

Global Solutions of Semilinear Parabolic Equations with Drift Term on Riemannian Manifolds

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Abstract

We study existence and non-existence of global solutions to the semilinear heat equation with a drift term and a power-like source term u^p , on Cartan-Hadamard manifolds. Under suitable assumptions on Ricci and sectional curvatures, we show that, for any $p > 1$, global solutions cannot exist if the initial datum is large enough. Furthermore, under appropriate conditions on the drift term, global existence is obtained for any $p > 1$, if the initial datum is sufficiently small. We also deal with Riemannian manifolds whose Ricci curvature tends to zero at infinity sufficiently fast. We show that for any non trivial initial datum, for certain p depending on the Ricci curvature's bound, global solutions cannot exist. On the other hand, for determined values of p , depending on the vector field b , global solutions exist, for sufficiently small initial data.

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1 Introduction

We investigate existence and non-existence of nonnegative global solutions of Cauchy problems for semilinear parabolic equations of the following form:

$$\begin{cases} \partial_t u = \Delta u + \langle b(x), \nabla u \rangle + u^p & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}; \end{cases} \quad (1.1)$$

here M is a Cartan-Hadamard manifold of dimension n , endowed with a metric tensor g , Δ and ∇ denote the Laplace-Beltrami operator and, respectively, the

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gradient with respect to g , $p > 1$, b is a vector field defined on M , u_0 is a given nonnegative bounded initial datum. For all vector fields X, Y belonging to the tangent bundle of M , we set

$$\langle X, Y \rangle := g(X, Y).$$

Here $T > 0$ is the maximal existence time; when $T = +\infty$, we say that the solution is global. Problem (1.1) can be regarded as a model for a reaction-diffusion process, which takes place on a Riemannian manifold, where u stands for the temperature, u^p is a nonlinear source term and the drift $\langle b, \nabla u \rangle$ can be determined by an external flow field.

Problem (1.1) with $b \equiv 0$ has been widely studied in the literature, both in the Euclidean space (see, e.g. [7], [13], [16], [17]) and on Riemannian manifolds (see, e.g., [12], [14], [10], [11], [20], [21], [22], [23]). In particular, in [3] the hyperbolic space is considered. Some results of [3] have been generalized to Cartan-Hadamard manifolds with strictly negative sectional curvature in [20], [21]. It is showed that for any $p > 1$, if u_0 is small enough, then problem (1.1) with $b \equiv 0$ admits a global nonnegative bounded solution.

Problem (1.1) with $M = \mathbb{R}^n$ has been investigated, e.g., in [1, 4]. In particular, in [4] it is proved that for any $p > 1$, for sufficiently large initial conditions, global solutions cannot exist. On the other hand, if, for some $\nu > -n$,

$$\langle b(x), x \rangle \geq \nu \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$p > 1 + \frac{2}{n + \nu},$$

then problem (1.1) has a global solution (see [4, Theorem 3.3.3]). In [4] also more general operators are considered and many other results are established.

In the present paper, we consider: (*j*) Cartan-Hadamard manifolds with radial Ricci curvature bounded from below and strictly negative sectional curvature; (*jj*) Cartan-Hadamard manifolds whose radial Ricci curvature can be negative, but tends to 0 at infinity fast enough.

Consider the case (*j*). We prove that, under suitable assumptions on b , if the initial data u_0 is big enough, then for any $p > 1$ global solutions to problem (1.1) cannot exist. To prove this result, we use a modification of the method exploited in [15] (see also [4]). Note that the curvature assumptions give certain bounds on the volume's growth of geodesics balls, which are essential to employ this method on a complete non-compact Riemannian manifold. Moreover, we will construct and use a function $\varphi \in C^2(M \setminus \partial B_{R_0}) \cap C^1(M)$ satisfying

$$\Delta \varphi - \langle b(x), \nabla \varphi \rangle + [\lambda - \operatorname{div} b(x)] \varphi \geq 0 \quad \text{for all } x \in M \setminus \partial B_{R_0}, \quad (1.2)$$

for some $R_0 > 0, \lambda > 0$. The non-existence theorem for large initial data also applies for $b = 0$. Hence it is a complementary result with respect to those given in [3, 20, 21] for $b = 0$.

On the other hand, under suitable assumptions on the vector field b , we show that, for any $p > 1$, there exists a global solution to problem (1.1), if the initial datum is small enough. A crucial point in the proof of this result is the construction of a (weak) supersolution to equation

$$\Delta w + \langle b(x), \nabla w \rangle + \lambda w = 0 \quad \text{in } M, \quad (1.3)$$

for suitable $\lambda > 0$. Observe that differently from the Euclidean case, any $p > 1$ is included. Note that such result applies in particular when $b \equiv 0$. In this case, it is in agreement with the results in [20], [21].

Consider now the case (jj) . We show that for any $p > 1$, if u_0 is big enough, then problem (1.1) does not admit global solutions. Moreover, we obtain that, for some $\underline{p} > 1$ depending on the Ricci curvature's bound, for every $1 < p < \underline{p}$, for any initial datum $u_0 \not\equiv 0$, problem (1.1) cannot have global solutions. In addition, under suitable assumptions on b , for some $\bar{p} > 1$ depending on b , for every $p > \bar{p}$, problem (1.1) has global solutions, for initial data u_0 small enough.

