix_d(h_j(n))^{p_d}}$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m=0}^{\infty} \frac{1}{m!} \Big[ix_1(h_j(n))^{p_1} + ix_2(h_j(n))^{p_2} + \ldots + ix_d(h_j(n))^{p_d} \Big]^m$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu_1 + \mu_2 + \ldots + \mu_d = m} \binom{m}{\mu_1, \mu_2, \ldots, \mu_d} y_1^{\mu_1} y_2^{\mu_2} \ldots y_d^{\mu_d}.$$

where we have set

$$y_\ell := ix_\ell(h_j(n))^{p_\ell}, \text{ for } \ell = 1, \ldots d \text{ with } d \in \mathbb{N}.$$

We define the infinite order differential operator

$$\mathcal{U}(x_1, x_2, \ldots, x_d, D_\xi) := \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu_1 + \mu_2 + \ldots + \mu_d = m} \mathcal{H}_{m, \mu_1, \mu_2, \ldots, \mu_d}(x_1, x_2, \ldots, x_d, D_\xi), \tag{21}$$

where

$$\mathcal{H}_{m, \mu_1, \mu_2, \ldots, \mu_d}(x_1, x_2, \ldots, x_d, D_\xi) := \binom{m}{\mu_1, \mu_2, \ldots, \mu_d} \frac{(ix_1)^{\mu_1} (ix_1)^{\mu_1} \ldots (ix_d)^{\mu_d}}{i^{p_1\mu_1 + p_2\mu_2 + \ldots + p_d\mu_d}} D_\xi^{p_1\mu_1 + p_2\mu_2 + \ldots + p_d\mu_d}$$

and with similar computations as in Proposition 2.8 we can prove that the operator

$$\mathcal{U}(x_1, x_2, \ldots, x_d, D_\xi) : A_1 \mapsto A_1$$

is continuous. Observing that

$$f_n(x_1, x_1, \ldots, x_d) = \mathcal{U}(x_1, x_2, \ldots, x_d, D_\xi) \sum_{j=0}^{n} Z_j(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}$$

and proceeding as in the case of two variables we get the statement. $\qquad \square$

4 Supershifts in several variables

The procedure to define superoscillatory functions can be extended to the case of supershifts. Recall that the supershift property of a function extends the notion of superoscillations and that this concept turned out to be a crucial ingredient for the study of the evolution of superoscillatory functions as initial conditions of the Schrödinger equation (or of other field equations).

Definition 4.1 (*Supershift*) Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval with $[-1, 1] \subset \mathcal{I}$ and let $\varphi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$, be a continuous function on $\mathcal{I}$. We set

$$\varphi_h(x) := \varphi(h, x), \quad h \in \mathcal{I}, \quad x \in \mathbb{R}$$

and we consider a sequence of points $(h_j(n))$ such that

$$h_j(n) \in [-1, 1] \text{ for } j = 0, \ldots, n \text{ and } n \in \mathbb{N}_0.$$

We define the functions

$$\psi_n(x) = \sum_{j=0}^{n} c_j(n) \varphi_{h_j(n)}(x), \tag{22}$$

where $(c_j(n))$ is a sequence of complex numbers for $j = 0, ..., n$ and $n \in \mathbb{N}_0$. If

$$\lim_{n \to \infty} \psi_n(x) = \varphi_a(x)$$

for some $a \in \mathcal{I}$ with $|a| > 1$, we say that the function $\psi_n(x)$, for $x \in \mathbb{R}$, admits a supershift.

Remark 4.2 The term supershift comes from the fact that the interval $\mathcal{I}$ can be arbitrarily large (it can also be $\mathbb{R}$) and that the constant a can be arbitrarily far away from the interval $[-1, 1]$ where the functions $\varphi_{h_{j,n}}(\cdot)$ are computed, see (22).

Remark 4.3 Superoscillations are a particular case of supershift. In fact, for the sequence (F_n) in (1), we have $h_j(n) = 1 - 2j/n$, $\varphi_{h_j(n)}(x) = e^{i(1-2j/n)x}$ and $c_j(n)$ are the coefficients $C_j(n, a)$ defined in (2).

Problem 2.2, for the supershift case, is formulated as follows.

