

POLY SLICE MONOGENIC FUNCTIONS, CAUCHY FORMULAS AND THE PS -FUNCTIONAL CALCULUS

DANIEL ALPAY, FABRIZIO COLOMBO, KAMAL DIKI, AND IRENE SABADINI

ABSTRACT. Since 2006 the theory of slice hyperholomorphic functions and the related spectral theory on the S -spectrum have had a very fast development. This new spectral theory based on the S -spectrum has applications, for example, in the formulation of quaternionic quantum mechanics, in Schur analysis and in fractional diffusion problems. In this paper we introduce and study the theory of poly slice monogenic functions, also proving some Cauchy type integral formulas. Then we introduce the associated functional calculus, called PS -functional calculus, which is the polyanalytic version of the S -functional calculus and which is based on the notion of S -spectrum. We study some different formulations of the calculus and we prove some of its properties, among which the product rules.

AMS Classification 30G30, 47A10, 47A60.

Keywords: Poly slice monogenic function, Cauchy formulas, PS -resolvent operators, modified S -resolvent operators, spectral theory on the S -spectrum.

CONTENTS

| | |
|--|----|
| 1. Introduction | 1 |
| 2. Preliminary material | 3 |
| 3. Poly slice monogenic functions | 7 |
| 4. Cauchy formulas and product of poly slice monogenic functions | 13 |
| 4.1. Cauchy formulas with kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$ | 13 |
| 4.2. Cauchy formulas with kernels $\Pi_\ell S_L^{-1}$ and $\Pi_\ell S_R^{-1}$ | 16 |
| 4.3. Poly \otimes_L -product and \otimes_R -product | 19 |
| 5. Formulations of the PS -functional calculus via the HS -resolvent operators | 20 |
| 6. Formulations of the PS -functional calculus via the modified S -resolvent operators | 29 |
| 7. Equivalence of the two definitions of the PS -functional calculus and the product rules | 33 |
| References | 38 |

1. INTRODUCTION

The theory of polyanalytic functions is an interesting topic in complex analysis. It extends the concept of holomorphic functions to nullsolutions of higher order powers of the Cauchy-Riemann operator. Precisely, n -analytic or polyanalytic functions are nullsolutions of the n -power of the Cauchy-Riemann operator. They were introduced in 1908 by Kolossov see [43] to study elasticity problems. This stream of research was then continued by his student Muskhelishvili and led to the book [47]. A rather complete introduction to polyanalytic functions is in [14, 15]. In more recent times this class of functions was studied by various authors and with no pretense of completeness we mention the works of Abreu, Agranovsky, Begehr, Feichtinger, Vasilevski [1, 2, 3, 16, 50] and the references therein. Some famous Hilbert spaces of holomorphic functions that were extended to the setting of polyanalytic functions are the Bergman and Fock spaces, see for example [5, 6, 14] and the references therein.

Polyanalytic functions are important not only from the theoretical point of view, but also in the theory of signals since they allow to encode n independent analytic functions into a single polyanalytic one using a special decomposition. This idea is similar to the problem of multiplexing signals. This is related to the

construction of the polyanalytic Segal-Bargmann transform mapping $L^2(\mathbb{R})$ onto the poly-Fock space, see [5].

In quantum mechanics polyanalytic functions are relevant for the study of the Landau levels associated to Schrödinger operators, see [5, 13]. They were used also in [4] to study sampling and interpolation problems on Fock spaces using time frequency analysis techniques such as short-time Fourier transform (STFT) or Gabor transforms. This allows to extend Bargmann theory to the polyanalytic setting using Gabor analysis.

The theory of slice hyperholomorphic functions started its full development from the beginning of this century [27, 38]. It has had a quite fast developments due to several authors and the main results, regarding the quaternionic setting, are contained in the books [10, 24, 36, 37] and the references therein, while for the Clifford algebra setting we refer the interested reader to the book [30] and its bibliography. Nowadays the function theory has expanded in several directions but it is in operators theory where it has found its most profound applications and several monographs have been published in the last decade [9, 10, 22, 23, 30]. The slice monogenic functions were introduced in [25, 26, 27, 28, 29] also in collaboration with D. C. Struppa, and in this paper we generalize this class of functions to the poly analytic setting.

In order to state our results we need to explain the context in which we work and to highlight the importance of this branch of operator theory which is called quaternionic and Clifford operator theory. First of all we point out that the appropriate definition of the quaternionic spectrum for a quaternionic linear operator has been open problem at least since the paper [19] of G. Birkhoff, J. von Neumann, in 1936 on the logic of quantum mechanics, where the authors showed that quantum mechanics can be formulated also on quaternions. Moreover, consider a generalization of the gradient operators such as

$$T = ia(x_1, x_2, x_3)\partial_{x_1} + jb(x_1, x_2, x_3)\partial_{x_2} + kc(x_1, x_2, x_3)\partial_{x_3}$$

where a, b and c are given real valued functions of the variables $(x_1, x_2, x_3) \in \mathbb{R}^3$, and i, j, k are the imaginary units of the quaternions. It is very interesting to observe that the spectral theory for vector operators like the gradient operator ∇ , or its generalizations such as the operator T defined above, has been unclear since long time, even before 1936.

Regarding the quaternionic spectral theorem we observe that some attempts have been done after the paper of G. Birkhoff, J. von Neumann, but all the approaches suffered of the lack of an appropriate notion of quaternionic spectrum. The turning point came in 2006 when it was introduced the S -spectrum and the S -functional calculus which are a cornerstone of quaternionic and Clifford operator theory. The S -spectrum was identified by purely hypercomplex analysis techniques and not on physical considerations, as it is widely explained in the introduction of [23].

The spectral theorem for quaternionic normal operators based on the S -spectrum was finally proved in 2015 by Alpay, Colombo and Kimsey and published in 2016, see [7]. This theorem is the most important tool for the formulation of quaternionic quantum mechanics and more recently there have been several efforts to study the perturbations of quaternionic normal operators in [20], moreover, the theory of quaternionic spectral operators has been developed in [33]. The theory of characteristic operator function has started its development in this setting not too long ago and the main advances can be found in the book [9].

There are several applications of the spectral theory on the S -spectrum to fractional diffusion and fractional evolution problems because it is possible to define the fractional powers of vector operators so that we can generate fractional Fourier laws, see [22]. With this strategy we are able to write the fractional heat equation modifying just the Fourier's law and preserving the conservation of energy law.

Among the most developed areas in the slice hyperholomorphic setting there is the theory of slice hyperholomorphic reproducing kernel Hilbert spaces and more in general quaternionic Schur analysis has been largely investigated in the last decade. The material is spread over several papers, but the interested reader can find several results in the book [10] and in the references therein.

The importance of hypercomplex analysis in operators theory is not limited just to the slice hypercomplex setting. In fact, using the classical theory of monogenic functions, see [34, 35, 40], A. McIntosh and his collaborators [42, 44] developed the monogenic functional calculus, which is based on the notion of monogenic spectrum. This monogenic calculus contains as particular cases the Weyl functional calculus

and the Taylor functional calculus for commuting operators, see the book [45]. In harmonic analysis, the monogenic functional calculus plays a crucial role as shown in the book [48] and the references therein. Moreover, the monogenic function theory has applications in boundary value problems [41], and in Clifford wavelets, singular integrals, and Hardy spaces as one can see in the book [46].

To complete this short introduction on the hypercomplex spectral theories we recall that there is a link between the slice hyperholomorphic functions and the monogenic functions via the Fueter-Sce-Qian extension theorem and that there is a link between the two spectral theories. This link is the so-called F -functional calculus that generates a version of the monogenic functional calculus using the notion of S -spectrum, see [32]. We recall that also the Radon transform is a bridge between the monogenic and generalized slice monogenic functions, see [21].

This introductory part explains how the results of this paper have to be seen in the framework of hyperholomorphic function theories and the associated spectral theories.

In fact, here we extend the slice monogenic function theory and its S -functional calculus to the poly slice monogenic setting, namely to the set of (suitable) functions in the kernel of the M th-power of the Cauchy-Riemann operator. The quaternionic counterpart of the function theory started with the recent works [11, 12], while the corresponding functional calculus is introduced in this paper for the first time. In fact, in this paper we begin a systematic study of the function theory, also proving the Cauchy formulas. These formulas can be written using different Cauchy kernels which extend the one in the complex case. However, the noncommutative context requires suitable techniques in order to prove the results. Furthermore, the components of the kernels in their poly slice monogenic decomposition have different behavior at infinity.

Then, we define the so-called PS -functional calculus. It is the poly slice monogenic version of the S -functional calculus and it coincides with it when the order is 1. This calculus is based on the S -spectrum, see Definition 5.2, and it applies to $(n + 1)$ -tuples of noncommuting operators (T_0, T_1, \dots, T_n) written as the paravector operator $T = T_0 + T_1 e_1 + \dots + T_n e_n$, where e_1, \dots, e_n are the units of the Clifford algebra \mathbb{R}_n . The quaternionic case is obviously a particular case. We prove several results and a crucial tool is given by suitable modified S -resolvent operators for which we could also prove the resolvent equations. For some results, like the product rules we assumed commutativity of the components of the operators.

The contents of the paper are organized as follows. In Section 2 we recall some preliminary results on the theory of slice monogenic functions. In Section 3 we develop the theory of poly slice monogenic functions and we show some properties. In particular, we prove a slice monogenic integral representation of poly slice monogenic functions that will be used for a representation of the PS -functional calculus. In Section 4 we prove the Cauchy formulas and we define a product of poly slice monogenic functions. In Section 5 we give the formulations of the PS -functional calculus via the PS -resolvent operators and the poly slice monogenic Cauchy formula. In Section 6 we define and study the formulations of the PS -functional calculus via some suitably modified S -resolvent operators. In Section 7 we show the equivalence of the two definitions of the PS -functional calculus and we prove the product rules.

2. PRELIMINARY MATERIAL

In this section we recall some preliminary material useful to extend the theory of polyanalytic functions to the slice monogenic setting. The classical polyanalytic functions are those functions $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ of class $\mathcal{C}^M(\Omega)$, for $M \in \mathbb{N}$, such that

$$\overline{\partial}_i^M f(z) = 0, \quad \text{for all } z = u + iv \in \Omega, \quad i = \sqrt{-1}$$

where

$$\overline{\partial}_i^M = \frac{1}{2^M} (\partial_u + i \partial_v)^M. \tag{1}$$

A Cauchy-type formula for polyanalytic functions appeared for the first time in Théodoresco's doctoral thesis [49] and recalled in the paper [15]:

Theorem 2.1 ([15], Theorem 1.3). *If a function f is polyanalytic of order M in a closed domain G bounded by a rectifiable closed contour Γ , then the value of f at any point z of the domain G is expressed, using values*

of the function itself and its formal derivatives at points t of the boundary Γ , by the formula

$$f(z) = \frac{1}{2\pi i} \sum_{\ell=0}^{M-1} \int_{\Gamma} \frac{1}{\ell!(t-z)} (\bar{z}-\bar{t})^{\ell} \frac{\partial^{\ell} f}{\partial \bar{t}^{\ell}} dt. \quad (2)$$

The formula contains a finite sum in which appear the kernels

$$\pi_{\ell}(z, t) = \frac{1}{\ell!(t-z)} (\bar{z}-\bar{t})^{\ell}, \quad \ell = 0, \dots, M-1.$$

Another polyanalytic Cauchy formula is given by

$$f(z) = \int_{\partial\Omega} \sum_{\ell=0}^{M-1} (-2)^{\ell} L_{\ell}(w-z) dw \bar{\partial}_i^{\ell} f(w),$$

where $\partial\Omega$ is the boundary of the smooth bounded domain Ω in \mathbb{C} , the infinitesimal arc length is denoted by dw , and we have set

$$L_{\ell}(z) := \frac{1}{2\pi i} \frac{(\operatorname{Re}(z))^{\ell}}{z^{\ell}}, \quad \ell = 0, \dots, M-1.$$

We observe that when we define $\bar{\partial}_i^M$ in (1) without the prefactor $1/2^M$, then in the Cauchy formula the coefficients $(-2)^{\ell}$ have to be replaced by $(-1)^{\ell}$.

In the quaternionic setting or, more in general, in the Clifford algebra setting, one can extend the notion of holomorphic functions by considering functions in the kernel of a generalized Cauchy-Riemann operator, thus obtaining the so-called regular or monogenic functions, or of its n -power, thus obtaining poly-regular functions or poly-monogenic functions, see [17, 18].

We now recall the main definitions on Clifford algebras and the main facts on slice monogenic functions that are necessary to introduce and develop the theory of poly slice monogenic functions. Let \mathbb{R}_n be the real Clifford algebra over n imaginary units e_1, \dots, e_n satisfying the relations $e_{\ell}e_m + e_me_{\ell} = 0$, $\ell \neq m$, $e_{\ell}^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = \{\ell_1 \dots \ell_r\} \in \mathcal{P}\{1, 2, \dots, n\}$, $\ell_1 < \dots < \ell_r$ is a multi-index and $e_A = e_{\ell_1}e_{\ell_2} \dots e_{\ell_r}$, $e_{\emptyset} = 1$. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^n x_{\ell}e_{\ell} \in \mathbb{R}_n$ called paravector and the real part x_0 of x will also be denoted by $\operatorname{Re}(x)$. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^n x_{\ell}e_{\ell}$. We denote by \mathbb{S} the sphere

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \dots + e_nx_n \mid x_1^2 + \dots + x_n^2 = 1\};$$

for $\mathbf{j} \in \mathbb{S}$ we obviously have $\mathbf{j}^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ we put $\mathbf{j}x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$. The set

$$[x] := \{y \in \mathbb{R}^{n+1} : y = x_0 + \mathbf{j}|\underline{x}|, \mathbf{j} \in \mathbb{S}\}$$

is an $(n-1)$ -dimensional sphere in \mathbb{R}^{n+1} . The vector space $\mathbb{R} + \mathbf{j}\mathbb{R}$ passing through 1 and $\mathbf{j} \in \mathbb{S}$ will be denoted by $\mathbb{C}_{\mathbf{j}}$ and an element belonging to $\mathbb{C}_{\mathbf{j}}$ will be indicated by $u + \mathbf{j}v$, for $u, v \in \mathbb{R}$. With an abuse of notation we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ can be interpreted as a function of the paravector x .

In this paper we use the definition of poly slice monogenic functions that is the generalization of slice monogenic functions in the spirit of the Fueter-Sce-Qian mapping theorem, see [32]. This definition is the most appropriate for operator theory and the reason is widely explained in several papers and in the books [22, 23]. The corresponding function theory in real alternative algebras is developed in [39]. The same definition will be used for poly slice monogenic functions in the next section.

Definition 2.2. Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if $[x] \in U$ for every $x \in U$.

Definition 2.3 (Slice monogenic functions). Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \rightarrow \mathbb{R}_n$ is called a left slice function, if it is of the form

$$f(x) = f_0(u, v) + \mathbf{j}f_1(u, v) \quad \text{for } x = u + \mathbf{j}v \in U$$

with the two functions $f_0, f_1 : \mathcal{U} \rightarrow \mathbb{R}_n$ that satisfy the compatibility conditions

$$f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v). \quad (3)$$

If in addition f_0 and f_1 are \mathcal{C}^1 and satisfy the Cauchy-Riemann equations

$$\begin{aligned} \partial_u f_0(u, v) - \partial_v f_1(u, v) &= 0 \\ \partial_v f_0(u, v) + \partial_u f_1(u, v) &= 0 \end{aligned} \quad (4)$$

then f is called left slice monogenic. A function $f : U \rightarrow \mathbb{R}_n$ is called a right slice function if it is of the form

$$f(x) = f_0(u, v) + f_1(u, v)j \quad \text{for } x = u + jv \in U$$

with the two functions $f_0, f_1 : \mathcal{U} \rightarrow \mathbb{R}_n$ that satisfy (3). If f_0 and f_1 are \mathcal{C}^1 and satisfy the Cauchy-Riemann equations (4) then f is called right slice monogenic.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic.

We denote the sets of left, right and intrinsic slice monogenic functions on U by $\mathcal{SM}_L(U)$, $\mathcal{SM}_R(U)$ and $\mathcal{N}(U)$, respectively.

Remark 2.4. The set $\mathcal{N}(U)$ is contained in both $\mathcal{SM}_L(U)$ and $\mathcal{SM}_R(U)$.

Definition 2.5. We define the notion of j -derivative by means of the operator:

$$\partial_j := \frac{1}{2} (\partial_u - j\partial_v).$$

For consistency, we will denote by

$$\bar{\partial}_j = \frac{1}{2} (\partial_u + j\partial_v)$$

the Cauchy- Riemann operator associated with the complex plane \mathbb{C}_j , for $j \in \mathbb{S}$.

Using the notations we have just introduced, the condition of left slice monogenicity will be expressed, in short, by $\bar{\partial}_j f = 0$. Right slice monogenicity will be expressed, with an abuse of notation, by $f \bar{\partial}_j = 0$.

Definition 2.6. Let U be an open set in \mathbb{R}^{n+1} and let $f : U \rightarrow \mathbb{R}_n$ be a slice monogenic function. Its slice derivative $\partial_S(f)$ is defined as

$$\partial_S(f) = \begin{cases} \partial_j(f)(x) & x = u + jv, v \neq 0 \\ \partial_u f(u) & u \in \mathbb{R}. \end{cases} \quad (5)$$

Lemma 2.7 (Splitting Lemma). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $f \in \mathcal{SM}_L(U)$. For every $j = j_1 \in \mathbb{S}$ let j_2, \dots, j_n be a completion to a basis of \mathbb{R}_n satisfying the defining relations $j_r j_s + j_s j_r = -2\delta_{rs}$. Then there exist 2^{n-1} holomorphic functions $\mathcal{F}_A : U \cap \mathbb{C}_j \rightarrow \mathbb{C}_j$ such that for every $z = u + jv$ we have*

$$f_j(z) = \sum_{|A|=0}^{n-1} \mathcal{F}_A(z) j_A, \quad j_A = j_{i_1} \dots j_{i_s},$$

where $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $j_\emptyset = 1$. When $f \in \mathcal{SM}_R(U)$, then the splitting lemma becomes

$$f_j(z) = \sum_{|A|=0}^{n-1} j_A \mathcal{F}_A(z), \quad j_A = j_{i_1} \dots j_{i_s}.$$

The following formula is an immediate consequence of the definition of slice functions, see [39].

