

EIGENVALUES OF THE LAPLACIAN WITH MOVING MIXED BOUNDARY CONDITIONS: THE CASE OF DISAPPEARING DIRICHLET REGION

VERONICA FELLI, BENEDETTA NORIS, AND ROBERTO OGNIBENE

ABSTRACT. In this work we consider the homogeneous Neumann eigenvalue problem for the Laplacian on a bounded Lipschitz domain and a singular perturbation of it, which consists in prescribing zero Dirichlet boundary conditions on a small subset of the boundary. We first describe the sharp asymptotic behaviour of a perturbed eigenvalue, in the case in which it is converging to a simple eigenvalue of the limit Neumann problem. The first term in the asymptotic expansion turns out to depend on the Sobolev capacity of the subset where the perturbed eigenfunction is vanishing. Then we focus on the case of Dirichlet boundary conditions imposed on a subset which is scaling to a point; by a blow-up analysis for the capacity potentials, we detect the vanishing order of the Sobolev capacity of such shrinking Dirichlet boundary portion.

Keywords. Asymptotics of Laplacian eigenvalues; Mixed boundary conditions; Singular perturbation of domain; Capacity.

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1. INTRODUCTION

The present work concerns the eigenvalue problem for the Laplacian with mixed Dirichlet-Neumann homogeneous boundary conditions. More in particular, our attention is devoted to the study of the behaviour of an eigenvalue of the mixed problem when the region where Dirichlet boundary conditions are prescribed is disappearing, in a suitable sense that will be specified later. Actually, the methods developed in the present paper to derive eigenvalue asymptotics under perturbed mixed boundary conditions turn out to be quite flexible and capable of treating also more general kinds of perturbation, e.g. the eigenvalue problem for the Neumann-Laplacian with a shrinking hole in the interior of the domain where homogeneous Dirichlet boundary conditions are assigned.

Let us introduce some basic assumptions and the functional setting. Let $\Omega \subseteq \mathbb{R}^N$ (with $N \geq 2$) be an open, bounded, Lipschitz and connected set and let $K \subseteq \overline{\Omega}$ be compact. Let $c \in L^\infty(\mathbb{R}^N)$ be such that

$$(1.1) \quad c(x) \geq c_0 > 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \text{for some } c_0 \in \mathbb{R}.$$

We define the bilinear form $q = q_c : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ as

$$(1.2) \quad q(u, v) := \int_{\Omega} (\nabla u \cdot \nabla v + cuv) \, dx \quad \text{for any } u, v \in H^1(\Omega).$$

For simplicity of notation we denote by $q(\cdot)$ also the quadratic form corresponding to (1.2), i.e. $q(u) = q(u, u)$. Thanks to assumption (1.1), the square root of the quadratic form $q(\cdot)$ is a norm on $H^1(\Omega)$, equivalent to the standard one

$$\|u\|_{H^1(\Omega)} = \left(\int_{\Omega} (|\nabla u|^2 + u^2) \, dx \right)^{1/2}.$$

We also introduce the Sobolev space $H_{0,K}^1(\Omega)$ defined as the closure in $H^1(\Omega)$ of $C_c^\infty(\overline{\Omega} \setminus K)$. We observe that, if $\partial\Omega$ is smooth and K is a regular submanifold of $\partial\Omega$, the space $H_{0,K}^1(\Omega)$ can be characterized as

$$H_{0,K}^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } K\},$$

where $u = 0$ on K , for functions in $H^1(\Omega)$, is meant in the trace sense, see [6]. Defining q_K as the restriction of the form q to $H_{0,K}^1(\Omega)$, we say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of q_K if there exists $u \in H_{0,K}^1(\Omega)$, $u \neq 0$, called *eigenfunction*, such that

$$(1.3) \quad q_K(u, v) = \lambda(u, v)_{L^2(\Omega)} \quad \text{for all } v \in H_{0,K}^1(\Omega),$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is the usual scalar product in $L^2(\Omega)$. From classical spectral theory we have that problem (1.3) admits a diverging sequence of positive eigenvalues

$$0 < \lambda_1(\Omega; K) < \lambda_2(\Omega; K) \leq \dots \leq \lambda_n(\Omega; K) \leq \dots,$$

where each one is repeated as many times as its multiplicity. Moreover, we denote by $(\varphi_n(\Omega; K))_n$ a sequence of eigenfunctions, which we choose so that it forms an orthonormal family in $L^2(\Omega)$. Hereafter we denote, for any integer $n \in \mathbb{N}_*$,

$$(1.4) \quad \lambda_n := \lambda_n(\Omega; \emptyset), \quad \varphi_n := \varphi_n(\Omega; \emptyset),$$

where $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$. We notice that the connectedness of the domain Ω is not a restrictive assumption, since the spectrum of q_K in a non connected domain is the union of the spectra on the single connected components. We also point out that the assumption (1.1) is not substantial and it can be dropped, since, up to a translation of the spectrum, we can recover a coercive form as in (1.2). Besides, we notice that, in the particular case when K is the empty set and $c(x) \equiv c > 0$, $(\lambda_n(\Omega; \emptyset) - c)_n$ coincides with the sequence of eigenvalues of the standard Laplacian with homogeneous Neumann boundary condition. If K is smooth (e.g. if K is the closure of a smooth open set of \mathbb{R}^N or a regular submanifold of $\partial\Omega$), problem (1.3) admits the following classical formulation

$$(1.5) \quad \begin{cases} -\Delta u + cu = \lambda u, & \text{in } \Omega \setminus K, \\ u = 0, & \text{on } K, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \setminus K. \end{cases}$$

When $K \subseteq \partial\Omega$, (1.5) is an elliptic problem with mixed Dirichlet-Neumann homogeneous boundary conditions and one can interpret the spectrum $(\lambda_n(\Omega; K))_n$ as the square roots of the frequencies of oscillation of an elastic, vibrating membrane, whose boundary is clamped on K and free in the rest of $\partial\Omega$.

In this paper we start from the unperturbed situation corresponding to the Neumann eigenvalue problem, i.e. the case $K = \emptyset$, and then we introduce a singular perturbation of it, which consist in considering a ‘‘small’’, nonempty $K \subseteq \overline{\Omega}$ and zero Dirichlet boundary conditions on it. Our aim is to study the eigenvalue variation due to this perturbation and to find the sharp asymptotics of the perturbed eigenvalue, in the limit when K is ‘‘disappearing’’ as a function of a certain parameter.

A detailed analysis in dimension 2 has been performed in [14], where Gadyl’shin investigated the case in which the perturbing set is a segment of length $\epsilon \rightarrow 0^+$ contained in the boundary of the domain: a (possibly multiple) eigenvalue of the limit Neumann problem is considered and the full asymptotic expansion of the perturbed eigenvalues is provided, see Theorems 1 and 2 in [14]. These expansions strongly depend on the vanishing order of the limit eigenfunctions in the point of the boundary where the segment is concentrating. In [14] a complete pointwise expansion of the perturbed eigenfunctions is provided as well. Moreover, in [13], Gadyl’shin considered, again in dimension 2, the complementary problem, i.e. the case in which the portion of the boundary where Neumann conditions are prescribed is vanishing. For this problem, in [13], the splitting of a multiple, limit eigenvalue is proved, together with a full asymptotic expansion of the perturbed eigenvalues. In the same framework, in [2] simple limit eigenvalues were considered, with the derivation of a more explicit expression of the coefficient of the leading term in the expansion stated in [13]. Besides, [2] contains a blow-up convergence result for the scaled, perturbed eigenfunction and some applications to eigenvalue problems for operators with Aharonov-Bohm potentials.

On the other hand, for arbitrary dimensions, [9] and [1] treated the eigenvalue problem for the Dirichlet-Laplacian under perturbations consisting in making a hole in the interior of the domain and letting it shrink. In particular, in [1] (and in its nonlocal counterpart [3]), where arguments

and techniques that inspired the ones developed in the present work were introduced, the sharp asymptotic behaviour of perturbed eigenvalues is described. We also mention [8], which concerns the spectral stability of the first eigenvalue, in both cases of shrinking Neumann and Dirichlet part, and [16], where again the first eigenvalue of the mixed problem is considered, but in a nonlocal framework.

2. STATEMENT OF THE MAIN RESULTS

In this section we give the basic definitions used in our work and we present our main results. Let us recall that, throughout the paper, Ω denotes an open, bounded, Lipschitz and connected subset of \mathbb{R}^N , where $N \geq 2$.

As in [1, 3, 9], the quantity that measures the “smallness” of the perturbation set $K \subseteq \overline{\Omega}$, which is suitable for the development of an eigenvalue stability theory for our problem, is a notion of capacity, as defined below.

Definition 2.1. Let $K \subseteq \overline{\Omega}$ be compact. We define the *relative Sobolev capacity* of K in $\overline{\Omega}$ as follows

$$\text{Cap}_{\overline{\Omega}}(K) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) dx : u \in H^1(\Omega), u - 1 \in H_{0,K}^1(\Omega) \right\}.$$

We refer to [5] for the mathematical description of this set function. A first taste of the fact that the relative Sobolev capacity defined above is a good perturbation parameter for our purposes is given by Proposition 3.3, which states that the space $H_{0,K}^1(\Omega)$ coincides with $H^1(\Omega)$ if and only if $\text{Cap}_{\overline{\Omega}}(K) = 0$. This means that zero capacity sets are negligible for H^1 functions. Furthermore, the following theorem yields continuity of perturbed eigenvalues when $\text{Cap}_{\overline{\Omega}}(K) \rightarrow 0$.

Theorem 2.2. Let $K \subseteq \overline{\Omega}$ be compact and let $\lambda_n(\Omega; K)$ be an eigenvalue of problem (1.3) for some $n \in \mathbb{N}_*$. Let also λ_n be as in (1.4). Then there exist $C > 0$ and $\delta > 0$ (independent of K) such that, if $\text{Cap}_{\overline{\Omega}}(K) < \delta$, then

$$0 \leq \lambda_n(\Omega; K) - \lambda_n \leq C (\text{Cap}_{\overline{\Omega}}(K))^{1/2}.$$

We observe that the left inequality is an easy consequence of the Min-Max variational characterization of the eigenvalues, namely

$$(2.1) \quad \lambda_n(\Omega; K) = \min \left\{ \max_{u \in V_n} \frac{q(u)}{\|u\|_{L^2(\Omega)}^2} : V_n \subseteq H_{0,K}^1(\Omega) \text{ } n\text{-dimensional subspace} \right\}.$$

In order to state the first main result of this paper, we introduce the following notion of convergence of sets.

Definition 2.3. Let $K \subseteq \overline{\Omega}$ be compact and let $\{K_\epsilon\}_{\epsilon > 0}$ be a family of compact subsets of $\overline{\Omega}$. We say that K_ϵ is *concentrating* at K , as $\epsilon \rightarrow 0$, if for any open set $U \subseteq \mathbb{R}^N$ such that $K \subseteq U$ there exists $\epsilon_U > 0$ such that $K_\epsilon \subseteq U$ for all $\epsilon \in (0, \epsilon_U)$.

We observe that the “limit” set of a concentrating family is not unique. Indeed, if K_ϵ is concentrating at K , then it is also concentrating at any compact set \tilde{K} such that $K \subseteq \tilde{K} \subseteq \overline{\Omega}$. Nevertheless, in the cases considered in the present paper (e.g. when the limit set has zero capacity) this notion of convergence is the one that ensures the continuity of the capacity (see Proposition 3.8); furthermore, it is related to the convergence of sets in the sense of Mosco, see [1]. An example of family of concentrating sets is given by a decreasing family of compact sets, see Example 3.7.