The paper is organized as follows. In section 2 we recall some useful preliminary notions from Riemannian Geometry and we made the main assumptions. Cartan-Hadamard manifolds of type (j) are treated in Section 3, for non-existence of global solutions, and in Section 4, for existence of global solutions. Moreover, Cartan-Hadamard manifolds of type (jj) are treated in Section 5, for non-existence of global solutions, and in Section 6, for existence of global solutions.

2 Mathematical framework

2.1 Preliminaries from Riemannian Geometry

In this section we recall some useful notions and results from Riemannian Geometry (see e.g. [2], [8], [9]). We consider *Cartan-Hadamard* manifolds, i.e. simply connected complete noncompact Riemannian manifolds with nonpositive sectional curvatures. On a Cartan-Hadamard manifold M , for any point $o \in M$, the *cut locus* of o , $\text{Cut}(o)$, is empty. Thus M is a manifold with a pole. Thus, for any $x \in M \setminus \{o\}$ one can define the *polar coordinates* (r, θ) with respect to o , where

$$r \equiv r(x) := \text{dist}(x, o).$$

For any $x_0 \in M$ and for any $R > 0$ we set

$$B_R(x_0) := \{x \in M : \text{dist}(x, x_0) < R\}.$$

We set $B_R \equiv B_R(o)$.

The Riemannian metric in $M \setminus \{o\}$ in polar coordinates reads

$$g = dr^2 + A_{ij}(r, \theta) d\theta^i d\theta^j,$$

where $(\theta^1, \dots, \theta^{n-1})$ are coordinates in \mathbb{S}^{n-1} and (A_{ij}) is a positive definite matrix. The Laplace-Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r, \theta) \frac{\partial}{\partial r} + \Delta_{S_r}, \quad (2.1)$$

where

$$\mathcal{F}(r, \theta) = \Delta r(x) \quad \text{for any } x \in M \setminus \{o\},$$

Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B_r$. The *area* element on S_r is

$$d\mu'|_{S_r} = \sqrt{A} d\theta^1 \dots d\theta^{n-1},$$

where $A := \det(A_{ij})$, while the *volume* element is

$$d\mu = d\mu' dr.$$

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$g = dr^2 + \psi^2(r) d\theta^2, \quad (2.2)$$

where $\psi \in \mathcal{A}$ with

$$\mathcal{A} := \{f \in C^\infty((0, \infty)) \cap C^1([0, \infty)) : f'(0) = 1, f(0) = 0, f > 0 \text{ in } (0, \infty)\}.$$

Observe that for $\psi(r) = r$, $M = \mathbb{R}^n$, while for $\psi(r) = \sinh r$, M is the n -dimensional hyperbolic space \mathbb{H}^n .

For any $x \in M \setminus \{o\}$, denote by $\text{Ric}_o(x)$ the *Ricci curvature* at x in the radial direction $\frac{\partial}{\partial r}$.

Let ω denote any pair of tangent vectors from $T_x M$ having the form $(\frac{\partial}{\partial r}, X)$, where X is a unit vector orthogonal to $\frac{\partial}{\partial r}$. We denote by $K_\omega(x)$ the *sectional curvature* at the point x of the plane determined by ω ; it is also called *sectional radial curvature*.

If M_ψ is a model manifold, then for any $x = (r, \theta) \in M_\psi \setminus \{o\}$

$$K_\omega(x) = -\frac{\psi''(r)}{\psi(r)},$$

and

$$\text{Ric}_o(x) = -(n-1) \frac{\psi''(r)}{\psi(r)}.$$

2.2 Assumptions

We always assume that M is a Cartan-Hadamard manifold of dimension $n \geq 2$. In addition, we make one the following assumptions:

$$\begin{cases} \text{(i)} & \mathbf{K}_\omega(x) \leq -h_1^2 \text{ for some } h_1 > 0, \text{ for any } x \in M \setminus \{o\}; \\ \text{(ii)} & \mathbf{Ric}_o(x) \geq -(n-1)h_2^2 \text{ for some } h_2 > h_1, \text{ for any } x \in M \setminus \{o\}; \end{cases} \quad (A_0)$$

or

$$\mathbf{Ric}_o(x) \geq -(n-1) \frac{\bar{\beta}}{1+r^2(x)} \text{ for some } \bar{\beta} > 0, \text{ for any } x \in M \setminus \{o\}. \quad (A_1)$$

Concerning the initial datum u_0 and the vector field b we always assume:

$$\begin{cases} \text{(i)} & b \in C^1(M); \\ \text{(ii)} & u_0 \in C(M), u_0 \geq 0. \end{cases} \quad (H)$$

In view of assumption (A_0) , by Laplace and Hessian comparison results (see, e.g., [2], [8]),

$$(n-1)h_1 \coth(h_1 r) \leq \mathcal{F}(r, \theta) \leq (n-1)h_2 \coth(h_2 r) \quad \text{for any } x \equiv (r, \theta) \in M \setminus \{o\}. \quad (2.3)$$