Theorem 2.8 (The Structure Formula or Representation Formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set.*

(I) *Let $f \in \mathcal{SM}_L(U)$. Then, for any vector $x = u + j_x v \in U$, the following formula holds:*

$$f(x) = \frac{1}{2} [1 - jj_x] f(u + jv) + \frac{1}{2} [1 + jj_x] f(u - jv), \quad \text{for all } u + jv \in U, \quad j \in \mathbb{S}. \quad (6)$$

(II) *Let $f \in \mathcal{SM}_R(U)$. Then, for any vector $x = u + j_x v \in U$, the following formula holds:*

$$f(x) = \frac{1}{2} f(u + jv) [1 - jj_x] + \frac{1}{2} f(u - jv) [1 + jj_x], \quad \text{for all } u + jv \in U, \quad j \in \mathbb{S}. \quad (7)$$

Remark 2.9. Using the representation formula we can write the slice monogenic Cauchy kernels in terms of the complex Cauchy kernel. For example for $S_L^{-1}(s, x)$ we have

$$S_L^{-1}(s, x) = \frac{1}{2} [1 - \mathbf{j}_x] \frac{1}{s - z} + \frac{1}{2} [1 + \mathbf{j}_x] \frac{1}{s - \bar{z}}$$

where we set $x = u + \mathbf{j}_x v$, $z = u + \mathbf{j}v$, $s = s_0 + \mathbf{j}s_1$, and where \mathbf{j} is the imaginary unit of the complex plane \mathbb{C}_j .

For slice monogenic functions we have two equivalent ways to write the Cauchy kernels.

Proposition 2.10. *If $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$, then*

$$-(x^2 - 2x\operatorname{Re}(s) + |s|^2)^{-1}(x - \bar{s}) = (s - \bar{q})(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1} \quad (8)$$

and

$$(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1}(s - \bar{x}) = -(x - \bar{s})(x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1}. \quad (9)$$

So we can give the following definition to distinguish the two representations of the Cauchy kernels.

Definition 2.11. Let $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$.

- We say that $S_L^{-1}(s, x)$ is written in the form I if

$$S_L^{-1}(s, x) := -(x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S_L^{-1}(s, x)$ is written in the form II if

$$S_L^{-1}(s, x) := (s - \bar{x})(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1}.$$

- We say that $S_R^{-1}(s, x)$ is written in the form I if

$$S_R^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1}.$$

- We say that $S_R^{-1}(s, x)$ is written in the form II if

$$S_R^{-1}(s, x) := (s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1}(s - \bar{x}).$$

Lemma 2.12. *Let $x, s \in \mathbb{R}^{n+1}$ with $s \notin [x]$. The left slice monogenic Cauchy kernel $S_L^{-1}(s, x)$ is left slice monogenic in x and right slice monogenic in s . The right slice monogenic Cauchy kernel $S_R^{-1}(s, x)$ is left slice monogenic in s and right slice monogenic in x .*

Definition 2.13 (Slice Cauchy domain). An axially symmetric open set $U \subset \mathbb{R}^{n+1}$ is called a slice Cauchy domain, if $U \cap \mathbb{C}_j$ is a Cauchy domain in \mathbb{C}_j for any $\mathbf{j} \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if, for any $\mathbf{j} \in \mathbb{S}$, the boundary $\partial(U \cap \mathbb{C}_j)$ of $U \cap \mathbb{C}_j$ is the union a finite number of non-intersecting piecewise continuously differentiable Jordan curves in \mathbb{C}_j .

Theorem 2.14 (Cauchy formulas). *Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain, let $\mathbf{j} \in \mathbb{S}$ and set $ds_j = ds(-\mathbf{j})$. If f is a (left) slice monogenic function on a set that contains \bar{U} then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j f(s), \quad \text{for any } x \in U. \quad (10)$$

If f is a right slice monogenic function on a set that contains \bar{U} , then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s) ds_j S_R^{-1}(s, x), \quad \text{for any } x \in U. \quad (11)$$

These integrals depend neither on U nor on the imaginary unit $\mathbf{j} \in \mathbb{S}$.

Theorem 2.15 (Cauchy formulas on unbounded slice Cauchy domains). *Let $U \subset \mathbb{R}^{n+1}$ be an unbounded slice Cauchy domain and let $\mathbf{j} \in \mathbb{S}$. If $f \in \mathcal{SM}_L(\bar{U})$ and $f(\infty) := \lim_{|x| \rightarrow \infty} f(x)$ exists and is finite, then*

$$f(x) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j f(s) \quad \text{for any } x \in U.$$

If $f \in \mathcal{SM}_R(\bar{U})$ and $f(\infty) := \lim_{|x| \rightarrow \infty} f(x)$ exists and is finite, then

$$f(x) = f(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s) ds_j S_R^{-1}(s, x) \quad \text{for any } x \in U.$$

3. POLY SLICE MONOGENIC FUNCTIONS

We can now give the definition of poly slice monogenic functions and develop the corresponding function theory, which will be used to define the *PS*-functional calculus. This definition is the monogenic counterpart of Definition 3.17 in [11].

Definition 3.1 (Poly slice monogenic functions). Let $M \in \mathbb{N}$ and denote by $\mathcal{C}^M(U)$ the set of continuously differentiable functions with all their derivatives up to order M on an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. We let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $F : U \rightarrow \mathbb{R}_n$ is called a left slice function, if it is of the form

$$F(x) = F_0(u, v) + jF_1(u, v) \quad \text{for } x = u + jv \in U$$

with the two functions $F_0, F_1 : \mathcal{U} \rightarrow \mathbb{R}_n$ that satisfy the compatibility condition

$$F_0(u, -v) = F_0(u, v), \quad F_1(u, -v) = -F_1(u, v). \quad (12)$$

If in addition F_0 and F_1 are in $\mathcal{C}^M(U)$ and satisfy the poly Cauchy-Riemann equations of order $M \in \mathbb{N}$

$$\frac{1}{2^M}(\partial_u + j\partial_v)^M(F_0(u, v) + jF_1(u, v)) = 0, \quad \text{for all } j \in \mathbb{S} \quad (13)$$

then F is called left poly slice monogenic function of order $M \in \mathbb{N}$. A function $F : U \rightarrow \mathbb{R}_n$ is called a right slice function if it is of the form

$$F(x) = F_0(u, v) + F_1(u, v)j \quad \text{for } x = u + jv \in U$$

with two functions $F_0, F_1 : \mathcal{U} \rightarrow \mathbb{R}^{n+1}$ that satisfy (12). If in addition F_0 and F_1 are in $\mathcal{C}^M(U)$ and satisfy the poly Cauchy-Riemann equations of order $M \in \mathbb{N}$

$$(F_0(u, v) + F_1(u, v)j)\frac{1}{2^M}(\partial_u + j\partial_v)^M = 0, \quad \text{for all } j \in \mathbb{S} \quad (14)$$

then F is called right poly slice monogenic of order $M \in \mathbb{N}$. We will denote by $\mathcal{PS}_L^M(U)$ and $\mathcal{PS}_R^M(U)$ the set of left and right poly slice monogenic functions on the open set U , respectively.

If F is a left (or right) slice function such that F_0 and F_1 are real-valued, then F is called intrinsic. By $\mathcal{PN}^M(U)$ we denote the set of poly intrinsic slice monogenic functions.

Remark 3.2. For the definition of slice monogenic functions we required that the pair (f_0, f_1) satisfies the Cauchy-Riemann system. In the case of poly slice monogenic functions we follow the same idea by observing that (for functions F of class $\mathcal{C}^M(U)$)

$$\frac{1}{2^M}(\partial_u + j\partial_v)^M F = \sum_{k=0}^M \binom{M}{k} (\partial_u)^{M-k} (\partial_v)^k j^k F = \sum_{k=0}^M \binom{M}{k} D_{M-k,k} j^k F$$

where $D_{M-k,k} = (\partial_u)^{M-k} (\partial_v)^k$. Due to the arbitrariness of j , the condition that F is left poly slice monogenic translates into a system of two differential equations of order M for the pair (F_0, F_1) of \mathbb{R}_n -valued functions (which reduces to the Cauchy-Riemann system when $M = 1$):

$$\begin{aligned} & \left(\sum_{k=0(\bmod 4)}^M \binom{M}{k} D_{M-k,k} - \sum_{k=2(\bmod 4)}^M \binom{M}{k} D_{M-k,k} \right) F_0(u, v) \\ & + \left(- \sum_{k=1(\bmod 4)}^M \binom{M}{k} D_{M-k,k} + \sum_{k=3(\bmod 4)}^M \binom{M}{k} D_{M-k,k} \right) F_1(u, v) = 0, \\ & \left(\sum_{k=1(\bmod 4)}^M \binom{M}{k} D_{M-k,k} - \sum_{k=3(\bmod 4)}^M \binom{M}{k} D_{M-k,k} \right) F_0(u, v) \\ & + \left(\sum_{k=0(\bmod 4)}^M \binom{M}{k} D_{M-k,k} - \sum_{k=2(\bmod 4)}^M \binom{M}{k} D_{M-k,k} \right) F_1(u, v) = 0. \end{aligned} \quad (15)$$

With obvious meaning of the symbols, we have

$$\begin{aligned} D_1 F_0(u, v) - D_2 F_1(u, v) &= 0 \\ D_2 F_0(u, v) + D_1 F_1(u, v) &= 0 \end{aligned}$$

However, since this system is rather complicated to write, we prefer to use the above Definition 3.1.

We point out that the definition of poly slice monogenic functions extends to functions with values in a Clifford Banach module in a very natural way. As in the quaternionic case, one can give the very useful definition of (strong) slice monogenicity, see [10] for vector-valued functions.

Definition 3.3 (Poly slice monogenic functions vector-valued). Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}.$$

A function $f : U \rightarrow X_L$ with values in a left Clifford Banach module X_L is called a left slice function, if is of the form

$$F(x) = F_0(u, v) + jF_1(u, v) \quad \text{for } x = u + jv \in U$$

with two functions $F_0, F_1 : \mathcal{U} \rightarrow X_L$ that satisfy the compatibility condition (12). If in addition F_0 and F_1 are in $\mathcal{C}^M(U)$ and satisfy the poly Cauchy-Riemann equations (13), then F is called (*strongly*) *left poly slice monogenic*.

A function $f : U \rightarrow X_R$ with values in a right Clifford-Banach module is called a right slice function if it is of the form

$$F(x) = F_0(u, v) + F_1(u, v)j \quad \text{for } x = u + jv \in U$$

with two functions $F_0, F_1 : \mathcal{U} \rightarrow X_R$ that satisfy the compatibility condition (12). If in addition F_0 and F_1 are in $\mathcal{C}^M(U)$ and satisfy the poly Cauchy-Riemann equations (14), then f is called (*strongly*) *right poly slice monogenic*.

We have the following proposition.

Proposition 3.4. Let $M \in \mathbb{N}$ and let $F \in \mathcal{PS}_L^M(U)$ (resp. $\mathcal{PS}_R^M(U)$ or $\mathcal{PN}^M(U)$). Then $F \in \mathcal{PS}_L^{M+M'}(U)$ (resp. $\mathcal{PS}_R^{M+M'}(U)$ or $\mathcal{PN}^{M+M'}(U)$) for all $M' \in \mathbb{N}$.

Proof. It is a direct consequences of the definition. □

Remark 3.5. The restriction of a function F to the complex plane \mathbb{C}_j is denoted by F_j .

Theorem 3.6 (Poly splitting lemma and poly decomposition). Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set.

(Ia) (Poly splitting lemma). Let $F \in \mathcal{PS}_L^M(U)$. Then, for every $j = j_1 \in \mathbb{S}$ let j_2, \dots, j_n be a completion to a basis of \mathbb{R}_n satisfying the defining relations $j_r j_s + j_s j_r = -2\delta_{rs}$. Then, there exist 2^{n-1} polyanalytic functions $\mathcal{F}_A : U \cap \mathbb{C}_j \rightarrow \mathbb{C}_j$ such that, for every $z = u + jv$, we have

$$F_j(z) = \sum_{|A|=0}^{n-1} \mathcal{F}_A(z) j_A, \quad j_A = j_{i_1} \dots j_{i_s},$$

where $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $j_\emptyset = 1$.

(Ib) (Poly decomposition). The function $F \in \mathcal{PS}_L^M(U)$ if and only if there exist uniquely determined functions $f_0, \dots, f_{M-1} \in \mathcal{SM}_L(U)$ such that, for $f_{M-1} \neq 0$, we have the following decomposition

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(x), \quad \forall x \in U. \tag{16}$$

(IIa) Let $F \in \mathcal{PS}_R^M(U)$. Then, for every $j_1, \dots, j_n \in \mathbb{S}$ as above there exist 2^{n-1} polyanalytic functions $\mathcal{F}_A : U \cap \mathbb{C}_j \rightarrow \mathbb{C}_j$ such that, for every $z = u + jv$, we have

$$F_j(z) = \sum_{|A|=0}^{n-1} j_A \mathcal{F}_A(z), \quad j_A = j_{i_1} \dots j_{i_s},$$

where $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $j_\emptyset = 1$.

(IIb) The function $F \in \mathcal{PS}_R^M(U)$ if and only if there exist uniquely determined functions $f_0, \dots, f_{M-1} \in \mathcal{SM}_R(U)$ such that, for $f_{M-1} \neq 0$, we have the following decomposition

$$F(x) = \sum_{k=0}^{M-1} f_k(x) \bar{x}^k, \quad \forall x \in U. \quad (17)$$

Proof. We consider the case (Ia) and (Ib) since the other ones follows with similar arguments.

Step (Ia). For every $j = j_1 \in \mathbb{S}$ let j_2, \dots, j_n be a completion to a basis of \mathbb{R}_n satisfying the defining relations $j_r j_s + j_s j_r = -2\delta_{rs}$. Then there exist 2^{n-1} functions $\mathcal{F}_A : U \cap \mathbb{C}_j \rightarrow \mathbb{C}_j$ such that for every $z = u + jv$

$$F_j(z) = \sum_{|A|=0}^{n-1} \mathcal{F}_A(z) j_A, \quad j_A = j_{i_1} \dots j_{i_s},$$

where $F_j(z)$ is the restriction to $U \cap \mathbb{C}_j$, and $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $j_\emptyset = 1$.

But since F is a left poly slice monogenic function, the functions $\mathcal{F}_A(z)$ are complex polyanalytic functions of order M .

Step (Ib). We show that the function $F(x)$ defined in (16) is poly slice monogenic. First of all, we note that F is a slice function since it is the sum of slice functions. Then, by the definition of poly slice monogenicity and the product rule, we have that

$$\frac{1}{2^M} (\partial_u + j\partial_v)^M F(u + jv) = \sum_{k=0}^{M-1} \frac{1}{2^M} (\partial_u + j\partial_v)^M \left((u - jv)^k f_k(u + jv) \right) = 0.$$

Viceversa, let us assume that $F(x) = F(u + jv) = F_0(u, v) + jF_1(u, v)$ is a left poly slice monogenic function of order M , i.e. the pair (F_0, F_1) is an even-odd pair satisfying (15). By fixing a basis e_1, \dots, e_n of \mathbb{R}_n we can write $F_i = \sum_{|A|=0}^n F_{i,A} e_A$, $i = 0, 1$ where the functions $F_{i,A}$ are real-valued, and by the linear independence of the basis elements e_A , the system (15) can be rewritten in terms of the 2^n real components $F_{i,A}$ of F_i , $i = 0, 1$. Thus if $F(u + jv) = F_0(u, v) + jF_1(u, v)$ is left poly slice monogenic, each function $F_A = F_{0,A} + jF_{1,A}$ is a slice function and polyanalytic. By the classical result applied to the \mathbb{C}_j -valued function F_A , we have

$$F_A(u + jv) = \sum_{k=0}^{M-1} (u - jv)^k f_{k,A}(u + jv)$$

where the functions $f_{k,A}$ are \mathbb{C}_j -valued, satisfy the Cauchy-Riemann system and are even-odd in the variables u, v by direct verification. We thus obtain

$$\begin{aligned} F(u + jv) &= \sum_{|A|=0}^n F_A(u + jv) e_A = \sum_{|A|=0}^n \sum_{k=0}^{M-1} (u - jv)^k f_{k,A}(u + jv) e_A \\ &= \sum_{k=0}^{M-1} (u - jv)^k f_k(u + jv) \end{aligned}$$

where we set $f_k = \sum_{|A|=0}^n f_{k,A} e_A$. The functions f_k are evidently left slice monogenic and this concludes the proof. \square

Definition 3.7. The functions $f_0, \dots, f_{M-1} \in \mathcal{SM}_L(U)$ that appear in Theorem 3.6 in the poly decomposition (Ib)

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(x), \quad \forall x \in U,$$

are called the (left monogenic) components of the left poly slice monogenic function F . Similarly, we will call the functions $f_0, \dots, f_{M-1} \in \mathcal{SM}_R(U)$ in (IIb) the components of the right poly slice monogenic function F .

Assumption 3.8. In the sequel, when dealing with a function F in $\mathcal{PS}_L^M(U)$ or $\mathcal{PS}_R^M(U)$, we always assume that F is written via its poly decomposition.

Proposition 3.9. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain and let $M \in \mathbb{N}$. Then we have:

- (I) $F \in \mathcal{PS}_L^M(U)$ is intrinsic if and only if all its left slice monogenic components are also intrinsic.
- (II) $F \in \mathcal{PS}_R^M(U)$ is intrinsic if and only if all its right slice monogenic components are also intrinsic.

Proof. We prove (I). We use the poly decomposition in Theorem 3.6 to write

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(x),$$

where f_k are the left slice monogenic components for all $k = 0, \dots, M-1$. First, we observe that if all the functions f_k are intrinsic, then F will preserve any complex plane $U \cap \mathbb{C}_j$ that is to say that $F(U \cap \mathbb{C}_j) \subseteq \mathbb{C}_j$, so it is intrinsic.

For the converse, let us assume that $F = F_0 + jF_1$ is intrinsic so that F_0, F_1 are real valued. This means that in the decomposition $F = \sum_{|A|=0}^n (F_{0,A} + jF_{1,A})e_A$ there is only the term corresponding to $e_\emptyset = 1$ and expanding in the form $F(u + jv) = \sum_{k=0}^{M-1} (u - jv)^k f_k(u + jv)$ the functions f_k are \mathbb{C}_j -valued, see the proof of Theorem 3.6 (Ib), and so they take \mathbb{C}_j into itself and so they are intrinsic. \square

Remark 3.10. (I) Observe that since $f_k(x)$ is slice monogenic, for $\ell, k \in \mathbb{N}$, for $k \geq \ell$, we have that

$$\bar{\partial}_j^\ell (\bar{x}^k f_k(x)) = \bar{\partial}_j^\ell (\bar{x}^k) f_k(x) = k(k-1)(k-2) \dots (k-\ell+1) \bar{x}^{k-\ell} f_k(x) = \frac{k!}{(k-\ell)!} \bar{x}^{k-\ell} f_k(x),$$

and $\bar{\partial}_j^\ell (\bar{x}^k f_k(x)) = 0$, for $k < \ell$.