In order to sharply describe the eigenvalue variation, the following definition of capacity associated to a H^1 -function plays a fundamental role.

Definition 2.4. For any $f \in H^1(\Omega)$ and $K \subseteq \overline{\Omega}$ compact, we define the *relative Sobolev f -capacity* of K in $\overline{\Omega}$ as follows

$$(2.2) \quad \text{Cap}_{\overline{\Omega},c}(K, f) := \inf \{q(u) : u \in H^1(\Omega), u - f \in H_{0,K}^1(\Omega)\}.$$

We remark that, if $K \subseteq \partial\Omega$, the above definition actually only depends on the trace of f on $\partial\Omega$, which belongs to $H^{1/2}(\partial\Omega)$, and in particular on its values on K , if K is regular. Moreover one can prove that, if a family of compact sets $K_\epsilon \subseteq \overline{\Omega}$ is concentrating to a compact $K \subseteq \overline{\Omega}$ such that $\text{Cap}_{\overline{\Omega}}(K) = 0$, then $\text{Cap}_{\overline{\Omega},c}(K_\epsilon, f) \rightarrow 0$ as $\epsilon \rightarrow 0$ for all $f \in H^1(\Omega)$, see Proposition 3.8 and Remark 3.4.

Hereafter we assume $n_0 \in \mathbb{N}_*$ to be such that

$$(2.3) \quad \lambda_0 := \lambda_{n_0} \text{ is simple}$$

and we denote as

$$(2.4) \quad \varphi_0 := \varphi_{n_0},$$

a corresponding $L^2(\Omega)$ -normalized eigenfunction. Our first main result is the following sharp asymptotic expansion of the eigenvalue variation.

Theorem 2.5. *Let $\{K_\epsilon\}_{\epsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating to $K \subseteq \overline{\Omega}$ compact such that $\text{Cap}_{\overline{\Omega}}(K) = 0$. Let λ_0, φ_0 be as in (2.3), (2.4) respectively and let $\lambda_\epsilon := \lambda_{n_0}(\Omega; K_\epsilon)$. Then*

$$\lambda_\epsilon - \lambda_0 = \text{Cap}_{\overline{\Omega},c}(K_\epsilon, \varphi_0) + o(\text{Cap}_{\overline{\Omega},c}(K_\epsilon, \varphi_0))$$

as $\epsilon \rightarrow 0$.

In order to give some relevant examples of explicit expansions, in the last part of the present work we provide the sharp asymptotic behavior of the function $\epsilon \mapsto \text{Cap}_{\overline{\Omega},c}(K_\epsilon, \varphi_0)$ appearing above, in a particular case. More precisely, we consider a family $\{K_\epsilon\}_{\epsilon>0} \subseteq \overline{\Omega}$ which is concentrating at a point $\bar{x} \in \overline{\Omega}$ in an appropriate way, that resembles the situation where a fixed set is being scaled and it is therefore maintaining the same shape while shrinking to the point. Hereafter we illustrate these results by distinguishing the cases $\bar{x} \in \partial\Omega$ and $\bar{x} \in \Omega$. Without losing generality we can assume that $\bar{x} = 0$. We perform this analysis under the assumption $N \geq 3$, since a detailed study of the case $N = 2$, with $K_\epsilon, K \subseteq \partial\Omega$, has been already pursued in [14]; nevertheless, our method, which is based on a blow-up analysis for the capacity potentials, could be adapted to the 2-dimensional case by using a logarithmic Hardy inequality to derive energy estimates, instead of the Hardy-type inequality of Lemma 6.7, which does not hold in dimension 2.

2.1. Sets scaling to a boundary point. We first focus on the case in which the perturbing compact sets $K_\epsilon \subseteq \overline{\Omega}$ are concentrating to a point of the boundary of Ω , which, up to a translation, can be assumed to be the origin. In this situation, we assume that the boundary $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of $0 \in \partial\Omega$, namely

$$(2.5) \quad \begin{aligned} &\text{there exists } r_0 > 0 \text{ and } g \in C^{1,1}(B'_{r_0}) \text{ such that} \\ &B_{r_0} \cap \Omega = \{x \in B_{r_0} : x_N > g(x')\}, \\ &B_{r_0} \cap \partial\Omega = \{x \in B_{r_0} : x_N = g(x')\}, \end{aligned}$$

where $B_{r_0} = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : |x| < r_0\}$ is the ball in \mathbb{R}^N centered at the origin with radius r_0 , $x' = (x_1, \dots, x_{N-1})$ and $B'_{r_0} = \{(x', x_N) \in B_{r_0} : x_N = 0\}$. It is not restrictive to assume that $\nabla g(0) = 0$, i.e. that $\partial\Omega$ is tangent to the coordinate hyperplane $\{x_N = 0\}$ in the origin. Let us introduce the following class of diffeomorphisms that “straighten” the boundary near 0:

$$(2.6) \quad \begin{aligned} \mathcal{C} := \{ &\Phi : \mathcal{U} \rightarrow B_R : \mathcal{U} \text{ is an open neighbourhood of } 0, R > 0, \\ &\Phi \text{ is a diffeomorphism of class } C^{1,1}(\mathcal{U}; B_R), \Phi(0) = 0, \\ &J_\Phi(0) = I_N, \Phi(\mathcal{U} \cap \Omega) = \mathbb{R}_+^N \cap B_R \text{ and } \Phi(\mathcal{U} \cap \partial\Omega) = B'_R\}, \end{aligned}$$

where $\mathbb{R}_+^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$ and I_N is the identity $N \times N$ matrix. Let us assume that, for any $\epsilon \in (0, 1)$, $K_\epsilon \subseteq \overline{\Omega}$ is a compact set and the family $\{K_\epsilon\}_\epsilon$ satisfies the following

properties:

$$(2.7) \quad \text{there exists } M \subseteq \overline{\mathbb{R}_+^N} \text{ compact such that } \Phi(K_\epsilon)/\epsilon \subseteq M \quad \text{for all } \epsilon \in (0, 1),$$

$$(2.8) \quad \begin{aligned} & \text{there exists } K \subseteq \overline{\mathbb{R}_+^N} \text{ compact such that} \\ & \mathbb{R}^N \setminus (\Phi(K_\epsilon)/\epsilon) \rightarrow \mathbb{R}^N \setminus K \quad \text{in the sense of Mosco, as } \epsilon \rightarrow 0, \end{aligned}$$

for some $\Phi \in \mathcal{C}$, where $\Phi(K_\epsilon)/\epsilon := \{x/\epsilon : x \in \Phi(K_\epsilon)\}$. With reference to [10, 18], we recall below the definition of convergence of sets in the sense of Mosco.

Definition 2.6. Let $\epsilon \in (0, 1)$ and let $U_\epsilon, U \subseteq \mathbb{R}^N$ be open sets. We say that U_ϵ is converging to U in the sense of Mosco as $\epsilon \rightarrow 0$ if the following two properties hold:

- (i) the weak limit points (as $\epsilon \rightarrow 0$) in $H^1(\mathbb{R}^N)$ of every family of functions $\{u_\epsilon\}_\epsilon \subseteq H^1(\mathbb{R}^N)$, such that $u_\epsilon \in H_0^1(U_\epsilon)$ for every $\epsilon > 0$, belong to $H_0^1(U)$;
- (ii) for every $u \in H_0^1(U)$ there exists a family $\{u_\epsilon\}_\epsilon \subseteq H^1(\mathbb{R}^N)$ such that $u_\epsilon \in H_0^1(U_\epsilon)$ for every $\epsilon > 0$ and $u_\epsilon \rightarrow u$ in $H^1(\mathbb{R}^N)$, as $\epsilon \rightarrow 0$.

We may also say that $H_0^1(U_\epsilon)$ is converging to $H_0^1(U)$ in the sense of Mosco.

In order to clarify hypotheses (2.7) and (2.8) we adduce below a bunch of examples in which they hold for subsets K_ϵ of $\partial\Omega$.

Examples 2.7.

- (i) The easiest case is when $\partial\Omega$ is flat in a neighbourhood of the origin and

$$K_\epsilon := \epsilon K = \{\epsilon x : x \in K\},$$

for a certain fixed $K \subseteq \mathbb{R}^{N-1}$ compact. Here we can choose as Φ the identity so that $\Phi(K_\epsilon)/\epsilon \equiv K$, which clearly satisfies both hypotheses (2.7) and (2.8).

- (ii) Another interesting example (always in the case of flat boundary) is when Φ is the identity and K_ϵ/ϵ is a perturbation of a compact set. More precisely, let $K_1, K_2 \subseteq \mathbb{R}^{N-1}$ be two compact sets containing the origin and let $f : (0, 1) \rightarrow (0, +\infty)$ be such that $f(s)/s \rightarrow 0$ as $s \rightarrow 0$. If we consider

$$K_\epsilon := \epsilon K_1 + f(\epsilon) K_2 = \{\epsilon x + f(\epsilon) y : x \in K_1, y \in K_2\}$$

then $\Phi(K_\epsilon)/\epsilon$ fulfills (2.7) and (2.8) with $K = K_1$. We remark that it is possible to generalize this idea and produce other examples.

- (iii) In the case of non-flat boundary, we have that conditions (2.7) and (2.8) hold e.g. when K_ϵ is the image through Φ^{-1} , for some $\Phi \in \mathcal{C}$, of sets like the ones in (i)–(ii). A remarkable case is when $\Phi(x', x_N) = (x', x_N - g(x'))$ in a neighbourhood of the origin, so that the restriction of Φ to $\partial\Omega$ is the orthogonal projection of $\partial\Omega$ onto its tangent hyperplane at 0. Hence assumption (2.8) is satisfied, for example, if the each set K_ϵ is a compact subset of $\partial\Omega$ whose orthogonal projection on the hyperplane tangent to $\partial\Omega$ at 0 is of the form ϵK , for K being a compact subset of \mathbb{R}^{N-1} .

- (iv) Finally, it is easy to prove that assumptions (2.7) and (2.8) hold for

$$K_\epsilon := B_\epsilon \cap \partial\Omega, \quad \text{with } K = \mathbb{R}^{N-1} \cap \overline{B_1}.$$

The last ingredient needed to detect the sharp asymptotics of the Sobolev capacity of shrinking sets is the notion of vanishing order for the limit eigenfunction. Let

$$L^2(\mathbb{S}_+^{N-1}) := \left\{ \psi : \mathbb{S}_+^{N-1} \rightarrow \mathbb{R} : \psi \text{ is measurable and } \int_{\mathbb{S}_+^{N-1}} |\psi|^2 dS < \infty \right\},$$

where $\mathbb{S}_+^{N-1} := \{(x_1, \dots, x_N) \in \mathbb{R}^N : |x| = 1, x_N > 0\}$. Moreover let

$$H^1(\mathbb{S}_+^{N-1}) := \{\psi \in L^2(\mathbb{S}_+^{N-1}) : \nabla_{\mathbb{S}_+^{N-1}} \psi \in L^2(\mathbb{S}_+^{N-1})\}.$$

The following proposition asserts that the limit eigenfunction φ_0 behaves like a harmonic polynomial near the origin.

