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TORIC IDEALS ASSOCIATED WITH GAP-FREE GRAPHS

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ABSTRACT. In this paper we prove that every toric ideal associated with a gap-free graph G has a squarefree lexicographic initial ideal. Moreover, in the particular case when the complementary graph of G is chordal (i.e. when the edge ideal of G has a linear resolution), we show that there exists a reduced Gröbner basis $\mathcal{G}$ of the toric ideal of G such that all the monomials in the support of $\mathcal{G}$ are squarefree. Finally, we show (using work by Herzog and Hibi) that if I is a monomial ideal generated in degree 2, then I has a linear resolution if and only if all powers of I have linear quotients, thus extending a result by Herzog, Hibi and Zheng.

1. INTRODUCTION

Algebraic objects depending on combinatorial data have attracted a lot of interest among both algebraists and combinatorialists: some valuable sources to learn about this research area are the books by Stanley [24], Villarreal [27], Miller and Sturmfels [12], and Herzog and Hibi [7]. It is often a challenge to establish relationships between algebraic and combinatorial properties of these objects.

Let G be a simple graph and consider its vertices as variables of a polynomial ring over a field K. We can associate with each edge e of G the squarefree monomial M_e of degree 2 obtained by multiplying the variables corresponding to the vertices of the edge. With this correspondence in mind, we can now introduce some algebraic objects associated with the graph G:

- the *edge ideal* $I(G)$ is the monomial ideal generated by $\{M_e \mid e \text{ is an edge of } G\}$;
- the *toric ideal* I_G is the kernel of the presentation of the K-algebra $K[G]$ generated by $\{M_e \mid e \text{ is an edge of } G\}$.

An important result by Fröberg [5] gives a combinatorial characterization of those graphs G whose edge ideal $I(G)$ admits a linear resolution: they are exactly the ones whose complementary graph G^c is chordal. Another strong connection between the realms of commutative algebra and combinatorics is the one which links initial ideals of the toric ideal I_G to triangulations of the edge polytope of G, see Sturmfels's book [25] and the recent article by Haase, Paffenholz, Piechnik and Santos [6]. Furthermore, Gröbner bases of I_G have been studied among others by Ohsugi and Hibi [21] and Tatakis and Thoma [26]. A necessary condition for I_G to have a squarefree initial

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ideal is the normality of $K[G]$, which was characterized combinatorially by Ohsugi and Hibi [19] and Simis, Vasconcelos and Villarreal [23]. Normality, though, is not sufficient: Ohsugi and Hibi [16] gave an example of a graph G such that $K[G]$ is normal but all possible initial ideals of I_G are not squarefree.

An interesting class of graphs is the one consisting of the so-called *gap-free graphs* (following Dao, Huneke and Schweig's notation in [3]), i.e. graphs such that any two edges with no vertices in common are linked by at least one edge. Unfortunately, these graphs do not have a standard name in the literature. Just to name a few possibilities:

- graph theorists refer to gap-free graphs as "$2K_2$-free graphs" and so do Hibi, Nishiyama, Ohsugi and Shikama in [9];
- Nevo and Peeva call them "C_4-free graphs" in [13] and [14];
- Ohsugi and Hibi use the phrase "graphs whose complement is weakly chordal" in [18];
- Corso and Nagel call bipartite gap-free graphs "Ferrers graphs" in [2].

The main goal of this paper is to prove that the toric ideal I_G has a squarefree lexicographic initial ideal, provided the graph G is gap-free (Theorem 3.9): moreover, the corresponding reduced Gröbner basis consists of circuits. In the particular case when $I(G)$ has a linear resolution (Theorem 3.6) we are actually able to prove that the reduced Gröbner basis $\mathcal{G}$ we describe consists of circuits such that all monomials (both leading and trailing) in the support of $\mathcal{G}$ are squarefree, thus extending a result of Ohsugi and Hibi [17] on multipartite complete graphs.

In [8] Herzog, Hibi and Zheng proved that the following conditions are equivalent:

(a) $I(G)$ has a linear resolution;

(b) $I(G)$ has linear quotients;

(c) $I(G)^k$ has a linear resolution for all $k \geq 1$.

It is quite natural to ask (see for instance the article by Hoefel and Whieldon [11]) whether these conditions are in turn equivalent to the fact that

(d) $I(G)^k$ has linear quotients for all $k \geq 1$.

In Theorem 2.6 we prove that this is indeed the case, as can be deduced from results in [7]. Note that all the equivalences between conditions (a), (b), (c), (d) above hold more generally for monomial ideals generated in degree 2 which are not necessarily squarefree.

The computer algebra system CoCoA [1] gave us the chance of performing computations which helped us to produce conjectures about the behaviour of the objects studied.