(II) Similarly, for $\ell, k \in \mathbb{N}$, for $k \geq \ell$, we have

$$(f_k(x) \bar{x}^k) \bar{\partial}_j^\ell = f_k(x) (\bar{x}^k) \bar{\partial}_j^\ell = k(k-1)(k-2) \dots (k-\ell+1) f_k(x) \bar{x}^{k-\ell} = \frac{k!}{(k-\ell)!} f_k(x) \bar{x}^{k-\ell},$$

and $(f_k(x) \bar{x}^k) \bar{\partial}_j^\ell = 0$, for $k < \ell$.

The next result was already proved for quaternions, we revise its proof in more details here for the sake of completeness.

Proposition 3.11. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and $M \in \mathbb{N}$. Then, denoting by Fg or gF the pointwise product, we have the following statements.

- (Ia) Let $F \in \mathcal{PN}^M(U)$ and $g \in \mathcal{SM}_L(U)$. Then, Fg belongs to $\mathcal{PM}_L^M(U)$.
- (Ib) Let $F \in \mathcal{PM}_L^M(U)$ and $g \in \mathcal{N}(U)$. Then, gF belongs to $\mathcal{PM}_L^M(U)$.
- (IIa) Let $F \in \mathcal{PN}^M(U)$ and $g \in \mathcal{SM}_R(U)$. Then, gF belongs to $\mathcal{PM}_R^M(U)$.
- (IIb) Let $F \in \mathcal{PM}_R^M(U)$ and $g \in \mathcal{N}(U)$. Then, Fg belongs to $\mathcal{PM}_R^M(U)$.
- (III) Let $F \in \mathcal{PN}^M(U)$ and $g \in \mathcal{N}(U)$. Then, $gF = Fg$ belongs to $\mathcal{PN}^M(U)$.

Proof. We prove just step (Ia). In much the same way we can prove the other points. So we assume that $F \in \mathcal{PN}^M(U)$, $g \in \mathcal{SM}_L(U)$ and $j \in \mathbb{S}$ and we set $x = u + jv$. We will prove that

$$\bar{\partial}_j^M (Fg)(u + jv) = 0. \tag{18}$$

Indeed, first we note that since F is intrinsic, we have $F(U \cap \mathbb{C}_j) \subset \mathbb{C}_j$. In particular, by the Leibniz rule, the equality

$$\bar{\partial}_j (Fg)(u + jv) = F \bar{\partial}_j (g)(u + jv) + \bar{\partial}_j (F) g(u + jv)$$

holds. We note that since F is poly slice monogenic of order M and g is slice monogenic we have $\bar{\partial}_j (g) = 0$ and $\bar{\partial}_j (F) \neq 0$. Thus, we obtain

$$\bar{\partial}_j (Fg)(u + jv) = \bar{\partial}_j (F) g(u + jv).$$

Then, since F is intrinsic we can use the Leibniz rule M times and get

$$\bar{\partial}_j^M (Fg)(u + jv) = \bar{\partial}_j^{M-1} (F) \bar{\partial}_j (g)(u + jv) + \bar{\partial}_j^M (F) g(u + jv).$$

Therefore, it follows that the formula (18) holds since $F \in \mathcal{PN}^M(U)$ and $g \in \mathcal{SM}_L(U)$. Hence, the pointwise product Fg is poly slice monogenic of order M on U . \square

Using the explicit poly decomposition of a given poly slice monogenic function F of order M , see Assumption 3.8, we can give it an integral representation using the Cauchy formula for the slice monogenic components $\{f_k(x)\}_{k=0,\dots,M-1}$.

Corollary 3.12 (Slice monogenic integral representation of poly slice monogenic functions). *Let $U \subseteq \mathbb{R}^{n+1}$ be a slice Cauchy domain. Let $\mathbf{j} \in \mathbb{S}$ and set $ds_{\mathbf{j}} = ds(-\mathbf{j})$.*

(I) *If F is a (left) poly slice monogenic function of order M on a set that contains \overline{U} then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{k=0}^{M-1} \overline{x}^k S_L^{-1}(s, x) ds_{\mathbf{j}} f_k(s), \quad \text{for any } x \in U. \quad (19)$$

(II) *If F is a right slice monogenic function on a set that contains \overline{U} , then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{k=0}^{M-1} f_k(s) ds_{\mathbf{j}} S_R^{-1}(s, x) \overline{x}^k, \quad \text{for any } x \in U. \quad (20)$$

The integrals in (19) and (20) depend neither on U nor on the imaginary unit $\mathbf{j} \in \mathbb{S}$.

Proof. We consider the case (19) since (20) can be obtained with similar considerations. So (19) follows by replacing the Cauchy formula for the slice monogenic components $\{f_k(x)\}_{k=0,\dots,M-1}$ given by

$$f_k(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} S_L^{-1}(s, x) ds_{\mathbf{j}} f_k(s), \quad \text{for any } x \in U$$

into the poly decomposition formula $F(x) = \sum_{k=0}^{M-1} \overline{x}^k f_k(x)$. \square

Similarly we can prove the case of unbounded domains.

Theorem 3.13 (Slice monogenic integral representation of poly slice monogenic functions on unbounded slice Cauchy domains). *Let $U \subset \mathbb{R}^{n+1}$ be an unbounded slice Cauchy domain and let $\mathbf{j} \in \mathbb{S}$. Let $F \in \mathcal{PS}_L(\overline{U})$ with poly slice decomposition*

$$F(x) = \sum_{k=0}^{M-1} \overline{x}^k f_k(x)$$

and the components $f_k \in \mathcal{SM}_L(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then, we have

$$F(x) = \sum_{k=0}^{M-1} \overline{x}^k f_k(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{k=0}^{M-1} \overline{x}^k S_L^{-1}(s, x) ds_{\mathbf{j}} f_k(s) \quad \text{for any } x \in U.$$

Let $F \in \mathcal{PS}_R(\overline{U})$ with poly slice decomposition

$$F(x) = \sum_{k=0}^{M-1} f_k(x) \overline{x}^k$$

and the components $f_k \in \mathcal{SM}_R(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then, we have

$$F(x) = \sum_{k=0}^{M-1} f_k(\infty) \overline{x}^k + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{k=0}^{M-1} f(s) ds_{\mathbf{j}} S_R^{-1}(s, x) \overline{x}^k \quad \text{for any } x \in U.$$

Remark 3.14. Observe that we have two possibilities to write the Cauchy kernels, using the form I or the form II.

The following result is a direct consequence of the form of the slice function, i.e.,

$$F(x) = F_0(u, v) + jF_1(u, v) \quad \text{for } x = u + jv \in U$$

with the two functions $F_0, F_1 : \mathcal{U} \rightarrow \mathbb{R}_n$ that satisfy the compatibility condition (see the definition of poly slice monogenic functions). We recall it for the reader's convenience.

Theorem 3.15 (Poly Structure (or Poly Representation) Formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain.*

(I) *Let $F \in \mathcal{PS}_L^M(U)$. Then, for any vector $x = u + j_x v \in U$, the following formula holds:*

$$F(x) = \frac{1}{2} [1 - j_x] F(u + jv) + \frac{1}{2} [1 + j_x] F(u - jv), \quad \text{for all } u + jv \in U, \quad j \in \mathbb{S}. \quad (21)$$

(II) *Let $F \in \mathcal{PS}_R^M(U)$. Then, for any vector $x = u + j_x v \in U$, the following formula holds:*

$$F(x) = \frac{1}{2} F(u + jv) [1 - j_x] + \frac{1}{2} F(u - jv) [1 + j_x], \quad \text{for all } u + jv \in U, \quad j \in \mathbb{S}. \quad (22)$$

In the following we use the notation $\overline{\mathbb{R}^{n+1}} = \mathbb{R}^{n+1} \cup \{\infty\}$. Then, we first recall the Runge's theorem for slice monogenic functions which was proved in [31]. We refer to this paper also for the terminology.

Theorem 3.16. *Let K be an axially symmetric compact set in \mathbb{R}^{n+1} , and let A be a set having a point in each connected component of $\overline{\mathbb{R}^{n+1}} \setminus K$. For any axially symmetric open set $U \supset K$, for every $f \in \mathcal{SM}_L(U)$ and for every $\varepsilon > 0$ there exists a rational function r whose poles are spheres in A such that*

$$|f(x) - r(x)| < \varepsilon,$$

for all $x \in K$. Similar considerations hold for every $f \in \mathcal{SM}_R(U)$.

Now, we are going to use Theorem 3.16 to prove the poly slice monogenic counterpart of the Runge's theorem. First, we give the definition of poly slice monogenic rational function.

Definition 3.17. We say that a left poly slice monogenic function $R(x)$ is rational if the left slice monogenic components $r_k(x)$ in the decomposition

$$R(x) = \sum_{k=0}^{M-1} \bar{x}^k r_k(x), \quad \forall x \in U \quad (23)$$

are rational. We say that a right poly slice monogenic function $R(x)$ is rational if the right slice monogenic components $r_k(x)$ in the decomposition

$$R(x) = \sum_{k=0}^{M-1} r_k(x) \bar{x}^k, \quad \forall x \in U \quad (24)$$

are rational.

Theorem 3.18 (Poly slice Runge's theorem). *Let $M \in \mathbb{N}$. Let K be an axially symmetric compact set in \mathbb{R}^{n+1} , and let A be a set having a point in each connected component of $\overline{\mathbb{R}^{n+1}} \setminus K$. For any axially symmetric domain $U \supset K$, for every $F \in \mathcal{PS}_L^M(U)$ and for every $\varepsilon > 0$ there exists a left poly rational function R whose poles are spheres in A such that*

$$|F(x) - R(x)| < \varepsilon,$$

for all $x \in K$. The same approximation holds for $F \in \mathcal{PS}_R^M(U)$. In the case F is intrinsic the rational functions R are also intrinsic.

Proof. We show just the case when $F \in \mathcal{PS}_L^M(U)$, the other statements follows in much the same way. We note that K is a compact, in particular K is bounded so that it is contained in some ball centered at the origin, i.e., $K \subset \overline{B(0, \rho)}$, for some $\rho > 0$. Furthermore, since F is poly slice monogenic of order M on the domain U , we know the validity of the poly decomposition (Ib) in Theorem 3.6 with all $\{f_k\}_{0 \leq k \leq M-1}$ that are slice monogenic functions on U and $f_{M-1} \neq 0$. In particular, since K is contained in U , we also have

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(x), \quad \forall x \in K. \quad (25)$$

Then, we can apply the Runge's theorem 3.16 on each slice monogenic function f_k . Thus, for all $k = 0, \dots, M-1$, we know that, for every $\varepsilon > 0$, there exist a rational function r_k whose poles are spheres in A such that

$$|f_k(x) - r_k(x)| < \varepsilon \left(\frac{1}{\sum_{j=0}^{M-1} \rho^j} \right), \quad (26)$$

for every $x \in K$. Now, we consider the poly slice monogenic rational function given by

$$R(x) = \sum_{k=0}^{M-1} \bar{x}^k r_k(x), \quad \forall x \in U.$$

Let $\varepsilon > 0$. Then, we have

$$|F(x) - R(x)| = \left| \sum_{k=0}^{M-1} \bar{x}^k (f_k(x) - r_k(x)) \right| \leq \sum_{k=0}^{M-1} |x|^k |f_k(x) - r_k(x)|,$$

for every $x \in K$. Therefore, we apply the inequality (26) to each slice monogenic component and get

$$|F(x) - R(x)| < \frac{\varepsilon}{\sum_{k=0}^{M-1} \rho^k} \left(\sum_{k=0}^{M-1} |x|^k \right), \quad (27)$$

for every $x \in K$. However, we know that $K \subset \overline{B(0, \rho)}$. So we have $|x| \leq \rho$, for any $x \in K$. In particular, this shows that

$$\sum_{k=0}^{M-1} |x|^k \leq \sum_{k=0}^{M-1} \rho^k,$$

for every $x \in K$. Therefore, inserting this fact in the inequality (27) we obtain that $|F(x) - R(x)| < \varepsilon$, for every $x \in K$. \square

4. CAUCHY FORMULAS AND PRODUCT OF POLY SLICE MONOGENIC FUNCTIONS

In this section we develop the Cauchy formulas for poly slice monogenic functions. These will be used in the next section to define the PS -functional calculus for noncommuting operators.

Lemma 4.1 (Poly Cauchy integral formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be a slice Cauchy domain, assume that $F \in \mathcal{PS}_L^M(U)$ and $G \in \mathcal{PS}_R^M(U)$ for some $M \in \mathbb{N}$. Then we have*

$$\int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-1)^\ell G(s) \bar{\partial}_j^{M-\ell-1} ds_j \bar{\partial}_j^\ell F(s) = 0, \quad (28)$$

where $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$ for $j \in \mathbb{S}$.

Proof. By writing $G = \sum_{|A|=0}^n e_A G_A$, $F = \sum_{|A|=0}^n F_A e_A$, see the proof of Theorem 3.6, (Ib), the result follows from the analogue theorem for polyanalytic functions of a complex variable. \square

4.1. Cauchy formulas with kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$. The poly slice monogenic Cauchy kernel are described in the next result:

Definition 4.2 (Cauchy kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$). Let $\ell \in \mathbb{N}$. We define the left poly slice monogenic Cauchy kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$ of order $\ell + 1$ by

$$\begin{aligned} P_\ell S_L^{-1}(s, x) &:= \frac{(\operatorname{Re}(s-x))^\ell}{\ell!} S_L^{-1}(s, x) \\ &= -\frac{(\operatorname{Re}(s-x))^\ell}{\ell!} (x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1} (x - \bar{s}), \quad s \notin [x] \end{aligned}$$

and right poly slice monogenic Cauchy kernels of order ℓ is defined by

$$\begin{aligned} P_\ell S_R^{-1}(s, x) &:= \frac{(\operatorname{Re}(s-x))^\ell}{\ell!} S_{R,\ell}^{-1}(s, x) \\ &= -\frac{(\operatorname{Re}(s-x))^\ell}{\ell!} (x-\bar{s})(x^2-2\operatorname{Re}(s)x+|s|^2)^{-1}, \quad s \notin [x]. \end{aligned}$$

Remark 4.3. Observe that, in both cases, we have used the slice monogenic Cauchy kernels written in form I so that the PS -functional calculus will work for paravector operators with noncommuting components. For the function theory it is also possible to use slice monogenic Cauchy kernels written in form II, but in this case to define the PS -functional calculus we are limited to commuting operators.

Lemma 4.4. *Let $\ell \in \mathbb{N}$ and let $x, s \in \mathbb{R}^{n+1}$ with $s \notin [x]$. The kernel $P_\ell S_L^{-1}(s, x)$ is left poly slice monogenic in x and right poly slice monogenic in s of order $\ell+1$. The kernel $P_\ell S_R^{-1}(s, x)$ is left poly slice monogenic in s and right poly slice monogenic in x of order $\ell+1$.*

Proof. For $\ell \in \mathbb{N}$. For all $s, x \in \mathbb{R}^{n+1}$, we set

$$P_\ell(s, x) = \frac{(\operatorname{Re}(s-x))^\ell}{\ell!}.$$

Then, $P_\ell(s, x)$ is a slice function, intrinsic and poly slice monogenic of order $\ell+1$ with respect to the variables x and s . By a direct computation, for $s \notin [x]$, the product $P_\ell(s, x)S_L^{-1}(s, x)$ is left poly slice monogenic in x and right poly slice monogenic in s of order $\ell+1$. Using similar arguments we treat $P_\ell(s, x)S_R^{-1}(s, x)$. \square

Theorem 4.5 (The poly slice Cauchy formulas with kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$). *Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain. For $\mathbf{j} \in \mathbb{S}$ set $ds_{\mathbf{j}} = ds(-\mathbf{j})$ and $\bar{\partial}_{\mathbf{j}} := \frac{1}{2}(\partial_u + \mathbf{j}\partial_v)$.*

(I) *If F is a left poly slice monogenic function on a set that contains \bar{U} then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_{\mathbf{j}} \bar{\partial}_{\mathbf{j}}^\ell F(s), \quad \text{for any } x \in U. \quad (29)$$

(II) *If F is a right poly slice monogenic function on a set that contains \bar{U} , then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{\ell=0}^{M-1} (-2)^\ell F(s) \bar{\partial}_{\mathbf{j}}^\ell ds_{\mathbf{j}} P_\ell S_R^{-1}(s, x), \quad \text{for any } x \in U. \quad (30)$$

The integrals (29) and (30) depend neither on U nor on the imaginary unit $\mathbf{j} \in \mathbb{S}$.