Problem 4.4 *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, ..., n$, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Suppose that for every $x \in \mathbb{R}$ the function $h \mapsto G(hx)$ extends to a holomorphic and entire function in h. Consider the functions*

$$f_n(x) = \sum_{j=0}^{n} Y_j(n, a) G(h_j(n)x), \quad x \in \mathbb{R}$$

where $h \mapsto G(hx)$ depends on the parameter $x \in \mathbb{R}$. Determine the coefficients $Y_j(n)$ in such a way that

$$f_n^{(p)}(0) = (a)^p G^{(p)}(0) \quad for \quad p = 0, 1, ..., n. \tag{23}$$

The solution of Problem 4.4, obtained in [11], is summarized in the following theorem.

Theorem 4.5 *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, ..., n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$ for every $i \neq j$ and $G^{(p)}(0) \neq 0$ for all $p = 0, 1, ..., n$, then there exists a unique solution $Y_j(n, a)$ of the linear system (23) and it is given by*

$$Y_j(n, a) = \prod_{k=0, \, k \neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right),$$

so that

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, \, k \neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) G(h_j(n)x), \quad x \in \mathbb{R}.$$

Remark 4.6 In the following we will consider those functions G and sequences $h_j(n)$ for which the holomorphic extension $f_n(z)$ of $f_n(x)$ converges in A_1 to $G(az)$.

We can now extend the notion of supershift of a function in several variables.

Definition 4.7 (*Supershifts in several variables*) Let $|a| > 1$. For $d \in \mathbb{N}$ with $d \geq 2$, we assume that $(x_1, ..., x_d) \in \mathbb{R}^d$. Let $(h_{j,\ell}(n))$, $j = 0, ..., n$ for $n \in \mathbb{N}_0$, be real-valued sequences for $\ell = 1, ..., d$ such that for

$$\sup_{j=0,...,n, \, n \in \mathbb{N}_0} |h_{j,\ell}(n)| \leq 1, \quad \text{for } \ell = 1, ..., d$$

and let $G_\ell(\lambda)$, for $\ell = 1, ..., d$, be entire holomorphic functions. We say that the sequence

$$F_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} c_j(n) G_1(x_1 h_{j,1}(n)) G_2(x_2 h_{j,2}(n)) \ldots G_d(x_d h_{j,d}(n)), \tag{24}$$

where $(c_j(n))_{j,n}$, $j = 0, \ldots, n$, for $n \in \mathbb{N}_0$ is a complex-valued sequence, admits the supershift property if

$$\lim_{n \to \infty} F_n(x_1, \ldots, x_d) = G_1(x_1 a) G_2(x_2 a) \ldots G_d(x_d a).$$

Theorem 4.8 (The case of two variables) *Let* $|a| > 1$ *and let*

$$f_n(x) := \sum_{j=0}^{n} Z_j(n, a)e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{25}$$

be a superoscillating function as in Definition 2.1 and assume that its holomorphic extension to the entire functions $f_n(z)$ *converges to* e^{iaz} *in the space* A_1*. Get* G_1 *and* G_2 *be holomorphic entire functions whose series expansion is given by*

$$G_1(\lambda) = \sum_{m=0}^{\infty} g_{1,m}\lambda^m, \quad G_2(\lambda) = \sum_{m=0}^{\infty} g_{2,m}\lambda^m \tag{26}$$

and define

$$F_n(x, y) = \sum_{k=0}^{n} Z_k(n, a)G_1(xh_j(n))G_2(yh_j(n)),$$

where $Z_k(n, a)$ *are given as in (25). Then* $F_n(x, y)$ *admits the supershift property that is*

$$\lim_{n \to \infty} F_n(x, y) = G_1(xa)G_2(ya).$$

Proof We consider

$$F_n(x, y) = \sum_{j=0}^{n} Z_j(n, a)G_1(xh_j(n))G_2(yh_j(n))$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m_1=0}^{\infty} g_{m_1} \sum_{m_2=0}^{\infty} g_{m_2} x^{m_1} y^{m_2} (h_j(n))^{m_1+m_2}.$$

