

BERNOULLI FREE BOUNDARY PROBLEM FOR THE INFINITY LAPLACIAN

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ABSTRACT. We study the interior Bernoulli free boundary problem for the infinity Laplacian. Our results cover existence, uniqueness, and characterization of solutions (above a threshold representing the “infinity Bernoulli constant”), their regularity, and their relationship with the solutions to the interior Bernoulli problem for the p -Laplacian.

1. INTRODUCTION

This paper concerns the following interior Bernoulli-type problem:

$$(P)_\lambda \quad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = \lambda & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

where Ω is an open bounded connected domain in \mathbb{R}^n ($n \geq 2$), and Δ_∞ is the infinity Laplacian, defined by

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u \quad \forall u \in C^2(\Omega).$$

Before presenting our results, we wish to put them into context by saying few words on related literature.

1.1. Bernoulli problem for the p -Laplacian. The analogue of problem $(P)_\lambda$ for the p -Laplacian, namely

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega^+(u), \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = \lambda & \text{on } F(u), \end{cases}$$

corresponds to the classical Bernoulli problem when $p = 2$, and by now it has been widely studied also in the nonlinear case of an arbitrary $p > 1$. It is motivated by several physical and industrial applications, including fluid dynamics, optimal insulation, and electro-chemical machining (see [34] for a more precise description). The main questions are the existence and uniqueness of solutions, the geometric properties of the free boundary $F(u)$, and especially its regularity (for an overview on these topics, we address to [14, 33]). When Ω is convex and regular, it was proved by Henrot and Shahgholian that there exists a positive constant $\lambda_{\Omega,p}$, called the *Bernoulli constant for the p -Laplacian*, such that the interior p -Bernoulli problem admits a non-constant solution if and only if $\lambda \geq \lambda_{\Omega,p}$; this solution is in general not unique, it has convex level sets, and its free boundary $F(u)$ is of class $C^{2,\alpha}$ (see [15, 35]).

Date: May 8, 2019.

2010 Mathematics Subject Classification. Primary 49K20, Secondary 35J70, 35J40.

Key words and phrases. Bernoulli problem, infinity Laplacian, capacitary potential, distance function.

When Ω is an arbitrary domain, not necessarily convex, one way of finding solutions is to use the approach introduced by Alt and Caffarelli in the linear case $p = 2$ in the seminal work [2]. It amounts to minimizing the integral functional

$$(1) \quad J_p^\lambda(u) := \frac{1}{p} \int_{\Omega} \left(\frac{|\nabla u|}{\lambda} \right)^p + \frac{p-1}{p} |\{u > 0\}|$$

over the space $W_1^{1,p}(\Omega) := 1 + W_0^{1,p}(\Omega)$.

This minimization problem admits a non-constant solution if and only if $\lambda \geq \Lambda_{\Omega,p}$, where $\Lambda_{\Omega,p}$ is a positive constant satisfying $\Lambda_{\Omega,p} \geq \lambda_{\Omega,p}$ [25]. A non-constant minimizer of J_p^λ over $W_1^{1,p}(\Omega)$ solves the p -Bernoulli problem provided the free boundary condition $|\nabla u| = \lambda$ is understood in a suitable weak sense (cf. (25)). The free boundary $F(u)$ turns out to be a locally analytic hyper-surface, except for a \mathcal{H}^{n-1} -negligible singular set (in the vast literature about the free boundary regularity, we limit ourselves to quote as main contributions [2, 10, 30] for the case $p = 2$ and [26, 27, 51] for general p).

1.2. Free boundary problems for the infinity Laplacian. This highly nonlinear and strongly degenerated operator was discovered by Aronsson in the sixties [3]. However, the study of boundary value problems for the infinity Laplacian started only in the early nineties, with the advent of viscosity solutions theory. Bhattacharya, DiBenedetto and Manfredi were the first to consider the Dirichlet problem for infinity harmonic functions and to prove the existence of a solution in the viscosity sense [7]; shortly afterwards, a fundamental contribution came by Jensen [40], who proved the validity of the comparison principle for the infinity Laplacian, yielding the uniqueness of solutions (see also [4, 5]). The last decade has seen a renewed and increasing interest around the infinity Laplacian, also due to its connections with differential games. With no attempt of completeness, among the topics under investigation in this growing field let us mention: inhomogeneous equations [8, 48], regularity of solutions [20, 31, 32, 47, 57], ground states [23, 39, 41, 59], overdetermined problems [18, 19, 21], tug-of-war games [44, 50]. In this scenario, the study of free boundary problems involving the infinity Laplacian seems to be rather at its early stage. To the best of our knowledge, only the following exterior version, in the complement $\omega := \mathbb{R}^n \setminus \overline{\Omega}$ of an open bounded convex set Ω , of Bernoulli problem has been considered in the literature (see [49]):

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \omega^+(u) := \{x \in \omega : u(x) > 0\}, \\ u = 1 & \text{on } \partial\omega, \\ |\nabla u| = a(x) & \text{on } F(u). \end{cases}$$

In particular, when Ω is a regular convex set and $a(x) \equiv \lambda$, the situation looks relatively simple: a unique explicit solution exists, given by $1 - \frac{1}{\lambda} \text{dist}(x, \partial\omega)$. It satisfies the condition $|\nabla u| = \lambda$ in the classical sense along its free boundary, which in this case is of class C^1 . Further, such solution can be identified with the pointwise limit, as $p \rightarrow +\infty$, of the unique solutions u_p to the exterior Bernoulli problem for the p -Laplacian.

On the variational side, let us mention that the asymptotics as $p \rightarrow +\infty$ of integral energies associated with the exterior p -Bernoulli problem (loosely speaking, functionals of the type (1) with Ω replaced by its complement) has been studied in [43]. In a somewhat close spirit, the limiting behaviour as $p \rightarrow +\infty$ of the solutions of the minimization problems for the p -Dirichlet integral with a positive boundary datum and a constraint on the volume of the support, has been studied in [54]. Still, in the theme of free boundary problems for the infinity Laplacian, see also [53, 55, 58].

1.3. Notion of solution. A delicate point before starting the analysis of problem $(P)_\lambda$ is to establish what is meant by a solution. Clearly the PDE has to be understood in the viscosity sense. Going further we point out that, contrarily to the case of the exterior problem mentioned above, for solutions to problem $(P)_\lambda$ the free boundary need not be globally C^1 . Consequently, a solution is not expected to be differentiable up to the boundary (see [37, 38]), so that also the free boundary condition cannot be interpreted in a pointwise, classical way. Thus, even at the boundary, a viscosity interpretation seems to be the most convenient one in order to manage both existence and uniqueness questions. More precisely, throughout the paper we interpret solutions to $(P)_\lambda$ to mean viscosity solutions defined in the next definition, introduced by De Silva [28, Definitions 2.2 and 2.3], and has been adopted in several subsequent works (see for instance [29, 45, 46]).

If $u, v: \Omega \rightarrow \mathbb{R}$ are two functions and $x \in \Omega$, by $u \prec_x v$ we mean that $u(x) = v(x)$ and $u(y) \leq v(y)$ in a neighborhood of x . Moreover, in the following definition for any test function φ of class C^2 , we set $\varphi^+ := \max\{\varphi, 0\}$.

Definition 1. A non-negative function $u \in C(\overline{\Omega})$ is a viscosity solution to $(P)_\lambda$ if

- (a) u is infinity harmonic at every $x \in \Omega^+(u)$, i.e., for any test function φ of class C^2 ,
 - (a1) if $u \prec_x \varphi$, then $-\Delta_\infty \varphi(x) \leq 0$;
 - (a2) if $\varphi \prec_x u$, then $-\Delta_\infty \varphi(x) \geq 0$;
- (b) the Dirichlet condition $u = 1$ holds pointwise on $\partial\Omega$;
- (c) the free boundary condition holds at every $y \in F(u)$, i.e., for any test function φ of class C^2 with $\nabla\varphi(y) \neq 0$,
 - (c1) if $\varphi^+ \prec_y u$, then $|\nabla\varphi(y)| \leq \lambda$;
 - (c2) if $u \prec_y \varphi^+$, then $|\nabla\varphi(y)| \geq \lambda$.

It is clear from the definition that $u = 0$ on $\Omega \setminus \Omega^+(u)$, hence, in particular, on $F(u)$.

We point out that a solution in the sense of Definition 1, is also a solution in the sense proposed by Caffarelli in [11, Definition 1] (see also [12, 13]). The converse is a priori not true, because a touching ball as in Caffarelli's definition does not exist necessarily at all points of the free boundary. Some of our results (e.g., Proposition 2 and Proposition 3) remain true if solutions are intended in the sense of [11]. However, Definition 1 à la De Silva seems to be the one which allow us to deal in an optimal way with the existence question (in particular, in the proof of Theorem 16 (b)).

1.4. Synopsis of the results. We carry over a detailed analysis of problem $(P)_\lambda$, which covers existence, uniqueness, and characterization of solutions, their regularity, and their relationship with the solutions to the interior Bernoulli problem for the p -Laplacian. We postpone to a companion paper [22] the study of the variational problem which is naturally associated with $(P)_\lambda$, namely the minimization of the supremal functional

$$J_\lambda(u) := \|\nabla u\|_\infty + \lambda|\{u > 0\}|$$

over the space of functions $u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ which are equal to 1 on $\partial\Omega$.