2. Notation and known facts

First of all, let us fix some notation. K will always be a field and G a simple graph with vertices $V(G) = \{1, \ldots, n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. We can associate

to each edge $e = \{i, j\}$ the degree 2 monomial (called *edge monomial*) $M_e := x_i x_j \in K[x_1, \dots, x_n]$ and hence we can consider the *edge ideal* $I(G) := (M_{e_1}, \dots, M_{e_m})$ and the subalgebra $K[G] := K[M_{e_1}, \dots, M_{e_m}]$. In the following we will denote by I_G the *toric ideal associated with* G, i.e. the kernel of the surjection

$$\begin{array}{ccc} K[y_1, \dots, y_m] & \twoheadrightarrow & K[G] \\ y_i & \mapsto & M_{e_i} \end{array}$$

Since the algebraic objects we defined are not influenced by isolated vertices of G, we will always assume without loss of generality that G does not have any isolated vertex. We will now introduce some terminology and state some well-known results about toric ideals of graphs: for reference, see for instance [7, Section 10.1].

A collection of (maybe repeated) consecutive edges

$$\Gamma = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{q-1}, v_q\}\}$$

(also denoted by $\{v_0 \to v_1 \to v_2 \to \dots \to v_{q-1} \to v_q\}$) is called a *walk* of G. If $v_0 = v_q$, the walk is *closed*. If q is even (respectively odd), the walk is an even (respectively odd) walk. A *path* is a walk having all distinct vertices; a *cycle* is the closed walk most similar to a path, i.e. such that vertices $v_0, \dots, v_{q-1}$ are all distinct. A *bow-tie* is a graph consisting of two vertex-disjoint odd cycles joined by a single path. Given a walk Γ, we will denote by $|\Gamma|$ the subgraph of G whose vertices and edges are exactly the ones appearing in Γ. If no confusion occurs, we will often write walks in more compact ways, such as by decomposing them into smaller walks. If Γ is a walk, $-\Gamma$ denotes the walk obtained from Γ by reversing the order of the edges.

If $\Gamma = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{2q-1}, v_{2q}\}\}$ is an even closed walk, one can associate with Γ a binomial $b_\Gamma \in K[y_1, \dots, y_m]$ in the following way:

$$b_\Gamma := \prod_{i=1}^{q} y_{\{v_{2i-2}, v_{2i-1}\}} - \prod_{i=1}^{q} y_{\{v_{2i-1}, v_{2i}\}},$$

where, if $e \in E(G)$, by y_e we mean the variable which is mapped to M_e by the standard surjection. A *subwalk* Γ' of Γ is an even closed walk such that all even edges of Γ' are also even edges of Γ and all odd edges of Γ' are also odd edges of Γ. An even closed walk Γ is called *primitive* if it does not have any proper subwalk. The set of binomials corresponding to primitive walks of a graph G coincides with the so-called *Graver basis* of I_G (see for instance [25]) and is denoted by Gr_G.

Remark 2.1. Note that, given a primitive walk Γ, one can paint – using two colours – the edges of $|\Gamma|$ so that those appearing in an even position in Γ are assigned the same colour and those appearing in an odd position are assigned the other one. If an edge were assigned both colours, then the walk Γ would not be primitive: deleting inside both monomials one instance of the variable corresponding to that edge, one could construct a proper subwalk of Γ.

The *support* of a binomial $b = u - v \in I_G$ is the union of the supports of the monomials u and v, that is to say the variables that appear in u and v. A binomial $b \in I_G$ is called a *circuit* if it is irreducible and has minimal support, i.e. there does not exist $b' \in I_G$ such that $\mathrm{supp}(b') \subsetneq \mathrm{supp}(b)$. The set of circuits of I_G is denoted by C_G.

Let I be an ideal of $S := K[x_1, \ldots, x_n]$. A Gröbner basis $\mathcal{G}$ of I with respect to a term order τ is called *reduced* if every element of $\mathcal{G}$ is monic, the leading terms of $\mathcal{G}$ minimally generate $in_\tau(I)$ and no trailing term of $\mathcal{G}$ lies in $in_\tau(I)$. Such a basis is unique and is denoted by $\mathrm{RGB}_\tau(I)$. Generally speaking, changing the term order τ yields a different reduced Gröbner basis: we will denote by $\mathrm{UGB}(I)$ the *universal Gröbner basis* of I, i.e. the union of all reduced Gröbner bases of I.

Proposition 2.2 ([25, Proposition 4.11])**.** *One has that* $\mathrm{C}_G \subseteq \mathrm{UGB}(I_G) \subseteq \mathrm{Gr}_G$.