Moreover, there exists $C_0 > 0$ such that

$$\mu'(S_R) \leq C_0 e^{(n-1)h_2 R} \quad \text{for any } R > 0. \quad (2.4)$$

On the other hand, in view of (A_1) , again by Laplace and Hessian comparison results (see, e.g., [5, Lemma 5.4] and remarks at pages 411-412 in [19]),

$$\frac{(n-1)}{r} \leq \mathcal{F}(r, \theta) \leq \frac{C_1(n-1)}{r} \quad \text{for any } x \equiv (r, \theta) \in M \setminus \{o\}, \quad (2.5)$$

for some $C_1 > 1$. Moreover, there exist $0 < C_2 < C_3$ such that

$$C_2 R^{n-1} \leq \mu'(S_R) \leq C_3 R^{\gamma(n-1)} \quad \text{for any } R > 0, \quad (2.6)$$

where

$$\gamma = \frac{1 + \sqrt{1 + 4\bar{\beta}}}{2}. \quad (2.7)$$

3 Non-existence of global solutions under assumption (A_0)

Let $a_1 > 0, C_1 > 0, a_2 > 0, C_2 > 0, R_0 > 0$. Define

$$\begin{aligned} \tilde{\varphi}(x) &\equiv \tilde{\varphi}(r) := C_1 e^{-a_1 r} \quad \text{for all } x \in M \setminus B_{R_0}; \\ \hat{\varphi}(x) &\equiv \hat{\varphi}(r) := C_2 e^{-a_2 r^2} \quad \text{for all } x \in B_{R_0}; \\ \varphi &:= \begin{cases} \tilde{\varphi} & \text{in } M \setminus B_{R_0} \\ \hat{\varphi} & \text{in } B_{R_0}. \end{cases} \end{aligned} \quad (3.8)$$

We always take

$$C_2 = C_1 e^{-a_1 R_0 + a_2 R_0^2},$$

$$a_2 = \frac{a_1}{2},$$

thus

$$\varphi \in C^2(M \setminus \partial B_{R_0}) \cap C^1(M).$$

As usual, we denote by $\frac{\partial}{\partial r}$ the unitary vector field in the *radial* direction. Consider the function $\underline{b}: M \rightarrow \mathbb{R}$ defined by

$$\underline{b}(x) := \left\langle b(x), \frac{\partial}{\partial r} \right\rangle, \quad x \in M. \quad (3.9)$$

Clearly, $\underline{b}(x) \frac{\partial}{\partial r}$ is the *radial component* of the vector field $b(x)$, $x \in M$.

Lemma 3.1. *Assume (A_0) , (H) . Suppose that there exist $b_0 \in \mathbb{R}$, $\hat{C} \geq 0$ such that*

$$\underline{b}(x) \geq \underline{b}_0 \quad \text{for all } x \in M, \quad (3.10)$$

$$\operatorname{div} b(x) \leq \hat{C} \quad \text{for all } x \in M. \quad (3.11)$$

Let $\lambda > \hat{C}$ and

$$0 < a_1 \leq \min \left\{ \frac{\lambda - \hat{C}}{(n-1)h_2 \coth(h_2 R_0) + \underline{b}_0}, \frac{\lambda - \hat{C}}{1 + R_0[(n-1)h_2 \coth(h_2 R_0) + \underline{b}_0]} \right\}. \quad (3.12)$$

Then the function φ defined in (3.8) satisfies (1.2).

Proof. In view of (2.1), (2.3) and (3.12) we have, for all $x \in M \setminus \overline{B}_{R_0}$,

$$\begin{aligned} & \Delta \varphi - \langle b(x), \nabla \varphi \rangle - \operatorname{div} b(x) \varphi + \lambda \varphi \\ &= \varphi''(r) + \mathcal{F}(r, \theta) \varphi'(r) - \underline{b}(x) \varphi'(r) + (\lambda - \hat{C}) \varphi(r) \\ &\geq e^{-a_1 r} \left\{ a_1^2 + a_1 [-(n-1)h_2 \coth(h_2 R_0) + \underline{b}(x)] + \lambda - \hat{C} \right\} \\ &\geq e^{-a_1 r} \left\{ a_1^2 + a_1 [-(n-1)h_2 \coth(h_2 R_0) + \underline{b}_0] + \lambda - \hat{C} \right\} \\ &\geq e^{-a_1 r} \left\{ -a_1 [(n-1)h_2 \coth(h_2 R_0) + \underline{b}_0] + \lambda - \hat{C} \right\} \geq 0. \end{aligned} \quad (3.13)$$

Furthermore, since $a_2 = \frac{a_1}{2}$, again by (2.1), (2.3) and (3.12), for all $x \in B_{R_0}$,

$$\begin{aligned} & \Delta \varphi - \langle b(x), \nabla \varphi \rangle - \operatorname{div} b(x) \varphi + \lambda \varphi \\ &= \varphi''(r) + \mathcal{F}(r, \theta) \varphi'(r) - \underline{b}(x) \varphi'(r) + (\lambda - \hat{C}) \varphi(r) \\ &\geq e^{-a_2 r^2} \left\{ 4a_2^2 r^2 - 2a_2 + 2a_2 r [-(n-1)h_2 \coth(h_2 r) + \underline{b}(x)] + \lambda - \hat{C} \right\} \\ &\geq e^{-a_2 r^2} \left\{ -2a_2 [1 + R_0((n-1)h_2 \coth(h_2 R_0) + \underline{b}_0)] + \lambda - \hat{C} \right\} \geq 0. \end{aligned} \quad (3.14)$$

From (3.13) and (3.14) we get the thesis. \square

Let $a_1 > (n-1)h_2$, so $\varphi \in L^1(M)$. Set

$$c := \frac{1}{\|\varphi\|_{L^1(M)}}. \quad (3.15)$$

Theorem 3.2. *Assume (A_0) , (H) , (3.11). Let $b \in L^\infty(M)$, $\lambda > \hat{C}$ and φ be as in Lemma 3.1 with*

$$a_1 > (n-1)h_2. \quad (3.16)$$

Let u be a solution of problem (1.1) with $u \in L^\infty(M \times (0, \tau))$ for each $0 < \tau < T$ and $\partial_t u(t) \varphi \in L^1(M)$ for every $t \in (0, T)$. Suppose that

$$c \int_M u_0(x) \varphi(x) d\mu > \lambda^{\frac{1}{p-1}}, \quad (3.17)$$

with c given by (3.15). Then problem (1.1) does not admit global solutions.