• *Existence.* By analogy with the case of the p -Laplacian, we define the ∞ -Bernoulli constant of Ω as

$$(2) \quad \lambda_{\Omega,\infty} := \inf \left\{ \lambda > 0 : (P)_\lambda \text{ admits a non-constant solution} \right\}.$$

Then we identify $\lambda_{\Omega,\infty}$ with the reciprocal of the inradius R_Ω of Ω . Indeed, for $\lambda < 1/R_\Omega$, problem $(P)_\lambda$ does not admit any non-constant solution (Theorem 16 (b)). The proof is based on a gradient estimate obtained via the gradient flow for infinity harmonic

functions (Proposition 2). Conversely, for $\lambda \geq 1/R_\Omega$, we get existence. More precisely, it is convenient to distinguish between genuine and non-genuine solutions, according to whether the set $\{u = 0\}$ has positive Lebesgue measure or not. For any $\lambda \geq 1/R_\Omega$, it is easily seen that problem $(P)_\lambda$ admits many non-genuine solutions, given by the infinity harmonic potentials (see Definition 4 below) of suitable compact subsets with empty interior contained in the set of points $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \geq 1/\lambda$ (Proposition 10). So the interesting feature is the existence of a genuine solution: if $\lambda > 1/R_\Omega$, we show that it is given precisely by the infinity harmonic potential $w_{1/\lambda}$ of the set $\overline{\Omega_{1/\lambda}}$, being $\Omega_{1/\lambda}$ the set of points $x \in \Omega$ such that $\text{dist}(x, \partial\Omega) > 1/\lambda$ (Theorem 16 (a)). This is obtained by constructing suitable upper and lower bounds for $w_{1/\lambda}$, and taking advantage of the simple behaviour of infinity harmonic potentials along rays of the distance function (see Section 2.3).

- *Uniqueness.* For $\lambda > 1/R_\Omega$, we obtain uniqueness of genuine solutions under two assumptions on the set $\Omega_{1/\lambda}$: connectedness and “open regularity” (Theorem 17); moreover, we show that these assumptions are sharp (Examples 21 and 22). It turns out that they are satisfied for example when Ω is convex. Remarkably, such uniqueness result on convex domains distinguishes the case of the ∞ -Laplacian from the case of the p -Laplacian, when we have multiplicity of solutions also in case of the ball.

- *Characterization of solutions.* For $\lambda \geq 1/R_\Omega$, we show that u is a solution to $(P)_\lambda$ if and only if it is the infinity harmonic potential of a set K belonging to a suitable family of compact subsets of $\overline{\Omega_{1/\lambda}}$. This result (Theorem 26) gives a complete picture of solutions to $(P)_\lambda$ in case Ω is an arbitrary domain.

- *Regularity.* As a by-product of the results described so far, combined with well-known facts about the regularity of infinity harmonic functions, we obtain that, for $\lambda \geq 1/R_\Omega$, any genuine solution is everywhere differentiable in $\Omega^+(u)$ (and $C^{1,\alpha}$ in dimension $n = 2$). Furthermore, the free boundary essentially shares the same regularity properties of the level set $\{\text{dist}(x, \partial\Omega) = 1/\lambda\}$ of the distance function. More precisely, if we denote by $\Sigma(\Omega)$ the cut locus of Ω (i.e., the closure of the set of points where the distance from $\partial\Omega$ is not differentiable), then $F(u) \setminus \Sigma(\Omega)$ is locally $C^{1,1}$. As a particular case, if $\lambda > 1/\text{dist}(\partial\Omega, \Sigma(\Omega))$, then $F(u)$ is of class $C^{1,1}$ and, if in addition $\partial\Omega$ is of class $C^{k,\alpha}$ for some $k \geq 2$, then $F(u)$ is of class $C^{k,\alpha}$ (see e.g. [24, Theorem 6.10]).

- *Relationship with the p -Bernoulli problem.* We show that, if Ω is convex and regular, both the p -Bernoulli constants $\lambda_{\Omega,p}$ and $\Lambda_{\Omega,p}$ defined as in Section 1.1 above converge to $\lambda_{\Omega,\infty} = 1/R_\Omega$ in the limit as $p \rightarrow +\infty$ (Corollary 28). Moreover, if u_p are solutions to the interior p -Bernoulli problem, we prove that they converge uniformly to the solution to problem $(P)_\lambda$ provided such solution is unique, and provided u_p are *variational* solutions, namely they are minimizers of functionals (1) over $W_1^{1,p}(\Omega)$ (Theorem 31).

1.5. Open problems. Let us conclude this Introduction by addressing some among the many interesting questions related to the results contained in this paper:

- (i) Is it possible to extend at least some of our results to the case of non-constant boundary data?
- (ii) Does the unique solution to problem $(P)_\lambda$ on a convex domain have convex level sets?
- (iii) In cases when there are multiple genuine solutions, does it exist a *minimal* genuine solution, and how can it be characterized?

- (iv) When the solution to problem $(P)_\lambda$ is not unique, is it still true that the variational solutions u_p to the interior p -Bernoulli problem converge in the limit as $p \rightarrow +\infty$, and what is their limit?

2. SOME PRELIMINARY RESULTS

In this section we collect some material which will be useful throughout the paper. To be self-contained, we start by giving a quick recall of some basic facts about infinity harmonic functions, for which we refer to [4, 16, 17].

Then we establish some general properties of (non-constant) solutions to $(P)_\lambda$ and of infinity harmonic potentials, which will play a crucial role in the sequel.

Let us firstly introduce some notation. We shall write for brevity $d(x) := \text{dist}(x, \partial\Omega)$, $x \in \bar{\Omega}$. Moreover, we denote by $R_\Omega := \max_{\bar{\Omega}} d$ the inradius of Ω , and for any $r \in [0, R_\Omega]$, we set

$$\begin{aligned}\Omega_r &= \{d > r\} := \{x \in \Omega : d(x) > r\}, \\ \{d \geq r\} &:= \{x \in \Omega : d(x) \geq r\}, \\ D_r &:= \Omega \setminus \bar{\Omega}_r.\end{aligned}$$

For every $x \in \bar{\Omega}$ we denote by

$$(3) \quad \Pi_{\partial\Omega}(x) := \{z \in \partial\Omega : d(x) = |z - x|\}$$

the set of the closest points (or projections) of x on $\partial\Omega$.

2.1. About infinity harmonic functions. A function $u \in C(\Omega)$ is called *infinity subharmonic* (resp. *infinity superharmonic*) if it satisfies condition (a1) (resp. (a2)) in Definition 1. It is called *infinity harmonic* if it is both infinity subharmonic and superharmonic. An infinity harmonic function on Ω is differentiable at every point $x \in \Omega$ in any space dimension, and of class $C^{1,\alpha}(\Omega)$ in dimension $n = 2$ [31, 32, 56].

The following conditions are equivalent:

- (i) u is infinity harmonic in Ω ;
- (ii) u has the *absolutely minimizing Lipschitz* property, which means that u is locally Lipschitz in Ω and, for every open set $\omega \Subset \Omega$ and every $v \in C(\bar{\omega})$, with $v = u$ on $\partial\omega$, $\|\nabla u\|_{L^\infty(\omega)} \leq \|\nabla v\|_{L^\infty(\omega)}$. The space of functions u having this property is denoted by $AML(\Omega)$;
- (iii) the functions $w = u$ and $w = -u$ enjoy *comparison with cones from above* in Ω , which means that, for every open set $\omega \Subset \Omega$ and for every $a, b \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, it holds

$$w(x) \leq C(x) := a + b|x - x_0|, \quad \forall x \in \partial(\omega \setminus \{x_0\}) \quad \implies \quad w(x) \leq C(x), \quad \forall x \in \omega.$$

Let u be infinity harmonic in Ω , and let $\bar{B}_r(x) \subset \Omega$. Then

$$(4) \quad \max_{y \in \bar{B}_r(x)} u(y) = \max_{y \in \partial B_r(x)} u(y), \quad \min_{y \in \bar{B}_r(x)} u(y) = \min_{y \in \partial B_r(x)} u(y),$$

and the following relations hold:

$$(5) \quad |\nabla u(x)| \leq \max_{y \in \overline{B}_r(x)} \frac{u(y) - u(x)}{r} = \max_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r},$$

$$(6) \quad |\nabla u(x)| \leq - \min_{y \in \overline{B}_r(x)} \frac{u(y) - u(x)}{r} = - \min_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r}$$

(see [16, Lemma 4.3]). Moreover, if the maximum and minimum at the right-hand side of (5), (6) are attained respectively at $p, q \in \partial B_r(x)$, i.e. if

$$p, q \in \partial B_r(x) : \quad u(p) = \max_{y \in \partial B_r(x)} u(y), \quad u(q) = \min_{y \in \partial B_r(x)} u(y),$$

then the following *increasing slope estimates* hold:

$$(7) \quad |\nabla u(x)| \leq |\nabla u(p)|, \quad |\nabla u(x)| \leq |\nabla u(q)|$$

(see [16, Proposition 6.2]).

2.2. Properties of solutions to $(P)_\lambda$. Observe that, if u is a strictly positive solution to $(P)_\lambda$, then by uniqueness $u \equiv 1$. Hence, any non-constant solution to $(P)_\lambda$ must vanish at some point of Ω , i.e., $F(u) \neq \emptyset$.

Proposition 2 (gradient estimate). *Let $u \in C(\overline{\Omega})$ be a solution to $(P)_\lambda$. Then $|\nabla u(x)| \leq \lambda$ for every $x \in \Omega^+(u)$.*

Proof. If u is a constant solution then the result is trivial. Let u be a non-constant solution. Let $x_0 \in \Omega^+(u)$ and let us prove that $|\nabla u(x_0)| \leq \lambda$. Since the statement is trivial if $\nabla u(x_0) = 0$, let us assume that $\nabla u(x_0) \neq 0$. In this case, we claim that there exists a finite family x_0, x_1, \dots, x_N of points with the following properties:

$$(8) \quad x_0, \dots, x_{N-1} \in \{u \leq u(x_0)\} \cap \Omega^+(u), \quad x_N \in F(u),$$

$$(9) \quad |\nabla u(x_j)| \geq |\nabla u(x_{j-1})| \quad \forall j = 1, \dots, N-1, \quad u(x_{N-1}) \geq \text{dist}(x_{N-1}, F(u)) |\nabla u(x_{N-1})|.$$

Since $u(x_0) < 1$ and u is continuous, the sub-level $\mathcal{C} := \{u \leq u(x_0)\}$ is a compact subset of Ω . Hence we can find $\rho > 0$ such that $\mathcal{C} \subset \Omega_\rho$. Then we fix $r \in (0, \rho)$ and we proceed as follows.

Assume we are given $x_{j-1} \in \{u \leq u(x_0)\} \cap \Omega^+(u)$, and let us construct the point x_j .