Proposition 2.8. *Let Ω satisfy assumption (2.5) and let φ_0 be as in (2.4). Then there exists $\gamma \in \mathbb{N}$ (possibly 0) and $\Psi \in C^\infty(\mathbb{S}_+^{N-1})$, $\Psi \neq 0$ such that, for all $\Phi \in \mathcal{C}$, there holds*

$$(2.9) \quad \frac{\varphi_0(\Phi^{-1}(\epsilon x))}{\epsilon^\gamma} \rightarrow |x|^\gamma \Psi \left(\frac{x}{|x|} \right) \quad \text{in } H^1(B_R^+) \text{ as } \epsilon \rightarrow 0,$$

for all $R > 0$, where $B_R^+ := B_R \cap \mathbb{R}_+^N$.

Furthermore, for every $R > 0$,

$$(2.10) \quad \epsilon^{-N-2\gamma} \int_{\Omega \cap B_{R\epsilon}} \varphi_0^2(x) \, dx \rightarrow \int_{B_R^+} \psi_\gamma^2(x) \, dx \quad \text{as } \epsilon \rightarrow 0$$

and

$$(2.11) \quad \epsilon^{-N-2\gamma+2} \int_{\Omega \cap B_{R\epsilon}} |\nabla \varphi_0(x)|^2 \, dx \rightarrow \int_{B_R^+} |\nabla \psi_\gamma(x)|^2 \, dx \quad \text{as } \epsilon \rightarrow 0$$

where

$$(2.12) \quad \psi_\gamma(x) := |x|^\gamma \Psi \left(\frac{x}{|x|} \right).$$

We observe that the function Ψ appearing in (2.9) and (2.12) is necessarily a spherical harmonic of degree γ which is symmetric with respect to the equator $x_N = 0$, hence satisfying homogeneous Neumann boundary conditions on $\{x_N = 0\}$. More precisely, Ψ solves

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} \Psi = \gamma(N + \gamma - 2)\Psi, & \text{in } \mathbb{S}_+^{N-1}, \\ \nabla_{\mathbb{S}^{N-1}} \Psi \cdot \mathbf{e}_N = 0, & \text{on } \partial \mathbb{S}_+^{N-1}, \end{cases}$$

where $\mathbf{e}_N = (0, \dots, 0, 1)$. The integer number γ is called the *vanishing order* of φ_0 in the origin. We also mention [7, 12, 19, 20] for asymptotic behaviour of solutions to elliptic PDEs.

Remark 2.9. We observe that the restriction of Ψ to the $N - 2$ dimensional unit sphere $\partial \mathbb{S}_+^{N-1}$ cannot vanish everywhere. Indeed this would mean that the nonzero harmonic function ψ_γ defined in (2.12) vanishes on $\partial \mathbb{R}_+^N$ together with its normal derivative; but then the trivial extension of ψ_γ to the whole \mathbb{R}^N would violate the classical unique continuation principle (see [22]), thus giving rise to a contradiction.

For $N \geq 3$, let us introduce the Beppo Levi space $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ defined as the completion of $C_c^\infty(\overline{\mathbb{R}_+^N})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})} := \left(\int_{\mathbb{R}_+^N} |\nabla u|^2 \, dx \right)^{1/2}.$$

Furthermore, for any compact $K \subseteq \overline{\mathbb{R}_+^N}$, we define the space $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$ as the closure of $C_c^\infty(\overline{\mathbb{R}_+^N} \setminus K)$ in $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$. Thereafter we introduce a notion of capacity that will appear in the asymptotic expansion of $\text{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0)$, when $\epsilon \rightarrow 0$.

Definition 2.10. For any compact $K \subseteq \overline{\mathbb{R}_+^N}$ and for any $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ we define the *relative f -capacity* of K in $\overline{\mathbb{R}_+^N}$ as

$$\text{cap}_{\overline{\mathbb{R}_+^N}}(K, f) := \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^2 \, dx : u \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N}), u - f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K) \right\}.$$

If $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ is equal to 1 in a neighbourhood of K , we denote by

$$\text{cap}_{\overline{\mathbb{R}_+^N}}(K) := \text{cap}_{\overline{\mathbb{R}_+^N}}(K, f)$$

the relative capacity of K in $\overline{\mathbb{R}_+^N}$. The definition can be extended to functions $f \in H_{\text{loc}}^1(\overline{\mathbb{R}_+^N})$ by letting

$$\text{cap}_{\overline{\mathbb{R}_+^N}}(K, f) := \inf \left\{ \int_{\overline{\mathbb{R}_+^N}} |\nabla u|^2 \, dx : u \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N}), u - \eta_K f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K) \right\},$$

where $\eta_K \in C_c^\infty(\overline{\mathbb{R}_+^N})$ is such that $\eta_K = 1$ in a neighbourhood of K .

Remark 2.11. We remark that the relative Sobolev capacity in $\overline{\mathbb{R}_+^N}$ of a compact set $K \subseteq \partial\mathbb{R}_+^N$, here denoted by $\text{cap}_{\overline{\mathbb{R}_+^N}}(K)$, actually coincides with half of the capacity of K , in the classical sense (see [17, Chapter 2.1]), defined as

$$\text{cap}_{\mathbb{R}^N}(K) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u - \eta_K \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K) \right\}.$$

Moreover we notice that $\text{cap}_{\overline{\mathbb{R}_+^N}}(K)$ coincides, up to a constant, with the Gagliardo $\frac{1}{2}$ -fractional capacity of K in \mathbb{R}^{N-1} , see e.g. [3] for the definition.

In this framework we are able to state the second main result of our paper, which concerns the sharp behaviour of the function $\epsilon \mapsto \text{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0)$ as $\epsilon \rightarrow 0^+$.

Theorem 2.12. *Let $N \geq 3$. Assume (2.5) holds true. Let $\{K_\epsilon\}_{\epsilon > 0} \subseteq \overline{\Omega}$ be a family of compact sets concentrating at $\{0\} \subseteq \partial\Omega$ as $\epsilon \rightarrow 0$ and let (2.7)-(2.8) hold for some $\Phi \in \mathcal{C}$ and for some compact set $K \subseteq \overline{\mathbb{R}_+^N}$ satisfying $\text{cap}_{\overline{\mathbb{R}_+^N}}(K) > 0$. Let φ_0 be as in (2.4) and let γ, ψ_γ be as in (2.9)-(2.12). Then*

$$\text{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0) = \epsilon^{N+2\gamma-2} (\text{cap}_{\overline{\mathbb{R}_+^N}}(K, \psi_\gamma) + o(1)), \quad \text{as } \epsilon \rightarrow 0,$$

with $\text{cap}_{\overline{\mathbb{R}_+^N}}(K, \psi_\gamma)$ being as in Definition 2.10.

Combining Theorems 2.12 and 2.5 we directly obtain the following corollary.

Corollary 2.13. *Under the same assumptions and with the same notations of both Theorems 2.12 and 2.5, we have that*

$$\lambda_\epsilon - \lambda_0 = \epsilon^{N+2\gamma-2} (\text{cap}_{\overline{\mathbb{R}_+^N}}(K, \psi_\gamma) + o(1)), \quad \text{as } \epsilon \rightarrow 0.$$

The expansion stated in Corollary 2.13 provides the sharp asymptotics of the eigenvalue variation if $\text{cap}_{\overline{\mathbb{R}_+^N}}(K, \psi_\gamma) > 0$. This happens e.g. whenever $K \subseteq \partial\mathbb{R}_+^N$ is a compact set such that $\text{cap}_{\overline{\mathbb{R}_+^N}}(K) > 0$, as proved in Proposition 3.12; we observe that the validity of such result strongly relies on the position of the nodal set of ψ_γ with respect to the set K .

On the other hand, if $K \subseteq \partial\mathbb{R}_+^N$ is compact, we have that $\text{cap}_{\overline{\mathbb{R}_+^N}}(K) > 0$ if its $N-1$ dimensional Lebesgue measure is nonzero, see Proposition 3.13.

2.2. Sets scaling to an interior point. Although the present study was mainly motivated by our interest in the eigenvalue asymptotics for moving mixed Dirichlet-Neumann boundary conditions, our techniques also apply to another class of perturbations, without any substantial difference, in view of the various possibilities embraced by Theorem 2.5. In particular, it is possible to state a result analogous to Theorem 2.12 in the case in which the perturbing sets K_ϵ are concentrating at a point that lies in the interior of Ω . In this case the limit problem is the one with homogeneous Neumann boundary conditions on $\partial\Omega$ and the perturbed problem can be thought of as Ω without a “small” hole, on which zero Dirichlet boundary conditions are prescribed. We assume that $0 \in \Omega$ is the “limit” of the concentrating subsets K_ϵ and we ask assumptions similar to (2.7)-(2.8) to be satisfied, that is

$$(2.13) \quad \text{there exists } M \subseteq \mathbb{R}^N \text{ compact such that } K_\epsilon/\epsilon \subseteq M \quad \text{for all } \epsilon \in (0, 1),$$

$$(2.14) \quad \begin{aligned} &\text{there exists } K \subseteq \mathbb{R}^N \text{ compact such that} \\ &\mathbb{R}^N \setminus (K_\epsilon/\epsilon) \rightarrow \mathbb{R}^N \setminus K \quad \text{in the sense of Mosco, as } \epsilon \rightarrow 0. \end{aligned}$$

As before, these assumptions are fulfilled, for instance, in the case $K_\epsilon := \epsilon K$, for a certain compact $K \subseteq \mathbb{R}^N$ such that $K_\epsilon \subseteq \Omega$ for every $\epsilon \in (0, 1)$. Since $0 \in \Omega$ is an interior point, from classical regularity results for elliptic equations (see e.g. [20]), there exist $\kappa \in \mathbb{N}$ and a spherical harmonic Z of degree κ such that

$$-\Delta_{\mathbb{S}^{N-1}} Z = \kappa(N + \kappa - 2)Z \quad \text{in } \mathbb{S}^{N-1}$$

and

$$(2.15) \quad \frac{\varphi_0(\epsilon x)}{\epsilon^\kappa} \rightarrow \zeta_\kappa(x) := |x|^\kappa Z\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_R) \text{ as } \epsilon \rightarrow 0,$$

for all $R > 0$. We can now state the last main result of our paper, which is analogous to Theorem 2.12.

Theorem 2.14. *Let $N \geq 3$ and $\{K_\epsilon\}_{\epsilon>0} \subseteq \Omega$ be a family of compact sets concentrating at $\{0\} \subseteq \Omega$ as $\epsilon \rightarrow 0$. Let (2.13)-(2.14) hold for some compact set $K \subseteq \mathbb{R}^N$ satisfying $\text{cap}_{\mathbb{R}^N}(K) > 0$. Let φ_0 be as in (2.4) and κ, ζ_κ be as in (2.15). Then*

$$\text{Cap}_{\bar{\Omega},c}(K_\epsilon, \varphi_0) = \epsilon^{N+2\kappa-2}(\text{cap}_{\mathbb{R}^N}(K, \zeta_\kappa) + o(1)), \quad \text{as } \epsilon \rightarrow 0,$$

where and $\text{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ is the standard Newtonian ζ_κ -capacity of K (see Definition 3.10).

We point out that, in general, $\text{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ may not be strictly positive, since K , still having positive capacity, may happen to be a subset of the zero level set of ζ_κ . In Lemma 3.11 we provide sufficient conditions for $\text{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$ to be strictly positive: e.g. this happens when K has nonzero capacity whereas the intersection of K with the nodal set of ζ_κ has zero capacity. We refer to [11, Theorem 4.15] for a sufficient condition for $\text{cap}_{\mathbb{R}^N}(K) > 0$: more precisely, we have that $\text{cap}_{\mathbb{R}^N}(K) > 0$ if its N -dimensional Lebesgue measure is nonzero.