We now consider the auxiliary complex variable ξ and we note that

$$\lambda^\ell = \frac{1}{i^\ell} D_\xi^\ell e^{i\xi\lambda}\Big|_{\xi=0} \quad \text{for} \quad \lambda \in \mathbb{C}, \quad \ell \in \mathbb{N}, \tag{27}$$

where D_ξ is the derivative with respect to ξ and $|_{\xi=0}$ denotes the restriction to $\xi = 0$, we have

$$F_n(x, y) = \sum_{j=0}^{n} Z_j(n, a) \sum_{m_1=0}^{\infty} g_{m_1} \sum_{m_2=0}^{\infty} g_{m_2} x^{m_1} y^{m_2} [h_j(n)]^{m_1+m_2}$$

$$= \sum_{j=0}^{n} Z_j(n, a) \sum_{m_1=0}^{\infty} g_{m_1} \sum_{m_2=0}^{\infty} g_{m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} D_\xi^{m_1+m_2} e^{i\xi h_j(n)}\Big|_{\xi=0}$$

$$= \sum_{m_1=0}^{\infty} g_{m_1} \sum_{m_2=0}^{\infty} g_{m_2} x^{m_1} y^{m_2} \frac{1}{i^{m_1+m_2}} D_\xi^{m_1+m_2} \sum_{j=0}^{n} Z_j(n, a)e^{i\xi h_j(n)}\Big|_{\xi=0}.$$

We now use the operator $\mathcal{V}(x, y, D_\xi)$ defined in (13) so that we can write

$$F_n(x, y) = \mathcal{V}(x, y, D_\xi) \sum_{j=0}^{n} Z_j(n, a)e^{i\xi h_j(n)}\Big|_{\xi=0}.$$

Here we use Proposition 2.10 in order to compute the limit and this concludes the proof. □

Remark 4.9 The notion of supershift and the previous results can be extended to the case of several variables.

Data availability statement The authors declare that there are no data associated with the research in this paper.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Aharonov, Y., Albert, D., Vaidman, L.: How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. Phys. Rev. Lett. **60**, 1351–1354 (1988)
2. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: Schrödinger evolution of superoscillations with δ- and δ'-potentials. Quantum Stud. Math. Found. **7**(3), 293–305 (2020)
3. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: Green's function for the Schrödinger equation with a generalized point interaction and stability of superoscillations. J. Differ. Equ. **277**, 153–190 (2021)
4. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: A unified approach to Schrödinger evolution of superoscillations and supershifts. Preprint arXiv:2102.11795, accepted in J. Evol. Equ
5. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Evolution of superoscillations in the Klein-Gordon field. Milan J. Math. **88**(1), 171–189 (2020)
6. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: How superoscillating tunneling waves can overcome the step potential. Ann. Phys. **414**, 168088, 19 (2020)
7. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: On the Cauchy problem for the Schrödinger equation with superoscillatory initial data. J. Math. Pure Appl. **99**, 165–173 (2013)
8. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences in several variables. J. Fourier Anal. Appl. **22**, 751–767 (2016)
9. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: The mathematics of superoscillations. Mem. Am. Math. Soc. **247**(1174), v+107 (2017)
10. Aharonov, Y., Colombo, F., Struppa, D.C., Tollaksen, J.: Schrödinger evolution of superoscillations under different potentials. Quant. Stud. Math. Found. **5**, 485–504 (2018)
11. Aharonov, Y., Colombo, F., Sabadini, I., Shushi, T., Struppa, D. C., Tollaksen, J.: A new method to generate superoscillating functions and supershifts. Proc. R. Soc. A. **477**(2249), Paper No. 20210020, 12 pp (2021)
12. Aharonov, Y., Rohrlich, D.: Quantum paradoxes: quantum theory for the perplexed. Wiley, Weinheim (2005)
13. Aharonov, Y., Sabadini, I., Tollaksen, J., Yger, A.: Classes of superoscillating functions. Quant. Stud. Math. Found. **5**, 439–454 (2018)
14. Aharonov, Y., Shushi, T.: A new class of superoscillatory functions based on a generalized polar coordinate system. Quant. Stud. Math. Found. **7**, 307–313 (2020)
15. Alpay, D., Colombo, F., Sabadini, I., Struppa, D.C.: Aharonov-Berry superoscillations in the radial harmonic oscillator potential. Quant. Stud. Math. Found. **7**, 269–283 (2020)
16. Aoki, T., Colombo, F., Sabadini, I., Struppa, D.C.: Continuity of some operators arising in the theory of superoscillations. Quant. Stud. Math. Found. **5**, 463–476 (2018)
17. Aoki, T., Colombo, F., Sabadini, I., Struppa, D.C.: Continuity theorems for a class of convolution operators and applications to superoscillations. Ann. Mat. Pura Appl. **197**, 1533–1545 (2018)
18. Aoki, T., Ishimura, R., Okada, Y., Struppa, D.C., Uchida, S.: Characterisation of continuous endomorphisms of the space of entire functions of a given order. Complex. Var Ell. Equ. **66**, 1439–1450 (2021)
19. Behrndt, J., Colombo, F., Schlosser, P.: Evolution of Aharonov–Berry superoscillations in Dirac δ-potential. Quant. Stud. Math. Found. **6**, 279–293 (2019)
20. Berenstein, C.A., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, New York (1995)