In Theorem 3.2 the solution to (1.1) is meant in the classical sense. Note that, as it will be apparent from the proof, the same conclusion holds for supersolutions.

Proof. Consider a family $\{\tilde{\zeta}_R\}_{R>0} \subset C_c^\infty([0, +\infty))$ of cut-off functions such that, for any $R > 0$,

$$\tilde{\zeta}_R = \begin{cases} 1 & \text{in } [0, R] \\ 0 & \text{in } [2R, +\infty) \end{cases},$$

$$0 \leq \tilde{\zeta}_R \leq 1, \quad |\tilde{\zeta}'_R| \leq \frac{C}{R}, \quad |\tilde{\zeta}''_R| \leq \frac{C}{R^2} \text{ for any } r \in [0, +\infty),$$

for some $C > 0$. Set

$$\tilde{\zeta}_R(x) = \tilde{\zeta}_R(r(x)), \quad x \in M.$$

Therefore,

$$|\nabla \tilde{\zeta}_R(x)| \leq \frac{C}{R} \chi_{B_{2R} \setminus B_R}(x) \quad \text{for any } x \in M. \quad (3.18)$$

Furthermore, in view of (2.1), (2.3),

$$|\Delta \tilde{\zeta}_R(x)| \leq \frac{C}{R} \chi_{B_{2R} \setminus B_R}(x) \quad \text{for any } x \in M. \quad (3.19)$$

From (2.4) and (3.16) we can infer that $\varphi \in L^1(M)$. Define

$$\psi(t) := \int_M c u(x, t) \varphi(x) d\mu, \quad t \in (0, T).$$

From (1.1) we obtain

$$\int_M c \varphi \zeta_R \partial_t u d\mu = \int_M c \varphi \zeta_R \Delta u d\mu + \int_M c \varphi \zeta_R \langle b, \nabla u \rangle d\mu + \int_M c \zeta_R \varphi u^p d\mu. \quad (3.20)$$

Let ν be the outer normal unit vector to ∂B_{R_0} . Since $\varphi \in C^2(M \setminus \partial B_{R_0}) \cap C^1(M)$, we get

$$\begin{aligned}
\int_M \varphi \zeta_R \Delta u d\mu &= \int_{B_{R_0}} \varphi \zeta_R \Delta u d\mu + \int_{B_{R_0}^c} \varphi \zeta_R \Delta u d\mu \\
&= - \int_{B_{R_0}} \langle \nabla u, \nabla \varphi_1 \rangle \zeta_R d\mu + \int_{\partial B_{R_0}} \varphi_1 \zeta_R \frac{\partial u}{\partial \nu} dS \\
&\quad - \int_{B_{R_0}^c} \langle \nabla u, \nabla \varphi_2 \rangle \zeta_R d\mu - \int_{\partial B_{R_0}} \varphi_2 \zeta_R \frac{\partial u}{\partial \nu} dS \\
&\quad - \int_M \langle \nabla u, \nabla \zeta_R \rangle \varphi d\mu \\
&= \int_{B_{R_0}} u \Delta \varphi_1 \zeta_R d\mu - \int_{\partial B_{R_0}} u \zeta_R \frac{\partial \varphi_1}{\partial \nu} dS \\
&\quad + \int_{B_{R_0}^c} u \Delta \varphi_2 \zeta_R d\mu + \int_{\partial B_{R_0}} u \zeta_R \frac{\partial \varphi_2}{\partial \nu} dS \\
&\quad + 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu \\
&= \int_{B_{R_0}} u \Delta \varphi_1 \zeta_R d\mu + \int_{B_{R_0}^c} u \Delta \varphi_2 \zeta_R d\mu \\
&\quad + 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu \\
&= \int_{B_{R_0}} u \zeta_R \Delta \varphi_1 d\mu + \int_{B_{R_0}^c} u \zeta_R \Delta \varphi_2 d\mu \\
&\quad + 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu.
\end{aligned} \tag{3.21}$$

Observe that in view of our assumptions, Lemma 3.1 can be applied. Since $u \geq 0, \varphi > 0, \zeta_R \geq 0$, from (1.2) and (3.21) we obtain

$$\begin{aligned}
\int_M \varphi \zeta_R \Delta u d\mu &\geq -\lambda \int_{B_{R_0}} u \varphi_1 \zeta_R d\mu + \int_{B_{R_0}} \langle b(x), \nabla \varphi_1 \rangle u \zeta_R d\mu \\
&\quad - \lambda \int_{B_{R_0}^c} u \varphi_2 \zeta_R d\mu + \int_{B_{R_0}^c} \langle b(x), \nabla \varphi_2 \rangle u \zeta_R d\mu \\
&\quad + \int_M u \varphi \zeta_R \operatorname{div} b d\mu + 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu \\
&= -\lambda \int_M u \varphi \zeta_R d\mu + \int_M \langle b(x), \nabla \varphi \rangle u \zeta_R d\mu + \int_M u \varphi \zeta_R \operatorname{div} b d\mu \\
&\quad + 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu.
\end{aligned} \tag{3.22}$$