The second inclusion of Proposition 2.2 means that every reduced Gröbner basis $\mathcal{G}$ of I_G consists of binomials coming from primitive walks of G. Consider the set of monomials (both leading and trailing) in such a basis: if they are all squarefree, we will say that $\mathcal{G}$ is *doubly squarefree*.

Complete characterizations of both C_G (Villarreal [28]) and $\mathrm{UGB}(I_G)$ (Tatakis and Thoma [26]) are known. We recall the characterization of C_G (using the phrasing in Ohsugi and Hibi's article [20]) as a reference.

Proposition 2.3. *A binomial $b \in I_G$ is a circuit of G if and only if $b = b_\Gamma$, where Γ is one of the following even closed walks:*

1. *an even cycle;*
2. *$\{C_1, C_2\}$ where C_1 and C_2 are odd cycles with exactly one common vertex;*
3. *$\{C_1, p, C_2, -p\}$ where C_1 and C_2 are vertex-disjoint odd cycles and p is a path running from a vertex of C_1 to a vertex of C_2.*

Definition 2.4. Let $I \subseteq S := K[x_1, \ldots, x_n]$ be a graded ideal generated in degree d.

- If the minimal free resolution of I as an S-module is linear until the k-th step, i.e. $\mathrm{Tor}_i^S(I, K)_j = 0$ for all $i \in \{0, \ldots, k\}$, $j \neq i + d$, we say that I is *k-step linear*.
- If I is k-step linear for every $k \geq 1$, we say I has a *linear resolution*.
- If I is minimally generated by $f_1, \ldots, f_s$ and for every $1 < i \leq s$ one has that $(f_1, f_2, \ldots, f_{i-1}) :_S (f_i)$ is generated by elements of degree 1, then $[f_1, \ldots, f_s]$ is called a *linear quotient ordering* and I is said to have *linear quotients*.
- If $I = I(G)$ for some graph G and I has one of the properties above, we say that G has that property.

Proposition 2.5 ([7, Proposition 8.2.1])**.** *Let $I \subseteq K[x_1, \ldots, x_n]$ be a graded ideal generated in degree d. Then*

$$I \text{ has linear quotients } \Rightarrow I \text{ has a linear resolution.}$$

We now recall an important result by Herzog, Hibi and Zheng ([8]) about the connection between linear quotients and linear resolution in the case when I is a monomial ideal generated in degree 2. Condition (d) below did not appear in the original paper: its equivalence to other conditions, though, can be obtained quickly using results in [7].

Theorem 2.6. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a monomial ideal generated in degree 2. Then the following conditions are equivalent:*

(a) I has a linear resolution;

(b) I has linear quotients;

(c) I^k has a linear resolution for all $k \geq 1$;

(d) I^k has linear quotients for all $k \geq 1$.

Proof. The implications $(c) \Rightarrow (a)$ and $(d) \Rightarrow (b)$ are obvious, while $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$ follow from Proposition 2.5. It is then enough to prove that $(a) \Rightarrow (d)$, but this follows at once from [7, Theorems 10.1.9 and 10.2.5] (since the lexicographic order $<_{\text{lex}}$ introduced in Theorem 10.2.5 is of the kind appearing in Theorem 10.1.9). □

Remark 2.7. Theorem 10.2.5 and the proof of Theorem 10.1.9 in [7] (or, as an alternative, just the proof of the implication $(a) \Rightarrow (b)$ in Theorem 10.2.6) tell us also that, if I is a monomial ideal of degree 2 having a linear resolution and $\{m_1, m_2, \ldots, m_s\}$ is a minimal set of monomial generators for I, then there exists a permutation σ of $\{1, \ldots, s\}$ such that $[m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(s)}]$ is a linear quotient ordering for I. As a consequence, if I is the edge ideal of some graph G having a linear resolution, there exists a way of ordering the edge monomials so that they form a linear quotient ordering.

We thank Aldo Conca for pointing out the following result:

Proposition 2.8. *Let $f_1, \ldots, f_s$ be distinct homogeneous elements of degree d in $S := K[x_1, \ldots, x_n]$ which are minimal generators for the ideal $(f_1, \ldots, f_s)$. The following conditions are equivalent:*

(a) $[f_1, \ldots, f_s]$ is a linear quotient ordering;

(b) the ideal $(f_1, \ldots, f_i)$ is 1-step linear for all $i \leq s$.

Proof. Let us prove that $(a) \Rightarrow (b)$. Let $i \leq s$. If $[f_1, \ldots, f_s]$ is a linear quotient ordering, than $[f_1, \ldots, f_i]$ is too and hence, by Proposition 2.5, the ideal $(f_1, \ldots, f_i)$ has a linear resolution; in particular, it is 1-step linear.