If $\overline{B}_r(x_{j-1}) \subset \Omega^+(u)$, then we let $x_j \in \overline{B}_r(x_{j-1})$ be such that

$$u(x_j) = \min_{y \in \overline{B}_r(x_{j-1})} u(y).$$

By definition, we have immediately $u(x_j) \leq u(x_{j-1})$, so that $x_j \in \mathcal{C} \cap \Omega^+(u)$. Moreover, since u is infinity-harmonic in $\Omega^+(u)$, by (4) and (7) it turns out that $x_j \in \partial B_r(x_{j-1})$ and $|\nabla u(x_j)| \geq |\nabla u(x_{j-1})|$.

If $\overline{B}_r(x_{j-1})$ is not contained in $\Omega^+(u)$, by our choice of r we have necessarily $\overline{B}_r(x_{j-1}) \cap F(u) \neq \emptyset$. (Indeed, since $x_{j-1} \in \mathcal{C} \subset \Omega_\rho$ and $r \in (0, \rho)$, we have $\overline{B}_r(x_{j-1}) \cap \partial \Omega = \emptyset$.) In this case, we set $N = j$, ending the construction, and we let $x_N \in F(u)$ be a closest point of x_{N-1} to $F(u)$. Setting $\delta := \text{dist}(x_{N-1}, F(u)) = |x_N - x_{N-1}|$ and taking into account $u(x_N) = 0 = \min_{y \in \overline{B}_\delta(x_{N-1})} u(y)$, by (6) we obtain

$$|\nabla u(x_{N-1})| \leq - \min_{y \in \overline{B}_\delta(x_{N-1})} \frac{u(y) - u(x_{N-1})}{\delta} = \frac{u(x_{N-1})}{\delta}.$$

It remains to show that our construction always stops in a finite number of steps. Specifically, for every $j = 1, \dots, N-1$, applying again (6), we obtain

$$|\nabla u(x_{j-1})| \leq - \min_{y \in \overline{B}_r(x_{j-1})} \frac{u(x) - u(x_{j-1})}{r} = \frac{u(x_{j-1}) - u(x_j)}{r};$$

hence

$$u(x_j) \leq u(x_{j-1}) - r|\nabla u(x_{j-1})| \leq u(x_{j-1}) - r|\nabla u(x_0)|,$$

so that in a finite number of steps we arrive at $F(u)$ thanks to the assumption $\nabla u(x_0) \neq 0$. Now, let us consider the open ball $B_\delta(x_{N-1}) \subset \Omega^+(u)$. By comparison with cones [17, Theorem 3.1], we have

$$(10) \quad u(x) \geq \varphi(x) := u(x_{N-1}) \left(1 - \frac{1}{\delta}|x - x_{N-1}|\right) \quad \forall x \in B_\delta(x_{N-1}).$$

Since $x_N \in \partial B_\delta(x_{N-1})$, by (10) we have that

$$\varphi^+ \prec_{x_N} u, \quad |\nabla \varphi(x_N)| = \frac{u(x_{N-1})}{\delta} (\neq 0).$$

Then, by applying first Definition 1(c1) and then the inequalities (9), we finally get

$$\lambda \geq \frac{u(x_{N-1})}{\delta} \geq |\nabla u(x_{N-1})| \geq |\nabla u(x_0)|,$$

and the proof is completed. \square

Proposition 3 (free boundary location). *Let $u \in C(\overline{\Omega})$ be a non-constant solution to $(P)_\lambda$. Then $\text{dist}(F(u), \partial\Omega) \geq \frac{1}{\lambda}$ (or, equivalently, $\{u = 0\} \subseteq \{d \geq \frac{1}{\lambda}\}$). If, in addition, $\text{int}\{u = 0\} \neq \emptyset$, then $\text{dist}(F(u), \partial\Omega) = \frac{1}{\lambda}$.*

Proof. Let $x \in F(u)$ and let $y \in \Pi_{\partial\Omega}(x) \subset \partial\Omega$ be a closest point to $\partial\Omega$. If $]y, x[\cap F(u) \neq \emptyset$, let $x_0 \in]y, x[\cap F(u)$ be the nearest point of $]y, x[\cap F(u)$ to $\partial\Omega$, otherwise let $x_0 := x$. By Proposition 2, we have

$$1 = u(y) - u(x_0) \leq \lambda d(x_0),$$

hence

$$d(x) \geq d(x_0) \geq \frac{1}{\lambda},$$

i.e. $x \in \{d \geq \frac{1}{\lambda}\}$. Hence, $F(u) \subseteq \{d \geq \frac{1}{\lambda}\}$, i.e. $\text{dist}(F(u), \partial\Omega) \geq \frac{1}{\lambda}$.

Let us prove that, if $\text{int}\{u = 0\} \neq \emptyset$, then also the opposite inequality holds. Let $r := \text{dist}(\partial\Omega, F(u))$. The function $v(x) := \frac{1}{r} \text{dist}(x, F(u))$ is infinity superharmonic in $\Omega^+(u)$ (see e.g. [42, p. 212]), and satisfies $v = 0$ on $F(u)$ and $v \geq 1$ on $\partial\Omega$. Hence, by the comparison principle for infinity harmonic functions [40, Theorem 2.22], we have that $v \geq u$ in $\Omega^+(u)$. Since $\text{int}\{u = 0\} \neq \emptyset$, there exists a ball $B = B_\rho(y) \subset \Omega \setminus \Omega^+(u)$ that is tangent to $F(u)$ at some point $x_0 \in F(u)$. Hence,

$$u(x) \leq v(x) \leq \frac{|x - x_0|}{r} \leq \frac{|x - y| - \rho}{r} =: \varphi(x), \quad x \in \Omega^+(u),$$

so that

$$u \prec_{x_0} \varphi^+, \quad |\nabla \varphi(x_0)| = \frac{1}{r} (\neq 0),$$

and by Definition 1(c2) we conclude that $1/r \geq \lambda$. \square

2.3. Properties of infinity-harmonic potentials.

Definition 4. Given a non-empty compact set $K \subset \Omega$, the infinity-harmonic potential of K relative to Ω is the unique viscosity solution w_K to the problem

$$(11) \quad \begin{cases} -\Delta_\infty w_K = 0, & \text{in } \Omega \setminus K, \\ w_K = 1, & \text{on } \partial\Omega, \\ w_K = 0, & \text{on } K. \end{cases}$$

Remark 5. Since $\Omega \setminus K$ may be disconnected, some words to explain the well-posedness of the above definition are in order. Let us write the open set $\Omega \setminus K$ as the union of its connected components $\{A^\alpha : \alpha \in I\}$. For every $\alpha \in I$, we have that $\partial A^\alpha \subseteq \partial\Omega \cup K$, and the function $f^\alpha : \partial A^\alpha \rightarrow \mathbb{R}$ defined by

$$(12) \quad f^\alpha := \begin{cases} 1, & \text{on } \partial A^\alpha \cap \partial\Omega, \\ 0, & \text{on } \partial A^\alpha \cap K, \end{cases}$$

is continuous on ∂A^α (being constant on each connected component of ∂A^α). Therefore, for every $\alpha \in I$, there exists a unique solution $w^\alpha \in C(\overline{A^\alpha})$ to the Dirichlet problem

$$\begin{cases} -\Delta_\infty w^\alpha = 0, & \text{in } A^\alpha, \\ w^\alpha = f^\alpha, & \text{on } \partial A^\alpha \end{cases}$$

(see [4, Theorems 3.1 and 6.1]). Consequently, problem (11) admits a unique solution, which is precisely the function $w_K \in C(\overline{\Omega})$ defined by $w_K = w^\alpha$ on $\overline{A^\alpha}$, $\alpha \in I$.

Remark 6. It is clear from Definition 4 that the set K is contained in $\{w_K = 0\}$. We point out that the inclusion may be strict. For instance, this happens when $\Omega = B_2(0)$ and $K = \partial B_1(0)$: in this case, K is strictly contained in $\{w_K = 0\} = \overline{B}_1$.

In general, given a non-empty compact set $K \subset \Omega$, necessary and sufficient conditions for the equality $K = \{w_K = 0\}$ can be given by looking at the connected components A^α of $\Omega \setminus K$ introduced in Remark 5. Letting

$$(13) \quad I_0 := \{\alpha \in I : \partial A^\alpha \subseteq K\}$$

we have

Lemma 7. *For a given non-empty compact set $K \subset \Omega$, the following properties are equivalent:*

- 1) $\{w_K = 0\} = K$;
- 2) the set I_0 defined in (13) is empty;
- 3) every point $x \in \Omega \setminus K$ can be joined to $\partial\Omega$ through a path in $\overline{\Omega} \setminus K$.

Proof. The equivalence between 2) and 3) follows immediately from the fact that the points in $\overline{\Omega} \setminus K$ which can be joined to $\partial\Omega$ through a path in $\overline{\Omega} \setminus K$ are precisely the points in $\overline{\Omega} \setminus K$ which belong to some set A^α with $\alpha \in I \setminus I_0$.

The fact that each of these conditions is equivalent to 1) follows by observing that

$$(14) \quad \{w_K = 0\} = K \cup \bigcup_{\alpha \in I_0} A^\alpha, \quad \{w_K > 0\} = \bigcup_{\alpha \in I \setminus I_0} A^\alpha.$$

Indeed, it is clear that $f^\alpha \equiv 0$ for $\alpha \in I_0$. On the other hand, for every x_0 belonging to a set A^α with $\alpha \in I \setminus I_0$, $w_K(x_0)$ is strictly positive, because such a point x_0 can be joined

to $\partial\Omega$ through a path in $\overline{\Omega} \setminus K$. Consequently, the value $w_K(x_0)$ can be estimated from below by a positive constant according to the next result (which is essentially taken from [6, Lemma 3.2]). \square

Proposition 8 (Harnack inequality). *Let $K \subset \Omega$ be a non-empty compact set, and let w_K be its infinity-harmonic potential relative to Ω . Let $x_0 \in A^\alpha$, with $\alpha \in I \setminus I_0$, and let γ be a path in $\overline{\Omega} \setminus K$ connecting x_0 to $\partial\Omega$. Then*

$$(15) \quad w_K(x_0) \geq e^{-L/\delta},$$

where L is the length of γ , and δ is the distance from γ to K .