The paper is organized as follows: in Section 3 we focus on the notion of capacity (as given in Definitions 2.1 and 2.10) and we prove some important properties (such as Propositions 3.12 and 3.13), also in relation with the notion of concentration of sets. In Section 4 we prove the continuity of eigenvalues $\lambda_n(\Omega; K)$ with respect to $\text{Cap}_{\bar{\Omega}}(K)$, i.e. Theorem 2.2. In Section 5 we prove our first main result Theorem 2.5. In Section 6 we prove our second main result Theorem 2.12 and finally, in Section 7, we prove Theorem 2.14.

2.3. Notation. Let us fix some notation we use throughout the paper:

- $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$;
- $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and $S_R = \partial B_R$ for, respectively, balls and spheres centered at the origin;
- $\mathbb{R}_+^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$ for the upper half space;
- we may identify $\mathbb{R}^{N-1} := \partial \mathbb{R}_+^N$;
- $B_R^+ := B_R \cap \mathbb{R}_+^N$ and $S_R^+ := \partial B_R^+ \cap \mathbb{R}_+^N$ for half balls and half spheres;
- $\mathbb{S}^{N-1} := S_1$ and $\mathbb{S}_+^{N-1} := S_1^+$ denote respectively the unitary sphere and upper unitary half-sphere;
- $B'_R := B_R \cap \partial \mathbb{R}_+^N$.

3. PRELIMINARIES ON CONCENTRATION OF SETS AND CAPACITY

In this section we focus on the notions of capacity and concentration of sets (see Definitions 2.1 and 2.3): we prove some basic properties and we investigate their mutual relations. We start by mentioning the existence of a capacity potential, that is to say a function that achieves the infimum in the definition of capacity. We omit the proof since it follows the classical one.

Proposition 3.1 (Capacity is Achieved). *Let $K \subseteq \bar{\Omega}$ be compact, $f \in H^1(\Omega)$ and $c \in L^\infty(\Omega)$ satisfying (1.1). The f -capacity of K , as introduced in Definition 2.4, is uniquely achieved, i.e. there exists a unique $V_{K,f,c} \in H^1(\Omega)$ which satisfies*

$$V_{K,f,c} - f \in H_{0,K}^1(\Omega) \quad \text{and} \quad \text{Cap}_{\bar{\Omega},c}(K, f) = q(V_{K,f,c}).$$

Since in the following the function c is fixed, for the sake of brevity we will write

$$V_{K,f} := V_{K,f,c}$$

omitting the dependence on c in the notation. We observe that $V_{K,f}$ satisfies

$$\begin{cases} -\Delta V_{K,f} + cV_{K,f} = 0, & \text{in } \Omega \setminus K, \\ \frac{\partial V_{K,f}}{\partial \nu} = 0, & \text{on } \partial\Omega \setminus K, \\ V_{K,f} = f, & \text{on } K, \end{cases}$$

in a weak sense, that is $V_{K,f} - f \in H_{0,K}^1(\Omega)$ and

$$(3.1) \quad q(V_{K,f}, \varphi) = \int_{\Omega} (\nabla V_{K,f} \cdot \nabla \varphi + cV_{K,f}\varphi) dx = 0 \quad \text{for all } \varphi \in H_{0,K}^1(\Omega).$$

Remark 3.2. In the particular case $c, f \equiv 1$ (as in Definition 2.1), we have that the potential $V_K := V_{K,1} \in H^1(\Omega)$ satisfies

$$(3.2) \quad V_K - 1 \in H_{0,K}^1(\Omega) \quad \text{and} \quad \int_{\Omega} (\nabla V_K \cdot \nabla \varphi + V_K \varphi) dx = 0 \quad \text{for all } \varphi \in H_{0,K}^1(\Omega).$$

It is easy to verify that $V_K^-, (V_K - 1)^+ \in H_{0,K}^1(\Omega)$, so that we can choose $\varphi = V_K^-$ and $\varphi = (V_K - 1)^+$ as test functions in the above equation, thus obtaining that $V_K^- \equiv 0$ and $(V_K - 1)^+ \equiv 0$, i.e.

$$(3.3) \quad 0 \leq V_K(x) \leq 1 \quad \text{for a.e. } x \in \Omega.$$

The following proposition asserts that the Sobolev spaces $H^1(\Omega)$ and $H_{0,K}^1(\Omega)$ coincide if and only if the set K has zero capacity, and draws conclusions on the eigenvalues of (1.3).

Proposition 3.3. *Let $K \subseteq \overline{\Omega}$ be compact. The following three assertions are equivalent:*

- (i) $\text{Cap}_{\overline{\Omega}}(K) = 0$;
- (ii) $H^1(\Omega) = H_{0,K}^1(\Omega)$;
- (iii) $\lambda_n(\Omega; K) = \lambda_n$ for every $n \in \mathbb{N}_*$.

Proof. In order to prove that (i) implies (ii) it is sufficient to prove that $H^1(\Omega) \subseteq H_{0,K}^1(\Omega)$ since the converse is trivial. We actually prove that $C^\infty(\overline{\Omega}) \subseteq H_{0,K}^1(\Omega)$ and the claim follows by density. By assumption (i), there exists $\{u_n\}_{n \geq 1} \subset H^1(\Omega)$ such that $u_n - 1 \in H_{0,K}^1(\Omega)$ for every $n \in \mathbb{N}_*$ and $\|u_n\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Let $u \in C^\infty(\overline{\Omega})$ and let us consider the sequence $\{u(1 - u_n)\}_{n \geq 1} \subset H_{0,K}^1(\Omega)$. We claim that $u(1 - u_n) \rightarrow u$ in $H^1(\Omega)$. Indeed

$$\begin{aligned} \|u - u(1 - u_n)\|_{H^1(\Omega)}^2 &= \|uu_n\|_{H^1(\Omega)}^2 \\ &\leq 2 \int_{\Omega} (u^2 |\nabla u_n|^2 + u_n^2 |\nabla u|^2) dx + \int_{\Omega} u^2 u_n^2 dx \\ &\leq 4 \max\{\|u\|_{L^\infty(\Omega)}^2, \|\nabla u\|_{L^\infty(\Omega)}^2\} \|u_n\|_{H^1(\Omega)}^2 \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

We now prove that (ii) implies (i). Let us consider the equation (3.2) solved by V_K . Since $H_{0,K}^1(\Omega) = H^1(\Omega)$, we can choose $\varphi = V_K$ in (3.2) and then reach the conclusion.

Finally, let us show that (ii) is equivalent to (iii). The fact that (ii) implies (iii) follows from the min-max characterization (2.1). Conversely, suppose that (iii) holds, i.e. $\lambda_n(\Omega; K) = \lambda_n$ for every $n \in \mathbb{N}_*$. Then for every $n \in \mathbb{N}_*$ there exists an eigenfunction belonging to $H_{0,K}^1(\Omega)$ associated to λ_n . By the Spectral Theorem, there exists an orthonormal basis of $H^1(\Omega)$ made of $H_{0,K}^1$ -functions, which implies that property (ii) holds. \square

Remark 3.4. An inspection of the proof of Proposition 3.3 shows that (ii) actually implies that $\text{Cap}_{\overline{\Omega},c}(K, f) = 0$ for all $f \in H^1(\Omega)$, and so

$$\text{Cap}_{\overline{\Omega}}(K) = 0 \quad \text{if and only if} \quad \text{Cap}_{\overline{\Omega},c}(K, f) = 0 \quad \text{for all } f \in H^1(\Omega).$$

Moreover, for any $f \in H^1(\Omega)$, we trivially have that $\text{Cap}_{\overline{\Omega},c}(K, f) = 0$ if and only if $V_{K,f} = 0$.

Example 3.5 (Capacity of a Point). Let $x_0 \in \overline{\Omega}$, then $\text{Cap}_{\overline{\Omega}}(\{x_0\}) = 0$.

Proof. If $N \geq 3$, let $v_n \in C^\infty(\overline{\Omega})$ be such that $v_n(x) = 1$ if $x \in B(x_0, \frac{1}{n}) \cap \overline{\Omega}$, $v_n(x) = 0$ if $x \in \overline{\Omega} \setminus B(x_0, \frac{2}{n})$, $0 \leq v_n(x) \leq 1$ and $|\nabla v_n(x)| \leq 2n$ for all $x \in \overline{\Omega}$. It is easy to prove that $q(v_n) \rightarrow 0$, as $n \rightarrow \infty$ thus concluding the proof for $N \geq 3$. If $N = 2$, we can instead consider $v_n \in H^1(\Omega)$ defined as $v_n(x) = 1$ if $x \in B(x_0, \frac{1}{n}) \cap \overline{\Omega}$, $v_n(x) = 0$ if $x \in \overline{\Omega} \setminus B(x_0, \frac{1}{\sqrt{n}})$, $v_n(x) = (\log n)^{-1}(-\log n - 2 \log |x - x_0|)$ if $x \in \overline{\Omega} \cap (B(x_0, \frac{1}{\sqrt{n}}) \setminus B(x_0, \frac{1}{n}))$. It is easy to prove that $q(v_n) \rightarrow 0$, as $n \rightarrow \infty$ thus concluding the proof for $N = 2$. \square

Remark 3.6. Let $\{K_\epsilon\}_{\epsilon>0}, K \subseteq \overline{\Omega}$ be compact sets such that K_ϵ is concentrating at K as $\epsilon \rightarrow 0$. Then, for any $\varphi \in C_c^\infty(\overline{\Omega} \setminus K)$, there exists $\epsilon_\varphi > 0$ such that $\varphi \in C_c^\infty(\overline{\Omega} \setminus K_\epsilon)$ for all $\epsilon < \epsilon_\varphi$. Moreover $\bigcap_{\epsilon>0} K_\epsilon \subseteq K$.

Example 3.7. An example of concentrating sets is a family of compact sets decreasing as $\epsilon \rightarrow 0$. Precisely, let $\{K_\epsilon\}_{\epsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ such that $K_{\epsilon_2} \subseteq K_{\epsilon_1}$ for any $\epsilon_2 \leq \epsilon_1$ and let $K \subseteq \overline{\Omega}$ be a compact set such that $K = \bigcap_{\epsilon>0} K_\epsilon$. Then, arguing by contradiction, thanks to Bolzano-Weierstrass Theorem in \mathbb{R}^N , it is easy to prove that K_ϵ is concentrating at K .

With the next proposition we emphasize what is the relation between the notion of concentration of sets and that of convergence of capacities: it turns out that convergence holds if the limit set has zero capacity.

Proposition 3.8. Let $K \subseteq \overline{\Omega}$ be a compact set and let $\{K_\epsilon\}_{\epsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating at K . If $\text{Cap}_{\overline{\Omega}}(K) = 0$ then

$$V_{K_\epsilon, f} \rightarrow V_{K, f} \quad \text{in } H^1(\Omega) \quad \text{and} \quad \text{Cap}_{\overline{\Omega}, c}(K_\epsilon, f) \rightarrow \text{Cap}_{\overline{\Omega}, c}(K, f) \quad \text{as } \epsilon \rightarrow 0$$

for all $f \in H^1(\Omega)$ and all $c \in L^\infty(\Omega)$ satisfying (1.1). This result holds true, in particular, for the Sobolev capacity (see Definition 2.1) and its potentials, corresponding to the case in which $c \equiv 1$.