To prove that $(b) \Rightarrow (a)$, let $i \in \{2, \ldots, s\}$. Consider the exact sequence

$$0 \to \operatorname{Ker} \varepsilon \to S(-d)^i \xrightarrow{\varepsilon} (f_1, \ldots, f_i) \to 0,$$

where ε is the map which sends e_j to f_j for all $j \in \{1, \ldots, i\}$. Then, by hypothesis, $\operatorname{Ker} \varepsilon$ is generated in degree 1. Since $(f_1, \ldots, f_{i-1}) :_S (f_i)$ is isomorphic to the i-th projection of $\operatorname{Ker} \varepsilon$, we are done. □

In what follows, we will denote by G^c the *complementary graph* of G, i.e. the graph which has the same vertex set of G and whose edges are exactly the non-edges of G.

The next result by Eisenbud, Green, Hulek and Popescu proves that, in our context, the algebraic concept of k-step linearity can be characterized in a purely combinatorial manner.

Proposition 2.9 ([4, Theorem 2.1])**.** *Let G be a graph and let $k \geq 1$. The following conditions are equivalent:*

- *G is k-step linear;*
- *G^c does not contain any induced cycle of length i for any $4 \leq i \leq k+3$.*

As a corollary, we recover the important result by Fröberg characterizing combinatorially graphs with a linear resolution.

Corollary 2.10 ([5])**.** *Let G be a graph. Then G has a linear resolution if and only if G^c is chordal, i.e. G^c does not contain any induced cycle of length greater than or equal to 4.*

Following the notation in [3], we will call a graph G *gap-free* if for any $\{v_1, v_2\}$, $\{w_1, w_2\}$ in $E(G)$ (where v_1, v_2, w_1, w_2 are all distinct) there exist $i, j \in \{1, 2\}$ such that $\{v_i, w_j\} \in E(G)$. In other words, in a gap-free graph any two edges with no vertices in common are linked by at least a bridge.

Remark 2.11. It is easy to see that G is gap-free if and only if G^c does not contain any induced cycle of length 4. It then follows from Proposition 2.9 that G is gap-free if and only if G is 1-step linear.

The following theorem holds more generally for affine semigroup algebras.

Theorem 2.12. *Let G be a graph.*

1. (Hochster [10]) *If $K[G]$ is normal, then it is Cohen-Macaulay.*
2. (Sturmfels [25, Proposition 13.15]) *If I_G admits a squarefree initial ideal with respect to some term order τ, then $K[G]$ is normal (and hence Cohen-Macaulay).*

The problem of normality of graph algebras (and, as a consequence, of edge ideals, see [23, Corollary 2.8]) was addressed and completely solved by Ohsugi and Hibi [19] and Simis, Vasconcelos and Villarreal [23]. One of the main results they found is the following:

Theorem 2.13. *A connected graph G is such that $K[G]$ is normal if and only if G satisfies the* odd cycle condition*, i.e. for every couple of disjoint minimal odd cycles $\{C_1, C_2\}$ in G there exists an edge linking C_1 and C_2.*

Ohsugi and Hibi [16] also found an example of a graph G such that $K[G]$ is normal but $in_\tau(I_G)$ is not squarefree for every choice of τ, hence the condition in Theorem 2.12.2 is sufficient but not necessary.

Remark 2.14. There is a strong connection between squarefree initial ideals of I_G and unimodular regular triangulations of the edge polytope of G. To get more information about this topic, see [25] and the recent work [6], in particular Section 2.4.

3. Results

We start by stating a result about the shape of primitive walks. This is a modification of [21, Lemma 2.1]: note that primitive walks were completely characterized by

Reyes, Tatakis and Thoma in [22, Theorem 3.1] and by Ogawa, Hara and Takemura in [15, Theorem 1]. In the rest of the paper we will often talk of primitive walks of type (i), (ii), (iii) referring to the classification below.

Lemma 3.1. *Let Γ be a primitive walk. Then Γ is one of these:*

(i) an even cycle;

(ii) $\{C_1, C_2\}$ where C_1 and C_2 are odd cycles with exactly one common vertex;

(iii) $\{C_1, p_1, C_2, p_2, \ldots, C_h, p_h\}$ where the p_i's are paths of length greater than or equal to one and the C_i's are odd cycles such that $C_{i \pmod h}$ and $C_{i+1 \pmod h}$ are vertex-disjoint for every i.