Proof. By possibly taking a slightly larger value of δ (but smaller than $\text{dist}(\gamma, K)$), it is not restrictive to assume that γ is a polygonal curve. Moreover, for $m \in \mathbb{N}$ large enough, we can assume that the polygonal has exactly $m + 1$ vertices $x_0, x_1, \dots, x_m = y$ with $|x_j - x_{j-1}| = L/m$ for every $j = 1, \dots, m$. By possibly moving a bit the point y (shortening the curve), we can also assume that y is a closest point in $\partial\Omega$ from x_{m-1} . Since $x_{j-1} \in B_\delta(x_j) \subset \Omega \setminus K$ for every $j = 1, \dots, m$, by comparison with cones, we have

$$w_K(x_{j-1}) \geq w_K(x_j) \left(1 - \frac{|x_j - x_{j-1}|}{\delta}\right) = w_K(x_j) \left(1 - \frac{L}{m\delta}\right),$$

so that

$$w_K(x_0) \geq w_K(y) \left(1 - \frac{L}{m\delta}\right)^m.$$

Since $w_K(y) = 1$ and m can be taken arbitrarily large, we finally get (15). \square

We conclude with a useful characterization of the infinity harmonic potential w_K along rays connecting K with $\partial\Omega$:

Proposition 9 (potential along rays). *Let $K \subset \Omega$ be a non-empty compact set, and let w_K be the infinity-harmonic potential of K relative to Ω . If $y \in \partial\Omega$ and $z \in K$ are two points such that $|y - z| = \text{dist}(\partial\Omega, K)$, then w_K is affine on the segment $[y, z]$.*

Proof. Set $R := \text{dist}(\partial\Omega, K)$. Since w_K enjoys comparison with cones from below, we have

$$w_K(x) \geq f(x) := 1 - \frac{|x - y|}{R}, \quad \forall x \in B_R(y) \cap \Omega.$$

On the other hand, the function $g(x) := \frac{1}{R} \text{dist}(x, K)$ is infinity superharmonic in $\Omega \setminus K$, with $g = 0$ on K and $g \geq 1$ on $\partial\Omega$, hence $g \geq w_K$ by the comparison principle for infinity harmonic functions. Since $f = g$ on the segment $[y, z]$, the statement follows. \square

3. EXISTENCE

We start the analysis of existence of solutions to problem $(P)_\lambda$ by observing that, for any $\lambda \geq 1/R_\Omega$, it admits many solutions whose zero level set is Lebesgue negligible. Inspired by the results of the previous section, they are found among infinity harmonic potentials w_K of suitably chosen compact sets K contained in $\{d \geq \frac{1}{\lambda}\}$. Recall that the zero set of w_K can be characterized as in (14); in particular, by Lemma 7, we have that $\{w_K = 0\} = K$ if and only if the set I_0 defined in (13) is empty.

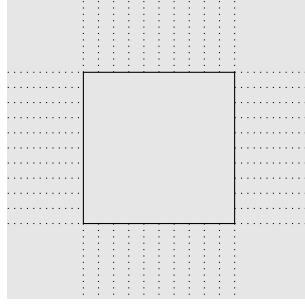


FIGURE 1. The sets $\Omega = (-2, 2) \times (-2, 2)$ (grey), $\partial\Omega_1$ (black), and \widehat{D}_1 (dotted)

Proposition 10. *Let $\lambda \geq 1/R_\Omega$, and let $K \subseteq \{d \geq \frac{1}{\lambda}\}$ be a non-empty compact set. Assume that*

$$(16) \quad \text{int}(K) = \emptyset \quad \text{and} \quad I_0 = \emptyset.$$

Then the infinity-harmonic potential w_K of K relative to Ω is a solution to $(P)_\lambda$.

Proof. By (16) and (14), the set $\{w_K = 0\}$ agrees with K and has empty interior, so that $\{w_K = 0\} = K = F(w_K)$. Thus, we have to show that the free boundary condition in Definition 1 is satisfied at every point $x_0 \in K$.

Since $K \subseteq \{d \geq \frac{1}{\lambda}\}$, by comparison with cones we have that $w_K(x) \leq \lambda|x - x_0|$ for every $x \in \Omega$. If $\varphi^+ \prec_{x_0} w_K$, then necessarily $|\nabla\varphi(x_0)| \leq \lambda$, hence condition (c1) in Definition 1 is satisfied.

If $w_K \prec_{x_0} \varphi^+$, then $\varphi \geq 0$ in Ω , because $\varphi^+ \geq w_K > 0$ in $\Omega \setminus \{w_K = 0\}$ and $\overline{\Omega} \setminus \{w_K = 0\} = \overline{\Omega}$. Since $\varphi(x_0) = 0$, then x_0 is a minimum point for the regular function φ , hence we can conclude that $\nabla\varphi(x_0) = 0$, and also condition (c2) in Definition 1 is satisfied. \square

Motivated by Proposition 10, we give the following definition.

Definition 11 (Genuine solutions). We say that a solution u to $(P)_\lambda$ is genuine if the set $\{u = 0\}$ has non-empty interior (and non-genuine otherwise).

Remark 12. In the special case $\lambda = \frac{1}{R_\Omega}$, problem $(P)_\lambda$ admits only non-genuine solutions. Indeed, we know from Proposition 3 that, for every solution u to problem $(P)_\lambda$, $F(u)$ is contained in the high ridge $\{d(x) = R_\Omega\}$ and hence the set $\{u = 0\}$ has necessarily empty interior.

We are now going to deal with the existence of genuine solutions to $(P)_\lambda$, for $\lambda > \frac{1}{R_\Omega}$. To that aim, we introduce two more definitions.

Definition 13. Given $r \in (0, R_\Omega)$, we define w_r as the infinity harmonic potential of $\overline{\Omega}_r$ relative to Ω , namely the unique solution to

$$\begin{cases} \Delta_\infty w_r = 0 & \text{in } D_r := \Omega \setminus \overline{\Omega}_r \\ w_r = 1 & \text{on } \partial\Omega \\ w_r = 0 & \text{in } \overline{\Omega}_r. \end{cases}$$

Definition 14. Given $r \in (0, R_\Omega]$, we set

$$\widehat{D}_r := \bigcup_{y \in \partial\Omega_r} \{y, z[: z \in \Pi_{\partial\Omega}(y)\},$$

where $\Pi_{\partial\Omega}(y) \subset \partial\Omega$ is the set of projections of y defined in (3).

Remark 15. Notice that, by definition, \widehat{D}_r is a subset of D_r , with possibly strict inclusion (think for instance to the case when Ω is a square, see Figure 1).

Theorem 16. (a) For every $\lambda > \frac{1}{R_\Omega}$, the function $w_{\frac{1}{\lambda}}$ is a genuine solution to problem $(P)_\lambda$; moreover it satisfies the estimates

$$(17) \quad 1 - \lambda d(x) \leq w_{\frac{1}{\lambda}}(x) \leq \lambda \operatorname{dist}(x, \partial\Omega_{\frac{1}{\lambda}}) \quad \text{in } \overline{D_{\frac{1}{\lambda}}}, \quad \text{with equalities in } \widehat{D_{\frac{1}{\lambda}}}.$$

(b) For every $\lambda \in (0, \frac{1}{R_\Omega})$, problem $(P)_\lambda$ does not admit non-constant solutions.

Proof. Throughout the proof, since λ is fixed, we set for brevity

$$w := w_{\frac{1}{\lambda}}, \quad D := D_{\frac{1}{\lambda}}, \quad \widehat{D} := \widehat{D_{\frac{1}{\lambda}}}.$$

Let us first show that w satisfies the inequalities in (17). This can be proved using the same arguments of Proposition 9. More precisely, the function $v(x) := 1 - \lambda d(x)$ is infinity subharmonic (since d is infinity superharmonic), and satisfies the equality $v = w$ on both $\partial\Omega$ and $\partial\Omega_{\frac{1}{\lambda}}$. By the comparison principle for infinity harmonic functions, it follows that $w \geq v$ in \overline{D} .

Similarly, the function $z(x) := \lambda \operatorname{dist}(\cdot, \partial\Omega_{\frac{1}{\lambda}})$ is infinity superharmonic, and satisfies $z = w = 0$ on $\partial\Omega_{\frac{1}{\lambda}}$, $w \leq z$ on $\partial\Omega$. Again by the comparison principle for infinity harmonic functions, we infer that $w \leq z$ in \overline{D} .

In order to obtain that the inequalities in (17) hold as equalities in \widehat{D} , we firstly notice that $\|\nabla w\|_\infty = \lambda$. Indeed, the inequality $\|\nabla w\|_\infty \geq \lambda$ follows immediately from the estimate

$$\|\nabla w\|_\infty \geq \sup \left\{ \frac{|w(x) - w(y)|}{|x - y|} : x \in \partial\Omega, y \in \partial\Omega_{\frac{1}{\lambda}} \right\};$$

the converse one follows from the fact that w has the AML property in D , which entails in particular $\|\nabla w\|_\infty \leq \|\nabla v\|_\infty = \lambda$.

Now assume by contradiction that the strict inequality $w > v$ holds at some point $x \in \widehat{D}$. If x belongs to the segment $]y, z[$, with $y \in \partial\Omega_{\frac{1}{\lambda}}$ and $z \in \Pi_{\partial\Omega}(y)$, we have that

$$\|\nabla w\|_\infty \geq \frac{|w(x) - w(y)|}{|x - y|} = \frac{w(x) - v(x)}{|x - y|} > \frac{v(x) - v(y)}{|x - y|} = \frac{|v(x) - v(y)|}{|x - y|} = \lambda.$$

Here, in the last equality we have exploited the fact that $d(x) - d(y) = |x - y|$. Indeed, if $x \in]y, z[\subset \widehat{D}_r$, with $y \in \partial\Omega_r$ and $z \in \Pi_{\partial\Omega}(y)$, it holds that $d(x) = r - |x - y|$ and $\operatorname{dist}(x, \partial\Omega_r) = r - |x - z|$, which implies in particular

$$(18) \quad r - d(x) = |x - y| = r - |x - z| = \operatorname{dist}(x, \partial\Omega_r).$$

We have thus contradicted the equality $\|\nabla w\|_\infty = \lambda$, and we conclude that $w(x) = v(x)$. Since, by (18), $v(x) = z(x)$ on \widehat{D} , the proof of (17) is achieved.