Proof. Consider (I). It is a consequence of the Lemma 4.1 and of Theorem 3.15. We now use the poly splitting lemma (Ia). Let $F \in \mathcal{PS}_L^M(U)$, and for every $\mathbf{j} = \mathbf{j}_1 \in \mathbb{S}$ let $\mathbf{j}_2, \dots, \mathbf{j}_n$ be a completion to a basis of \mathbb{R}_n satisfying the defining relations $\mathbf{j}_r \mathbf{j}_s + \mathbf{j}_s \mathbf{j}_r = -2\delta_{rs}$. Then there exist 2^{n-1} polyanalytic functions $\mathcal{F}_A : U \cap \mathbb{C}_{\mathbf{j}} \rightarrow \mathbb{C}_{\mathbf{j}}$ such that, for every $z = u + \mathbf{j}v$, we have

$$F_{\mathbf{j}}(z) = \sum_{|A|=0}^{n-1} \mathcal{F}_A(z) \mathbf{j}_A, \quad \mathbf{j}_A = \mathbf{j}_{i_1} \dots \mathbf{j}_{i_s},$$

where $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $\mathbf{j}_0 = 1$. So we can write

$$\mathcal{F}_A(z) = \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{2\pi \mathbf{j}} \frac{1}{(s-z)} \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} ds \bar{\partial}_{\mathbf{j}}^\ell \mathcal{F}_A(s), \quad z \in U \cap \mathbb{C}_{\mathbf{j}}$$

and also

$$F_{\mathbf{j}}(z) = \sum_{|A|=0}^{n-1} \mathcal{F}_A(z) \mathbf{j}_A = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{s-z} \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} ds_{\mathbf{j}} \bar{\partial}_{\mathbf{j}}^\ell \sum_{|A|=0}^{n-1} \mathcal{F}_A(s) \mathbf{j}_A, \quad z \in U \cap \mathbb{C}_{\mathbf{j}}.$$

Thus we can write $F_{\mathbf{j}}(z)$ and $F_{\mathbf{j}}(\bar{z})$ as

$$F_{\mathbf{j}}(z) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{s-z} \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} ds_{\mathbf{j}} \bar{\partial}_{\mathbf{j}}^\ell F(s), \quad z \in U \cap \mathbb{C}_{\mathbf{j}},$$

and

$$F_j(\bar{z}) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{s-\bar{z}} \frac{(\operatorname{Re}(s-\bar{z}))^\ell}{\ell!} ds_j \bar{\partial}_j^\ell F(s), \quad \bar{z} \in U \cap \mathbb{C}_j,$$

where we set $x = u + j_x v$, $z = u + jv$, $s = s_0 + js_1$, and where j is the imaginary unit of the complex plane \mathbb{C}_j on which we integrate. Now we use the poly monogenic representation formulas and the two Cauchy formulas for $F_j(z)$ and $F_j(\bar{z})$ on the complex plane \mathbb{C}_j . Specifically, we use

$$F(x) = \frac{1}{2} [1 - j_x j] F_j(z) + \frac{1}{2} [1 + j_x j] F_j(\bar{z})$$

so we have

$$\begin{aligned} F(x) &= \frac{1}{2} [1 - j_x j] \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{s-z} \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} ds_j \bar{\partial}_j^\ell F(s) \\ &\quad + \frac{1}{2} [1 + j_x j] \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{s-\bar{z}} \frac{(\operatorname{Re}(s-\bar{z}))^\ell}{\ell!} ds_j \bar{\partial}_j^\ell F(s), \end{aligned}$$

and also

$$\begin{aligned} F(x) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{2} [1 - j_x j] \frac{1}{s-z} \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} ds_j \bar{\partial}_j^\ell F(s) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{1}{2} [1 + j_x j] \frac{1}{s-\bar{z}} \frac{(\operatorname{Re}(s-\bar{z}))^\ell}{\ell!} ds_j \bar{\partial}_j^\ell F(s). \end{aligned}$$

Finally we get

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell \frac{(\operatorname{Re}(s-x))^\ell}{\ell!} S_L^{-1}(s, x) ds_j \bar{\partial}_j^\ell F(s)$$

where we have replaced the poly slice monogenic Cauchy kernel

$$\frac{(\operatorname{Re}(s-x))^\ell}{\ell!} S_L^{-1}(s, x) = \frac{1}{2} [1 - j_x j] \frac{(\operatorname{Re}(s-z))^\ell}{\ell!} \frac{1}{s-z} + \frac{1}{2} [1 + j_x j] \frac{(\operatorname{Re}(s-\bar{z}))^\ell}{\ell!} \frac{1}{s-\bar{z}}$$

written via the poly slice monogenic representation formula. \square

Definition 4.6. We say that a poly slice monogenic function is poly slice monogenic function at infinity if its slice monogenic components are slice monogenic at infinity.

Theorem 4.7 (Poly slice monogenic Cauchy formulas on unbounded slice Cauchy domains with kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$). *Let $U \subset \mathbb{R}^{n+1}$ be an unbounded slice Cauchy. For $j \in \mathbb{S}$ set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$.*

(I) *Let $F \in \mathcal{PS}_L(\bar{U})$ with poly slice decomposition*

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(x)$$

and the components $f_k \in \mathcal{SM}_L(\bar{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then, we have

$$F(x) = \sum_{k=0}^{M-1} \bar{x}^k f_k(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \bar{\partial}_j^\ell F(s) \quad \text{for any } x \in U.$$

(II) *Let $F \in \mathcal{PS}_R(\bar{U})$ with poly slice decomposition*

$$F(x) = \sum_{k=0}^{M-1} f_k(x) \bar{x}^k$$

and the components $f_k \in \mathcal{SM}_R(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then we have

$$F(x) = \sum_{k=0}^{M-1} f_k(\infty) \overline{x}^k + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell F(s) \overline{\partial}_j^\ell ds_j P_\ell S_R^{-1}(s, x) \quad \text{for any } x \in U.$$

Proof. Let us consider (I). For sufficiently large $r > 0$, the set $U_r := U \cap B_r(0)$ is a bounded slice Cauchy domain with $x \in U_r$ and $\mathbb{H} \setminus U_r \subset U$. By Theorem 4.5

$$\begin{aligned} F(x) &= \frac{1}{2\pi} \int_{\partial(U_r \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) \\ &\quad + \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s), \quad \text{for any } x \in U. \end{aligned}$$

The Cauchy theorem for poly slice monogenic functions implies that we can vary r without changing the value of the second integral. Letting r tend to infinity, we find that the monogenic components converge to $f_k(\infty)$ and we obtain the statement, since

$$\lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) = \sum_{k=0}^{M-1} \overline{x}^k f_k(\infty).$$

□

Remark 4.8. There is a more direct proof of the Cauchy formulas with the Cauchy kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$. In fact, the integrals

$$J_k(x) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{k-1} (-2)^\ell P_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s)$$

can be computed directly. As an example consider $k = 2$.

$$\begin{aligned} J_2(x) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j \left(\overline{s} f_1(s) + f_0(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} (-2)(\operatorname{Re}(s) - x_0) S_L^{-1}(s, x) ds_j f_1(s) \end{aligned}$$

using the relation $S_L^{-1}(s, x)s - xS_L^{-1}(s, x) = 1$ and the Cauchy theorem we get

$$J_2(x) = \frac{1}{2\pi} \overline{x} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j f_1(s) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j f_0(s)$$

so we obtain $J_2(x) = \overline{x} f_1(x) + f_0(x)$.

4.2. Cauchy formulas with kernels $\Pi_\ell S_L^{-1}$ and $\Pi_\ell S_R^{-1}$. As one may easily compute, the components of $P_\ell S_L^{-1}$ (and of $P_\ell S_R^{-1}$) in the poly slice monogenic decomposition do not have a finite limit for $|x| \rightarrow \infty$, for every fixed s and so they are not poly slice monogenic at infinity. Thus we introduce other kernels whose components are poly slice monogenic at infinity.

Definition 4.9 (The Cauchy kernels $\Pi_\ell S_L^{-1}$ and $\Pi_\ell S_R^{-1}$). Let $\ell \in \mathbb{N}$. We define the left poly slice monogenic Cauchy kernels of order $\ell + 1$ by

$$\begin{aligned} \Pi_\ell S_L^{-1}(s, x) &:= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \overline{x}^k S_L^{-1}(s, x) (-\overline{s})^{\ell-k} \\ &= -\frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \overline{x}^k (x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1} (x - \overline{s}) (-\overline{s})^{\ell-k}, \quad s \notin [x] \end{aligned}$$

and right poly slice monogenic Cauchy kernels of order $\ell + 1$ is defined by

$$\begin{aligned}\Pi_\ell S_R^{-1}(s, x) &:= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-\bar{s})^{\ell-k} S_R^{-1}(s, x) \bar{x}^k \\ &= -\frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-\bar{s})^{\ell-k} (x - \bar{s})(x^2 - 2\operatorname{Re}(s)x + |s|^2)^{-1} \bar{x}^k, \quad s \notin [x].\end{aligned}$$

Lemma 4.10. *Let $\ell \in \mathbb{N}$ and let $x, s \in \mathbb{R}^{n+1}$ with $s \notin [x]$. The kernel $\Pi_\ell S_L^{-1}(s, x)$ is left poly slice monogenic in x and right poly slice monogenic in s of order $\ell + 1$. The kernel $\Pi_\ell S_R^{-1}(s, x)$ is left poly slice monogenic in s and right poly slice monogenic in x of order $\ell + 1$.*

Proof. Consider the kernel $\Pi_\ell S_L^{-1}(s, x)$. The statement is a direct consequence of the definition of poly slice monogenicity and of the poly decomposition of poly slice monogenic functions. In fact, $\Pi_\ell S_L^{-1}(s, x)$ is of the form

$$\Pi_\ell S_L^{-1}(s, x) = \sum_{k=0}^{\ell} \psi_\ell(s, x) \bar{s}^{\ell-k}$$

where the components

$$\psi_\ell(s, x) := \frac{1}{\ell!} \binom{\ell}{k} \bar{x}^k S_L^{-1}(s, x) (-1)^{\ell-k}$$

are right poly slice monogenic function in the variable s for $s \notin [x]$. The order $\ell + 1$ is clear from the definition. To see that the kernel $\Pi_\ell S_L^{-1}(s, x)$ is left poly slice monogenic in x it is more convenient to write $S_L^{-1}(s, x)$ in the form II and reasoning in a similar way. The case of $\Pi_\ell S_R^{-1}(s, x)$ follows with similar considerations. \square

Lemma 4.11. *For $s \notin [x]$ the kernel $\Pi_\ell S_L^{-1}(s, x)$ is the unique left (resp. right) poly slice monogenic extension in x (resp. s) of the kernel $\pi_\ell(z, s) = \frac{1}{\ell!} (\bar{z} - \bar{s})^\ell (s - z)^{-1}$, $z, s \in \mathbb{C}_j$, for $z \neq s$. Analogously, for $s \notin [x]$, the kernel $\Pi_\ell S_R^{-1}(s, x)$ is the unique right (resp. left) poly slice monogenic extension in x (resp. s) of the kernel $\pi_\ell(z, s) = \frac{1}{\ell!} (\bar{z} - \bar{s})^\ell (s - z)^{-1}$, $z, s \in \mathbb{C}_j$, $z \neq s$.*

Proof. When $x = z \in \mathbb{C}_j$ it is evident that the restriction of $\Pi_\ell S_L^{-1}(s, x)$ to \mathbb{C}_j is $\pi_\ell(z, s)$. Since $\Pi_\ell S_L^{-1}(s, x) = \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \bar{x}^k S_L^{-1}(s, x) (-\bar{s})^{\ell-k}$ and its slice monogenic components are unique, the assertion follows. \square

Theorem 4.12 (Cauchy formulas with the kernels $\Pi_\ell S_L^{-1}$ and $\Pi_\ell S_R^{-1}$). *Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain. For $j \in \mathbb{S}$ set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$.*

(I) *If F is a left poly slice monogenic function on a set that contains \bar{U} then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, x) ds_j \bar{\partial}_j^\ell F(s), \quad \text{for any } x \in U. \quad (31)$$

(II) *If F is a right poly slice monogenic function on a set that contains \bar{U} , then*

$$F(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, x), \quad \text{for any } x \in U. \quad (32)$$

The integrals (29) and (30) depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. It follows in much the same way as the case of the poly slice Cauchy formulas with kernels $P_\ell S_L^{-1}$ and $P_\ell S_R^{-1}$ (see Theorem 4.5) using the poly splitting lemmas by taking $x, s \in \mathbb{C}_j$. Then the assertion follows by the poly representation formula, the polyanalytic Cauchy kernel π_ℓ used by Théodoresco see (2) and Lemma 4.11. \square

Theorem 4.13 (Poly slice monogenic Cauchy formulas on unbounded slice Cauchy domains). *Let $U \subset \mathbb{R}^{n+1}$ be an unbounded slice Cauchy. For $j \in \mathbb{S}$ set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$.*

(I) Let $F \in \mathcal{PS}_L(\overline{U})$ with poly slice decomposition

$$F(x) = \sum_{k=0}^{M-1} \overline{x}^k f_k(x)$$

and the components $f_k \in \mathcal{SM}_L(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then, we have

$$F(x) = \sum_{k=0}^{M-1} \overline{x}^k f_k(\infty) + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) \quad \text{for any } x \in U.$$

(II) Let $F \in \mathcal{PS}_R(\overline{U})$ with poly slice decomposition

$$F(x) = \sum_{k=0}^{M-1} f_k(x) \overline{x}^k$$

and the components $f_k \in \mathcal{SM}_R(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. Then we have

$$F(x) = \sum_{k=0}^{M-1} f_k(\infty) \overline{x}^k + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \overline{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, x) \quad \text{for any } x \in U.$$

Proof. Consider (I). Recall that we assume the components $f_k \in \mathcal{SM}_L(\overline{U})$ of $F \in \mathcal{PS}_L(\overline{U})$ are such that the limits $\lim_{|x| \rightarrow \infty} f_k(x) = f_k(\infty)$ exist and are finite for all $k = 0, \dots, M-1$. For sufficiently large $r > 0$, the set $U_r := U \cap B_r(0)$ is a bounded slice Cauchy domain with $x \in U_r$ and $\mathbb{R}^{n+1} \setminus U_r \subset U$. By 4.5

$$\begin{aligned} F(x) &= \frac{1}{2\pi} \int_{\partial(U_r \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s), \quad \text{for any } x \in U. \end{aligned}$$

The Cauchy theorem for poly slice monogenic functions implies that we can vary r without changing the value of the second integral. Letting r tend to infinity, we find that it equals $\sum_{k=0}^{M-1} \overline{x}^k f_k(\infty)$ and we obtain the statement. \square

Remark 4.14. Also for the Cauchy formulas with the kernels $\Pi_\ell S_L^{-1}$ and $\Pi_\ell S_R^{-1}$ the integrals

$$I_k(r) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{k-1} \Pi_\ell S_L^{-1}(s, x) ds_j \overline{\partial}_j^\ell F(s)$$

can be computed directly. As an example consider the case $k = 2$, so we have

$$\begin{aligned} I_2(x) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, x) ds_j \left(\overline{s} f_1(s) + f_0(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \left(S_L^{-1}(s, x)(-\overline{s}) + \overline{x} S_L^{-1}(s, x) \right) ds_j f_1(s) \end{aligned}$$

and so we get $I_2(x) = \overline{x} f_1(x) + f_0(x)$ using the Cauchy formula of slice monogenic functions.

4.3. Poly \otimes_L -product and \otimes_R -product. We conclude this section with the product of poly slice monogenic functions. We note that given two poly slice monogenic functions F and G of orders N and M , we can use the poly decomposition formulas to define a natural product. This product take out of the class but it will be useful for the product rule of the PS -functional calculus.

Definition 4.15 (Poly \otimes_L -product and \otimes_R -product and pointwise product). Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric open set and $M, N \geq 1$.

(I) Let $F \in \mathcal{PS}_L^N(U)$ and $G \in \mathcal{PS}_L^M(U)$ and let

$$F(x) = \sum_{k=0}^{N-1} \bar{x}^k f_k(x) \quad \text{and} \quad G(x) = \sum_{k=0}^{M-1} \bar{x}^k g_k(x), \quad \text{for all } x \in U,$$

be their poly decompositions for f_0, \dots, f_{N-1} and $g_0, \dots, g_{M-1} \in \mathcal{SM}_L(U)$. We defined the poly \otimes_L -product of F and G by:

$$(F \otimes_L G)(x) := \sum_{\ell=0}^{N+M-2} \bar{x}^\ell \left(\sum_{k+h=\ell} (f_k *_L g_h)(x) \right), \quad (33)$$

where $*_L$ is the $*$ -product of left slice monogenic functions.

(II) Let $F \in \mathcal{PS}_R^N(U)$ and $G \in \mathcal{PS}_R^M(U)$ and let

$$F(x) = \sum_{k=0}^{N-1} f_k(x) \bar{x}^k \quad \text{and} \quad G(x) = \sum_{k=0}^{M-1} g_k(x) \bar{x}^k, \quad \text{for all } x \in U,$$

be their poly decompositions for f_0, \dots, f_{N-1} and $g_0, \dots, g_{M-1} \in \mathcal{SM}_R(U)$. We defined the poly \otimes_R -product of F and G by:

$$(F \otimes_R G)(x) := \sum_{\ell=0}^{N+M-2} \left(\sum_{k+h=\ell} (f_k *_R g_h)(x) \right) \bar{x}^\ell, \quad (34)$$

where $*_R$ is the star-product of right slice monogenic functions.

(III) Let $F \in \mathcal{PN}_L^N(U)$ and $G \in \mathcal{PS}_L^M(U)$ and let

$$F(x) = \sum_{k=0}^{N-1} \bar{x}^k f_k(x) \quad \text{and} \quad G(x) = \sum_{k=0}^{M-1} \bar{x}^k g_k(x), \quad \text{for all } x \in U,$$

be their poly decompositions for $f_0, \dots, f_{N-1} \in \mathcal{N}(U)$ and $g_0, \dots, g_{M-1} \in \mathcal{SM}_L(U)$. The pointwise product of F and G is defined by:

$$(FG)(x) := \sum_{\ell=0}^{N+M-2} \bar{x}^\ell \left(\sum_{k+h=\ell} (f_k g_h)(x) \right). \quad (35)$$

Similarly we define the pointwise product for $F \in \mathcal{PS}_R^N(U)$ and $G \in \mathcal{PN}_R^M(U)$.

Remark 4.16. It is clear that the pointwise product of two poly slice monogenic functions does not preserve the poly slice monogenicity, but, as we will see, it does when we consider intrinsic functions.

Theorem 4.17. Let $U \in \mathbb{R}^{n+1}$ be an axially symmetric open set and $M, N \geq 1$.

(I) Let $F \in \mathcal{PS}_L^N(U)$ and $G \in \mathcal{PS}_L^M(U)$ and let

$$(F \otimes_L G)(x) := \sum_{\ell=0}^{N+M-2} \bar{x}^\ell \left(\sum_{k+h=\ell} (f_k *_L g_h)(x) \right), \quad (36)$$

the poly \otimes_L -product of F and G . Then $(F \otimes_L G) \in \mathcal{PS}_L^{M+N-1}(U)$.

(II) Let $F \in \mathcal{PS}_R^N(U)$ and $G \in \mathcal{PS}_R^M(U)$ and let

$$(F \otimes_R G)(x) := \sum_{\ell=0}^{N+M-2} \left(\sum_{k+h=\ell} (f_k *_R g_h)(x) \right) \bar{x}^\ell, \quad (37)$$

be the poly \otimes_R -product of F and G . Then $(F \otimes_R G) \in \mathcal{PS}_R^{M+N-1}(U)$.

Proof. Let us prove (I). With similar computations we get (II). Observe that, in (36), the function

$$h_\ell(x) := \sum_{k+h=\ell} (f_k *_L g_h)(x)$$

is slice monogenic in U for all ℓ , by definition, so the function

$$(F \circledast_L G)(x) := \sum_{\ell=0}^{N+M-2} \bar{x}^\ell h_\ell(x),$$

belongs to $\mathcal{PS}_R^{M+N-1}(U)$ thanks to the poly decomposition theorem. \square

As a consequence of the previous result we have the case of intrinsic functions which will be used for the product rule for the PS -functional calculus.