Furthermore,

$$\begin{aligned}
& \int_M \varphi \zeta_R \langle b, \nabla u \rangle d\mu \\
&= - \int_M u \varphi \zeta_R \operatorname{div} b d\mu - \int_M u \zeta_R \langle b, \nabla \varphi \rangle d\mu - \int_M u \varphi \langle b, \nabla \zeta_R \rangle d\mu.
\end{aligned} \tag{3.23}$$

By (3.20), (3.22) and (3.23),

$$\begin{aligned}
\int_M c \varphi \zeta_R \partial_t u d\mu &\geq - \int_M \lambda u \varphi \zeta_R d\mu + \int_M \langle b(x), \nabla \varphi \rangle u \zeta_R d\mu \\
&+ 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu + \int_M u \varphi \Delta \zeta_R d\mu \\
&- \int_M \langle b(x), \nabla \varphi \rangle u \zeta_R d\mu - \int_M \langle b(x), \nabla \zeta_R \rangle u \varphi d\mu \\
&+ \int_M c \zeta_R \varphi u^p d\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_M c \varphi \zeta_R \partial_t u d\mu + \lambda \int_M u \varphi \zeta_R d\mu &\geq 2 \int_M u \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu \\
&+ \int_M u \varphi \Delta \zeta_R d\mu - \int_M u \varphi \langle b, \nabla \zeta_R \rangle d\mu + \int_M c \zeta_R \varphi u^p d\mu.
\end{aligned} \tag{3.24}$$

In view of (2.4), (6.21) and (3.16), for every $t \in (0, T)$, we get, for some $C > 0$,

$$\begin{aligned}
\left| \int_M u(x, t) \langle \nabla \varphi, \nabla \zeta_R \rangle d\mu \right| &\leq \frac{\|u(t)\|_\infty C}{R} \int_{B_{2R}} |\nabla \varphi| d\mu \\
&\leq \frac{\|u(t)\|_\infty C}{R} \int_0^{2R} e^{[(n-1)h_2 - a_1]r} dr \xrightarrow{R \rightarrow +\infty} 0.
\end{aligned} \tag{3.25}$$

From (2.4), (6.20) and (3.16), for every $t \in (0, T)$, we obtain, for some $C > 0$,

$$\left| \int_M u \varphi \Delta \zeta_R d\mu \right| \leq \frac{\|u(t)\|_\infty C}{R} \int_0^{2R} e^{[-a_1 + (n-1)h_2]r} dr \xrightarrow{R \rightarrow +\infty} 0. \tag{3.26}$$

In view of (2.4) and (3.16), for every $t \in (0, T)$, we get, for some $C > 0$,

$$\left| \int_M u \varphi \langle b, \nabla \zeta_R \rangle d\mu \right| \leq \frac{\|u(t)\|_\infty \|b\|_\infty C}{R} \int_0^R e^{[(n-1)h_2 - a_1]r} dr \xrightarrow{R \rightarrow +\infty} 0. \tag{3.27}$$

Since, for every $t \in (0, T)$, $u(t) \in L^\infty(M)$ and $\varphi \in L^1(M)$, by the dominated convergence theorem,

$$\int_M u \varphi \zeta_R d\mu \xrightarrow{R \rightarrow +\infty} \int_M c \varphi u d\mu = \psi(t), \tag{3.28}$$

and

$$\int_M c\zeta_R \varphi u^p d\mu \xrightarrow{R \rightarrow +\infty} \int_M c\varphi u^p d\mu \geq \psi^p(t); \quad (3.29)$$

here use of Jensen's inequality and (3.15) have been made. Since, for every $t \in (0, T)$, $\partial_t u(t)\varphi \in L^\infty(M)$, by the dominated convergence theorem,

$$\int_M c\varphi \zeta_R \partial_t u d\mu \xrightarrow{R \rightarrow +\infty} \int_M c\varphi \partial_t u d\mu = \psi'(t). \quad (3.30)$$

By (3.24)-(3.30) and (3.17)

$$\psi'(t) \geq \psi^p(t) - \lambda\psi(t), \quad \psi(0) > \lambda^{\frac{1}{p-1}}.$$

This easily implies that there exists $T > 0$ such that

$$\lim_{t \rightarrow T^-} \psi(t) = +\infty.$$

Hence the thesis follows. \square

4 Existence of global solutions under assumption

(A_0)

Let $\underline{b}(x)$ be defined as in (3.9). Assume that

$$\underline{b}(x) \geq \underline{b}_1 > -(n-1)h_1 \quad \text{for all } x \in M. \quad (4.1)$$

Set

$$\sigma := (n-1)h_1 + \underline{b}_1.$$

So, $\sigma > 0$. Suppose that

$$0 < \lambda < \frac{\sigma^2}{4}. \quad (4.2)$$

Hence we can find $a \in \mathbb{R}$ such that

$$0 < a < \frac{\sigma + \sqrt{\sigma^2 - 4\lambda}}{2}. \quad (4.3)$$

Define

$$w(x) \equiv w(r) := e^{-ar}, \quad x \in M.$$

Proposition 4.1. *Let assumptions (A_0) and (H) be satisfied. Let conditions (4.1)-(4.3) be fulfilled. Then w is a weak supersolution of (1.3).*