(a) We are now in a position to prove that w solves problem $(P)_\lambda$, which amounts to show that it satisfies the free boundary condition (c) of Definition 1 along the free boundary $F(w) = \partial\Omega_{\frac{1}{\lambda}}$.

Let $x_0 \in \partial\Omega_{\frac{1}{\lambda}}$, let $\varphi^+ \prec_{x_0} w$, with $p := \nabla\varphi(x_0) \neq 0$. By the upper bound inequality in (17), we have

$$\varphi(x) \leq w(x) \leq \lambda \operatorname{dist}(x, \partial\Omega_{\frac{1}{\lambda}}) \quad \forall x \in D,$$

hence

$$\varphi(x_0 + tp) \leq \lambda \operatorname{dist}(x_0 + tp, \partial\Omega_{\frac{1}{\lambda}}) \leq \lambda t |p|, \quad t > 0 \text{ small.}$$

Dividing by $t > 0$ and taking the limit as $t \rightarrow 0^+$ we get $|p|^2 \leq \lambda |p|$, hence $|p| \leq \lambda$, so that (c1) holds.

Let us now consider condition (c2) at a point $x_0 \in \partial\Omega_{\frac{1}{\lambda}}$. Let $y \in \Pi_{\partial\Omega}(x_0)$. By (17), the function w is affine with slope λ on the segment $]x_0, y[\subset \widehat{D}$. If φ is a test function as in condition (c2), setting $\nu := (y - x_0)/|y - x_0|$, we have that

$$|\nabla\varphi(x_0)| \geq \lim_{t \rightarrow 0^+} \frac{\varphi(x_0 + t\nu)}{t} \geq \lim_{t \rightarrow 0^+} \frac{w(x_0 + t\nu)}{t} = \lambda,$$

and (c2) follows.

(b) We observe that, if u is a non-constant solution to $(P)_\lambda$ (for an arbitrary $\lambda > 0$), it holds that

$$(19) \quad \sup_{x \in \Omega^+(u)} |\nabla u(x)| \geq 1/R_\Omega.$$

Indeed, if we assume that $|\nabla u(x)| \leq L < 1/R_\Omega$ for every $x \in \Omega^+(u)$, then we obtain

$$u(x) \geq 1 - Ld(x) \geq 1 - LR_\Omega > 0 \quad \forall x \in \overline{\Omega^+(u)},$$

a contradiction.

Statement (b) is a direct consequence of Proposition 3, since $F(u) \neq \emptyset$ for non-constant solutions. \square

4. UNIQUENESS

Prior to starting the analysis of the uniqueness of solutions for problem $(P)_\lambda$, we emphasize that one has to restrict attention to the class of genuine solutions and to choose $\lambda > 1/R_\Omega$. Indeed, if these requirements are dropped, by applying the results of the previous section we readily get the following conclusions:

- For $\lambda > 1/R_\Omega$, according to Proposition 10 there exist infinitely many non-genuine solutions to $(P)_\lambda$, corresponding to the infinity harmonic potentials of any compact set $K \subseteq \{d \geq \frac{1}{\lambda}\}$ satisfying (16).
- For $\lambda = 1/R_\Omega$, we know that all the solutions to $(P)_\lambda$ are non-genuine (*cf.* Remark 12). Moreover, it is easy to see that any compact set K contained in the high ridge of Ω satisfies (16). Therefore, there exist either one or multiple non-constant solutions to $(P)_\lambda$ respectively when the high ridge is a singleton or not.

We are thus led to formulate the question as:

When uniqueness of genuine solutions to $(P)_\lambda$ occurs for $\lambda > 1/R_\Omega$?

Our answer is given in the statement below.

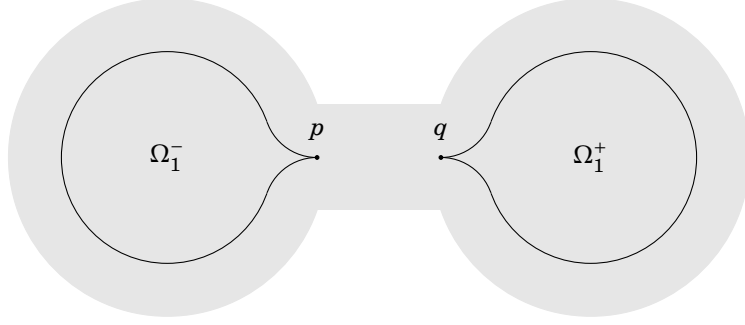
Theorem 17 (Uniqueness of genuine solutions). *Let $\lambda > 1/R_\Omega$. Assume that*

(H1) $\Omega_{\frac{1}{\lambda}}$ *is connected;*

(H2) $\overline{\Omega_{\frac{1}{\lambda}}} = \{d \geq \frac{1}{\lambda}\}$.

Then $w_{\frac{1}{\lambda}}$ is the unique genuine solution to $(P)_\lambda$.

Corollary 18. *Assume Ω is convex. For every $\lambda > 1/R_\Omega$, $w_{\frac{1}{\lambda}}$ is the unique genuine solution to problem $(P)_\lambda$.*

FIGURE 2. The set Ω of Example 21 (grey) and $\partial\Omega_1$ (black)

Remark 19. (About the connectness assumption (H1)). When Ω is convex, assumption (H1) is satisfied because also Ω_r is convex for every $r \in [0, R_\Omega)$. For general Ω , (H1) is satisfied if $\frac{1}{\lambda} < \text{dist}(\partial\Omega, \Sigma(\Omega))$, $\Sigma(\Omega)$ being the cut locus of Ω , namely the closure of the set of points where the distance from $\partial\Omega$ is not differentiable. Indeed, if $r < \text{dist}(\partial\Omega, \Sigma(\Omega))$, then $\Sigma(\Omega) \subset \Omega_r$ and $\Sigma(\Omega_r) = \Sigma(\Omega)$. By Theorem 5.3 in [1], Ω and Ω_r have the same homotopy class as $\Sigma(\Omega)$. Since Ω is connected by assumption, then also $\Sigma(\Omega)$ and Ω_r are connected.

Remark 20. (About the regularity assumption (H2)). When Ω is convex, assumption (H2) is satisfied because $\overline{\Omega_r}$ agrees with $\{d \geq r\}$ for every $r \in [0, R_\Omega)$. For general Ω , we have the inclusion $\overline{\Omega_r} \subseteq \{d \geq r\}$, which may be possibly strict (see for instance Example 22 below). Assumption (H2) can be also rephrased by asking that the set $C := \{d \geq \frac{1}{\lambda}\}$ satisfies $C = \overline{\text{int } C}$. In topology, sets satisfying this last condition are known as *regular closed sets*. It is clear from the definition that such sets are closed in the usual sense, and have a non-empty interior if they are not empty.

Assumptions (H1) and (H2) are sharp, as we can have multiple genuine solutions as soon as $\Omega_{\frac{1}{\lambda}}$ is not connected and/or $\overline{\Omega_{\frac{1}{\lambda}}} \neq \{d \geq \frac{1}{\lambda}\}$. This fact is illustrated in Examples 21 and 22 below.

Example 21 (Multiplicity of genuine solutions without (H1)). If $\Omega_{\frac{1}{\lambda}}$ is not connected, then problem $(P)_\lambda$ may have more than one genuine solution. Let us show this phenomenon with an explicit example. Let $\Omega \subset \mathbb{R}^2$ be the set

$$\Omega := B_3((-4, 0)) \cup B_3((4, 0)) \cup ((-4, 4) \times (-1, 1))$$

(see Figure 2), and let $\lambda = 1$.

The set Ω_1 is not connected, since it is the disjoint union of two connected components $\Omega_1^- := \Omega_1 \cap \{x_1 < 0\}$ and $\Omega_1^+ := \Omega_1 \cap \{x_1 > 0\}$.

We have proved in Theorem 16 that the function w_1 is a solution to $(P)_1$.

Furthermore, we claim that the infinity-harmonic potentials of $\overline{\Omega_1^\pm}$ relative to Ω are both solutions to $(P)_1$. Let us prove this claim when u is the infinity-harmonic potential of $\overline{\Omega_1^-}$. By Proposition 9 we have that $u(x) = w_1(x)$ on the set

$$A^- := \{x = (x_1, x_2) \in \Omega \setminus \overline{\Omega_1^-} : x_1 < 2\sqrt{2} - 4\}.$$

Hence, we already know that u satisfies the free boundary condition of Definition 1 at all points $x_0 \in F(u) = \partial\Omega_1^-, x_0 \neq p := (2\sqrt{2} - 4, 0)$. It remains to prove that the free boundary condition is satisfied at p . Since p has two projections $y_\pm := (2\sqrt{2} - 4, \pm 1)$ on $\partial\Omega$, it does

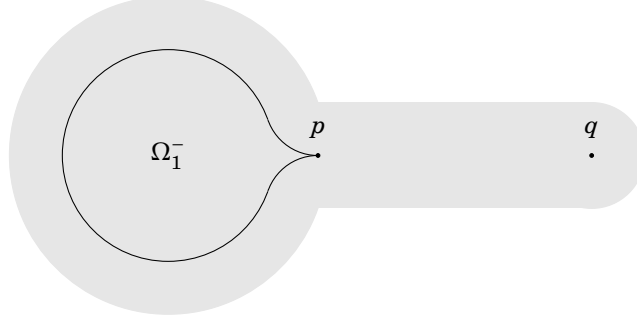


FIGURE 3. The set Ω of Example 22 (grey) and $\partial\Omega_1$ (black)

not exist a smooth function φ such that $u \prec_p \varphi^+$. On the other hand, if φ is a smooth function such that $\varphi^+ \prec_p u$, then necessarily $|\nabla\varphi(p)| \leq 1$, since $u(x) \leq \text{dist}(x, \overline{\Omega_1^-})$. This proves that u is a solution to $(P)_1$.