Proof. Since $q(V_{K_\epsilon, f}) \leq q(f)$ for all $\epsilon > 0$, then $\{V_{K_\epsilon, f}\}_\epsilon$ is bounded in $H^1(\Omega)$ and so there exists $W \in H^1(\Omega)$ such that, along a sequence $\epsilon_n \rightarrow 0$,

$$V_{K_{\epsilon_n}, f} \rightharpoonup W \quad \text{weakly in } H^1(\Omega) \quad \text{as } n \rightarrow \infty,$$

that is

$$(3.4) \quad \int_{\Omega} (\nabla V_{K_{\epsilon_n}, f} \cdot \nabla \varphi + c V_{K_{\epsilon_n}, f} \varphi) dx \rightarrow \int_{\Omega} (\nabla W \cdot \nabla \varphi + c W \varphi) dx \quad \text{for all } \varphi \in H^1(\Omega).$$

Therefore, taking into account Remark 3.6 and the equation solved by $V_{K_\epsilon, f}$ (3.1), we have that

$$(3.5) \quad \int_{\Omega} (\nabla W \cdot \nabla \varphi + c W \varphi) dx = 0$$

for all $\varphi \in C_c^\infty(\overline{\Omega} \setminus K)$ and then, by density, for all $\varphi \in H_{0, K}^1(\Omega)$. Moreover, taking $\varphi = V_{K, f} - f$ (respectively $\varphi = V_{K_\epsilon, f} - f$) in the equation (3.1) for $V_{K, f}$ (respectively $V_{K_\epsilon, f}$), we obtain

$$(3.6) \quad \text{Cap}_{\overline{\Omega}, c}(K, f) = \int_{\Omega} (\nabla V_{K, f} \cdot \nabla f + c V_{K, f} f) dx,$$

respectively

$$(3.7) \quad \text{Cap}_{\overline{\Omega}, c}(K_\epsilon, f) = \int_{\Omega} (\nabla V_{K_\epsilon, f} \cdot \nabla f + c V_{K_\epsilon, f} f) dx.$$

From Proposition 3.3, we have that $H_{0, K}^1(\Omega) = H^1(\Omega)$ and then (3.5) yields $W = V_{K, f} = 0$; on the other hand, from (3.7) and (3.6) it follows that

$$\text{Cap}_{\overline{\Omega}}(K_{\epsilon_n}, f) \rightarrow \text{Cap}_{\overline{\Omega}, c}(K, f) = 0 \quad \text{as } n \rightarrow \infty.$$

Urysohn's Subsequence Principle concludes the proof. \square

The following lemma is a fundamental step in the proof of our main results. It states that, when a sequence of sets is concentrating, as $\epsilon \rightarrow 0$, at a zero capacity set, the squared $L^2(\Omega)$ -norm of the associated capacitary potentials is negligible, as $\epsilon \rightarrow 0$, with respect to the capacity.

Lemma 3.9. *Let $K \subseteq \overline{\Omega}$ be compact and let $\{K_\epsilon\}_{\epsilon>0}$ be a family of compact subsets of $\overline{\Omega}$ concentrating at K . If $\text{Cap}_{\overline{\Omega}}(K) = 0$, then*

$$\int_{\Omega} |V_{K_\epsilon, f}|^2 dx = o(\text{Cap}_{\overline{\Omega}, c}(K_\epsilon, f)) \quad \text{as } \epsilon \rightarrow 0$$

for all $f \in H^1(\Omega)$ and all $c \in L^\infty(\Omega)$ satisfying (1.1).

Proof. Assume by contradiction that, for a certain $f \in H^1(\Omega)$, there exists $\epsilon_n \rightarrow 0$ and $C > 0$ such that

$$\int_{\Omega} |V_{K_{\epsilon_n}, f}|^2 dx \geq \frac{1}{C} \text{Cap}_{\overline{\Omega}, c}(K_{\epsilon_n}, f).$$

Let

$$W_n := \frac{V_{K_{\epsilon_n}, f}}{\|V_{K_{\epsilon_n}, f}\|_{L^2(\Omega)}}.$$

Then $\|W_n\|_{L^2(\Omega)} = 1$ and

$$\|\nabla W_n\|_{L^2(\Omega)}^2 + \int_{\Omega} c W_n^2 dx = \frac{\text{Cap}_{\overline{\Omega}, c}(K_{\epsilon_n}, f)}{\|V_{K_{\epsilon_n}, f}\|_{L^2(\Omega)}^2} \leq C.$$

Hence $\{W_n\}_n$ is bounded in $H^1(\Omega)$ and so there exists $W \in H^1(\Omega)$ such that $W_n \rightharpoonup W$ weakly in $H^1(\Omega)$, up to a subsequence, as $n \rightarrow \infty$. By compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we have that $\|W\|_{L^2(\Omega)} = 1$. Using Remark 3.6, we can pass to the limit in the equation satisfied by W_n and then obtain

$$\int_{\Omega} (\nabla W \cdot \nabla \varphi + c W \varphi) dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\overline{\Omega} \setminus K).$$

On the other hand, since $\text{Cap}_{\overline{\Omega}}(K) = 0$, in view of Proposition 3.3 $C_c^\infty(\overline{\Omega} \setminus K)$ is dense in $H^1(\Omega)$ and so $W = 0$, thus a contradiction arises. \square

We are now going to prove that the term $\text{cap}_{\overline{\mathbb{R}_+^N}}(K, \psi_\gamma)$, appearing in the expansion stated in Corollary 2.13, is nonzero whenever $K \subseteq \partial\mathbb{R}_+^N$ is a compact set such that $\text{cap}_{\overline{\mathbb{R}_+^N}}(K) > 0$; to this aim we prove a more general lemma concerning the standard (Newtonian) capacity of a set, whose definition we recall below. For any open set $U \subseteq \mathbb{R}^N$, we denote by $\mathcal{D}^{1,2}(U)$ the completion of $C_c^\infty(U)$ with respect to the $L^2(U)$ -norm of the gradient.

Definition 3.10. For $N \geq 3$, let $K \subseteq \mathbb{R}^N$ be a compact set and let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighbourhood of K . If $f \in H_{\text{loc}}^1(\mathbb{R}^N)$, the following quantity

$$\text{cap}_{\mathbb{R}^N}(K, f) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u - f\eta_K \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K) \right\}$$

is called the f -capacity of K . For $f = 1$, $\text{cap}_{\mathbb{R}^N}(K) := \text{cap}_{\mathbb{R}^N}(K, 1)$ is called the capacity of K (as already introduced in Remark 2.11).

Lemma 3.11. *Let $N \geq 3$ and $K \subseteq \mathbb{R}^N$ be a compact set such that $\text{cap}_{\mathbb{R}^N}(K) > 0$. Let $f \in C^\infty(\mathbb{R}^N)$ and let $Z_f := \{x \in \mathbb{R}^N : f(x) = 0\}$. If $\text{cap}_{\mathbb{R}^N}(Z_f \cap K) < \text{cap}_{\mathbb{R}^N}(K)$, then*

$$\text{cap}_{\mathbb{R}^N}(K, f) > 0.$$

Proof. In this proof we make use of some properties of the classical Newtonian capacity of a set and we refer to [17, Chapter 2] for the details. Let us consider a sequence of bounded open sets $U_n \subseteq \mathbb{R}^N$ such that

$$Z_f \cap K \subseteq U_{n+1} \subseteq U_n, \quad \text{for all } n \text{ and } \bigcap_{n \geq 1} \overline{U}_n = Z_f \cap K.$$

Let $K_n := K \setminus U_n$. Since $K \subseteq K_n \cup \overline{U}_n$, by subadditivity and monotonicity of the capacity

$$\text{cap}_{\mathbb{R}^N}(K_n) \geq \text{cap}_{\mathbb{R}^N}(K) - \text{cap}_{\mathbb{R}^N}(\overline{U}_n).$$

Moreover, since $\bigcap_{n \geq 1} \overline{U}_n = Z_f \cap K$, then $\text{cap}_{\mathbb{R}^N}(Z_f \cap K) = \lim_{n \rightarrow \infty} \text{cap}_{\mathbb{R}^N}(\overline{U}_n)$, and so

$$(3.8) \quad \text{cap}_{\mathbb{R}^N}(K_n) > 0$$

for large n , by the assumption $\text{cap}_{\mathbb{R}^N}(K) - \text{cap}_{\mathbb{R}^N}(Z_f \cap K) > 0$. Now we claim that

$$(3.9) \quad \text{cap}_{\mathbb{R}^N}(K, |f|) > 0.$$

Let us fix a sufficiently large n in order for (3.8) to hold and let us set

$$c_n := \frac{1}{2} \inf_{K_n} |f| = \frac{1}{2} \min_{K_n} |f| > 0.$$

By definition of K_n we have $|f| \geq 2c_n > c_n$ on K_n and therefore, by continuity, $|f| > c_n$ in an open neighbourhood of K_n . Let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighbourhood of K and let $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be an arbitrary function such that $u_n - \eta_K |f| \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K_n)$. We define

$$v_n := \min\{1, u_n/c_n\} \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We have that $v_n - \eta_{K_n} \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K_n)$, where $\eta_{K_n} \in C_c^\infty(\mathbb{R}^N)$ is equal to 1 in a neighbourhood of K_n . Therefore v_n is an admissible competitor for $\text{cap}_{\mathbb{R}^N}(K_n)$ and also, by truncation, the energy of v_n is lower than the energy of u_n/c_n . Hence

$$\text{cap}_{\mathbb{R}^N}(K_n) \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leq \int_{\mathbb{R}^N} \frac{|\nabla u_n|^2}{c_n^2} dx.$$

By arbitrariness of u_n , we have that $\text{cap}_{\mathbb{R}^N}(K_n, |f|) \geq c_n^2 \text{cap}_{\mathbb{R}^N}(K_n) > 0$. Moreover, by monotonicity, $\text{cap}_{\mathbb{R}^N}(K, |f|) \geq \text{cap}_{\mathbb{R}^N}(K_n, |f|) > 0$ and so (3.9) is proved. Finally, we claim that

$$(3.10) \quad \text{cap}_{\mathbb{R}^N}(K, f) \geq \text{cap}_{\mathbb{R}^N}(K, |f|).$$

Indeed, if $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is such that $\xi - \eta_K f \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$, then $|\xi| - \eta_K |f| \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$. Hence

$$\text{cap}_{\mathbb{R}^N}(K, |f|) \leq \int_{\mathbb{R}^N} |\nabla |\xi||^2 dx = \int_{\mathbb{R}^N} |\nabla \xi|^2 dx$$

for all $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\xi - \eta_K f \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus K)$, which implies (3.10). Combining (3.9) and (3.10) we can conclude the proof. \square

As an application of the previous lemma, we obtain the following result.

Proposition 3.12. *For $N \geq 3$, let $K \subseteq \partial \mathbb{R}_+^N$ be a compact set such that $\text{cap}_{\mathbb{R}_+^N}(K) > 0$ and let ψ_γ be as in (2.12). Then*

$$\text{cap}_{\mathbb{R}_+^N}(K, \psi_\gamma) > 0.$$

Proof. Let $Z_{\psi_\gamma} = \{x \in \mathbb{R}^N : \psi_\gamma(x) = 0\}$ as in the statement of Lemma 3.11. We notice that $\text{cap}_{\mathbb{R}^N}(K) = 2\text{cap}_{\mathbb{R}_+^N}(K) > 0$ (see Remark 2.11), so that the first assumption of Lemma 3.11 holds. Concerning the second assumption, we have that $\text{cap}_{\mathbb{R}^N}(Z_{\psi_\gamma} \cap K) = 2\text{cap}_{\mathbb{R}_+^N}(Z_{\psi_\gamma} \cap K) = 0$, since the set $Z_{\psi_\gamma} \cap K$ is $(N-2)$ -dimensional, in view of Remark 2.9, and $(N-2)$ -dimensional sets have zero capacity in \mathbb{R}^N , see e.g. [17, Theorem 2.52]. Then Lemma 3.11 provides $\text{cap}_{\mathbb{R}^N}(K, \psi_\gamma) > 0$ and the proof follows by applying again Remark 2.11. \square

We conclude this section with the following lower bound of $\text{cap}_{\mathbb{R}_+^N}(K)$ in terms of its $N-1$ dimensional Lebesgue measure.