Proof. In view of (2.3) and (4.3), for any $x \in M \setminus \{o\}$,

$$\begin{aligned} \Delta w + \langle b(x), \nabla w(x) \rangle + \lambda w(x) \\ &= e^{-ar} [a^2 - a\mathcal{F}(r, \theta) - a\underline{b}(x) + \lambda] \\ &\leq e^{-ar} \{a^2 - a[(n-1)h_1 + \underline{b}_1] + \lambda\} \leq 0. \end{aligned} \quad (4.4)$$

Since $w'(0) < 0$, by a Kato's type inequality, w is a weak supersolution of equation (1.3) in the whole M . \square

Theorem 4.2. *Let assumptions (A_0) , (H) , (4.1) and (4.2) be satisfied. Let w be defined as in Proposition 4.1. Assume that*

$$0 \leq u_0 \leq \tilde{C}w \quad \text{in } M, \quad (4.5)$$

where

$$0 < \tilde{C} < \frac{1}{\|w\|_\infty} \lambda^{\frac{1}{p-1}}. \quad (4.6)$$

Then there exists a global solution u of problem (1.1); in addition, $u \in L^\infty(M \times (0, +\infty))$.

In order to prove Theorem 4.2, we adapt to the present situation some arguments used in [3, Theorem 3.1] (see also [21, Theorem 3.2]). Let $\{\Omega_j\}_{j \in \mathbb{N}}$ be a sequence of domains $\{\Omega_j\}_{j \in \mathbb{N}} \subseteq M$ such that $\bar{\Omega}_j \subseteq \Omega_{j+1}$ for every $j \in \mathbb{N}$, $\bigcup_{j=1}^\infty \Omega_j = M$, $\partial\Omega_j$ is smooth for every $j \in \mathbb{N}$. Furthermore, for every $j \in \mathbb{N}$ let $\zeta_j \in C_c^\infty(\Omega_j)$ such that $0 \leq \zeta_j \leq 1$, $\zeta_j \equiv 1$ in $\Omega_{j/2}$.

Proof of Theorem 4.2. For any $j \in \mathbb{N}$ there exists a unique classical solution u_j to problem

$$\begin{cases} \partial_t u = \Delta u + \langle b(x), \nabla u \rangle + u^p & \text{in } \Omega_j \times (0, T) \\ u = 0 & \text{in } \partial\Omega_j \times (0, T) \\ u = \zeta_j u_0 & \text{in } \Omega_j \times \{0\}. \end{cases} \quad (4.7)$$

Take the constant $\tilde{C} > 0$ given by (4.6). Let

$$\tilde{w}(x) := \tilde{C}w(x) \quad (x \in M),$$

and

$$\xi(t) = \left\{ 1 - \frac{1}{\lambda} \|\tilde{w}\|_\infty^{p-1} [1 - e^{-(p-1)\lambda t}] \right\}^{-\frac{1}{p-1}} \quad (t \in [0, \infty)).$$

Note that ξ is well-defined in $[0, \infty)$ due to (4.6). It is easily seen that ξ solves problem

$$\begin{cases} \xi' = \|\tilde{w}\|_\infty^{p-1} e^{-(p-1)\lambda t} \xi^p, & t \in (0, \infty) \\ \xi(0) = 1. \end{cases} \quad (4.8)$$

Select any λ satisfying (4.2). Define

$$\bar{u}(x, t) := e^{-\lambda t} \xi(t) \tilde{w}(x) \quad ((x, t) \in M \times [0, \infty)). \quad (4.9)$$

Due to (4.8), we have

$$\begin{aligned} & \partial_t \bar{u} - \Delta \bar{u} - \langle b(x), \nabla \bar{u} \rangle - \bar{u}^p \\ &= -\lambda e^{-\lambda t} \xi(t) \tilde{w}(x) + e^{-\lambda t} \|\tilde{w}\|_\infty^{p-1} e^{-(p-1)\lambda t} \xi^p(t) \tilde{w}(x) \end{aligned}$$

$$+\lambda e^{-\lambda t} \xi(t) \tilde{w}(x) - e^{-\lambda p t} \xi^p(t) \tilde{w}^p(x) \geq 0 \quad \text{weakly in } M \times (0, \infty).$$

So, \bar{u} is a weak supersolution of equation

$$\partial_t u = \Delta u + \langle b(x), \nabla u \rangle + u^p \quad \text{in } M \times (0, \infty). \quad (4.10)$$

Moreover, due to (4.5), for any $j \in \mathbb{N}$, \bar{u} is a bounded weak supersolution of problem (4.7). Obviously, for each $j \in \mathbb{N}$, $\underline{u} \equiv 0$ is a subsolution of problem (4.7). Hence, by the comparison principle, for every $j \in \mathbb{N}$ we obtain

$$0 \leq u_j \leq \bar{u} \quad \text{in } B_j \times (0, T). \quad (4.11)$$

By standard a priori estimates (see, e.g., [6], [18]), we can infer that there exists a subsequence $\{u_{j_{k_h}}\}$ of $\{u_{j_k}\}$, which converges in $C_{x,t}^{2,1}(K \times [\varepsilon, T])$ as $h \rightarrow +\infty$, for each compact subset $K \subset M$ and for each $\varepsilon \in (0, T)$, and in $C_{\text{loc}}(M \times [0, T])$, to some function $u \in C_{x,t}^{2,1}(M \times (0, T]) \cap C(M \times [0, T])$. Moreover, u is a classical solution problem (1.1). Furthermore, from (4.11) we get

$$0 \leq u \leq \bar{u} \quad \text{in } M \times (0, T).$$

Hence the thesis follows. \square

5 Non-existence of global solutions under assumption (A_1)

Let $a > 0$. Define

$$\eta(x) \equiv \eta(r) := e^{-ar^2} \quad \text{for all } x \in M. \quad (5.12)$$

Lemma 5.1. *Assume (A_1) , (H) . Suppose that (3.11) holds and that there exists $\sigma > 0$ such that*

$$\underline{b}(x) \geq -\frac{\sigma}{r(x)} \quad \text{for all } x \in M \setminus \{o\}, \quad (5.13)$$

with $\underline{b}(x)$ defined in (3.9). Let $a > 0$ and

$$\lambda \geq 2a[1 + C_1(n-1) + \sigma] + \hat{C}, \quad (5.14)$$

with C_1 given by (2.5). Then the function η defined in (5.12) satisfies (1.2).