One can also construct infinitely many other genuine solutions to $(P)_1$. Specifically, let $q := -p$, let C be a closed subset of $[p, q] \cup \overline{\Omega_1^+}$ with empty interior, and let $K := C \cup \overline{\Omega_1^-}$. Then the infinity-harmonic potential of K relative to Ω turns out to be a solution to $(P)_1$. Another symmetric family of genuine solutions can be constructed by taking C a closed subset of $[p, q] \cup \overline{\Omega_1^-}$ with empty interior and $K := C \cup \overline{\Omega_1^+}$. (For both families, the free boundary condition can be checked by arguing with minor modifications as done in the proof of Proposition 10).

Example 22 (Multiplicity of genuine solutions without (H2)). More than one genuine solution may occur also in case $\overline{\Omega_{\frac{1}{\lambda}}}$ is strictly contained in $\{d \geq \frac{1}{\lambda}\}$. To enlighten this fact, let us modify the above example by considering the set

$$\Omega := B_3((-4, 0)) \cup B_1((4, 0)) \cup ((-4, 4) \times (-1, 1)).$$

Again, we take $\lambda = 1$. In this case, $\{d \geq 1\} = \overline{\Omega_1} \cup [p, q] \neq \overline{\Omega_1}$, with $p := (2\sqrt{2} - 4, 0)$ and $q := (4, 0)$. Similarly as above, for every closed subset C of the segment $[p, q]$, the infinity-harmonic potential of $K := \overline{\Omega_1^-} \cup C$ relative to Ω is a solution to $(P)_\lambda$.

We now turn to the proof of Theorem 17. It is based on the characterization of the set $\text{int}\{u = 0\}$ (see Proposition 24 below). We start by proving a simple geometric lemma.

Lemma 23. *Let A be a non-empty open subset of Ω such that, for some constant $R > 0$,*

$$(20) \quad d(x) = \text{dist}(x, \partial A) + R, \quad \forall x \in A.$$

Then A is a union of connected components of Ω_R . In particular, if Ω_R is connected, then $A = \Omega_R$.

Proof. From (20) we have that $d(x) > R$ for every $x \in A$, hence $A \subseteq \Omega_R$.

We claim that $\partial A \subseteq \partial\Omega_R$. Specifically, let $y \in \partial A$. For every $\varepsilon > 0$ there exists a point $x \in A$ such that $|x - y| < \varepsilon$, so that, by (20),

$$\begin{aligned} d(y) &< d(x) + \varepsilon = \text{dist}(x, \partial A) + R + \varepsilon < R + 2\varepsilon, \\ d(y) &> d(x) - \varepsilon = \text{dist}(x, \partial A) + R - \varepsilon > R - 2\varepsilon, \end{aligned}$$

hence $d(y) = R$, and the claim is proved.

Let A' be a connected component of A , and let B a connected component of Ω_R such $A' \cap B \neq \emptyset$. By the previous claim, $\partial A' \cap B = \emptyset$, hence B can be written as the union of the two open sets A' and $B \setminus \overline{A'}$. Since B is connected, then necessarily $B \setminus \overline{A'} = \emptyset$ and $A' = B$. \square

Proposition 24. *Let $\lambda > 1/R_\Omega$ and let u be a solution to $(P)_\lambda$. Then $\text{int}\{u = 0\}$ is a (possibly empty) union of connected components of $\Omega_{\frac{1}{\lambda}}$.*

Proof. We are going to prove that, if the set $A := \text{int}\{u = 0\}$ is not empty, it satisfies the assumption (20) of Lemma 23 with $R = \frac{1}{\lambda}$.

Let $x \in A$, let $x_0 \in \Pi_{\partial A}(x)$ and let $r := |x - x_0|$, so that $B_r(x) \subset A$ and $x_0 \in F(u)$. Let us consider the function

$$\varphi(y) := \frac{|y - x| - r}{d(x) - r}, \quad y \in \Omega.$$

We have that $\varphi(y) \geq 0$ for every $y \in \partial A \subseteq F(u)$, and $\varphi(y) \geq 1$ for every $y \in \partial\Omega$. Hence, by comparison, $\varphi \geq u$ in $\Omega \setminus A$ and, in particular, $u \prec_{x_0} \varphi^+$. By Definition 1, it follows that

$$(21) \quad |\nabla\varphi(x_0)| = \frac{1}{d(x) - r} \geq \lambda.$$

Let $z \in \Pi_{\partial\Omega}(x)$. The point $y_0 := x + r \frac{z-x}{|x-z|}$ belongs to $\overline{B}_r(x) \subset \overline{A}$ and, by Proposition 3, $\overline{B}_r(x) \subseteq \overline{A} \subseteq \overline{\Omega_{\frac{1}{\lambda}}}$, so that

$$(22) \quad \frac{1}{\lambda} \leq d(y_0) = |y_0 - z| = |x - z| - |x - y_0| = d(x) - r.$$

From (21) and (22) it follows that $d(x) - r = \frac{1}{\lambda}$, i.e. the assumptions of Lemma 23 hold with $R = \frac{1}{\lambda}$. \square

We are now in a position to give:

Proof of Theorem 17. Let u be a genuine solution to $(P)_\lambda$, for some $\lambda > 1/R_\Omega$.

Since by assumption the interior of $\{u = 0\}$ is not empty, by Proposition 24 it is a union of connected components of $\Omega_{\frac{1}{\lambda}}$ and hence, by assumption (H1), it agrees with $\Omega_{\frac{1}{\lambda}}$.

On the other hand, by Proposition 3, the closed set $\{u = 0\}$ is contained in $\{d \geq \frac{1}{\lambda}\}$ and, by assumption (H2), we have $\{d \geq \frac{1}{\lambda}\} = \overline{\Omega_{\frac{1}{\lambda}}}$.

Summarizing, we have

$$\Omega_{\frac{1}{\lambda}} = \text{int}(\{u = 0\}) \subseteq \{u = 0\} \subseteq \left\{d \geq \frac{1}{\lambda}\right\} = \overline{\Omega_{\frac{1}{\lambda}}}.$$

Hence, $\{u = 0\} = \overline{\Omega_{\frac{1}{\lambda}}}$ and $u = w_{\frac{1}{\lambda}}$. \square

5. CHARACTERIZATION OF SOLUTIONS

In the following theorem we will characterize all solutions to $(P)_\lambda$ as the infinity-harmonic potentials of compact subsets of Ω .

Definition 25. For a fixed $\lambda \geq 1/R_\Omega$, let \mathcal{K}_λ be the family of all non-empty sets $K \subset \mathbb{R}^n$ satisfying the following properties:

- (i) K is a compact subset of $\{d \geq 1/\lambda\}$.
- (ii) If \tilde{K} is a connected component of K with non-empty interior, then $\text{int } \tilde{K}$ coincides with a connected component of $\Omega_{1/\lambda}$.

(iii) If $\Omega \setminus K$ is decomposed as in Section 2.3, then the set I_0 defined in (13) is empty (cf. Lemma 7 for equivalent conditions).

Theorem 26. *Let $\lambda \geq 1/R_\Omega$. Then a function $u \in C(\overline{\Omega})$ is a non-constant solution to $(P)_\lambda$ if and only if there exists a set $K \in \mathcal{K}_\lambda$ such that $u = w_K$.*

Proof. The case $\lambda = 1/R_\Omega$ is trivial (see Remark 12), so that we shall assume that $\lambda > 1/R_\Omega$.

Let $u \in C(\overline{\Omega})$ be a solution to $(P)_\lambda$. Let us prove that the set $K := \{u = 0\}$ belongs to the class \mathcal{K}_λ introduced in Definition 25, and that $u = w_K$.

Condition (i) is satisfied by Proposition 3.

Condition (ii) is clearly satisfied if u is a non-genuine solution, while it follows from Proposition 24 if u is a genuine solution.

Condition (iii) can be easily checked arguing by contradiction. Specifically, assume that the set I_0 defined in (13) is not empty. In this case, there exists a connected component A of $\Omega \setminus K$ such that $\partial A \subset K$. But then, by uniqueness, necessarily $u = 0$ on A , with $A \cap K = \emptyset$, against the definition of K .

We have thus proved that $K \in \mathcal{K}_\lambda$. Finally we observe that, since K satisfies condition (iii), we have $\{w_K = 0\} = K$ (cf. Lemma 7), and hence $u = w_K$.

Vice versa, let $K \in \mathcal{K}_\lambda$ and let us prove that w_K is a solution to $(P)_\lambda$.

By property (iii) in Definition 25, we have that $F(w_K) = \partial K$, hence it is enough to prove that the free boundary condition is satisfied at any point of ∂K . Let $x_0 \in \partial K$.

We have two possibilities: either $x_0 \notin \text{int } \overline{K}$, or $x_0 \in \partial B$, where B is a connected component of $\text{int } K$ (which thanks to property (ii) in Definition 25 is also a connected component of the open set $\Omega_{1/\lambda}$).

If $x_0 \notin \text{int } \overline{K}$, we are done by arguing exactly as in Proposition 10 (in particular, by exploiting property (i) in Definition 25).

If $x_0 \in \partial B$, we argue as in the proof of Theorem 16(a). More precisely, we prove firstly that the following inequalities analogous to (17) are satisfied:

$$(23) \quad 1 - \lambda d(x) \leq w_K(x) \leq \lambda \text{dist}(x, \partial B), \quad \forall x \in \overline{\Omega} \setminus K,$$

with equalities for every $x \in [x_0, y_0]$, being $y_0 \in \Pi_{\partial\Omega}(x_0)$. Then, by using (23), we obtain the free boundary condition at x_0 by proceeding in the same way as in the second part of the proof of Theorem 16(a). \square

6. ASYMPTOTICS OF p -BERNOULLI PROBLEMS AS $p \rightarrow +\infty$

In this section we explore the relation between problem $(P)_\lambda$ and the interior Bernoulli problem for the p -Laplacian. For the benefit of the reader, we start by revisiting in more detail some facts which in part have been already mentioned in the Introduction (some bibliographical references already given therein are skipped below).