Proposition 3.13. *Let $N \geq 3$ and $K \subseteq \partial \mathbb{R}_+^N$ be compact. Then there exists a constant $C > 0$ (only depending on N) such that*

$$(|K|_{N-1})^{\frac{N-2}{N-1}} \leq C \text{cap}_{\mathbb{R}_+^N}(K),$$

where $|\cdot|_{N-1}$ denotes the $N-1$ dimensional Lebesgue measure.

Proof. By definition of $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ and $\text{cap}_{\overline{\mathbb{R}_+^N}}(K)$, for every $\epsilon > 0$ there exists $u \in C_c^\infty(\overline{\mathbb{R}_+^N})$ such that $u = 1$ in an open neighbourhood U of K and

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 \, dx \leq \text{cap}_{\overline{\mathbb{R}_+^N}}(K) + \epsilon.$$

On the other hand

$$|K|_{N-1} \leq |U|_{N-1} = \int_U |u|^{\frac{2(N-1)}{N-2}} \, dS \leq \int_{\mathbb{R}^{N-1}} |u|^{\frac{2(N-1)}{N-2}} \, dS.$$

By combining the two previous inequalities with the embedding $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N}) \hookrightarrow L^{\frac{2(N-1)}{N-2}}(\mathbb{R}^{N-1})$ we can conclude the proof. \square

4. CONTINUITY OF THE EIGENVALUES WITH RESPECT TO THE CAPACITY

The aim of this section is to prove continuity of the eigenvalues $\lambda_n(\Omega; K)$, in the limit as $\text{Cap}_{\overline{\Omega}}(K) \rightarrow 0$.

Proof of Theorem 2.2. If $\text{Cap}_{\overline{\Omega}}(K) = 0$ the conclusion follows obviously from Proposition 3.3. Let us assume that $\text{Cap}_{\overline{\Omega}}(K) > 0$. Then, by definition of $\text{Cap}_{\overline{\Omega}}(K)$, there exists $v \in C^\infty(\overline{\Omega})$ such that $v - 1 \in C_c^\infty(\overline{\Omega} \setminus K)$ and $\|v\|_{H^1(\Omega)}^2 \leq 2 \text{Cap}_{\overline{\Omega}}(K)$. Letting $w = (1 - (1 - v)^+)^+$, we have that $w \in W^{1,\infty}(\Omega)$, $0 \leq w \leq 1$ a.e. in Ω , $w - 1 \in H_{0,K}^1(\Omega)$, and $\|w\|_{H^1(\Omega)}^2 \leq \|v\|_{H^1(\Omega)}^2 \leq 2 \text{Cap}_{\overline{\Omega}}(K)$.

Let $\varphi_1, \dots, \varphi_n$ be the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_n$ and let $\Phi_i := \varphi_i(1 - w)$, $i = 1, \dots, n$. It's easy to prove that $\Phi_i \in H_{0,K}^1(\Omega)$ for all $i = 1, \dots, n$. Let us consider the linear subspace of $H_{0,K}^1(\Omega)$

$$E_n := \text{span}\{\Phi_1, \dots, \Phi_n\}.$$

We claim that, if $\text{Cap}_{\overline{\Omega}}(K)$ is sufficiently small, $\{\Phi_i\}_{i=1}^n$ is linearly independent, thus implying that $\dim E_n = n$. In order to compute $q(\Phi_i, \Phi_j)$, we test the equation satisfied by φ_i with $\varphi_j(1 - w)^2$. It follows that

$$\int_{\Omega} [(1 - w)^2 \nabla \varphi_i \cdot \nabla \varphi_j + c(1 - w)^2 \varphi_i \varphi_j] \, dx = \int_{\Omega} [\lambda_i (1 - w)^2 \varphi_i \varphi_j + 2(1 - w) \varphi_j \nabla \varphi_i \cdot \nabla w] \, dx.$$

Thanks to the previous identity, we are able to compute

$$q(\Phi_i, \Phi_j) = \int_{\Omega} [\varphi_j(1 - w) \nabla \varphi_i \cdot \nabla w - \varphi_i(1 - w) \nabla \varphi_j \cdot \nabla w + \varphi_i \varphi_j |\nabla w|^2 + \lambda_i \varphi_i \varphi_j (1 - w)^2] \, dx.$$

From classical elliptic regularity theory (see e.g. [21, Proposition 5.3]) it is well-known that $\varphi_i \in L^\infty(\Omega)$. Then, thanks also to Hölder inequality and (1.1), we have that

$$|q(\Phi_i, \Phi_j) - \delta_{ij} \lambda_i| \leq C_1 [(\text{Cap}_{\overline{\Omega}}(K))^{1/2} + \text{Cap}_{\overline{\Omega}}(K)],$$

for a certain $C_1 > 0$ (depending only on $\|\varphi_i\|_{L^\infty(\Omega)}$ and λ_i , $i = 1, \dots, n$), where δ_{ij} is the *Kronecker's Delta*. The above inequality implies that

$$q(\Phi_i, \Phi_j) = \delta_{ij} \lambda_i + O((\text{Cap}_{\overline{\Omega}}(K))^{1/2}) \quad \text{as } \text{Cap}_{\overline{\Omega}}(K) \rightarrow 0,$$

hence there exists $\delta > 0$ such that, if $\text{Cap}_{\overline{\Omega}}(K) < \delta$, then Φ_1, \dots, Φ_n are linearly independent. Let us now compute the L^2 scalar products

$$\int_{\Omega} \Phi_i \Phi_j \, dx = \int_{\Omega} \varphi_i \varphi_j (1 - w)^2 \, dx = \delta_{ij} - 2 \int_{\Omega} \varphi_i \varphi_j w \, dx + \int_{\Omega} \varphi_i \varphi_j w^2 \, dx.$$

Arguing as before, by Hölder inequality we obtain that

$$\begin{aligned} \left| \int_{\Omega} \Phi_i \Phi_j \, dx - \delta_{ij} \right| &\leq 2\sqrt{2} \|\varphi_i \varphi_j\|_{L^2(\Omega)} (\text{Cap}_{\overline{\Omega}}(K))^{1/2} + 2 \|\varphi_i \varphi_j\|_{L^\infty(\Omega)} \text{Cap}_{\overline{\Omega}}(K) \\ &\leq C_2 [(\text{Cap}_{\overline{\Omega}}(K))^{1/2} + \text{Cap}_{\overline{\Omega}}(K)], \end{aligned}$$

for a certain $C_2 > 0$ (depending only on $\|\varphi_i\|_{L^\infty(\Omega)}$, $i = 1, \dots, n$), i.e.

$$\int_{\Omega} \Phi_i \Phi_j \, dx = \delta_{ij} + O((\text{Cap}_{\overline{\Omega}}(K))^{1/2}) \quad \text{as } \text{Cap}_{\overline{\Omega}}(K) \rightarrow 0.$$

Now, from the min-max characterization (2.1), we have that

$$\begin{aligned}
\lambda_n(\Omega; K) &\leq \max_{\substack{\alpha_1, \dots, \alpha_n \in \mathbb{R} \\ \sum_{i=1}^n \alpha_i^2 = 1}} \frac{q(\sum_{i=1}^n \alpha_i \Phi_i)}{\sum_{i,j=1}^n \alpha_i \alpha_j \int_{\Omega} \Phi_i \Phi_j \, dx} \\
&= \max_{\substack{\alpha_1, \dots, \alpha_n \in \mathbb{R} \\ \sum_{i=1}^n \alpha_i^2 = 1}} \frac{\sum_{i,j=1}^n \alpha_i \alpha_j q(\Phi_i, \Phi_j)}{\sum_{i,j=1}^n \alpha_i \alpha_j (\delta_{ij} + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2}))} \\
&= \max_{\substack{\alpha_1, \dots, \alpha_n \in \mathbb{R} \\ \sum_{i=1}^n \alpha_i^2 = 1}} \frac{\sum_{i=1}^n \alpha_i^2 \lambda_i + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2})}{1 + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2})} \\
&\leq \frac{\lambda_n + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2})}{1 + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2})} = \lambda_n + O((\text{Cap}_{\bar{\Omega}}(K))^{1/2})
\end{aligned}$$

as $\text{Cap}_{\bar{\Omega}}(K) \rightarrow 0$. \square

5. SHARP ASYMPTOTICS OF PERTURBED EIGENVALUES

This section is devoted to the proof of Theorem 2.5. To this aim, let us give a preliminary lemma concerning the inverse of the operator $-\Delta + c$, when it acts on functions that vanish on a compact set.

Lemma 5.1. *For $K \subseteq \bar{\Omega}$ compact, let $A_K : H_{0,K}^1(\Omega) \rightarrow H_{0,K}^1(\Omega)$ be the linear bounded operator defined by*

$$(5.1) \quad q(A_K(u), v) = (u, v)_{L^2(\Omega)} \quad \text{for every } u, v \in H_{0,K}^1(\Omega).$$

Then

- (i) A_K is symmetric, non-negative and compact; in particular, 0 belongs to its spectrum $\sigma(A_K)$.
- (ii) $\sigma(A_K) \setminus \{0\} = \{\mu_n(\Omega; K)\}_{n \in \mathbb{N}_*}$ and $\mu_n(\Omega; K) = 1/\lambda_n(\Omega; K)$ for every $n \in \mathbb{N}_*$.
- (iii) For every $\mu \in \mathbb{R}$ and $u \in H_{0,K}^1(\Omega) \setminus \{0\}$ it holds

$$(5.2) \quad (\text{dist}(\mu, \sigma(A_K)))^2 \leq \frac{q(A_K(u) - \mu u)}{q(u)}.$$

Proof. (i) A_K is clearly symmetric and non-negative; let us show that it is compact. We write $A_K = \mathcal{R} \circ \mathcal{I}$, where $\mathcal{I} : H_{0,K}^1(\Omega) \rightarrow (H_{0,K}^1(\Omega))^*$ is the compact immersion

$${}_{(H_{0,K}^1(\Omega))^*} \langle \mathcal{I}(u), v \rangle_{H_{0,K}^1(\Omega)} = \int_{\Omega} uv \, dx \quad \text{for every } u, v \in H_{0,K}^1(\Omega),$$

and $\mathcal{R} : (H_{0,K}^1(\Omega))^* \rightarrow H_{0,K}^1(\Omega)$ is the Riesz isomorphism (on $H_{0,K}^1(\Omega)$ endowed with the scalar product q) given by

$$q(\mathcal{R}(F), v) = {}_{(H_{0,K}^1(\Omega))^*} \langle F, v \rangle_{H_{0,K}^1(\Omega)}$$

for every $v \in H_{0,K}^1(\Omega)$ and $F \in (H_{0,K}^1(\Omega))^*$. Then A_K is compact and $0 \in \sigma(A_K)$ (see for example [15, Theorem 6.16]).