Proof. Let Γ be a primitive walk neither of type (i) nor (ii). Since Γ is primitive, there exists a cycle C_1 inside Γ (otherwise $\Gamma = \{p, -p\}$ where p is a path and hence all edges of p would appear both in odd and even position in Γ, thus violating the primitivity); moreover, since Γ is not of type (i), C_1 has to be odd. Let $C_1 = \{v_1 \to v_2 \to \ldots \to v_{2k+1} \to v_1\}$; then

$$\Gamma = \{v_1 \to v_2 \to \ldots \to v_{2k+1} \to v_1 = u_0 \to u_1 \to u_2 \to u_3 \to \ldots\}.$$

Let $s \geq 1$ be the least integer such that u_s coincides with one of the vertices in $\{u_0 = v_1, v_2, \ldots, v_{2k+1}, u_1, \ldots, u_{s-1}\}$.

- Suppose $u_s = v_i$ where $i \neq 1$. If $s = 1$ and $i \in \{2, 2k+1\}$, we get that the edge $\{v_1, v_i\}$ is both an even and an odd edge of Γ (contradiction). In all other cases, paint the edges appearing in Γ red and black alternately and note that, since $i \neq 1$, there are both a red and a black edge of C_1 starting from v_i. Then exactly one of $\{v_1 = u_0 \to u_1 \to \ldots \to u_{s-1} \to u_s = v_i \to v_{i+1} \to \ldots \to v_{2k+1} \to v_1 = u_0\}$ and $\{v_1 = u_0 \to u_1 \to \ldots \to u_{s-1} \to u_s = v_i \to v_{i-1} \to \ldots \to v_2 \to v_1 = u_0\}$ is an even closed subwalk, thus violating the primitivity of Γ. This gives us a contradiction.

- Suppose $u_s = u_i$ where $i \in \{0, \ldots, s-2\}$ (since G has no loops, $i \neq s-1$). Note that one actually has that $i < s-2$, since $i = s-2$ would imply that the edge $\{u_{s-1}, u_s\}$ is both an even and an odd edge of Γ (contradiction). Therefore there exists a cycle $C_2 = \{u_s = u_i \to u_{i+1} \to \ldots \to u_{s-1} \to u_s = u_i\}$ disjoint from C_1 by construction. Since Γ is primitive, C_2 must be odd; moreover, since Γ is not of type (ii), one has that $i \neq 0$. This means that we have found a path $p_1 = \{u_0 \to u_1 \to \ldots \to u_i\}$ linking the odd cycles C_1 and C_2. We can now repeat the whole procedure starting from the cycle C_2 to find a path p_2 and an odd cycle C_3 disjoint from C_2 and so on, hence proving the claim in a finite number of steps. □

Remark 3.2. The referee noted that an alternative proof of Lemma 3.1 may be given using [15, Theorem 1].

Remark 3.3. Note that, by Proposition 2.3, all binomials corresponding to primitive walks of type (i) and (ii) are circuits.

Notation 3.4. Let G be a graph with m edges and let τ be a term ordering on $K[y_1, \dots, y_m]$. With a slight abuse of notation, we will often say that $e \preceq_\tau e'$ instead of $y_e \preceq_\tau y_{e'}$ (where $e, e' \in E(G)$). Moreover, if H is a subgraph of G and τ is lexicographic, we will say that $e \in E(H)$ is the *leading edge* of H with respect to τ if $y_{e'} \preceq_\tau y_e$ for every $e' \in E(H)$.

Next we introduce the main technical lemma of the paper. Note that, when dealing with the vertices $v_1, \dots, v_s$ of a cycle, for the sake of simplicity we will often write v_i instead of $v_{i \pmod s}$.

Lemma 3.5. *Let Γ be a primitive closed walk of G of type (iii) and let τ be a lexicographic term order on $K[y_1, \dots, y_m]$. Let e be the leading edge of $|\Gamma|$ with respect to τ: by Lemma 3.1, e lies into a bow-tie $\{C_1, p, C_2\}$. Let $C_1 = \{v_1 \to v_2 \to \dots \to v_{2k+1} \to v_1\}$, $C_2 = \{v'_1 \to v'_2 \to \dots \to v'_{2\ell+1} \to v'_1\}$ and let v_1 and v'_1 be the starting and ending vertices of the path p. Suppose one of the following two conditions holds:*

(a) $e \in p$ and there exist i, j such that $\tilde{e} := \{v_i, v'_j\} \in E(G)$, $\tilde{e} \prec_\tau e$, $\tilde{e} \neq \{v_1, v'_1\}$;

(b) $e = \{v_i, v_{i+1}\}$ and there exists j such that at least one between $\{v_i, v'_j\}$ and $\{v_{i+1}, v'_j\}$ is an edge of G (call it $\tilde{e}$) such that $\tilde{e} \prec_\tau e$ and $\tilde{e} \neq \{v_1, v'_1\}$.

Then $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.

Proof. First of all, by Remark 2.1 the primitivity of the walk Γ allows us to paint the edges of $|\Gamma|$ red and black so that no two edges consecutive in Γ are painted the same colour. We can assume without loss of generality that the edge e is black.