Proof. In view of (2.1), (2.5), (5.13) and (5.14) we have, for all $x \in M \setminus \bar{B}_{R_0}$,

$$\begin{aligned} & \Delta \eta - \langle b(x), \nabla \eta \rangle - \text{div } b(x) \eta + \lambda \eta \\ & \geq \eta''(r) + \mathcal{F}(r, \theta) \eta'(r) - \underline{b}(x) \eta'(r) + (\lambda - \hat{C}) \varphi(r) \\ & \geq e^{-ar^2} \left\{ 4a^2 r^2 - 2a - 2ar \frac{(n-1)C_1}{r} + 2ar \underline{b}(x) + \lambda - \hat{C} \right\} \\ & \geq e^{-ar^2} \left\{ -2a[1 + C_1(n-1) + \sigma] + \lambda - \hat{C} \right\} \geq 0. \end{aligned}$$

\square

Due to (2.6), $\eta \in L^1(M)$. Set

$$k := \frac{1}{\int_M \eta(x) d\mu}. \quad (5.15)$$

Theorem 5.2. *Assume (A_1) , (H) , (3.11), (5.13), (5.14). Let $b \in L^\infty(M)$ and φ be as in Lemma 5.1. Let u be a solution of problem (1.1) with $u \in L^\infty(M \times (0, \tau))$ for each $0 < \tau < T$ and $\partial_t u(t) \varphi \in L^1(M)$ for every $t \in (0, T)$. Suppose that*

$$k \int_M u_0(x) \eta(x) d\mu > \lambda^{\frac{1}{p-1}}, \quad (5.16)$$

with k given by (5.15). Then problem (1.1) does not admit global solutions.

The proof of Theorem 5.2 is very similar to that of Theorem 3.2, and it is based on Lemma 5.1 instead of Lemma 3.1. Therefore it is omitted.

Corollary 5.3. *Assume (A_1) , (H) , and (3.11), with $\hat{C} = 0$. Let $u_0 \not\equiv 0, b \in L^\infty(M)$. Let u be a solution of problem (1.1) with $u \in L^\infty(M \times (0, \tau))$ for each $0 < \tau < T$ and $\partial_t u(t) \varphi \in L^1(M)$ for every $t \in (0, T)$. Suppose that*

$$1 < p < 1 + \frac{2}{\gamma(n-1) + 1}, \quad (5.17)$$

with γ given by (2.7). Then problem (1.1) does not admit global solutions.

Remark 5.4. *Observe that if $M = \mathbb{R}^n$, then (A_1) is fulfilled with $\bar{\beta} = 0$. So, $\gamma = 1$. Therefore, condition (5.17) gives*

$$1 < p < 1 + \frac{2}{n}.$$

This agrees with the results in [7], [13], [16]. Indeed, in \mathbb{R}^n also the equality sign included. However, by means of our methods we are not able to consider the equality sign in condition (5.17).

Proof. The conclusion follows from Theorem 5.2, if we show that condition (5.16) is fulfilled for $a > 0$ and

$$\lambda = 2a[1 + C_1(n-1) + \sigma].$$

Note that, for some $\tilde{C}_1 > 0$ (see, e.g., [4, formula (3.3.11)]), for any $a > 0$,

$$\int_0^\infty e^{-ar^2} r^{\gamma(n-1)} dr \leq \tilde{C}_1 a^{-\frac{n}{2} - \frac{(\gamma-1)(n-1)}{2}} = \tilde{C}_1 a^{-\frac{\gamma(n-1)+1}{2}}.$$

Hence, in view of (2.6) and (5.15), for some $\tilde{C}_2 > 0$,

$$1 = k \int_M \eta(x) d\mu \leq k \tilde{C}_2 \int_0^\infty e^{-ar^2} r^{\gamma(n-1)} dr \leq k \tilde{C}_2 \tilde{C}_1 a^{-\frac{\gamma(n-1)+1}{2}}.$$

Thus

$$k \geq \frac{a^{\frac{\gamma(n-1)+1}{2}}}{\tilde{C}_1 \tilde{C}_2},$$

and

$$k \int_M \eta(x) u_0(x) d\mu \geq \frac{a^{\frac{\gamma(n-1)+1}{2}}}{\tilde{C}_1 \tilde{C}_2} \int_M \eta(x) u_0(x) d\mu. \quad (5.18)$$