(ii) Again by [15, Theorems 6.16], $\sigma(A_K) \setminus \{0\}$ consists of isolated eigenvalues having finite multiplicity. Being $q(\cdot)$ a norm over $H_{0,K}^1(\Omega)$, we have that $\mu \neq 0$ is an eigenvalue of A_K if and only if there exists $u \in H_{0,K}^1(\Omega)$, $u \neq 0$, such that

$$q(A_K(u), v) = \mu q(u, v) \quad \text{for every } v \in H_{0,K}^1(\Omega),$$

so that $1/\mu = \lambda_n(\Omega; K)$ for some $n \in \mathbb{N}_*$.

(ii) This is a consequence of the Spectral Theorem (see for example [15, Theorem 6.21 and Proposition 8.20]). \square

We have now all the ingredients to give the proof of Theorem 2.5. It is inspired by [1, Theorem 1.4] (see also [3, Theorem 1.5]).

Proof of Theorem 2.5. Let us recall that, by assumption, $\lambda_0 = \lambda_{n_0} = \lambda_{n_0}(\Omega; \emptyset)$ is simple and that φ_0 is an associated $L^2(\Omega)$ -normalized eigenfunction. Recall also that $\lambda_\epsilon = \lambda_{n_0}(\Omega; K_\epsilon)$. For simplicity of notation we write $V_\epsilon := V_{K_\epsilon, \varphi_0}$ and $C_\epsilon := \text{Cap}_{\bar{\Omega}, c}(K_\epsilon, \varphi_0) = q(V_\epsilon)$. Moreover we let $\psi_\epsilon := \varphi_0 - V_\epsilon$, that is ψ_ϵ is the orthogonal projection of φ_0 on $H_{0, K_\epsilon}^1(\Omega)$ with respect to q . Indeed there holds

$$q(\psi_\epsilon - \varphi_0, \varphi) = 0 \quad \text{for all } \varphi \in H_{0, K_\epsilon}^1(\Omega).$$

We split the proof into three steps.

Step 1. We claim that

$$(5.3) \quad |\lambda_\epsilon - \lambda_0| = o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0.$$

For any $\varphi \in H_{0, K_\epsilon}^1(\Omega)$, being λ_0 an eigenvalue of q , we have

$$(5.4) \quad q(\psi_\epsilon, \varphi) - \lambda_0(\psi_\epsilon, \varphi)_{L^2(\Omega)} = q(\varphi_0, \varphi) - \lambda_0(\psi_\epsilon, \varphi)_{L^2(\Omega)} = \lambda_0(V_\epsilon, \varphi)_{L^2(\Omega)}.$$

According to the notation introduced in Lemma 5.1, we can rewrite (5.4) as

$$(5.5) \quad (\psi_\epsilon, \varphi)_{L^2(\Omega)} = \mu_0 q(\psi_\epsilon, \varphi) - (V_\epsilon, \varphi)_{L^2(\Omega)},$$

where $\mu_0 = \mu_{n_0}(\Omega; \emptyset) = \mu_{n_0}(\Omega; K) = 1/\lambda_0$. By (5.2) we have

$$(5.6) \quad (\text{dist}(\mu_0, \sigma(A_{K_\epsilon})))^2 \leq \frac{q(A_{K_\epsilon}(\psi_\epsilon) - \mu_0 \psi_\epsilon)}{q(\psi_\epsilon)}.$$

From Proposition 3.8 it follows that

$$|q(\varphi_0, V_\epsilon)| \leq \sqrt{q(\varphi_0)} \sqrt{q(V_\epsilon)} = \sqrt{\lambda_0} \sqrt{C_\epsilon} = o(1)$$

as $\epsilon \rightarrow 0$, so that, using the definition of ψ_ϵ , the denominator in the right hand side of (5.6) can be estimated as follows

$$(5.7) \quad q(\psi_\epsilon) = q(\varphi_0) + C_\epsilon - 2q(\varphi_0, V_\epsilon) = \lambda_0 + o(1)$$

as $\epsilon \rightarrow 0$. Concerning the numerator in the right hand side of (5.6), the definition of A_{K_ϵ} and relation (5.5) provide

$$q(A_{K_\epsilon}(\psi_\epsilon), \varphi) = (\psi_\epsilon, \varphi)_{L^2(\Omega)} = \mu_0 q(\psi_\epsilon, \varphi) - (V_\epsilon, \varphi)_{L^2(\Omega)},$$

for every $\varphi \in H_{0, K_\epsilon}^1$, so that, choosing $\varphi = A_{K_\epsilon}(\psi_\epsilon) - \mu_0 \psi_\epsilon$ in the previous identity, we arrive at

$$q(A_{K_\epsilon}(\psi_\epsilon) - \mu_0 \psi_\epsilon) = -(V_\epsilon, A_{K_\epsilon}(\psi_\epsilon) - \mu_0 \psi_\epsilon)_{L^2(\Omega)}.$$

The Cauchy-Schwartz inequality, assumption (1.1) and Lemma 3.9, together with the previous equality, provide

$$(q(A_{K_\epsilon}(\psi_\epsilon) - \mu_0 \psi_\epsilon))^{1/2} \leq \frac{1}{\sqrt{c_0}} \|V_\epsilon\|_{L^2(\Omega)} = o(C_\epsilon^{1/2})$$

as $\epsilon \rightarrow 0$. By combining the last inequality with (5.6) and (5.7) we see that

$$(5.8) \quad \text{dist}(\mu_0, \sigma(A_{K_\epsilon})) = o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0.$$

We know from Theorem 2.2 that $\lambda_n(\Omega; K_\epsilon) \rightarrow \lambda_n$ as $\epsilon \rightarrow 0$ and so, since λ_0 is assumed to be simple, also λ_ϵ is simple for $\epsilon > 0$ sufficiently small. Hence, denoting $\mu_\epsilon = \mu_{n_0}(\Omega; K_\epsilon) = 1/\lambda_\epsilon$, we have

$$\text{dist}(\mu_0, \sigma(A_{K_\epsilon})) = |\mu_0 - \mu_\epsilon|$$

for $\epsilon > 0$ small enough. Then, using (5.8),

$$|\lambda_0 - \lambda_\epsilon| = \lambda_0 \lambda_\epsilon |\mu_0 - \mu_\epsilon| = o(C_\epsilon^{1/2})$$

as $\epsilon \rightarrow 0$, so that claim (5.3) is proved.

Let now $\Pi_\epsilon: L^2(\Omega) \rightarrow L^2(\Omega)$ be the orthogonal projection onto the one-dimensional eigenspace corresponding to λ_ϵ , that is to say

$$\Pi_\epsilon \psi = (\psi, \varphi_\epsilon)_{L^2(\Omega)} \varphi_\epsilon \quad \text{for every } \psi \in L^2(\Omega),$$

where we denoted by φ_ϵ a $L^2(\Omega)$ -normalized eigenfunction associated to λ_ϵ .

Step 2. We claim that

$$(5.9) \quad q(\psi_\epsilon - \Pi_\epsilon \psi_\epsilon) = o(C_\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

Let

$$\Phi_\epsilon := \psi_\epsilon - \Pi_\epsilon \psi_\epsilon \quad \text{and} \quad \xi_\epsilon := A_{K_\epsilon}(\Phi_\epsilon) - \mu_\epsilon \Phi_\epsilon.$$

Using the fact that $\Pi_\epsilon \psi_\epsilon$ belongs to the eigenspace associated to λ_ϵ and relation (5.4), we have, for every $\varphi \in H_{0,K_\epsilon}^1$,

$$\begin{aligned} q(\Phi_\epsilon, \varphi) - \lambda_\epsilon(\Phi_\epsilon, \varphi)_{L^2(\Omega)} &= q(\psi_\epsilon, \varphi) - \lambda_0(\psi_\epsilon, \varphi)_{L^2(\Omega)} + (\lambda_0 - \lambda_\epsilon)(\psi_\epsilon, \varphi)_{L^2(\Omega)} \\ &= \lambda_0(V_\epsilon, \varphi)_{L^2(\Omega)} + (\lambda_0 - \lambda_\epsilon)(\psi_\epsilon, \varphi)_{L^2(\Omega)}. \end{aligned}$$

Thanks to the previous relation, with $\varphi = \xi_\epsilon$, and the definition of A_{K_ϵ} , we obtain

$$(5.10) \quad \begin{aligned} q(\xi_\epsilon) &= q(A_{K_\epsilon}(\Phi_\epsilon), \xi_\epsilon) - \mu_\epsilon q(\Phi_\epsilon, \xi_\epsilon) = -\mu_\epsilon [q(\Phi_\epsilon, \xi_\epsilon) - \lambda_\epsilon(\Phi_\epsilon, \xi_\epsilon)_{L^2(\Omega)}] \\ &= -\frac{\lambda_0}{\lambda_\epsilon}(V_\epsilon, \xi_\epsilon)_{L^2(\Omega)} - \frac{\lambda_0 - \lambda_\epsilon}{\lambda_\epsilon}(\psi_\epsilon, \xi_\epsilon)_{L^2(\Omega)}. \end{aligned}$$

Combining (5.5) and (5.7) we obtain that $\|\psi_\epsilon\|_{L^2(\Omega)} = 1 + o(1)$ as $\epsilon \rightarrow 0$. Therefore, from (5.10), taking into account (5.7), we deduce the existence of a constant C independent from ϵ such that

$$\sqrt{q(\xi_\epsilon)} \leq C (\|V_\epsilon\|_{L^2(\Omega)} + |\lambda_0 - \lambda_\epsilon|).$$

Thus, using the definition of ξ_ϵ , Lemma 3.9 and (5.3), we obtain that

$$(5.11) \quad q(A_{K_\epsilon}(\Phi_\epsilon) - \mu_\epsilon \Phi_\epsilon) = o(C_\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

Let

$$N_\epsilon = \{w \in H_{0,K_\epsilon}^1 : (w, \varphi_\epsilon)_{L^2(\Omega)} = 0\}.$$

Note that, by definition, $\Phi_\epsilon \in N_\epsilon$. Moreover, being φ_ϵ an eigenfunction associated to λ_ϵ , from the definition of A_{K_ϵ} in (5.1) it follows that

$$A_{K_\epsilon}(w) \in N_\epsilon \quad \text{for every } w \in N_\epsilon.$$

In particular, the following operator

$$\tilde{A}_\epsilon = A_{K_\epsilon}|_{N_\epsilon} : N_\epsilon \rightarrow N_\epsilon$$

is well defined. One can easily check that \tilde{A}_ϵ satisfies properties (i)-(iii) in Lemma 5.1; moreover $\sigma(\tilde{A}_\epsilon) = \sigma(A_{K_\epsilon}) \setminus \{\mu_\epsilon\}$. In particular, letting $\delta > 0$ be such that $(\text{dist}(\mu_\epsilon, \sigma(\tilde{A}_\epsilon)))^2 \geq \delta$ for every ϵ small enough, estimate (5.11), combined with (5.2), provides

$$q(\psi_\epsilon - \Pi_\epsilon \psi_\epsilon) = q(\Phi_\epsilon) \leq \frac{1}{\delta} (\text{dist}(\mu_\epsilon, \sigma(\tilde{A}_\epsilon)))^2 q(\Phi_\epsilon) \leq \frac{1}{\delta} q(\tilde{A}_\epsilon(\Phi_\epsilon) - \mu_\epsilon \Phi_\epsilon) = o(C_\epsilon)$$

as $\epsilon \rightarrow 0$, thus proving claim (5.9).