Corollary 4.18. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and $M, N \geq 1$.*

(I) *Let $F \in \mathcal{PN}_L^N(U)$ and $G \in \mathcal{PS}_L^M(U)$. Then we have*

$$(F \circledast_L G)(x) = F(x)G(x) = \sum_{\ell=0}^{N+M-2} \bar{x}^\ell \left(\sum_{k+h=\ell} f_k(x)g_h(x) \right), \quad (38)$$

and $FG \in \mathcal{PS}_L^{M+N-1}(U)$.

(II) *Let $F \in \mathcal{PS}_R^N(U)$ and $G \in \mathcal{PN}_R^M(U)$. Then we have*

$$(F \circledast_R G)(x) = F(x)G(x) = \sum_{\ell=0}^{N+M-2} \left(\sum_{k+h=\ell} f_k(x)g_h(x) \right) \bar{x}^\ell, \quad (39)$$

and $FG \in \mathcal{PS}_R^{M+N-1}(U)$.

Proof. It is a direct consequence of the product theorem for slice monogenic functions, i.e., it is the case when the slice monogenic $*$ -product becomes the pointwise product. \square

5. FORMULATIONS OF THE PS -FUNCTIONAL CALCULUS VIA THE HS -RESOLVENT OPERATORS

In the sequel, we will consider a Banach space V over \mathbb{R} with norm $\|\cdot\|$. It is possible to endow V with an operation of multiplication by elements of \mathbb{R}_n which gives a two-sided module over \mathbb{R}_n . A two-sided module V over \mathbb{R}_n is called a Banach module over \mathbb{R}_n , if there exists a constant $C \geq 1$ such that $\|va\| \leq C\|v\|\|a\|$ and $\|av\| \leq C\|a\|\|v\|$ for all $v \in V$ and $a \in \mathbb{R}_n$. By V_n we denote $V \otimes \mathbb{R}_n$ that turns out to be a two-sided Banach module over \mathbb{R}_n . An element in V_n is of the type $\sum_A v_A \otimes e_A$ (where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index). The multiplications of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined by $va = \sum_A v_A \otimes (e_{AA})$ and $av = \sum_A v_A \otimes (ae_A)$. For simplicity, we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Finally, we define $\|v\|_{V_n}^2 = \sum_A \|v_A\|_V^2$.

We denote by $\mathcal{B}(V)$ the space of bounded \mathbb{R} -homomorphisms of the Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$. Given $T_A \in \mathcal{B}(V)$, we can introduce the Clifford operator $T = \sum_A e_A T_A$ and its action on $v = \sum v_B e_B \in V_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The operator T is a right-module homomorphism which is a bounded linear map on V_n .

In the sequel, we will consider an important subclass of Clifford operators, the ones of the form $T = T_0 + \sum_{j=1}^n e_j T_j$ are called paravector operators, where $T_j \in \mathcal{B}(V)$ for $j = 0, 1, \dots, n$. The subset of paravector operators in $\mathcal{B}(V_n)$ will be denoted by $\mathcal{B}^{0,1}(V_n)$. For Clifford operators $T = \sum_A e_A T_A$ we define $\|T\|_{\mathcal{B}(V_n)} = \sum_A \|T_A\|_{\mathcal{B}(V)}$ and in particular when T is a paravector operator we have $\|T\|_{\mathcal{B}^{0,1}(V_n)} = \sum_j \|T_j\|_{\mathcal{B}(V)}$. Note that, in the sequel, we will omit the subscripts $\mathcal{B}^{0,1}(V_n)$ or $\mathcal{B}(V_n)$ in the norm of an operator. Note also that $\|TS\| \leq \|T\|\|S\|$, finally we denote by \mathcal{I} the identity operator. We recall this crucial result which is the heart of the spectral theory on the S -spectrum because it shows the notion of S -spectrum and of S -resolvent operators. The subset of $\mathcal{B}^{0,1}(V_n)$ that consists of those paravector operators $T = T_0 + \sum_{j=1}^n e_j T_j$ with commuting components T_0, \dots, T_n will be denoted by $\mathcal{BC}^{0,1}(V_n)$. Finally we will use the notation $\bar{T} = T_0 - \sum_{j=1}^n e_j T_j$.

We recall some facts on the S -functional calculus that we will use in the sequel.

Theorem 5.1. Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \mathbb{R}^{n+1}$ with $\|T\| < |s|$.

(i) The left S -resolvent series equals

$$\sum_{m=0}^{+\infty} T^m s^{-m-1} = -(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}).$$

(ii) The right S -resolvent series equals

$$\sum_{m=0}^{+\infty} s^{-m-1}T^m = -(T - \bar{s}\mathcal{I})(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I})^{-1}.$$

Definition 5.2. Let $T \in \mathcal{B}^{0,1}(V_n)$. For $s \in \mathbb{R}^{n+1}$, we set

$$\mathcal{Q}_s(T) := T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}.$$

We define the S -resolvent set $\rho_S(T)$ of T as

$$\rho_S(T) := \{s \in \mathbb{R}^{n+1} : \mathcal{Q}_s(T) \text{ is invertible in } \mathcal{B}(V_n)\}$$

and we define the S -spectrum $\sigma_S(T)$ of T as

$$\sigma_S(T) := \mathbb{R}^{n+1} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo S -resolvent operator of T at s .

Definition 5.3. Let $T \in \mathcal{B}^{0,1}(V_n)$. For $s \in \rho_S(T)$, we define the left S -resolvent operator as

$$S_L^{-1}(s, T) = -\mathcal{Q}_s(T)^{-1}(T - \bar{s}\mathcal{I}),$$

and the right S -resolvent operator as

$$S_R^{-1}(s, T) = -(T - \bar{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

Lemma 5.4. Let $T \in \mathcal{B}^{0,1}(V_n)$.

(I) The left S -resolvent operator $S_L^{-1}(s, T)$ is a $\mathcal{B}(V_n)$ -valued right-slice monogenic function of the variable s on $\rho_S(T)$.

(II) The right S -resolvent operator $S_R^{-1}(s, T)$ is a $\mathcal{B}(V_n)$ -valued left-slice monogenic function of the variable s on $\rho_S(T)$.

Definition 5.5. For $T \in \mathcal{B}^{0,1}(V_n)$, we denote by $\mathcal{SM}_L(\sigma_S(T))$, $\mathcal{SM}_R(\sigma_S(T))$ and $\mathcal{N}(\sigma_S(T))$, the set of all left, right and intrinsic slice monogenic functions with $\sigma_S(T) \subset U$, where U is a slice Cauchy domain such that $\bar{U} \subset \operatorname{dom}(f)$ and $\operatorname{dom}(f)$ is the domain of the function f .

Definition 5.6 (The S -functional calculus). Let $T \in \mathcal{B}^{0,1}(V_n)$ for $j \in \mathbb{S}$ set $ds_j = ds(-j)$. Then we have the formulations of the S -functional calculus. We define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j f(s), \quad \text{for all } f \in \mathcal{SM}_L(\sigma_S(T)), \quad (40)$$

and

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F(s) ds_j S_R^{-1}(s, T), \quad \text{for all } f \in \mathcal{SM}_R(\sigma_S(T)). \quad (41)$$

Remark 5.7. The definition of the S -functional calculus is well posed because the integrals (40) and (41) depend neither on U nor on the imaginary unit $j \in \mathbb{S}$. This is also independent of the fact that the components of the operator T commute or not among themselves.

The S -resolvent equation is useful to prove several properties of the S -functional calculus. So it is natural to ask if it is possible to obtain an analog of the classical resolvent equation

$$(\lambda I - A)^{-1}(\mu I - A)^{-1} = ((\lambda I - A)^{-1} - (\mu I - A)^{-1})(\mu - \lambda)^{-1}, \quad \lambda, \mu \in \mathbb{C} \setminus \sigma(A), \quad (42)$$

where A is a complex operator on a Banach space. The generalization to this non commutative setting, involves both the left and the right S -resolvent operators and the analogue of the term $((\lambda I - A)^{-1} - (\mu I - A)^{-1})(\mu - \lambda)^{-1}$, which is the difference of the resolvent operators $(\lambda I - A)^{-1} - (\mu I - A)^{-1}$ multiplied by the

Cauchy kernel $(\mu - \lambda)^{-1}$ for the S -functional calculus becomes the difference of the S -resolvent operators $S_R^{-1}(s, T) - S_L^{-1}(p, T)$ entangled in a suitable way with the slice monogenic Cauchy kernels. In fact we have:

Theorem 5.8 (The S -resolvent equation, see [8]). *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s, q \in \rho_S(T)$ with $q \notin [s]$. Set $\mathcal{Q}_s(q)^{-1} := (q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1}$. Then the equation*

$$S_R^{-1}(s, T)S_L^{-1}(q, T) = [(S_R^{-1}(s, T) - S_L^{-1}(q, T))q - \bar{s}(S_R^{-1}(s, T) - S_L^{-1}(q, T))] \mathcal{Q}_s(q)^{-1} \quad (43)$$

holds true. Equivalently, it can also be written as

$$S_R^{-1}(s, T)S_L^{-1}(q, T) = \mathcal{Q}_q(s)^{-1} [(S_L^{-1}(q, T) - S_R^{-1}(s, T))\bar{q} - s(S_L^{-1}(q, T) - S_R^{-1}(s, T))]. \quad (44)$$

The S -resolvent equation is a consequence of the left and the right S -resolvent equations:

Theorem 5.9. *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \rho_S(T)$. The left S -resolvent operator satisfies the left S -resolvent equation*

$$S_L^{-1}(s, T)s - TS_L^{-1}(s, T) = \mathcal{I} \quad (45)$$

and the right S -resolvent operator satisfies the right S -resolvent equation

$$sS_R^{-1}(s, T) - S_R^{-1}(s, T)T = \mathcal{I}. \quad (46)$$

We point out that the equations (45) and (46) cannot be considered the generalizations of the classical resolvent equation. Only the equations in Theorem 5.8 have the properties of the classical resolvent equation (42). The product rule is a consequence of the S -resolvent equation.

Theorem 5.10 (Product rule). *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SM}_L(\sigma_S(T))$ or let $f \in \mathcal{SM}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then*

$$(fg)(T) = f(T)g(T).$$

In this section we give the formulations of the poly slice monogenic version of the S -functional based on the poly slice Cauchy formulas. This calculus that will be indicated by PS -functional calculus

Definition 5.11 (The ΠS -resolvent operators $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$). *Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. We define the left poly S -resolvent operator (for short ΠS -resolvent operator) of order $\ell + 1$, for $\ell \in \mathbb{N}$ as*

$$\begin{aligned} \Pi_\ell S_L^{-1}(s, T) &:= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \bar{T}^k S_L^{-1}(s, T) (-\bar{s})^{\ell-k} \\ &= -\frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \bar{T}^k \mathcal{Q}_s(T)^{-1} (T - \bar{s}T) (-\bar{s})^{\ell-k}, \end{aligned}$$

and the right poly S -resolvent operator (for short right ΠS -resolvent operator)

$$\begin{aligned} \Pi_\ell S_R^{-1}(s, T) &:= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-\bar{s})^{\ell-k} S_R^{-1}(s, T) \bar{T}^k \\ &= -\frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-\bar{s})^{\ell-k} (T - \bar{s}T) \mathcal{Q}_s(T)^{-1} \bar{T}^k. \end{aligned}$$

where $\mathcal{Q}_s(T) := T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}$.

Lemma 5.12. *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $\ell \in \mathbb{N}$. Then we have:*

(I) *The poly left S -resolvent $\Pi_\ell S_L^{-1}(s, T)$ is a $\mathcal{B}(V_n)$ -valued right poly slice monogenic function of the variable s on $\rho_S(T)$ of order $\ell + 1$.*

(II) *The poly right S -resolvent $\Pi_\ell S_R^{-1}(s, T)$ is a $\mathcal{B}(V_n)$ -valued left poly slice monogenic function of the variable s on $\rho_S(T)$ of order $\ell + 1$.*

Proof. Consider case (I). It is a direct consequence of the definition because it is of the form

$$\Pi_\ell S_L^{-1}(s, T) = \sum_{k=0}^{\ell} \psi_\ell(s, T) \bar{s}^{\ell-k}$$

where the components

$$\psi_\ell(s, T) := \frac{1}{\ell!} \binom{\ell}{k} \bar{T}^k S_L^{-1}(s, T) (-1)^{\ell-k}$$

are right $\mathcal{B}(V_n)$ -valued slice monogenic function in the variable s on $\rho_S(T)$, recalling that $V_n = V \otimes \mathbb{R}_n$. Apply Theorem 3.6 which still holds for function $\mathcal{B}(V_n)$ -valued we get the statement. The order $\ell + 1$ is clear from the definition. Case (II) follows with the same considerations. \square

We can define the left and the right PS -resolvent equations for $T \in \mathcal{BC}^{0,1}(V_n)$, that is when T has commuting components. This will have consequences on the product rules.

Definition 5.13. Let $M \in \mathbb{N}$ and let $T \in \mathcal{B}^{0,1}(V_n)$. We denote by $\mathcal{PS}_L^M(\sigma_S(T))$, $\mathcal{PS}_R^M(\sigma_S(T))$ and $\mathcal{PN}^M(\sigma_S(T))$ the set of all left, right and intrinsic poly slice monogenic functions F of order M , respectively, with $\sigma_S(T) \subset U$, where U is a slice Cauchy domain such that $\bar{U} \subset \text{dom}(F)$ and $\text{dom}(F)$ is the domain of the function F .

Using the Cauchy formula of poly slice monogenic functions we give the definition of the PS -functional calculus.

Definition 5.14 (The PS -functional calculus (I)). Let $T \in \mathcal{B}^{0,1}(V_n)$, $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$, $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$ let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the PS -resolvent operators, for $\ell = 0, \dots, M-1$. We define

$$F(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s), \quad \text{for all } F \in \mathcal{PS}_L^M(\sigma_S(T)), \quad (47)$$

and

$$F(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T), \quad \text{for all } F \in \mathcal{PS}_R^M(\sigma_S(T)). \quad (48)$$

The following theorem shows that the definitions of the PS -functional calculus are well posed.

Theorem 5.15. Let $T \in \mathcal{B}^{0,1}(V_n)$, $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$, $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$ and let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the PS -resolvent operators, for $\ell = 0, \dots, M-1$. Then the integrals (47) and (48) depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. Recall that we work under Assumption 3.8. The independence of the integrals (47) and (48) from the open set U is standard. We treat the case of $F \in \mathcal{PS}_L^M(\sigma_S(T))$, for functions in $F \in \mathcal{PS}_R^M(\sigma_S(T))$ the proof is similar with obvious changes. If $\bar{U}' \not\subset U$, then $O := U \cap U'$ is a slice Cauchy domain that contains $\sigma_S(T)$. We can hence find a third slice Cauchy domain U'' with $\sigma_S(T) \subset U''$ and $\bar{U}'' \subset O = U \cap U'$. The above arguments show that the integrals over the boundaries of all three sets agree.

To show the independence of $j \in \mathbb{S}$ we choose two units $i, j \in \mathbb{S}$ and two slice Cauchy domains $U_q, U_s \subset \text{dom}(F)$ with $\sigma_S(T) \subset U_q$ and $\bar{U}_q \subset U_s$. (The subscripts q and s are chosen to indicate the respective variable of integration in the following computation). We start from the definition of the PS -functional calculus integrating on $\partial(U_s \cap \mathbb{C}_j)$:

$$F(T) := \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s),$$

where $F(s) = \sum_{k=0}^{M-1} \bar{s}^k f_k(s)$. We recall that

$$\bar{\partial}_j^\ell (\bar{s}^k f_k(s)) = \frac{k!}{(k-\ell)!} \bar{s}^{k-\ell} f_k(s), \quad \text{for } k \geq \ell$$

and $\bar{\partial}_j^\ell(\bar{s}^k f_k(s)) = 0$, for $k < \ell$. So we have that

$$\bar{\partial}_j^\ell \left(\sum_{k=0}^{M-1} \bar{s}^k f_k(s) \right) = \sum_{k=0}^{M-1} \bar{\partial}_j^\ell \left(\bar{s}^k f_k(s) \right) = \sum_{k=\ell}^{M-1} \frac{k!}{(k-\ell)!} \bar{s}^{k-\ell} f_k(s),$$

since for $k < \ell$ the terms $\bar{\partial}_j^\ell \left(\sum_{k=0}^{M-1} \bar{s}^k f_k(s) \right)$ are zero. Now consider $F(T)$ written as

$$F(T) = \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \sum_{k=\ell}^{M-1} \frac{k!}{(k-\ell)!} \bar{s}^{k-\ell} f_k(s).$$

We can write it more explicitly (putting a label $(\ell = 1, 2, \dots, M-1)$ in front of the integrals to identify them in the sequel) as

$$\begin{aligned} F(T) &= (\ell = 0) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_0 S_L^{-1}(s, T) ds_j \sum_{k=0}^{M-1} \bar{s}^k f_k(s), \\ &+ (\ell = 1) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_1 S_L^{-1}(s, T) ds_j \sum_{k=1}^{M-1} \frac{k!}{(k-1)!} \bar{s}^{k-1} f_k(s), \\ &+ (\ell = 2) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_2 S_L^{-1}(s, T) ds_j \sum_{k=2}^{M-1} \frac{k!}{(k-2)!} \bar{s}^{k-2} f_k(s), \\ &\dots \\ &+ (\ell = M-1) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_{M-1} S_L^{-1}(s, T) ds_j (M-1)! f_{M-1}(s). \end{aligned} \tag{49}$$

Now we replace the explicit expressions of the $\Pi_\ell S$ -resolvent operators

$$\begin{aligned} \Pi_0 S_L^{-1}(s, T) &= S_L^{-1}(s, T), \\ \Pi_1 S_L^{-1}(s, T) &= S_L^{-1}(s, T)(-\bar{s}) + \bar{T} S_L^{-1}(s, T), \\ \Pi_2 S_L^{-1}(s, T) &= \frac{1}{2!} \left(S_L^{-1}(s, T)(-\bar{s})^2 + 2\bar{T}^1 S_L^{-1}(s, T)(-\bar{s}) + \bar{T}^2 S_L^{-1}(s, T) \right), \\ &\dots \\ \Pi_{M-1} S_L^{-1}(s, T) &= \frac{1}{(M-1)!} \sum_{k=0}^{M-1} \binom{M-1}{k} \bar{T}^k S_L^{-1}(s, T)(-\bar{s})^{M-1-k}. \end{aligned} \tag{50}$$

and we get

$$\begin{aligned} F(T) &= (\ell = 0) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j \sum_{k=0}^{M-1} \bar{s}^k f_k(s), \\ &+ (\ell = 1) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(S_L^{-1}(s, T)(-\bar{s}) + \bar{T} S_L^{-1}(s, T) \right) ds_j \sum_{k=1}^{M-1} \frac{k!}{(k-1)!} \bar{s}^{k-1} f_k(s), \\ &+ (\ell = 2) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \frac{1}{2!} \left(S_L^{-1}(s, T)(-\bar{s})^2 + 2\bar{T}^1 S_L^{-1}(s, T)(-\bar{s}) + \bar{T}^2 S_L^{-1}(s, T) \right) ds_j \\ &\quad \times \sum_{k=2}^{M-1} \frac{k!}{(k-2)!} \bar{s}^{k-2} f_k(s) \\ &\dots \\ &+ (\ell = M-1) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\frac{1}{(M-1)!} \sum_{k=0}^{M-1} \binom{M-1}{k} \bar{T}^k S_L^{-1}(s, T)(-\bar{s})^{M-1-k} \right) ds_j \\ &\quad \times (M-1)! f_{M-1}(s). \end{aligned}$$