Hence condition (5.16) is verified, if

$$\frac{a^{\frac{\gamma(n-1)+1}{2}}}{\tilde{C}_1 \tilde{C}_2} \int_M \eta(x) u_0(x) d\mu > a^{\frac{1}{p-1}} [2(1 + C_1(n-1) + \sigma)]^{\frac{1}{p-1}},$$

that is

$$\int_M \eta(x) u_0(x) d\mu > \tilde{C}_1 \tilde{C}_2 a^{-\frac{\gamma(n-1)+1}{2} + \frac{1}{p-1}} [2(1 + C_1(n-1) + \sigma)]^{\frac{1}{p-1}}. \quad (5.19)$$

Since $u_0 \not\equiv 0$,

$$\liminf_{a \rightarrow 0^+} \int_M \eta(x) u_0(x) d\mu > 0;$$

furthermore, in view of (5.17), we have

$$\lim_{a \rightarrow 0^+} a^{-\frac{\gamma(n-1)+1}{2} + \frac{1}{p-1}} [2(1 + C_1(n-1) + \sigma)]^{\frac{1}{p-1}} = 0.$$

Hence, conditions (5.19) and (5.16) are satisfied for $a > 0$ sufficiently small. This completes the proof. \square

6 Existence of global solutions under assumption (A_1)

Let $\underline{b}(x)$ be defined as in (3.9). For any $C > 0, \alpha > 0, t_0 > 0$ define

$$\underline{u}(x, t) := C(t + t_0)^{-\alpha} e^{-\frac{C^2}{4(t+t_0)}} \quad \text{for any } x \in M, t > 0. \quad (6.20)$$

Proposition 6.1. *Let assumptions (A_1) and (H) be satisfied. Suppose that*

$$\underline{b}(x) \geq \frac{\nu}{r(x)} \quad \text{for all } x \in M \setminus \{o\}, \quad (6.21)$$

for some $-n < \nu \leq 0$, and that

$$p > 1 + \frac{2}{n + \nu}. \quad (6.22)$$

Then, for some $\alpha > 0, C > 0$, the function \bar{u} , defined in (6.20), verifies

$$\partial_t \bar{u} \geq \Delta \bar{u} + \langle b, \nabla \bar{u} \rangle + \bar{u}^p \quad \text{in } M \times (0, +\infty).$$

The proof of Proposition 6.1 is modelled after that of [4, Lemma 3.3.2].

Proof. From (2.1) and (2.5) we get, for every $(x, t) \in M \times (0, +\infty)$,

$$\begin{aligned}
& \partial_t \bar{u} - \Delta \bar{u} - \langle b(x), \nabla \bar{u} \rangle \\
&= C e^{-\frac{r^2}{4(t+t_0)}} (t+t_0)^{-\alpha-1} \left\{ -\alpha + \frac{1}{2} + \frac{r}{2} \mathcal{F}(r, \theta) + \frac{1}{2} \underline{b}(x)r \right\} \\
&= C e^{-\frac{r^2}{4(t+t_0)}} (t+t_0)^{-\alpha-1} \left\{ -\alpha + \frac{1}{2} + \frac{r}{2} \frac{n-1}{r} + \frac{\nu}{2} \right\} \\
&\geq C e^{-\frac{r^2}{4(t+t_0)}} (t+t_0)^{-\alpha-1} \frac{n+\nu-2\alpha}{2} = \frac{\bar{u}}{t+t_0} \frac{n+\nu-2\alpha}{2}.
\end{aligned} \tag{6.23}$$

Due to (6.22) we can find

$$0 < \epsilon < \frac{n+\nu}{2} - \frac{1}{p-1}.$$

Fix any $\epsilon \in (0, \frac{n+\nu}{2})$ and choose

$$\alpha = \frac{n+\nu}{2} - \epsilon. \tag{6.24}$$

Note that

$$\bar{u}(x, t) \leq C(t+t_0)^{-\alpha} \quad \text{for all } (x, t) \in M \times (0, +\infty).$$

Therefore

$$\frac{1}{t+t_0} \geq \left(\frac{\bar{u}(x, t)}{C} \right)^{\frac{1}{\alpha}} \quad \text{for all } (x, t) \in M \times (0, +\infty). \tag{6.25}$$

Choose

$$0 < C \leq \epsilon^{\frac{1}{\alpha}}. \tag{6.26}$$

From (6.23)-(6.26) we obtain

$$\partial_t \bar{u} - \Delta \bar{u} - \langle b, \nabla \bar{u} \rangle \geq \bar{u}^{1+\frac{1}{\alpha}} \quad \text{in } M \times (0, +\infty). \tag{6.27}$$

In view of (6.22) and (6.24), for $\epsilon > 0$ sufficiently small

$$1 + \frac{1}{\alpha} < p; \tag{6.28}$$

in addition,

$$0 < \bar{u} \leq 1 \quad \text{in } M \times (0, +\infty). \tag{6.29}$$

By (6.27)-(6.29),

$$\partial_t \bar{u} - \Delta \bar{u} - \langle b, \nabla \bar{u} \rangle \geq \bar{u}^p \quad \text{in } M \times (0, +\infty).$$

The proof is complete. \square

Theorem 6.2. *Let assumptions (A_1) and (H) be satisfied. Suppose that (6.21) and (6.22) hold. Assume that*

$$0 \leq u_0(x) \leq \bar{u}(x, 0) \quad \text{for all } x \in M,$$

where \bar{u} given by Proposition 6.1. Then there exists a global solution to problem (1.1).

Proof. The conclusion follows arguing as in the proof of Theorem 4.2 replacing the supersolution defined in (3.15) with that provided by Proposition 6.1. \square

Theorem 6.2 generalizes [4, Theorem 3.3.3], where $M = \mathbb{R}^n$ is treated.

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