The interior Bernoulli free boundary problem for the p -Laplacian, for a given $p > 1$, consists in finding a (non-constant) solution to

$$(24) \quad \begin{cases} \Delta_p u = 0 & \text{in } \Omega^+(u), \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = \lambda & \text{on } F(u). \end{cases}$$

Then the *Bernoulli constant for the p -Laplacian* is defined by

$$\lambda_{\Omega,p} := \inf\{\lambda > 0 : (24) \text{ admits a non-constant solution}\}.$$

Here a solution to (24) is meant as a function $u \in W_1^{1,p}(\Omega)$ such that, according to [26], the free boundary condition is satisfied in the following weak sense:

$$(25) \quad \lim_{\varepsilon \rightarrow 0} \int_{\partial\{u > \varepsilon\} \cap \Omega} (|\nabla u| - \lambda)\eta \cdot \nu = 0 \quad \forall \eta \in W_0^{1,p}(\Omega; \mathbb{R}^n),$$

where ν is the unit outward normal.

In particular, when Ω is a regular convex domain, the following results due to Henrot and Shahgholian hold:

- for every $\lambda \geq \lambda_{\Omega,p}$, problem (24) admits a classical non-constant solution $u \in C(\overline{\Omega^+(u)}) \cap C^2(\Omega^+(u))$, which has convex level sets; moreover, the free boundary $F(u)$ is of class $C^{2,\alpha}$ [36, Thm. 2.1], and the free boundary condition is satisfied in the pointwise sense

$$\lim_{\Omega^+(u) \ni y \rightarrow x} |\nabla u(y)| = \lambda \quad \forall x \in F(u).$$

- $\lambda_{\Omega,p}$ can be characterized, loosely speaking, as the infimum of positive λ such that the family of sub-solutions to (24) is not empty, and it satisfies the lower bound

$$(26) \quad \lambda_{\Omega,p} \geq 1/R_\Omega$$

[35, Thms. 3.1 and 3.2].

When Ω is an arbitrary domain, not necessarily convex, following the celebrated work [2] by Alt and Caffarelli, in order to find solutions to problem (24) one can consider the integral functionals

$$J_p^\lambda(u) := \frac{1}{p} \int_\Omega \left(\frac{|\nabla u|}{\lambda} \right)^p + \frac{p-1}{p} |\{u > 0\}|$$

and look for minimizers to

$$(27) \quad \min \left\{ J_p^\lambda(u) : u \in W_1^{1,p}(\Omega) \right\}, \quad W_1^{1,p}(\Omega) := 1 + W_0^{1,p}(\Omega).$$

Accordingly, the constant

$$\Lambda_{\Omega,p} := \inf \{ \lambda > 0 : (27) \text{ admits a non-constant solution} \},$$

can be regarded as a *variational Bernoulli constant for the p -Laplacian*. We have that:

- For every $\lambda \geq \Lambda_{\Omega,p}$, problem (27) admits a non-constant minimizer u (see [25, Thm. 1.1]); such minimizer turns out to be a solution to the Bernoulli problem (24), provided the free boundary condition $|\nabla u| = \lambda$ is understood in the weak sense (25) [26, Thm. 2.1]; moreover, the free boundary $F(u)$ is a locally analytic hyper-surface, except for a \mathcal{H}^{n-1} -negligible singular set [26, Cor. 9.2].
- As a consequence of the results recalled at the above item, we have that

$$(28) \quad \Lambda_{\Omega,p} \geq \lambda_{\Omega,p};$$

this inequality may be strict, as the explicit computation of both constants $\Lambda_{\Omega,p}$ and $\lambda_{\Omega,p}$ in case of the ball reveals [25, Section 4].

Being this a quick picture of the state of the art, in the light of the results proved in the previous sections for problem $(P)_\lambda$, it is natural to ask:

What is the asymptotics of the Bernoulli constants $\lambda_{\Omega,p}$ and $\Lambda_{\Omega,p}$ as $p \rightarrow +\infty$? Further, if for a fixed λ and p large enough there exists a non-constant solution u_p to (24), what is the limiting behaviour of u_p as $p \rightarrow +\infty$?

Regarding the asymptotics of the Bernoulli constants $\lambda_{\Omega,p}$ and their variational counterparts $\Lambda_{\Omega,p}$, we have:

Proposition 27. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. Then*

$$\limsup_{p \rightarrow +\infty} \lambda_{\Omega,p} \leq \limsup_{p \rightarrow +\infty} \Lambda_{\Omega,p} \leq 1/R_\Omega.$$

Proof. In view of the inequality (28), it is enough to prove that

$$\limsup_{p \rightarrow +\infty} \Lambda_{\Omega,p} \leq 1/R_\Omega.$$

To obtain this inequality we observe that, if we fix $\lambda > 1/R_\Omega$, for p large enough problem (27) admits a non-constant minimizer. Indeed, setting $v_\lambda := (1 - \lambda d)_+$, for $p \gg 1$ we have

$$J_p^\lambda(v_\lambda) = \frac{1}{p}|D_{1/\lambda}| + \frac{p-1}{p}|D_{1/\lambda}| = |D_{1/\lambda}| < \frac{p-1}{p}|\Omega| = J_p(1). \quad \square$$

Corollary 28. *Assume that Ω is convex with $\partial\Omega$ of class C^1 . Then*

$$\lim_{p \rightarrow +\infty} \lambda_{\Omega,p} = \lim_{p \rightarrow +\infty} \Lambda_{\Omega,p} = 1/R_\Omega.$$

Proof. It follows from (26) and Proposition 27. \square

Now, let Ω be convex and let $\lambda > 1/R_\Omega$. By Proposition 27, for p large enough there exists a non-constant solution u_p to (24). Moreover, by Corollary 18, problem $(P)_\lambda$ admits a unique solution given precisely by the infinity harmonic potential $w_{\frac{1}{\lambda}}$ of $\overline{\Omega_{\frac{1}{\lambda}}}$. Nevertheless, we cannot expect that, in general, u_p converge to $w_{\frac{1}{\lambda}}$ as $p \rightarrow +\infty$. To enlighten this fact and get a feeling of the situation, let us have a look at what happens when Ω is a ball.

Example 29 (The radial case). Let B_R be the ball of center 0 and radius R in \mathbb{R}^n , and let $\lambda > 1/R$. It is well-known that for $\lambda = \lambda_p(B_R)$ the Bernoulli problem (24) on B_R admits a unique solution, which is called parabolic, whereas for any $\lambda > \lambda_p(B_R)$ it admits two solutions, which are called hyperbolic and elliptic (as they are respectively decreasing and increasing with respect to the parameter λ).

Since we want to examine the asymptotic behaviour of these solutions as $p \rightarrow +\infty$, let us briefly recover their expressions. By a result of Reichel [52], a solution to problem (24) on B_R is necessarily radial. Hence, for $\rho \in (0, R)$ and $p > n$, we compute the p -harmonic function u_p in $B_R \setminus \overline{B}_\rho$ which satisfies the Dirichlet boundary conditions $u_p = 1$ on ∂B_R and $u_p = 0$ on ∂B_ρ . It is given by

$$(29) \quad u_p(x) = \frac{|x|^\alpha - \rho^\alpha}{R^\alpha - \rho^\alpha}, \quad \rho < |x| < R, \quad \alpha := \frac{p-n}{p-1}$$

(observe that, for $p > n$, the exponent α belongs to $(0, 1)$, and tends to 1 as $p \rightarrow +\infty$). We are interested in finding the values of $\rho \in (0, R)$ such that

$$(30) \quad |\nabla u_p(x)| = \lambda, \quad \text{for } |x| = \rho.$$

Since $|\nabla u_p(x)| = \alpha \frac{|x|^{\alpha-1}}{R^\alpha - \rho^\alpha}$, condition (30) is equivalent to

$$(31) \quad f_\alpha(\rho) := \lambda \rho^\alpha + \alpha \rho^{\alpha-1} - \lambda R^\alpha = 0.$$

It is immediate to check that f_α is strictly decreasing in $\left(0, \frac{1-\alpha}{\rho}\right)$ and strictly increasing in $\left(\frac{1-\alpha}{\rho}, R\right)$, so that

$$m_\alpha := \min_{(0,R)} f_\alpha = f_\alpha\left(\frac{1-\alpha}{\rho}\right) = \left(\frac{\lambda}{1-\alpha}\right)^{1-\alpha} - \lambda R^\alpha.$$

Moreover,

$$\lim_{\rho \rightarrow 0^+} f_\alpha(\rho) = +\infty, \quad f_\alpha(R) = \alpha R^{\alpha-1} > 0.$$

Hence, equation (31) has one solution if $m_\alpha = 0$, two solutions if $m_\alpha < 0$, no solutions if $m_\alpha > 0$. Observe that

$$m_\alpha \leq 0 \iff \lambda \geq \lambda_p(B_R) := \frac{1}{R} (1-\alpha)^{1-1/\alpha} = \frac{1}{R} \left(\frac{n-1}{p-1}\right)^{-(n-1)/(p-n)}.$$

In particular, for p large enough, since $\lim_{\alpha \rightarrow 1^-} m_\alpha = 1 - \lambda R < 0$, equation (31) has exactly two zeros ρ'_α and ρ''_α ; correspondingly, the sets $\partial B_{\rho'_\alpha}$ and $\partial B_{\rho''_\alpha}$ are the free boundaries of the so-called hyperbolic and elliptic solutions to (24).

Let us look at what happens as $p \rightarrow +\infty$. We know from the above computations that

$$0 < \rho'_\alpha < \frac{1-\alpha}{\lambda} < \rho''_\alpha < R.$$

This gives at once $\rho'_\alpha \rightarrow 0$ as $\alpha \rightarrow 1^-$. On the other hand it is easily seen that, for every $0 < \varepsilon < \min\{1/\lambda, R - 1/\lambda\}$, it holds

$$\lim_{\alpha \rightarrow 1^-} f_\alpha\left(R - \frac{1}{\lambda} - \varepsilon\right) = -\varepsilon \lambda < 0, \quad \lim_{\alpha \rightarrow 1^-} f_\alpha\left(R - \frac{1}{\lambda} + \varepsilon\right) = \varepsilon \lambda > 0,$$

so that $\rho''_\alpha \rightarrow R - \frac{1}{\lambda}$ as $\alpha \rightarrow 1^-$.