Now in the integral for $(\ell = 0)$

$$J_{(\ell=0)} := \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j \sum_{k=0}^{M-1} \bar{s}^k f_k(s)$$

we separate the term with $f_0(s)$ and we write it as

$$J_{(\ell=0)} := R_0 - \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j f_0(s)$$

where R_0 contains all the other terms. The terms for $(\ell = 1)$ are separated as

$$J_{(\ell=1)} := R_1 + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \bar{T} S_L^{-1}(s, T) ds_j f_1(s)$$

where R_1 contains all the other terms. We proceed in the same manner also for $(\ell = 2)$ and for the rest of the terms to get

$$\begin{aligned} J_{(\ell=2)} &:= R_2 + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \bar{T}^2 S_L^{-1}(s, T) ds_j f_2(s) \\ &\quad \dots \\ J_{(\ell=M-1)} &:= R_{M-1} + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \bar{T}^{M-1} S_L^{-1}(s, T) ds_j f_{M-1}(s). \end{aligned}$$

Finally consider the sum of the terms in $\sum_{j=0}^{M-1} R_j$. It turns out to be zero and this can be seen by gathering in the sum $\sum_{j=0}^{M-1} R_j$ the terms that contain $f_0, f_1 \dots f_{N-1}$. In each of these sums have the addends that cancel. This shows that we are left with

$$F(T) = \sum_{k=0}^{M-1} \bar{T}^k \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j f_k(s).$$

Repeating the above computation with the imaginary unit $i \in \mathbb{S}$ on $\partial(U_q \cap \mathbb{C}_i)$ we obtain

$$F(T) = \sum_{k=0}^{M-1} \bar{T}^k \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_i)} S_L^{-1}(q, T) dq_i f_k(q)$$

but since the S -functional calculus is independent from the imaginary units i and j in \mathbb{S} we get the statement. \square

Remark 5.16 (The case $M = 3$). For the reader's convenience, in order to understand in which way the terms in the proof of the previous result combine, we write explicitly the case $M = 3$. We recall that the explicit expressions of the $\Pi_\ell S$ -resolvent operators are given in (50) and the function $F(s)$ is of the form

$$F(s) = \bar{s}^2 f_2(s) + \bar{s} f_1(s) + f_0(s).$$

So we have

$$F(T) := \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^2 \Pi_\ell \bar{S}_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell \left(\bar{s}^2 f_2(s) + \bar{s} f_1(s) + f_0(s) \right)$$

and we split in 3 terms

$$\begin{aligned} F(T) &= (\ell = 0) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_0 \bar{S}_L^{-1}(s, T) ds_j \left(\bar{s}^2 f_2(s) + \bar{s} f_1(s) + f_0(s) \right) \\ &\quad + (\ell = 1) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_1 \bar{S}_L^{-1}(s, T) ds_j \left(2\bar{s} f_2(s) + f_1(s) \right) \\ &\quad + (\ell = 2) \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \Pi_2 \bar{S}_L^{-1}(s, T) ds_j \left(2f_2(s) \right) \end{aligned}$$

and, replacing the $\Pi_\ell S$ -resolvent operators, the addends become

$$\begin{aligned} J_{(\ell=0)} &:= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j \left(\bar{s}^2 f_2(s) + \bar{s} f_1(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j \left(f_0(s) \right) \end{aligned}$$

where the last integral gives $f_0(T)$, thanks to the S -functional calculus. Now consider

$$\begin{aligned} J_{(\ell=1)} &:= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(S_L^{-1}(s, T)(-\bar{s}) \right) ds_j \left(2\bar{s} f_2(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(S_L^{-1}(s, T)(-\bar{s}) \right) ds_j \left(f_1(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\bar{T} S_L^{-1}(s, T) \right) ds_j \left(2\bar{s} f_2(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\bar{T} S_L^{-1}(s, T) \right) ds_j \left(f_1(s) \right), \end{aligned}$$

where the last integral gives $\bar{T} f_1(T)$. For $J_{(\ell=2)}$ we have

$$\begin{aligned} J_{(\ell=2)} &:= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\frac{1}{2!} S_L^{-1}(s, T)(-\bar{s})^2 \right) ds_j \left(2f_2(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\frac{1}{2!} \cdot 2\bar{T} S_L^{-1}(s, T)(-\bar{s}) \right) ds_j \left(2f_2(s) \right) \\ &\quad + \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\frac{1}{2!} \bar{T}^2 S_L^{-1}(s, T) \right) ds_j \left(2f_2(s) \right), \end{aligned}$$

where the last integral give the term $\bar{T}^2 f_2(T)$. Finally consider all the terms that remain. Observe that all the terms in the sum with f_1 cancel and the terms that contain f_2 cancel out as well.

Remark 5.17. To prove that the integrals (47) and (48) do not depend on the imaginary unit $j \in \mathbb{S}$ we can also use the poly slice monogenic Cauchy formula. This strategy to show that the PS -functional calculus is well posed is more similar to the one used for the S -functional calculus. We just give the hints because the computations are longer with respect to the proof that we have given above. With the same notations on the domains as in the proof above we observe the following: the set $U_q^c := \mathbb{R}^{n+1} \setminus U_q$ is an unbounded axially symmetric domain with $\bar{U}_q^c \subset \rho_S(T)$. The left PS -resolvent operator is right poly slice monogenic on $\rho_S(T)$ and at infinity, since

$$\lim_{s \rightarrow \infty} S_L^{-1}(s, T) = \lim_{s \rightarrow \infty} \sum_{n=0}^{+\infty} T^n s^{-n-1} = 0$$

and the left S -resolvent operator is slice monogenic. Now observe that the ΠS -resolvent operator

$$\Pi_\ell S_L^{-1}(s, T) = \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \bar{T}^k S_L^{-1}(s, T)(-\bar{s})^{\ell-k}$$

has operator valued slice monogenic components, and they go to zero, i.e.,

$$\frac{(-1)^{\ell-k}}{\ell!} \binom{\ell}{k} \bar{T}^k S_L^{-1}(s, T) \rightarrow 0, \quad \text{for } s \rightarrow \infty.$$

So we can represent it with the Cauchy formula for unbounded slice Cauchy domains and it is right poly slice monogenic on the S -resolvent set $\rho_S(T)$. We have

$$\Pi_\ell S_L^{-1}(s, T) = \frac{1}{2\pi} \int_{\partial(U_q^c \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_\ell S_L^{-1}(q, T) \bar{\partial}_i^m dq_i \Pi_m S_R^{-1}(q, s)$$

because $\Pi_\ell S_L^{-1}(s, T)$ is poly slice monogenic up to order $M \geq \ell$, for any $s \in U_q^c$. From $S_R^{-1}(q, s) = -S_L^{-1}(s, q)$ we find that the Cauchy kernel

$$\Pi_m S_R^{-1}(s, q) = \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} (-\bar{s})^{m-k} S_R^{-1}(s, q) \bar{q}^k$$

can be written as

$$\begin{aligned} \Pi_m S_R^{-1}(q, s) &= \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} (-\bar{q})^{m-k} S_R^{-1}(q, s) \bar{s}^k \\ &= -\frac{1}{m!} \sum_{k=0}^{\ell} \binom{m}{k} (-\bar{q})^{m-k} S_L^{-1}(s, q) \bar{s}^k. \end{aligned}$$

So keeping in mind that $\partial(U_q^c \cap \mathbb{C}_i) = -\partial(U_q \cap \mathbb{C}_i)$ we have the chain of equalities

$$\begin{aligned} F(T) &:= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \left(\frac{1}{2\pi} \int_{\partial(U_q^c \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_\ell S_L^{-1}(q, T) \bar{\partial}_i^m dq_i \Pi_m S_R^{-1}(q, s) \right) ds_j \bar{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \left(\frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_\ell S_L^{-1}(q, T) \bar{\partial}_i^m dq_i \Pi_m S_L^{-1}(s, q) \right) ds_j \bar{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_\ell S_L^{-1}(q, T) \bar{\partial}_i^m dq_i \left(\frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_m S_L^{-1}(s, q) ds_j \bar{\partial}_j^\ell F(s) \right) \end{aligned}$$

where we have used Fubini's theorem and with some very long computations we obtain

$$F(T) = \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_m S_L^{-1}(q, T) \bar{\partial}_i^m dq_i F(q) = \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_i)} \sum_{m=0}^{M-1} \Pi_m S_L^{-1}(q, T) dq_i \bar{\partial}_i^m F(q).$$

So we conclude that the poly slice monogenic Cauchy formula gives the independence on the imaginary units in the sphere \mathbb{S} because we chose $\bar{U}_q \subset U_s$. The computations above to show all the cancellations are slightly more complicated with respect to the one in the prove that we have given above.

We start by proving some basic results related to the *PS*-functional calculus introduced in Definition 5.14

Proposition 5.18 (Linearity of the *PS*-functional calculus). *Let $T \in \mathcal{B}^{0,1}(V_n)$ and $M \geq 1$. Then, we have*

(1) *If $F, G \in \mathcal{PS}_L^M(\sigma_S(T))$, then we have:*

$$(F + G)(T) = F(T) + G(T), \quad (F\lambda)(T) = F(T)\lambda, \quad \text{for all } \lambda \in \mathbb{R}_n.$$

(2) *If $F, G \in \mathcal{PS}_R^M(\sigma_S(T))$, then we have:*

$$(F + G)(T) = F(T) + G(T), \quad (\lambda F)(T) = \lambda F(T), \quad \text{for all } \lambda \in \mathbb{R}_n.$$

Proof. This follows directly by construction of the *PS*-functional calculus in Definition 5.14. \square

Remark 5.19. So far, we always made use of the poly decomposition (16) of a function $F \in \mathcal{PS}_L^M(U)$ in terms of $f_0, \dots, f_{M-1} \in \mathcal{SM}_L(U)$. When the components are polynomials or power series $f_\ell(x)$ for $\ell = 0, \dots, M-1$, then

$$f_\ell(x) = \sum_{u=0}^{\infty} x^u A_{\ell,u}, \quad \text{with } A_{\ell,u} \in \mathbb{R}_n, \quad \text{for } u \in \mathbb{N}_0, \quad \ell = 0, \dots, M-1.$$

So, for $f_{M-1} \neq 0$, we have the decomposition

$$F(x) = \sum_{\ell=0}^{M-1} \bar{x}^\ell \sum_{u=0}^{\infty} x^u A_{\ell,u}, \quad \forall x \in U, \tag{51}$$

where U is the set of convergence of all the series $f_\ell(x)$, for $\ell = 0, \dots, M-1$. Analogous considerations hold for the right case (17), and we have

$$F(x) = \sum_{\ell=0}^{M-1} \left(\sum_{u=0}^{\infty} A_{\ell,u} x^u \right) \bar{x}^\ell, \quad \forall x \in U. \quad (52)$$

The following theorem shows the compatibility of the PS -functional calculus with the poly slice monogenic polynomials and series with respect to the slice monogenic components. That is when we consider, for example, the expansion (51), the PS -functional calculus gives

$$F(T) = \sum_{\ell=0}^{M-1} \bar{T}^\ell \sum_{u=0}^{\infty} T^u A_{\ell,u},$$

when F is defined on the S -spectrum of T .

Theorem 5.20. *Let $T \in \mathcal{B}^{0,1}(V_n)$ where we set $\bar{T} = T_0 - \sum_{j=1}^n e_j T_j$. Let $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$. Let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the ΠS -resolvent operators, for $\ell = 0, \dots, M-1$, and suppose $\sigma_S(T) \subset U$, where U is a slice Cauchy domain.*

(I) *Then when F is the series expansion in (51) we have*

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s) = \sum_{\ell=0}^{M-1} \bar{T}^\ell \sum_{u=0}^{\infty} T^u A_{k,u}.$$

(II) *Then when F is the series expansion in (52) we have*

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T) = \sum_{\ell=0}^{M-1} \left(\sum_{u=0}^{\infty} A_{k,u} T^u \right) \bar{T}^\ell.$$

Proof. It follows from the definition with some computations. \square

Remark 5.21. From Theorem 5.20 we see a first difference with respect to the S -functional calculus for intrinsic functions. In fact, for the PS -functional calculus it is not enough that F is an intrinsic function to have that

$$\sum_{\ell=0}^{M-1} \bar{T}^\ell \sum_{u=0}^{\infty} T^u A_{k,u} = \sum_{\ell=0}^{M-1} \left(\sum_{u=0}^{\infty} A_{k,u} T^u \right) \bar{T}^\ell$$

but we have to require that T has commuting components, i.e. $T \in \mathcal{BC}^{0,1}(V_n)$.

The following theorem is important to prove the product rule.

Theorem 5.22 (The intrinsic PS -functional calculus (I)). *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$. Let $F \in \mathcal{PN}^M(\sigma_S(T))$. Let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the ΠS -resolvent operators, for $\ell = 0, \dots, M-1$, and suppose $\sigma_S(T) \subset U$, where U is a slice Cauchy domain. Then the left and the right formulations of the PS -functional calculus define the same operators; i.e., we have*

$$\begin{aligned} F(T) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T). \end{aligned} \quad (53)$$

Proof. It is a consequence of the Theorem 3.18, when we approximate the intrinsic function F by a sequence of rational functions F_n we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F_n(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F_n(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T)$$

for $T \in \mathcal{BC}^{0,1}(V_n)$ and from the continuity of the PS -functional calculus we get the statement. \square

6. FORMULATIONS OF THE PS -FUNCTIONAL CALCULUS VIA THE MODIFIED S -RESOLVENT OPERATORS

There is an alternative way to define the PS -functional calculus: instead of using the Cauchy formula for poly slice monogenic functions, we can use the integral representation of slice monogenic function. In order to apply this strategy we need to define the modified S -resolvent operators and their related modified S -resolvent equation. As we have seen the PS -resolvent operators are poly slice monogenic, while the modified S -resolvent operators are slice monogenic.

We define the modified S -resolvent operator series.

Definition 6.1. Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \mathbb{R}^{n+1}$ with $\|T\| < |s|$ and $B \in \mathcal{B}(V_n)$. We call

$$\sum_{m=0}^{+\infty} T^m B s^{-m-1}, \quad \text{and} \quad \sum_{m=0}^{+\infty} s^{-m-1} B T^m$$

the modified left (resp. right) S -resolvent operator series expansions.

Theorem 6.2. Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $s \in \mathbb{R}^{n+1}$ with $\|T\| < |s|$ and let $B \in \mathcal{B}(V_n)$.

(i) The modified left S -resolvent operator series equals

$$\sum_{m=0}^{+\infty} T^m B s^{-m-1} = -(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I})^{-1}(TB - B\bar{s}).$$

(ii) The modified right S -resolvent operator series equals

$$\sum_{m=0}^{+\infty} s^{-m-1} B T^m = -(BT - \bar{s}B)(T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I})^{-1}.$$

Proof. It follows directly from the relations

$$\sum_{m=0}^{+\infty} q^m B s^{-1-m} = -(q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1}(qB - B\bar{s}), \quad (54)$$

and

$$\sum_{m=0}^{+\infty} s^{-1-m} B q^m = -(Bq - \bar{s}B)(q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1}, \quad (55)$$

which hold for $s, q \in \mathbb{R}^{n+1}$ with $|q| < |s|$, replacing q by T and $\|T\| < |s|$. See [8] for more details. \square

Definition 6.3 (The modified S -resolvent operators). Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $B \in \mathcal{B}(V_n)$. For $s \in \rho_S(T)$, we define the *modified left S -resolvent operator* as

$$S_L^{-1}(s, T; B) := -\mathcal{Q}_s(T)^{-1}(TB - B\bar{s}),$$

and the *modified right S -resolvent operator* as

$$S_R^{-1}(s, T; B) := -(BT - \bar{s}B)\mathcal{Q}_s(T)^{-1},$$

where $\mathcal{Q}_s(T) := T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}$.

Lemma 6.4. Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $B \in \mathcal{B}(V_n)$.

(I) The modified left S -resolvent operator $S_L^{-1}(s, T; B)$ is a $\mathcal{B}(V_n)$ -valued right-slice monogenic function of the variable s on $\rho_S(T)$.

(II) The modified right S -resolvent operator $S_R^{-1}(s, T; B)$ is a $\mathcal{B}(V_n)$ -valued left-slice monogenic function of the variable s on $\rho_S(T)$.

Proof. It can be proved by a direct computation as in the case of the S -resolvent operators. \square

Some more interesting relation can be obtained when $B \in \mathcal{B}(V_n)$ commutes with T and this fact will be crucial in the sequel.