21. Berry, M., et al.: Roadmap on superoscillations. J. Opt. **21**, 053002 (2019)
22. Berry, M.V., Anandan, J.S., Safko, J.L.: Faster than Fourier, in Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir, Aharonov, pp. 55–65. World Scientific, Singapore (1994)
23. Berry, M.: Exact nonparaxial transmission of subwavelength detail using superoscillations. J. Phys. A **46**, 205203 (2013)
24. Berry, M.V.: Representing superoscillations and narrow Gaussians with elementary functions. Milan J. Math. **84**, 217–230 (2016)
25. Berry, M.V., Popescu, S.: Evolution of quantum superoscillations, and optical superresolution without evanescent waves. J. Phys. A **39**, 6965–6977 (2006)
26. Berry, M.V., Shukla, P.: Pointer supershifts and superoscillations in weak measurements. J. Phys. A **45**, 015301 (2012)
27. Colombo, F., Gantner, J., Struppa, D.C.: Evolution by Schrödinger equation of Aharonov-Berry superoscillations in centrifugal potential. Proc. A **475**(2225), 20180390, 17 pp (2019)
28. Colombo, F., Pinton, S., Sabadini, I., Struppa, D.C.: The general theory of superoscillations and supershifts in several variables, preprint (2022)
29. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Gauss sums, superoscillations and the Talbot carpet. J. Math. Pure Appl. **9**(147), 163–178 (2021)
30. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Superoscillating sequences and hyperfunctions. Publ. Res. Inst. Math. Sci. **55**(4), 665–688 (2019)
31. Colombo, F., Struppa, D.C., Yger, A.: Superoscillating sequences towards approximation in S or S'-type spaces and extrapolation. J. Fourier Anal. Appl. **25**(1), 242–266 (2019)
32. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.:Superoscillating functions and the super-shift for generalized functions, to appear in Complex Analysis and Operator Theory (2022)
33. Colombo, F., Valente, G.: Evolution of Superoscillations in the Dirac Field. Found. Phys. **50**, 1356–1375 (2020)
34. Ferreira, P.J.S.G., Kempf, A.: Unusual properties of superoscillating particles. J. Phys. A **37**, 12067–76 (2004)
35. Ferreira, P.J.S.G., Kempf, A.: Superoscillations: faster than the Nyquist rate. IEEE Trans. Signal Process. **54**, 3732–3740 (2006)
36. Ferreira, P.J.S.G., Kempf, A., Reis, M.J.C.S.: Construction of Aharonov-Berry's superoscillations. J. Phys. A **40**, 5141–5147 (2007)
37. Kempf, A.: Four aspects of superoscillations. Quant. Stud. Math. Found. **5**, 477–484 (2018)
38. Kempf, A.: Black holes, bandwidths and Beethoven. J. Math. Phys. **41**(4), 2360–2374 (2000)
39. Lindberg, J.: Mathematical concepts of optical superresolution. J. Opt. **14**, 083001 (2012)
40. Pozzi, E., Wick, B.D.: Persistence of superoscillations under the Schrödinger equation. Evolut. Equ. Control Theory. https://doi.org/10.3934/eect.2021029
41. Šoda, B., Kempf, A.: Efficient method to create superoscillations with generic target behavior. Quant. Stud. Math. Found. **7**(3), 347–353 (2020)
42. Tang, E., Garg, L., Kempf, A.: Scaling properties of superoscillations and the extension to periodic signals. J. Phys. A **49**(33), 335202, 17 pp (2016)
43. Toraldo di Francia, G.: Super-gain antennas and optical resolving power. Nuovo Cimento Suppl. **9**, 426–438 (1952)
44. Yger, A.: Private Communication (2021)