(a) Paint $\tilde{e}$ red. We can suppose without loss of generality that $i \neq 1$: hence, exactly one of $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ is black. This means that exactly one of the two paths going from v_i to v_1 along C_1 has its first edge painted black: let w be this path. We now need to define a path w' going from v'_1 to v'_j.

- If $j \neq 1$, exactly one of $\{v'_{j-1}, v'_j\}$ and $\{v'_j, v'_{j+1}\}$ is black. Applying the same reasoning as before, let w' be the path going from v'_1 to v'_j along C_2 having its last edge painted black.
- If $j = 1$ and the last edge of p is red, let $w' = \{v'_1 \to v'_2 \to \dots \to v'_{2k+1} \to v'_1\}$ (in other words, the whole cycle C_2); if the last edge of p is black, let w' be the empty path in v'_1.

Let $\Gamma' = \{v'_j \xrightarrow{\tilde{e}} v_i \xrightarrow{w} v_1 \xrightarrow{p} v'_1 \xrightarrow{w'} v'_j\}$. By construction, Γ' is an even closed walk, since its edges are alternately red and black and the first and the last one have different colours. Moreover, it is easy to check that Γ' is primitive either of type (ii) (when $j = 1$ and the last edge of p is red) or of type (i) (in all other cases); hence, $\Gamma' \in \mathrm{Gr}_G$. Finally, since τ is a lexicographic term order, to get who the leading monomial of $b_{\Gamma'}$ is we just have to identify the leading edge of Γ': since $\tilde{e} \prec_\tau e$ by hypothesis and the rest of the edges of Γ' are edges of Γ, we get that the leading monomial of $b_{\Gamma'}$ is the one formed by black edges. Since the black edges of Γ' all lie in Γ, we have that $in_\tau(b_{\Gamma'})$ divides $in_\tau(b_\Gamma)$. Since $b_\Gamma \neq b_{\Gamma'}$, we have that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.

(b) Paint $\tilde{e}$ red and define w' in the same way as in part (a). Let w be defined the following way:

- if $\tilde{e} = \{v_i, v'_j\}$, let $w := \{v_{i+1} \to v_{i+2} \to \ldots \to v_{2k+1} \to v_1\}$ (if $i = 2k+1$, w is the empty path);
- if $\tilde{e} = \{v_{i+1}, v'_j\}$, let $w := \{v_i \to v_{i-1} \to \ldots \to v_2 \to v_1\}$ (if $i = 1$, w is the empty path).

Let

$$\Gamma' := \begin{cases} \{v_{i+1} \xrightarrow{w} v_1 \xrightarrow{p} v'_1 \xrightarrow{w'} v'_j \xrightarrow{\tilde{e}} v_i \xrightarrow{e} v_{i+1}\} & \text{if } \tilde{e} = \{v_i, v'_j\} \\ \{v_i \xrightarrow{w} v_1 \xrightarrow{p} v'_1 \xrightarrow{w'} v'_j \xrightarrow{\tilde{e}} v_{i+1} \xrightarrow{e} v_i\} & \text{if } \tilde{e} = \{v_{i+1}, v'_j\} \end{cases}$$

Reasoning the same way as in part (a), we get that Γ' is an even closed walk; moreover, it can be easily checked that Γ' is primitive either of type (ii) (when v'_1 belongs to $\tilde{e}$ and the last edge of p is red or when v_1 belongs to $\tilde{e}$, with no restrictions on the colour of the last edge of p) or type (i) (in all other cases), hence $b_{\Gamma'} \in \mathrm{Gr}_G$. For the same reasons as in part (a), we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$. □

Theorem 3.6. *Let G be a graph with linear resolution and let $[e_1, \ldots, e_m]$ be an ordering of the edges of G such that $[M_{e_1}, \ldots, M_{e_m}]$ is a linear quotient ordering for $I(G)$ (such an ordering exists by Remark 2.7). Let τ be the lexicographic order on $K[y_1, \ldots, y_m]$ such that $y_1 \prec_\tau y_2 \prec_\tau \ldots \prec_\tau y_m$. Then the reduced Gröbner basis of I_G with respect to τ is doubly squarefree.*

Proof. By Proposition 2.8, the linear quotient property is equivalent to asking that each subgraph $\{e_1, \ldots, e_i\}$ is 1-step linear, that is to say gap-free by Remark 2.11. Let Γ be a primitive walk such that at least one of the two monomials of b_Γ is not squarefree. This implies that Γ is primitive of type (iii). Hence, by Lemma 3.1, we know that the leading edge e of Γ lies into a bow-tie $\{C_1, p, C_2\}$. Let $G_{\preceq e}$ be the subgraph of G obtained by considering all the edges e' such that $e' \preceq_\tau e$. This means that $G_{\preceq e} = \{e_1, e_2, \ldots, e_s = e\}$; hence, $G_{\preceq e}$ is gap-free. Using the notation of Lemma 3.5, we have to consider two different cases.