Lemma 6.5. Let $T \in \mathcal{B}^{0,1}(V_n)$ and suppose that $B \in \mathcal{B}(V_n)$ commutes with T . Let $s \in \rho_S(T)$. Then we have

$$S_L^{-1}(s, T; B) = BS_L^{-1}(s, T),$$

and

$$S_R^{-1}(s, T; B) = S_R^{-1}(s, T)B.$$

Proof. Since $TB = BT$ the statement follows from the fact that $(TB - B\bar{s}) = B(T - \bar{s}I)$ and $\mathcal{Q}_s(T)^{-1}B = B\mathcal{Q}_s(T)^{-1}$. \square

Theorem 6.6. Let $T \in \mathcal{B}^{0,1}(V_n)$ and suppose that $B \in \mathcal{B}(V_n)$ commutes with T . Let $s \in \rho_S(T)$. The modified left S -resolvent operator satisfies the modified left S -resolvent equation

$$S_L^{-1}(s, T; B)s - TS_L^{-1}(s, T; B) = BS_L^{-1}(s, T)s - TBS_L^{-1}(s, T) = B \quad (56)$$

and the modified right S -resolvent operator satisfies the modified right S -resolvent equation

$$sS_R^{-1}(s, T; B) - S_R^{-1}(s, T; B)T = sS_R^{-1}(s, T)B - S_R^{-1}(s, T)BT = B. \quad (57)$$

Proof. Since $2\operatorname{Re}(s)$ and $|s|^2$ are real, they commute with the operator T . Therefore

$$T\mathcal{Q}_s(T) = \mathcal{Q}_s(T)T$$

and in turn

$$\mathcal{Q}_s(T)^{-1}T = T\mathcal{Q}_s(T)^{-1}.$$

Thus we have

$$\begin{aligned} S_L^{-1}(s, T; B)s - TS_L^{-1}(s, T; B) &= -\mathcal{Q}_s(T)^{-1}(TB - B\bar{s})s + T\mathcal{Q}_s(T)^{-1}(TB - B\bar{s}) \\ &= \mathcal{Q}_s(T)^{-1}(-(TB - B\bar{s})s + T(TB - B\bar{s})) \\ &= \mathcal{Q}_s(T)^{-1}B\mathcal{Q}_s(T) = B, \end{aligned}$$

where we have used the fact that B and T commute. The modified right S -resolvent equation follows by similar computations. \square

Theorem 6.7 (The modified S -resolvent equation). Let $T \in \mathcal{B}^{0,1}(V_n)$ and suppose that $B \in \mathcal{B}(V_n)$ commutes with T and let $s, q \in \rho_S(T)$ with $q \notin [s]$. Then the equation

$$S_R^{-1}(s, T)BS_L^{-1}(q, T) = [(S_R^{-1}(s, T)B - BS_L^{-1}(q, T))q - \bar{s}(S_R^{-1}(s, T)B - BS_L^{-1}(q, T))] \mathcal{Q}_s(q)^{-1} \quad (58)$$

holds true. Equivalently, it can also be written as

$$S_R^{-1}(s, T)BS_L^{-1}(q, T) = \mathcal{Q}_q(s)^{-1}[(BS_L^{-1}(q, T) - S_R^{-1}(s, T)B)\bar{q} - s(BS_L^{-1}(q, T) - S_R^{-1}(s, T)B)], \quad (59)$$

where $\mathcal{Q}_s(q) := q^2 - 2\operatorname{Re}(s)q + |s|^2$.

Proof. We show that

$$S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) = [(S_R^{-1}(s, T)B - BS_L^{-1}(q, T))q - \bar{s}(S_R^{-1}(s, T)B - BS_L^{-1}(q, T))], \quad (60)$$

which is equivalent to (58). The modified left S -resolvent equation (56) implies

$$BS_L^{-1}(q, T)q = BTS_L^{-1}(q, T) + B.$$

Applying this identity in the third and fifth equality and using the fact the $TB = BT$:

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= S_R^{-1}(s, T)BS_L^{-1}(q, T)(q^2 - 2s_0q + |s|^2) \\ &= S_R^{-1}(s, T)[BS_L^{-1}(q, T)q]q - 2s_0S_R^{-1}(s, T)[BS_L^{-1}(q, T)q] + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &= S_R^{-1}(s, T)[BTS_L^{-1}(q, T) + B]q - 2s_0S_R^{-1}(s, T)[BTS_L^{-1}(q, T) + B] + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &= S_R^{-1}(s, T)T[BS_L^{-1}(q, T)q] + S_R^{-1}(s, T)Bq - 2s_0S_R^{-1}(s, T)BTS_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B \\ &\quad + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &= S_R^{-1}(s, T)T[BTS_L^{-1}(q, T) + B] + S_R^{-1}(s, T)Bq - 2s_0S_R^{-1}(s, T)BTS_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B \\ &\quad + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T). \end{aligned}$$

So, the above relation becomes

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= S_R^{-1}(s, T)TBT S_L^{-1}(q, T) + S_R^{-1}(s, T)TB + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0S_R^{-1}(s, T)BT S_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T). \end{aligned}$$

Now we replace the term $S_R^{-1}(s, T)BT$ by $sS_R^{-1}(s, T)B - B$, two times, in the above formula using the right S -resolvent equation (57) written as

$$S_R^{-1}(s, T)BT = sS_R^{-1}(s, T)B - B.$$

We get the equality

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= [S_R^{-1}(s, T)BT]T S_L^{-1}(q, T) + [S_R^{-1}(s, T)TB] + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0[S_R^{-1}(s, T)BT]S_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \end{aligned}$$

and replacing (57) in the above equation one more time in two places indicated in parenthesis we get

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= [sS_R^{-1}(s, T)B - B]T S_L^{-1}(q, T) + [sS_R^{-1}(s, T)B - B] + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0[sS_R^{-1}(s, T)B - B]S_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \end{aligned}$$

so we get

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= s[S_R^{-1}(s, T)BT]S_L^{-1}(q, T) - BT S_L^{-1}(q, T) \\ &\quad + [sS_R^{-1}(s, T)B - B] + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0[sS_R^{-1}(s, T)B - B]S_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T). \end{aligned}$$

Replacing one more time we have

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= s[sS_R^{-1}(s, T)B - B]S_L^{-1}(q, T) \\ &\quad - BT S_L^{-1}(q, T) + [sS_R^{-1}(s, T)B - B] + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0[sS_R^{-1}(s, T)B - B]S_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B \\ &\quad + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T) \end{aligned}$$

so we obtain

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= s^2S_R^{-1}(s, T)BS_L^{-1}(q, T) - sBS_L^{-1}(q, T) \\ &\quad - BT S_L^{-1}(q, T) + sS_R^{-1}(s, T)B - B + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0sS_R^{-1}(s, T)BS_L^{-1}(q, T) + 2s_0BS_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B \\ &\quad + |s|^2S_R^{-1}(s, T)BS_L^{-1}(q, T). \end{aligned}$$

Now we gather some terms in order to get

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad - sBS_L^{-1}(q, T) - BT S_L^{-1}(q, T) + sS_R^{-1}(s, T)B - B + S_R^{-1}(s, T)Bq \\ &\quad + 2s_0BS_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B \end{aligned}$$

and

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad - sBS_L^{-1}(q, T) - BT S_L^{-1}(q, T) + sS_R^{-1}(s, T)B - B + S_R^{-1}(s, T)Bq \\ &\quad + 2s_0BS_L^{-1}(q, T) - 2s_0S_R^{-1}(s, T)B. \end{aligned}$$

Since $-sBS_L^{-1}(q, T) + 2s_0BS_L^{-1}(q, T) = \bar{s}BS_L^{-1}(q, T)$ we have

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad + \bar{s}BS_L^{-1}(q, T) - BT S_L^{-1}(q, T) + sS_R^{-1}(s, T)B - B + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0S_R^{-1}(s, T)B. \end{aligned}$$

Recalling the S -resolvent equation

$$BS_L^{-1}(q, T)q = BTS_L^{-1}(q, T) + B$$

we obtain

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad + \bar{s}BS_L^{-1}(q, T) - BS_L^{-1}(q, T)q + sS_R^{-1}(s, T)B + S_R^{-1}(s, T)Bq \\ &\quad - 2s_0S_R^{-1}(s, T)B \end{aligned}$$

and so

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad + \bar{s}BS_L^{-1}(q, T) - BS_L^{-1}(q, T)q - \bar{s}S_R^{-1}(s, T)B + S_R^{-1}(s, T)Bq. \end{aligned}$$

Finally we obtain

$$\begin{aligned} S_R^{-1}(s, T)BS_L^{-1}(q, T)\mathcal{Q}_s(q) &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad + \bar{s}BS_L^{-1}(q, T) - BS_L^{-1}(q, T)q - \bar{s}S_R^{-1}(s, T)B + S_R^{-1}(s, T)Bq \\ &= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s, T)BS_L^{-1}(q, T) \\ &\quad + (S_R^{-1}(s, T)B - BS_L^{-1}(q, T))q - \bar{s}(S_R^{-1}(s, T)B - BS_L^{-1}(q, T)) \end{aligned}$$

and since $s^2 - 2s_0s + |s|^2 = 0$ we get (60). With similar computations we can show that also (59) holds. \square

Definition 6.8. (The modified S -resolvent operators) Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. We define the modified left S -resolvent operator of order $\ell \in \mathbb{N}$ as

$$\bar{T}^\ell S_L^{-1}(s, T) := -\bar{T}^\ell \mathcal{Q}_s(T)^{-1}(T - \bar{s}\mathcal{I}), \quad (61)$$

and the modified right S -resolvent operator of order $\ell \in \mathbb{N}$ as

$$S_R^{-1}(s, T)\bar{T}^\ell := -(T - \bar{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}\bar{T}^\ell, \quad (62)$$

where $\mathcal{Q}_s(T) := T^2 - 2\operatorname{Re}(s)T + |s|^2\mathcal{I}$.

Remark 6.9. Observe that when T has commuting components the operator $B := \bar{T}^\ell$ commutes with T , so in this case, by Lemma 6.5, we have

$$S_L^{-1}(s, T; \bar{T}^\ell) = \bar{T}^\ell S_L^{-1}(s, T),$$

and

$$S_R^{-1}(s, T; \bar{T}^\ell) = S_R^{-1}(s, T)\bar{T}^\ell.$$

We are ready for an alternative, but equivalent, definition of the PS -functional calculus.

Definition 6.10 (The PS -functional calculus (II)). Let $T \in \mathcal{B}^{0,1}(V_n)$ for $j \in \mathbb{S}$ set $ds_j = ds(-j)$ and let $M \in \mathbb{N}$. Then we have the formulations (II) of the PS -functional calculus. We define

$$F(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \bar{T}^\ell S_L^{-1}(s, T) ds_j f_\ell(s), \quad \text{for all } F(s) = \sum_{\ell=0}^{M-1} \bar{s}^\ell f_\ell(s) \in \mathcal{PS}_L^M(\sigma_S(T)), \quad (63)$$

and

$$F(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} f_\ell(s) ds_j S_R^{-1}(s, T)\bar{T}^\ell, \quad \text{for all } F(s) = \sum_{\ell=0}^{M-1} f_\ell(s)\bar{s}^\ell \in \mathcal{PS}_R^M(\sigma_S(T)). \quad (64)$$

Theorem 6.11. *The definition of the PS -functional calculus (II) in (63) and (64) is well posed because the integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.*

Proof. The Definitions (63) and (64) are well posed since they are based on the S -functional calculus and Lemma 6.4 that assures that the modified S -resolvent operators preserve the slice monogenicity. \square

The linearity of the PS -functional calculus (II) is a direct consequence of the definition.

Proposition 6.12. Let $T \in \mathcal{B}^{0,1}(V_n)$ and $M \in \mathbb{N}$.

(1) If $F, G \in \mathcal{PS}_L^M(\sigma_S(T))$, then we have:

$$(F + G)(T) = F(T) + G(T), \quad (F\lambda)(T) = F(T)\lambda, \quad \text{for all } \lambda \in \mathbb{R}_n.$$

(2) If $F, G \in \mathcal{PS}_R^M(\sigma_S(T))$, then we have:

$$(F + G)(T) = F(T) + G(T), \quad (\lambda F)(T) = \lambda F(T), \quad \text{for all } \lambda \in \mathbb{R}_n.$$

Theorem 6.13. Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $M \in \mathbb{N}$. Then:

(I) When F is the series expansion in (51), we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \bar{T}^\ell S_L^{-1}(s, T) ds_j f_\ell(s) = \sum_{\ell=0}^{M-1} \bar{T}^\ell \sum_{u=0}^{\infty} T^u A_{k,u}.$$

(II) When F is the series expansion in (52), we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} f_\ell(s) ds_j S_R^{-1}(s, T) \bar{T}^\ell = \sum_{\ell=0}^{M-1} \left(\sum_{u=0}^{\infty} A_{k,u} T^u \right) \bar{T}^\ell.$$

We recall that $\bar{T} = T_0 - \sum_{j=1}^n e_j T_j$.

Proof. It is a consequence of the S -functional calculus. \square

Theorem 6.14 (The intrinsic PS -functional calculus (II)). Let $T \in \mathcal{BC}^{0,1}(V_n)$ and set $ds_j = ds(-j)$ and let F be an intrinsic poly monogenic function $F \in \mathcal{PN}^M(\sigma_S(T))$, for $M \in \mathbb{N}$ with components f_ℓ . Then the left and the right formulations of the PS -functional calculus (II) define the same operator, i.e., we have

$$F(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \bar{T}^\ell S_L^{-1}(s, T) ds_j f_\ell(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} f_\ell(s) ds_j S_R^{-1}(s, T) \bar{T}^\ell. \quad (65)$$

Proof. It is a consequence of the fact that $T \in \mathcal{BC}^{0,1}(V_n)$ and of the S -functional calculus for intrinsic functions, i.e.,

$$f_\ell(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j f_\ell(s) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f_\ell(s) ds_j S_R^{-1}(s, T),$$

holds. \square

7. EQUIVALENCE OF THE TWO DEFINITIONS OF THE PS -FUNCTIONAL CALCULUS AND THE PRODUCT RULES

According to Proposition 3.11 we have some possibilities to obtain the pointwise product of a poly slice monogenic function and a slice monogenic function that preserves the poly slice monogenicity. According to such proposition we obtain the related product rules. Using a product that takes out of the class of poly slice monogenic function of a given order, by Corollary 4.18, we obtain a more general product rule.

First we establish the equivalence of the definitions of the PS -functional calculus (I) and (II).

Theorem 7.1 (Equivalence of the definitions of the PS -functional calculus). Let $T \in \mathcal{B}^{0,1}(V_n)$, $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$, $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$. Let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the ΠS -resolvent operators. Then we have

$$\begin{aligned} F(T) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \bar{T}^\ell S_L^{-1}(s, T) ds_j f_\ell(s), \quad \text{for all } F(s) = \sum_{\ell=0}^{M-1} \bar{s}^\ell f_\ell(s) \in \mathcal{PS}_L^M(\sigma_S(T)), \end{aligned} \quad (66)$$

and

$$\begin{aligned}
F(T) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T) \\
&= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} f_\ell(s) ds_j S_R^{-1}(s, T) \bar{T}^\ell, \quad \text{for all } F(s) = \sum_{\ell=0}^{M-1} f_\ell(s) \bar{s}^\ell \in \mathcal{PS}_R^M(\sigma_S(T)).
\end{aligned} \tag{67}$$

Proof. It is a consequence of Runge's theorems for slice monogenic and poly slice monogenic functions and the continuity properties of the two formulations of the PS -functional calculus. \square

As a corollary of the above theorem we have the four possible representations of the operator $f(T)$ when F is an intrinsic poly slice monogenic function, i.e., when $F \in \mathcal{PN}^M(\sigma_S(T))$.

Theorem 7.2 (The formulations of PS -functional calculus for intrinsic functions). *Let $T \in \mathcal{BC}^{0,1}(V_n)$, $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$, $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$ and Let $M \in \mathbb{N}$, for $j \in \mathbb{S}$, set $ds_j = ds(-j)$ and $\bar{\partial}_j := \frac{1}{2}(\partial_u + j\partial_v)$. Let $\Pi_\ell S_L^{-1}(s, T)$ and $\Pi_\ell S_R^{-1}(s, T)$ be the ΠS -resolvent operators. Then, for every $F \in \mathcal{PN}^M(\sigma_S(T))$, we have*

$$\begin{aligned}
F(T) &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \Pi_\ell S_L^{-1}(s, T) ds_j \bar{\partial}_j^\ell F(s) \\
&= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} \bar{T}^\ell S_L^{-1}(s, T) ds_j f_\ell(s) \\
&= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} F(s) \bar{\partial}_j^\ell ds_j \Pi_\ell S_R^{-1}(s, T) \\
&= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{\ell=0}^{M-1} f_\ell(s) ds_j S_R^{-1}(s, T) \bar{T}^\ell.
\end{aligned} \tag{68}$$

Proof. It is a consequence of Runge's theorems for slice monogenic and poly slice monogenic functions and the continuity properties of the two formulations of the PS -functional calculus. \square

The following lemma will be crucial in the proof of the product rules.

Lemma 7.3 (See [8]). *Let $B \in \mathcal{B}(V_n)$. For any $q, s \in \mathbb{R}^{n+1}$ with $q \notin [s]$ and let f be an intrinsic slice monogenic function and let U be a slice Cauchy domain with $\bar{U} \subset \text{dom}(f)$. By setting*

$$\mathcal{Q}_s(q)^{-1} := (q^2 - 2\text{Re}(s)q + |s|^2)^{-1}$$

then we have

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s) ds_j (\bar{s}B - Bq) \mathcal{Q}_s(q)^{-1} = Bf(q)$$

for any q that belongs to U and any $j \in \mathbb{S}$.

Using Proposition 3.11 we obtain the first set of product rules for the PS -functional calculus.