Summarizing, the above analysis shows that the two families of p -harmonic functions which fit the Bernoulli free boundary condition (30) for $\rho = \rho'_p$ and $\rho = \rho''_p$ have respectively the following asymptotic behaviour: their free boundary degenerates or converge to the set $\partial\Omega_{\frac{1}{\lambda}}$, i.e.,

$$\rho'_p \rightarrow 0, \quad \rho''_p \rightarrow R - \frac{1}{\lambda},$$

and the functions u_p , as given by (29), converge to

$$w_R(x) = \frac{|x|}{R}, \quad x \in \overline{B}_R, \quad w_{\frac{1}{\lambda}}(x) = 1 - \lambda(R - |x|), \quad x \in \overline{B}_R \setminus B_{R-\frac{1}{\lambda}}.$$

In particular, only the elliptic family converges to the unique solution of $(P)_\lambda$. Let us remark that, for $\lambda \geq \lambda_p(B_R)$, contrary to the hyperbolic solutions, the elliptic ones are variational, namely they solve the minimization problem (27) on B_R (see [34, Sec. 5.3], [25, Sec. 4]).

Now, as suggested by the example of the ball, we give a convergence result for variational solutions. The reader may find a similar Γ -convergence result in the paper [43], where the authors deal with the asymptotic behaviour of variational energies related to the exterior Bernoulli boundary problem for the p -Laplacian as $p \rightarrow +\infty$.

Lemma 30. (i) For every function u belonging to the space

$$\mathcal{V}(\Omega) := \{u \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega) : u = 1 \text{ on } \partial\Omega\}.$$

the map $p \mapsto J_p^\lambda(u)$ is monotone nondecreasing.

(ii) In the limit as $p \rightarrow +\infty$, the sequence $(J_p^\lambda)_p$ Γ -converges, with respect to the weak topology of $W^{1,q}(\Omega)$, to the functional J_∞ given by

$$(32) \quad J_\infty(u) := \begin{cases} |\{u > 0\}|, & \text{if } \|\nabla u\|_\infty \leq \lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. The first part of the statement can be found in [43, Proposition 1], but we enclose a short proof for the sake of completeness. If $1 < p \leq q$, by applying Young's inequality $AB \leq (A^r/r) + (B^s/s)$, with $A = |\nabla u|/\lambda$, $B = 1$, $r = q/p$ and $s = r/(r-1)$, we obtain

$$J_p^\lambda(u) \leq \frac{1}{q} \int_\Omega \left(\frac{|\nabla u|}{\lambda} \right)^q dx + \left(\frac{q-p}{pq} + \frac{p-1}{p} \right) |\{u > 0\}| = J_q^\lambda(u).$$

The second part of the statement follows from the first one: it is enough to observe that the functional J_∞ is the "pointwise" limit of J_p^λ , and apply a well-known property of Γ -convergence (see [9, Remark 1.40(ii)]). \square

Theorem 31. Let $\lambda > 1/R_\Omega$. For p large enough, let $u_{\lambda,p}$ be a solution to the p -Bernoulli problem (24) which is found by solving the minimization problem (27). Then, there exists an increasing sequence (p_j) , diverging to $+\infty$, such that

$$u_{\lambda,p_j} \rightharpoonup u_\infty \text{ weakly in } W^{1,q}(\Omega) \quad \forall q > 1, \quad u_{\lambda,p_j} \rightarrow u_\infty \text{ uniformly in } \bar{\Omega},$$

where u_∞ is a solution of the ∞ -Bernoulli problem $(P)_\lambda$ satisfying

$$\text{int}\{u_\infty = 0\} = \{d > 1/\lambda\}.$$

Proof. Thanks to the assumption $\lambda > 1/R_\Omega$ and to Proposition 27, we know that for p large enough problem (27) admits a solution $u_{\lambda,p}$. As λ is fixed, we shall write for brevity J_p in place of J_p^λ and u_p in place of $u_{\lambda,p}$.

Let us first show that, for every fixed $q > 1$, the family (u_p) is uniformly bounded in $W^{1,q}(\Omega)$. Since

$$\frac{1}{p} \int_\Omega \left(\frac{|\nabla u_p|}{\lambda} \right)^p dx \leq J_p(u_p) \leq J_p(1) = \frac{p-1}{p} |\Omega| \leq |\Omega|,$$

we get

$$\|\nabla u_p\|_p \leq \lambda p^{1/p} |\Omega|^{1/p}.$$

Then, by Hölder's inequality, for every $p > q + 1$ it holds

$$(33) \quad \|\nabla u_p\|_q \leq \|\nabla u_p\|_p |\Omega|^{\frac{1}{q} - \frac{1}{p}} \leq \lambda p^{1/p} |\Omega|^{\frac{1}{q}} \leq C,$$

where $C > 0$ is a constant independent of p .

Using a diagonal argument, we can construct an increasing sequence $p_j \rightarrow +\infty$ such that u_{p_j} converges weakly in $W^{1,q}(\Omega)$ for every $q > 1$ and uniformly in $\bar{\Omega}$ to a function u_∞ . We claim that u_∞ is a solution to $(P)_\lambda$, satisfying $\text{int}\{u_\infty = 0\} = \{d > 1/\lambda\}$.

The fact that u_∞ is infinity harmonic in its positivity set is a standard consequence of the fact that u_{p_j} solve (27) with $p = p_j \rightarrow +\infty$, see for instance the arguments in [54, proof of Theorem 1].

Moreover, since $u_{p_j} = 1$ on $\partial\Omega$ for every j , from the uniform convergence it follows immediately that the same condition is satisfied by u_∞ .

Next we are going to show that the set

$$K := \{u_\infty = 0\}$$

satisfies $\text{int}(K) = \{d > 1/\lambda\}$, and that it belongs to the class \mathcal{K}_λ introduced in Definition 25.

From the second inequality in (33), we see that $\|\nabla u_\infty\|_{L^\infty(\Omega)} \leq \lambda$. Since $u = 1$ on $\partial\Omega$, we conclude that $u \in \mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega)$ is the space defined in Lemma 30. Therefore, we deduce as a first information on K the inclusion

$$(34) \quad K \subseteq \{d \geq 1/\lambda\}.$$

To go farther, we claim that u_∞ solves the minimization problem

$$(35) \quad \min \left\{ J_\infty(u) : u \in \mathcal{V}(\Omega) \right\},$$

where J_∞ is the functional defined by (32). Indeed, since $u_{p_j} \rightarrow u_\infty$ uniformly in $\bar{\Omega}$, for every fixed $\varepsilon > 0$, there exists an index $j_\varepsilon \in \mathbb{N}$ such that

$$|\{u_\infty > 0\}| < |\{u_{p_j} > 0\}| + \varepsilon, \quad \forall j > j_\varepsilon.$$

Then, for $j > j_\varepsilon$, it holds

$$(36) \quad \begin{aligned} J_{p_j}(u_\infty) &\leq \frac{1}{p_j} |\{u_\infty > 0\}| + \frac{p_j - 1}{p_j} |\{u_\infty > 0\}| \\ &\leq \frac{1}{p_j} |\Omega| + \frac{p_j - 1}{p_j} (|\{u_{p_j} > 0\}| + \varepsilon) \\ &\leq \frac{1}{p_j} |\Omega| + J_{p_j}(u_{p_j}) + \varepsilon. \end{aligned}$$

Thanks to the monotonicity property stated in Lemma 30, we can now pass to the limit as $j \rightarrow +\infty$ in (36). By the arbitrariness of $\varepsilon > 0$, and recalling that u_{p_j} are solutions to (27) (with $p = p_j$), we obtain, for every $u \in \mathcal{V}(\Omega)$,

$$J_\infty(u_\infty) = \lim_{j \rightarrow +\infty} J_{p_j}(u_\infty) \leq \liminf_{j \rightarrow +\infty} J_{p_j}(u_{p_j}) \leq \liminf_{j \rightarrow +\infty} J_{p_j}(u) = J_\infty(u),$$

so that u_∞ solves problem (35) as claimed. Consequently, by taking as a competitor the function $(1 - \lambda d)_+$, we deduce that $|\{u_\infty > 0\}| \leq |D_{\frac{1}{\lambda}}|$, or equivalently

$$(37) \quad |K| \geq |\{d \geq 1/\lambda\}|.$$

Since $|\nabla d| = 1$ a.e. in Ω , every level set of d has zero Lebesgue measure, so that $|\{d \geq 1/\lambda\}| = |\{d > 1/\lambda\}|$. Since $\text{int}\{d \geq 1/\lambda\} = \{d > 1/\lambda\}$, by combining conditions (34) and (37) we obtain that $\text{int} K = \{d > 1/\lambda\}$. As a consequence, K belongs to the family \mathcal{K}_λ introduced in Definition 25, so that, by Theorem 26, u_∞ is a solution to $(P)_\lambda$. \square

Corollary 32. *Let $\lambda > 1/R_\Omega$. Then, under the assumptions (H1)–(H2) of Theorem 17 (hence, in particular, when Ω is convex), in the limit as $p \rightarrow +\infty$ we have*

$$u_{\lambda,p} \rightharpoonup w_{\frac{1}{\lambda}} \text{ weakly in } W^{1,q}(\Omega) \quad \forall q > 1, \quad u_{\lambda,p_j} \rightarrow w_{\frac{1}{\lambda}}, \text{ uniformly in } \bar{\Omega},$$

where $w_{\frac{1}{\lambda}}$ is the infinity harmonic potential of $\bar{\Omega}_{\frac{1}{\lambda}}$, namely the unique genuine solution to the ∞ -Bernoulli problem $(P)_\lambda$.

Proof. From Theorem 31, there exists an increasing sequence $p_j \nearrow \infty$ such that $u_{\lambda,p_j} \rightarrow u_\infty$, with u_∞ solution to $(P)_\lambda$. Hence, by Theorem 17 we have that $u_\infty = w_{1/\lambda}$. By the same argument, any other converging subsequence must converge to $w_{1/\lambda}$. \square

Acknowledgments. We thank Bozhidar Velichkov for a useful discussion about the viscosity interpretation of free boundary conditions.

The authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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