- If $e \in p$, consider the edges $\{v_2, v_3\}$ and $\{v'_2, v'_3\}$. Since $G_{\preceq e}$ is gap-free, there exists an edge $\tilde{e} \in E(G)$ which links the edges we are considering and is such that $\tilde{e} \prec_\tau e$. By applying Lemma 3.5.(a), we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.
- If $e = \{v_i, v_{i+1}\}$, consider the edge $\{v'_2, v'_3\}$. Reasoning as before, we discover the existence of an edge $\tilde{e} \in E(G)$ linking these two edges and having the property that $\tilde{e} \prec_\tau e$: hence, by applying Lemma 3.5.(b), we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.

This ends the proof. □

As a corollary we recover a result by Ohsugi and Hibi [17] about complete multipartite graphs:

Corollary 3.7 ([17])**.** *If G is a complete multipartite graph, then there exists a doubly squarefree Gröbner basis of I_G.*

Proof. The complementary graph of a complete multipartite graph is a disjoint union of cliques and hence is chordal. Applying Theorem 3.6 yields the thesis. □

Remark 3.8. In Theorem 3.6 we actually proved that $\mathrm{RGB}_\tau(I_G)$ does not contain any binomials corresponding to primitive walks of type (iii). This means in particular that $\mathrm{RGB}_\tau(I_G)$ consists entirely of circuits (and hence I_G is generated by circuits, as one could have already noticed applying Theorem 2.6 in Ohsugi and Hibi's article [21]).

Theorem 3.9. *Let G be a gap-free graph and order its edges the following way: $\{v_{i_1}, v_{i_2}\} \preceq \{v_{j_1}, v_{j_2}\}$ if and only if $v_{i_1}v_{i_2} \preceq_\sigma v_{j_1}v_{j_2}$, where σ is an arbitrary graded reverse lexicographic order on $K[v_1, \dots, v_n]$. Rename the edges so that $e_1 \succ e_2 \succ \dots \succ e_m$. Let τ be the lexicographic order on $K[y_1, \dots, y_m]$ such that $y_1 \succ_\tau y_2 \succ_\tau \dots \succ_\tau y_m$. Then $in_\tau(I_G)$ is generated by squarefree elements.*

Proof. Let Γ be a primitive walk of G such that $in_\tau(b_\Gamma)$ is not squarefree and let $e = \{u_1, u_2\}$ be the leading edge of Γ with respect to τ. Then, since Γ has to be of type (iii), by Lemma 3.1 there exists a bow-tie $\{C_1, p, C_2\}$ containing e. We will use the notation of Lemma 3.5 to denote the edges of this bow-tie.

- If $e \in C_1$, then no edges of C_2 have vertices in common with e. In the following we will say that a vertex $v \in V(|\Gamma|)$ satisfies condition $(<)$ if

  $$v \prec_\sigma u_1, v \prec_\sigma u_2.$$

  Note that, by definition of σ and τ, if an edge $\{w_1, w_2\} \in E(|\Gamma|)$ shares no vertices with e, then at least one of w_1 and w_2 must satisfy condition $(<)$. Since no edges of C_2 share vertices with e, any pair of consecutive vertices in C_2 must include a vertex satisfying condition $(<)$: since C_2 is odd, by pigeonhole principle we get that there exists an edge e' of C_2 whose vertices both satisfy condition $(<)$.

  Since G is gap-free, there exists $\tilde{e} \in E(G)$ linking e and e': moreover, since both vertices of e' satisfy condition $(<)$, one has that $\tilde{e} \prec_\tau e$. If $\tilde{e} \neq \{v_1, v_1'\}$ then, by Lemma 3.5.(b), we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$. If $\tilde{e} = \{v_1, v_1'\}$, then $v_1 \in e$ and we have to consider two different cases.

  - If p is made of an even number of edges, then $\Gamma' := \{C_1, p, -\tilde{e}\}$ is a primitive walk of type (ii) such that $in_\tau(b_{\Gamma'})$ divides $in_\tau(b_\Gamma)$. Hence $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.
  - If p is made of an odd number of edges, then consider $\Gamma' := \{C_1, \tilde{e}, C_2, -\tilde{e}\}$. By Proposition 2.3, $b_{\Gamma'}$ is a circuit and hence Γ' is a primitive walk. Since $in_\tau(b_{\Gamma'})$ is squarefree and divides $in_\tau(b_\Gamma)$, we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.

- If $e \in p$, we have to discuss two different situations.