Theorem 7.4 (Product rule (first case)). *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $M \in \mathbb{N}$.*

- (Ia) *Let $F \in \mathcal{PN}^M(\sigma_S(T))$ and $g \in \mathcal{SM}_L(\sigma_S(T))$. Then $(Fg)(T) = F(T)g(T)$.*
- (Ib) *Let $F \in \mathcal{PM}_L^M(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then $(gF)(T) = g(T)F(T)$.*
- (IIa) *Let $F \in \mathcal{PN}^M(\sigma_S(T))$ and $g \in \mathcal{SM}_R(\sigma_S(T))$. Then $(gF)(T) = g(T)F(T)$.*
- (IIb) *Let $F \in \mathcal{PM}_R^M(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then $(Fg)(T) = F(T)g(T)$.*
- (III) *Let $F \in \mathcal{PN}^M(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then we have*

$$(gF)(T) = (Fg)(T) = F(T)g(T) = g(T)F(T).$$

Proof. We will show point (Ia). The other claims follow in much the same way. So for $F \in \mathcal{PN}^M(\sigma_S(T))$ and $g \in \mathcal{SM}_L(\sigma_S(T))$ using the PS -functional calculus (II) for F and the S -functional calculus for g we have

$$g(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j g(s), \quad \text{for all } f \in \mathcal{SM}_L(\sigma_S(T)).$$

Let U_q and U_s be bounded slice Cauchy domains that contain $\sigma_S(T)$ such that $\overline{U_q} \subset U_s$ and $\overline{U_s} \subset \text{dom}(F) \cap \text{dom}(g)$. The subscripts q and s refer to the respective variable of integration in the following computation. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. So for $F \in \mathcal{PN}^M(\sigma_S(T))$ by Theorem 6.14 we can represent F as

$$F(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \sum_{k=0}^{M-1} f_k(s) ds_j S_R^{-1}(s, T) \overline{T}^k.$$

When we consider the product

$$F(T)g(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j S_R^{-1}(s, T) \overline{T}^k \frac{1}{2\pi} \int_{\Gamma_q} S_L^{-1}(q, T) dq_j g(q)$$

so we write

$$F(T)g(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} S_R^{-1}(s, T) \overline{T}^k S_L^{-1}(q, T) dq_j g(q). \quad (69)$$

Now we use the modified S -resolvent equation with $B = \overline{T}^k$ that commute with T , since the paravector operator T has commuting components we have the modified S -resolvent equation

$$S_R^{-1}(s, T) \overline{T}^k S_L^{-1}(q, T) = \left[\left(S_R^{-1}(s, T) \overline{T}^k - \overline{T}^k S_L^{-1}(q, T) \right) q - \overline{s} \left(S_R^{-1}(s, T) \overline{T}^k - \overline{T}^k S_L^{-1}(q, T) \right) \right] \mathcal{Q}_s(q)^{-1} \quad (70)$$

we replace it (70) in (69) to get

$$\begin{aligned} F(T)g(T) &= \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \left[\left(S_R^{-1}(s, T) \overline{T}^k - \overline{T}^k S_L^{-1}(q, T) \right) q \right. \\ &\quad \left. - \overline{s} \left(S_R^{-1}(s, T) \overline{T}^k - \overline{T}^k S_L^{-1}(q, T) \right) \right] \mathcal{Q}_s(q)^{-1} dq_j g(q) \end{aligned}$$

but also

$$\begin{aligned} F(T)g(T) &= \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} S_R^{-1}(s, T) \overline{T}^k q \mathcal{Q}_s(q)^{-1} dq_j g(q) \\ &\quad - \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \overline{T}^k S_L^{-1}(q, T) q \mathcal{Q}_s(q)^{-1} dq_j g(q) \\ &\quad - \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \overline{s} S_R^{-1}(s, T) \overline{T}^k \mathcal{Q}_s(q)^{-1} dq_j g(q) \\ &\quad + \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \overline{s} \overline{T}^k S_L^{-1}(q, T) \mathcal{Q}_s(q)^{-1} dq_j g(q). \end{aligned}$$

Now the integrals with the right S -resolvent operator are zero, in fact we can write them as

$$\begin{aligned} &\frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} S_R^{-1}(s, T) \overline{T}^k q \mathcal{Q}_s(q)^{-1} dq_j g(q) \\ &= \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j S_R^{-1}(s, T) \overline{T}^k \frac{1}{2\pi} \int_{\Gamma_q} \left[q \mathcal{Q}_s(q)^{-1} dq_j g(q) \right] = 0 \end{aligned}$$

by the Cauchy theorem because $q\mathcal{Q}_s(q)^{-1}$ and $g(q)$ are slice monogenic. Similarly it is

$$-\frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \bar{s} S_R^{-1}(s, T) \bar{T}^k \mathcal{Q}_s(q)^{-1} dq_j g(q) = 0$$

so we remain with

$$\begin{aligned} F(T)g(T) &= -\frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \bar{T}^k S_L^{-1}(q, T) q \mathcal{Q}_s(q)^{-1} dq_j g(q) \\ &\quad + \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \bar{s} \bar{T}^k S_L^{-1}(q, T) \mathcal{Q}_s(q)^{-1} dq_j g(q) \end{aligned}$$

and also we obtain

$$F(T)g(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \left(\bar{s} \bar{T}^k S_L^{-1}(q, T) - \bar{T}^k S_L^{-1}(q, T) q \right) \mathcal{Q}_s(q)^{-1} dq_j g(q).$$

Using Fubini's theorem we get

$$F(T)g(T) = \frac{1}{2\pi} \int_{\Gamma_q} \left[\frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{M-1} f_k(s) ds_j \left(\bar{s} \bar{T}^k S_L^{-1}(q, T) - \bar{T}^k S_L^{-1}(q, T) q \right) \mathcal{Q}_s(q)^{-1} \right] dq_j g(q)$$

and by Lemma 7.3, setting $B := \bar{T}^k S_L^{-1}(q, T)$, we obtain

$$F(T)g(T) = \frac{1}{2\pi} \int_{\Gamma_q} \sum_{k=0}^{M-1} \bar{T}^k S_L^{-1}(q, T) f_k(q) dq_j g(q)$$

and finally

$$\begin{aligned} F(T)g(T) &= \frac{1}{2\pi} \int_{\Gamma_q} \sum_{k=0}^{M-1} \bar{T}^k S_L^{-1}(q, T) dq_j f_k(q) g(q) \\ &= \sum_{k=0}^{M-1} \bar{T}^k (f_k g)(T) = (Fg)(T), \end{aligned}$$

and this concludes the proof. \square

Based on the product of poly slice monogenic functions in Corollary 4.18 we prove the following product rule. This product rule requires that the paravector operator T has commuting components.

Theorem 7.5 (Product rule (second case)). *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $M, N \in \mathbb{N}$.*

(I) *Let $F \in \mathcal{N}_L^N(\sigma_S(T))$ and $G \in \mathcal{PS}_L^M(\sigma_S(T))$, then we have*

$$(FG)(T) = F(T)G(T). \tag{71}$$

(II) *Let $F \in \mathcal{PS}_R^N(\sigma_S(T))$ and $G \in \mathcal{N}_R^M(\sigma_S(T))$, then we have*

$$(FG)(T) = F(T)G(T). \tag{72}$$

Proof. We reason as in Theorem 7.4 taking the same contours of integration and pointing out just the differences. We show point (I). The other point (II) follows in much the same way. Let U_q and U_s be bounded slice Cauchy domains that contain $\sigma_S(T)$ such that $\bar{U}_q \subset U_s$ and $\bar{U}_s \subset \text{dom}(F) \cap \text{dom}(G)$. The subscripts q and s refer to the respective variable of integration in the following computation. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. So by Theorem 6.14 we can represent F as

$$F(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j S_R^{-1}(s, T) \bar{T}^k, \quad \text{for all } F \in \mathcal{PN}^N(\sigma_S(T))$$

and

$$G(T) = \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{T}^\ell S_L^{-1}(q, T) dq_j g_\ell(q), \quad \text{for all } F \in \mathcal{P}S_L^M(\sigma_S(T))$$

so when we consider the product

$$F(T)G(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j S_R^{-1}(s, T) \overline{T}^k \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{T}^\ell S_L^{-1}(q, T) dq_j g_\ell(q)$$

and also.

$$F(T)G(T) = \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} S_R^{-1}(s, T) \overline{T}^{k+\ell} S_L^{-1}(q, T) dq_j g_\ell(q).$$

We use the modified S -resolvent equation to get

$$\begin{aligned} F(T)G(T) &= \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} S_R^{-1}(s, T) \overline{T}^{k+\ell} q \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q) \\ &\quad - \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{T}^{k+\ell} S_L^{-1}(q, T) q \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q) \\ &\quad - \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{s} S_R^{-1}(s, T) \overline{T}^{k+\ell} \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q) \\ &\quad + \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{s} \overline{T}^{k+\ell} S_L^{-1}(q, T) \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q). \end{aligned}$$

Also here the two integrals that contain the right S -resolvent operators are zero so we obtain

$$\begin{aligned} F(T)G(T) &= - \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{T}^{k+\ell} S_L^{-1}(q, T) q \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q) \\ &\quad + \frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} f_k(s) ds_j \frac{1}{2\pi} \int_{\Gamma_q} \sum_{\ell=0}^{M-1} \overline{s} \overline{T}^{k+\ell} S_L^{-1}(q, T) \mathcal{Q}_s(q)^{-1} dq_j g_\ell(q). \end{aligned}$$

Using Fubini's theorem we finally have

$$\begin{aligned} F(T)G(T) &= \\ &= \frac{1}{2\pi} \int_{\Gamma_q} \left[\frac{1}{2\pi} \int_{\Gamma_s} \sum_{k=0}^{N-1} \sum_{\ell=0}^{M-1} f_k(s) ds_j \left(\overline{s} \overline{T}^{k+\ell} S_L^{-1}(q, T) - \overline{T}^{k+\ell} S_L^{-1}(q, T) q \right) \mathcal{Q}_s(q)^{-1} \right] dq_j g_\ell(q) \end{aligned}$$

setting $B := \overline{T}^{k+\ell} S_L^{-1}(q, T)$ in Lemma 7.3 we get

$$\begin{aligned} F(T)G(T) &= \frac{1}{2\pi} \int_{\Gamma_q} \sum_{k=0}^{N-1} \sum_{\ell=0}^{M-1} \overline{T}^{k+\ell} S_L^{-1}(q, T) f_k(q) dq_j g_\ell(q) \\ &= \frac{1}{2\pi} \int_{\Gamma_q} \sum_{k=0}^{N-1} \sum_{\ell=0}^{M-1} \overline{T}^{k+\ell} S_L^{-1}(q, T) dq_j (f_k g_\ell)(q) \\ &= \sum_{k=0}^{N-1} \sum_{\ell=0}^{M-1} \overline{T}^{k+\ell} (f_k g_\ell)(T) = (FG)(T), \end{aligned}$$

and this concludes the proof. \square

REFERENCES

- [1] L. D. Abreu, *Super-wavelets versus poly-Bergman spaces*, Int. Eq. Op. Theor., **73** (2012), 177–193.
- [2] L. D. Abreu, H. G. Feichtinger, *Function spaces of polyanalytic functions*, in Harmonic and complex analysis and its applications, Trends Math., Birkhäuser Springer, Cham, 2014, 1–38.
- [3] M. L. Agranovsky, *Characterization of polyanalytic functions by meromorphic extensions into chains of circles*, J. d’Analyse Math., **113** (2011), 305–329.
- [4] L.D. Abreu, Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions, *Applied and Computational Harmonic Analysis*, **29** (2010), 287–302.
- [5] L.D. Abreu and H.G. Feichtinger, Function spaces of polyanalytic functions, In *Harmonic and complex analysis and its applications*, pages 1–38. Springer, 2014.
- [6] D. Alpay, An advanced complex analysis problem book, *Topological vector spaces, functional analysis, and Hilbert spaces of analytic functions*. Birkhäuser Basel, 2015.
- [7] D. Alpay, F. Colombo, D.P. Kimsey, *The spectral theorem for quaternionic unbounded normal operators based on the S -spectrum*, J. Math. Phys., **57** (2016), no. 2, 023503, 27 pp.
- [8] D. Alpay, F. Colombo, J. Gantner, I. Sabadini, *A new resolvent equation for the S -functional calculus*, J. Geom. Anal., **25** (2015), 1939–1968.
- [9] D. Alpay, F. Colombo, I. Sabadini, *Quaternionic de Branges spaces and characteristic operator function*, SpringerBriefs in Mathematics, Springer, Cham, 2020.
- [10] D. Alpay, F. Colombo, I. Sabadini, *Slice Hyperholomorphic Schur Analysis*, Operator Theory: Advances and Applications, 256. Birkhäuser/Springer, Cham, 2016. xii+362 pp.
- [11] D. Alpay, K. Diki, and I. Sabadini. *On slice polyanalytic functions of a quaternionic variable*, Results Math., **74** (2019), no. 1, Paper No. 17, 25 pp.
- [12] D. Alpay, K. Diki, and I. Sabadini. *On the global operator and Fueter mapping theorem for slice polyanalytic functions*. Anal. Appl. (Singap.), **19** (2021), no. 6, 941–964.
- [13] N. Askour, A. Intissar, and Z. Mouayn. *Espaces de Bargmann généralisés et formules explicites pour leurs noyaux reproduisants*. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 325(7):707–712, 1997.
- [14] M.B. Balk. Polyanalytic functions. *Mathematical research*, 1991.
- [15] M. B. Balk, *Polyanalytic functions and their generalizations*, in Complex analysis, I, 195–253, Encyclopaedia Math. Sci., 85, Springer, Berlin, 1997.
- [16] H. Begehr, *Orthogonal decompositions of the function space $L_2(\overline{D}; \mathbb{C})$* , J. Reine Angew. Math. 549 (2002), 191–219.
- [17] F. Brackx. On (k) -monogenic functions of a quaternion variable. *Function theoretic methods in differential equations*, 8:22–44, 1976.
- [18] F. Brackx and R. Delanghe. Hypercomplex function theory and Hilbert modules with reproducing kernel. *Proc. London Math. Soc.*, 37(3):545–576, 1978.
- [19] G. Birkhoff, J. von Neumann, *The logic of quantum mechanics*, Ann. of Math., **37** (1936), 823–843.
- [20] P. Cerejeiras, F. Colombo, U. Kaehler, I. Sabadini, *Perturbation of normal quaternionic operators*, Trans. Amer. Math. Soc., **372** (2019), 3257–328.
- [21] F. Colombo, R. Lavicka, I. Sabadini, V. Soucek, *The Radon transform between monogenic and generalized slice monogenic functions*, Math. Ann., **363** (2015), 733–752.
- [22] F. Colombo, J. Gantner, *Quaternionic closed operators, fractional powers and fractional diffusion processes*, Operator Theory: Advances and Applications, 274. Birkhäuser/Springer, Cham, 2019. viii+322 pp.
- [23] F. Colombo, J. Gantner, D. P. Kimsey, *Spectral theory on the S -spectrum for quaternionic operators*, Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham, 2018. ix+356 pp.
- [24] F. Colombo, I. Sabadini, D.C. Struppa, *Entire Slice Regular Functions*, SpringerBriefs in Mathematics. Springer, Cham, 2016, v+118.
- [25] F. Colombo, I. Sabadini, *A structure formula for slice monogenic functions and some of its consequences*, Hypercomplex analysis, 101–114, Trends Math., Birkhäuser Verlag, Basel, 2009.
- [26] F. Colombo, I. Sabadini, *The Cauchy formula with s -monogenic kernel and a functional calculus for noncommuting operators*, J. Math. Anal. Appl, **373** (2011), 655–679.
- [27] F. Colombo, I. Sabadini, D.C. Struppa, *Slice monogenic functions*, Israel J. Math., **171** (2009), 385–403.
- [28] F. Colombo, I. Sabadini, D.C. Struppa, *An extension theorem for slice monogenic functions and some of its consequences*, Israel J. Math., **177** (2010), 369–389.
- [29] F. Colombo, I. Sabadini, D.C. Struppa, *Duality theorems for slice hyperholomorphic functions*, J. Reine Angew. Math., **645** (2010), 85–105.
- [30] F. Colombo, I. Sabadini, D.C. Struppa, *Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions*, Progress in Mathematics, 289. Birkhäuser/Springer Basel AG, Basel, 2011. vi+221 pp.
- [31] F. Colombo, I. Sabadini, D.C. Struppa, *The Runge theorem for slice hyperholomorphic functions*, Proc. Amer. Math. Soc., **139** (2011), 1787–1803.
- [32] F. Colombo, I. Sabadini, D.C. Struppa, *Michele Sce’s Works in Hypercomplex Analysis. A Translation with Commentaries*, Birkhäuser, 2020, Hardcover ISBN 978-3-030-50215-7.
- [33] J. Gantner, *Operator Theory on One-Sided Quaternionic Linear Spaces: Intrinsic S -Functional Calculus and Spectral Operators*, Mem. Amer. Math. Soc., **267** (2020), no. 1297, iii+101 pp.

- [34] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Progress in Mathematical Physics, Vol. 39, Birkhäuser, Boston, 2004.
- [35] R. Delanghe, F. Sommen, V. Soucek, *Clifford algebra and spinor-valued functions. A function theory for the Dirac operator*, Related REDUCE software by F. Brackx and D. Constaes. Mathematics and its Applications, 53. Kluwer Academic Publishers Group, Dordrecht, 1992. xviii+485 pp.
- [36] S. G. Gal, I. Sabadini, *Quaternionic approximation. With application to slice regular functions*, Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2019. x+221 pp.
- [37] G. Gentili, C. Stoppato, D. C. Struppa, *Regular functions of a quaternionic variable*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013.
- [38] G. Gentili, D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., **216** (2007), 279–301.
- [39] R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., **226** (2011), 1662–1691.
- [40] K. Gürlebeck, K. Habetha, and W. Sprößig, *Holomorphic functions in the plane and n-dimensional space*. Birkhäuser Verlag, Basel, 2008. Translated from the 2006 German original.
- [41] K. Gürlebeck, W. Sprössig, *Quaternionic analysis and elliptic boundary value problems*, International Series of Numerical Mathematics, 89. Birkhäuser Verlag, Basel, 1990. 253 pp.
- [42] B. Jefferies, A. McIntosh, J. Picton-Warlow, *The monogenic functional calculus*, Studia Math., **136** (1999), 99–119.
- [43] G. V. Kolossov, *Sur les problèmes d'élasticité à deux dimensions*, C. R. Acad. Sci., **146** (1908), 522–525.
- [44] A. McIntosh, A. Pryde, *A functional calculus for several commuting operators*, Indiana U. Math. J., **36** (1987), 421–439.
- [45] B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin, 2004.
- [46] M. Mitrea, *Clifford wavelets, singular integrals, and Hardy spaces*, Lecture Notes in Mathematics, 1575. Springer-Verlag, Berlin, 1994. xii+116 pp.
- [47] N. I. Muskhelishvili, *Some Basic Problems of Mathematical Elasticity Theory* (in Russian) (Nauka, Moscow, 1968).
- [48] T. Qian, P. Li, *Singular integrals and Fourier theory on Lipschitz boundaries*, Science Press Beijing, Beijing; Springer, Singapore, 2019. xv+306 pp.
- [49] N. Théodoresco, *La dérivée aréolaire et ses applications à la physique mathématique*, Paris, 1931.
- [50] N. L. Vasilevski, *On the structure of Bergman and poly-Bergman spaces*, Integr. Equat. Oper. Theory, **33** (1999), 471–488.

(DA) FACULTY OF MATHEMATICS, PHYSICS, AND COMPUTATION, SCHMID COLLEGE OF SCIENCE AND TECHNOLOGY, CHAPMAN UNIVERSITY, ONE UNIVERSITY DRIVE ORANGE, CALIFORNIA 92866, USA

Email address: `alpay@chapman.edu`

(FC) POLITECNICO DI MILANO, DIPARTIMENTO DI MATEMATICA, VIA E. BONARDI, 9, 20133 MILANO, ITALY

Email address: `fabrizio.colombo@polimi.it`

(KD) POLITECNICO DI MILANO, DIPARTIMENTO DI MATEMATICA, VIA E. BONARDI, 9, 20133 MILANO, ITALY

Email address: `kamal.diki@polimi.it`

(IS) POLITECNICO DI MILANO, DIPARTIMENTO DI MATEMATICA, VIA E. BONARDI, 9, 20133 MILANO, ITALY

Email address: `irene.sabadini@polimi.it`