  If p is made of more than one edge, then at least one of the cycles C_1 and C_2 has no vertices in common with e (let it be C_1 without loss of generality). Then, applying the same pigeonhole reasoning used in the previous case, we discover the existence of an edge e' of C_1 whose vertices both satisfy condition

(<). Since G is gap-free, there exists $\tilde{e}$ linking e' and $\{v_2', v_3'\}$. Since $\tilde{e} \prec_\tau e$ by construction, applying Lemma 3.5.(a) we get that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$.

The last case standing is the one where $p = \{\{v_1, v_1'\}\} = \{e\}$. Let $\hat{C}_1 := \{v_2, v_3, \ldots, v_{2k+1}\}$, $\hat{C}_2 := \{v_2', v_3', \ldots, v_{2\ell+1}'\}$. If there exist two consecutive vertices belonging to either $\hat{C}_1$ or $\hat{C}_2$ and satisfying condition (<), then we can apply Lemma 3.5.(a) to infer that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$. Suppose otherwise. Then condition (<) is satisfied alternately: to be more precise, we have that the vertices of $\hat{C}_1$ (or $\hat{C}_2$) satisfying condition (<) are either the ones with odd index or the ones with even index. We can suppose without loss of generality that $v_3, v_5, \ldots, v_{2k+1}, v_3', v_5', \ldots, v_{2\ell+1}'$ are the vertices in $\hat{C}_1 \cup \hat{C}_2$ satisfying condition (<). Consider the edges $\{v_2, v_3\}$ and $\{v_2', v_3'\}$. Since G is gap-free, these edges are surely linked by some edge $\tilde{e}$: if one of v_3 and v_3' belongs to $\tilde{e}$ we have that $\tilde{e} \prec_\tau e$ and hence, by Lemma 3.5.(a), we can conclude that $b_\Gamma \notin \mathrm{RGB}_\tau(I_G)$. What happens if $\tilde{e} = \{v_2, v_2'\}$? If $\tilde{e} \prec_\tau e$ we are done for the same reason as before. Suppose $\tilde{e} \succ_\tau e$. Then, by definition of τ, at least one of v_1 and v_1' (call it w) must be such that $w \prec_\sigma v_2$ and $w \prec_\sigma v_2'$. Since e is the leading edge of $|\Gamma|$, though, one has that $e \succ_\tau \{v_1, v_2\}$ and $e \succ_\tau \{v_1', v_2'\}$, hence $v_1' \succ_\sigma v_2$ and $v_1 \succ_\sigma v_2'$. This gives us a contradiction. □

Remark 3.10. The proof of Theorem 3.9 shows also that $\mathrm{RGB}_\tau(I_G)$ consists of circuits and hence (as we already knew by [21, Theorem 2.6]) I_G is generated by circuits. To see this, replace the hypothesis "Γ primitive walk such that $in_\tau(b_\Gamma)$ is not squarefree" with "Γ primitive walk of type (iii)" and note that the only primitive walks of type (iii) that may appear in $\mathrm{RGB}_\tau(I_G)$ are bow-ties (more precisely, just those with a connecting path of length one). Since binomials associated with bow-ties are circuits by Proposition 2.3, we are done.

Remark 3.11. In general, the construction appearing in Theorem 3.9 does not necessarily yield a doubly squarefree reduced Gröbner basis of I_G. For instance, consider the gap-free graph G with 6 vertices and the following edges:

$$e_1 = \{1,2\},\ e_2 = \{1,3\},\ e_3 = \{2,3\},\ e_4 = \{1,4\},\ e_5 = \{3,4\},$$
$$e_6 = \{1,5\},\ e_7 = \{4,5\},\ e_8 = \{2,6\},\ e_9 = \{3,6\},\ e_{10} = \{5,6\}.$$

Note that the edges are ordered from the biggest to the smallest in a reverse lexicographic way according to the vertex order $1 > 2 > 3 > 4 > 5 > 6$ (in the sense explained in the claim of Theorem 3.9). Let τ be the lexicographic order on $K[y_1, \ldots, y_{10}]$ such that $y_1 \succ_\tau y_2 \succ_\tau \ldots \succ_\tau y_{10}$. Then CoCoA computations yield

$$\mathrm{RGB}_\tau(I_G) = \{y_1y_{10} - y_6y_8,\ y_1y_5 - y_3y_4,\ y_1y_9 - y_2y_8,\ y_5y_{10} - y_7y_9,\ y_2y_7 - y_5y_6,$$
$$y_2y_{10} - y_6y_9,\ y_3y_4y_{10} - y_5y_6y_8,\ y_2y_5y_8 - y_3y_4y_9,\ \boxed{y_3y_4y_7y_9 - y_5^2y_6y_8}\}.$$

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