Step 3. We claim that

$$(5.12) \quad \lambda_\epsilon - \lambda_0 = C_\epsilon + o(C_\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

From the definition of ψ_ϵ , Lemma 3.9, and the previous step we deduce that

$$\|\varphi_0 - \Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)} \leq \|V_\epsilon\|_{L^2(\Omega)} + \|\psi_\epsilon - \Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)} = o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0,$$

which yields both

$$(5.13) \quad \|\Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)} = 1 + o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0$$

and, consequently,

$$(5.14) \quad \|\varphi_0 - \hat{\varphi}_\epsilon\|_{L^2(\Omega)} = o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0,$$

where $\hat{\varphi}_\epsilon = \Pi_\epsilon \psi_\epsilon / \|\Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)}$. Using the fact that $\hat{\varphi}_\epsilon$ is an eigenfunction associated to λ_ϵ and (5.5) with $\varphi = \hat{\varphi}_\epsilon$, we obtain

$$(5.15) \quad (\lambda_\epsilon - \lambda_0)(\psi_\epsilon, \hat{\varphi}_\epsilon)_{L^2(\Omega)} = \lambda_0(V_\epsilon, \hat{\varphi}_\epsilon)_{L^2(\Omega)}.$$

But actually

$$(5.16) \quad \lambda_0(V_\epsilon, \hat{\varphi}_\epsilon) = C_\epsilon + o(C_\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

Indeed, since ψ_ϵ and V_ϵ are orthogonal with respect to q and φ_0 is an eigenfunction corresponding to λ_0 , we have that

$$\begin{aligned} C_\epsilon &= q(V_\epsilon) = q(V_\epsilon, \varphi_0 - \psi_\epsilon) = q(V_\epsilon, \varphi_0) = \lambda_0(V_\epsilon, \varphi_0)_{L^2(\Omega)} \\ &= \lambda_0(V_\epsilon, \hat{\varphi}_\epsilon)_{L^2(\Omega)} + \lambda_0(V_\epsilon, \varphi_0 - \hat{\varphi}_\epsilon)_{L^2(\Omega)}, \end{aligned}$$

so that Lemma 3.9 and relation (5.14) allow us to prove (5.16). Concerning the left hand side of (5.15) we have, exploiting (5.13) and (5.9),

$$(\psi_\epsilon, \hat{\varphi}_\epsilon)_{L^2(\Omega)} = \frac{(\psi_\epsilon - \Pi_\epsilon \psi_\epsilon, \Pi_\epsilon \psi_\epsilon)_{L^2(\Omega)} + \|\Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)}^2}{\|\Pi_\epsilon \psi_\epsilon\|_{L^2(\Omega)}} = 1 + o(C_\epsilon^{1/2}) \quad \text{as } \epsilon \rightarrow 0.$$

By combining the last estimate with (5.15) and (5.16), we complete the proof. \square

6. SET SCALING TO A BOUNDARY POINT

Hereafter we assume $N \geq 3$. The purpose of this section is to find, in some particular cases, the explicit behaviour of the function $\epsilon \mapsto \text{Cap}_{\bar{\Omega}, \mathcal{C}}(K_\epsilon, \varphi_0)$ and therefore to give a more concrete connotation to the asymptotic expansion proved in Theorem 2.5. We consider a particular class of families of concentrating sets, that includes the case in which K_ϵ is obtained by rescaling a fixed compact set K by a factor $\epsilon > 0$.

First, we prove Proposition 2.8.

Proof of Proposition 2.8. The proof is organized as follows: we first derive (2.9) for a certain diffeomorphism in the class \mathcal{C} and then we prove that it holds true for any diffeomorphism in \mathcal{C} . Hence we start by taking into consideration a particular diffeomorphism $\Phi_{\text{AE}} \in \mathcal{C}$, first introduced in [4]. Let $g \in C^{1,1}(B'_{r_0})$ be, as in (2.5), the function that describes $\partial\Omega$ locally near the origin and let $\rho \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\text{supp } \rho \subseteq B'_1$, $\rho \geq 0$ in \mathbb{R}^{N-1} , $\rho \not\equiv 0$ and $-\nabla\rho(y') \cdot y' \geq 0$ in \mathbb{R}^{N-1} . Then, for any $\delta > 0$, let

$$\rho_\delta(y') = c_\rho^{-1} \delta^{-N+1} \rho\left(\frac{y'}{\delta}\right) \quad \text{with} \quad c_\rho = \int_{\mathbb{R}^{N-1}} \rho(y') \, dy',$$

be a family of mollifiers. Now, for $j = 1, \dots, N-1$ and $y_N > 0$, we let

$$u_j(y', y_N) := y_j - y_N \left(\rho_{y_N} \star \frac{\partial g}{\partial y_j} \right)(y'),$$

where \star denotes the convolution product. Moreover, we define

$$\psi_j(y', y_N) := \begin{cases} u_j(y', y_N), & \text{for } y_N > 0, \\ 4u_j(y', -\frac{y_N}{2}) - 3u_j(y', -y_N), & \text{for } y_N < 0. \end{cases}$$

One can prove that $\psi_j \in C^{1,1}(B_{r_0/2})$. Finally, we let $F: B_{r_0/2} \rightarrow \mathbb{R}^N$ be defined as follows

$$F(y', y_N) := (\psi_1(y', y_N), \dots, \psi_{N-1}(y', y_N), y_N + g(y')).$$

Computations show that the Jacobian matrix of F on the hyperplane $\{y_N = 0\}$ is as follows

$$J_F(y', 0) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\frac{\partial g}{\partial y_1}(y') \\ 0 & 1 & \cdots & 0 & -\frac{\partial g}{\partial y_2}(y') \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{\partial g}{\partial y_{N-1}}(y') \\ \frac{\partial g}{\partial y_1}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 \end{pmatrix},$$

and so $|\det J_F(0)| = 1 + |\nabla g(0)|^2 = 1$. Hence, by the inverse function theorem, F is invertible in a neighbourhood of the origin: namely there exists $r_1 \in (0, r_0/2)$ such that F is a diffeomorphism of

class $C^{1,1}$ from B_{r_1} to $\mathcal{U} = F(B_{r_1})$ for some \mathcal{U} open neighbourhood of 0. Moreover, it is possible to choose r_1 sufficiently small so that

$$\begin{aligned} F^{-1}(\mathcal{U} \cap \Omega) &= \mathbb{R}_+^N \cap B_{r_1} = B_{r_1}^+, \\ F^{-1}(\mathcal{U} \cap \partial\Omega) &= \partial\mathbb{R}_+^N \cap B_{r_1} = B_{r_1}'^+, \end{aligned}$$

which means that, near the origin, the image of Ω through F^{-1} has flat boundary (coinciding with $\partial\mathbb{R}_+^N$). In particular we have that the diffeomorphism

$$\Phi_{\text{AE}} : \mathcal{U} \rightarrow B_{r_1}, \quad \Phi_{\text{AE}} := F^{-1}$$

belongs to the class \mathcal{C} defined in (2.6).

For $y \in \Phi_{\text{AE}}(\mathcal{U} \cap \Omega) = B_{r_1}^+$, we let $\hat{\varphi}_0(y) := \varphi_0(\Phi_{\text{AE}}^{-1}(y))$; from the equation satisfied by φ_0 in Ω , we deduce that

$$(6.1) \quad \int_{B_{r_1}^+} (A(y) \nabla \hat{\varphi}_0(y) \cdot \nabla \varphi(y) + \hat{c}(y) \hat{\varphi}_0(y) \varphi(y)) \, dy = \lambda_0 \int_{B_{r_1}^+} p(y) \hat{\varphi}_0(y) \varphi(y) \, dy$$

for all $\varphi \in H_{0, S_{r_1}^+}^1(B_{r_1}^+)$, where $S_{r_1}^+ := \partial B_{r_1} \cap \overline{\mathbb{R}_+^N}$ and

$$(6.2) \quad \begin{aligned} A(y) &= J_{\Phi_{\text{AE}}}(\Phi_{\text{AE}}^{-1}(y)) J_{\Phi_{\text{AE}}}(\Phi_{\text{AE}}^{-1}(y))^T |\det J_{\Phi_{\text{AE}}}(\Phi_{\text{AE}}^{-1}(y))|^{-1}, \\ \hat{c}(y) &= c(\Phi_{\text{AE}}^{-1}(y)) |\det J_{\Phi_{\text{AE}}}(\Phi_{\text{AE}}^{-1}(y))|^{-1}, \\ p(y) &= |\det J_{\Phi_{\text{AE}}}(\Phi_{\text{AE}}^{-1}(y))|^{-1}. \end{aligned}$$

We point out that equation (6.1) is the weak formulation of the problem

$$\begin{cases} -\operatorname{div}(A(y) \nabla \hat{\varphi}_0(y)) + \hat{c}(y) \hat{\varphi}_0(y) = \lambda_0 p(y) \hat{\varphi}_0(y), & \text{in } B_{r_1}^+, \\ \nabla \hat{\varphi}_0(y) A(y) \cdot \nu(y) = 0, & \text{on } B_{r_1}'^+. \end{cases}$$

One can prove that A is symmetric and uniformly elliptic in $B_{r_1}^+$ (if r_1 is chosen sufficiently small); moreover, if we denote $A(y) = (a_{i,j}(y))_{i,j=1,\dots,N}$, then $a_{i,j} \in C^{0,1}(B_{r_1}^+ \cup B_{r_1}'^+)$ and

$$(6.3) \quad \begin{aligned} a_{i,i}(y', 0) &= 1 + |\nabla g(y')|^2 - \left(\frac{\partial g}{\partial y_i}(y') \right)^2, & \text{for all } i = 1, \dots, N-1, \\ a_{i,j}(y', 0) &= -\frac{\partial g}{\partial y_i}(y') \frac{\partial g}{\partial y_j}(y'), & \text{for all } i, j = 1, \dots, N-1, \, i \neq j, \\ a_{i,N}(y', 0) &= 0, & \text{for all } i = 1, \dots, N-1, \\ a_{N,N} &= 1. \end{aligned}$$

Therefore, if we consider an even reflection of $\hat{\varphi}_0$ (which we still denote as $\hat{\varphi}_0$) through the hyperplane $\{x_N = 0\}$ in B_{r_1} , then it satisfies, in this ball, an elliptic equation in divergence form with Lipschitz continuous second order coefficients. More in particular $\hat{\varphi}_0$ weakly satisfies

$$(6.4) \quad -\operatorname{div}(\bar{A}(y) \nabla \hat{\varphi}_0(y)) = h(y) \hat{\varphi}_0(y) \quad \text{in } B_{r_1}$$

where

$$\bar{A}(y) := \begin{cases} A(y_1, \dots, y_{N-1}, y_N), & \text{if } y_N > 0, \\ QA(y_1, \dots, y_{N-1}, -y_N)Q, & \text{if } y_N < 0, \end{cases}$$

with

$$Q := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

and $h \in L^\infty(B_{r_1})$. We point out that Lipschitz continuity of the coefficients of the matrix \bar{A} comes from the fact that $a_{i,j} \in C^{0,1}(B_{r_1}^+ \cup B'_{r_1})$ and that $a_{i,N}(y', 0) = 0$ for all $i < N$. Hence we deduce, from [20], that there exists a homogeneous harmonic polynomial ψ_γ of degree $\gamma \in \mathbb{N}$ such that

$$\hat{\varphi}_0(y) = \psi_\gamma(y) + R_\gamma(y)$$

where $\|R_\gamma\|_{H^1(B_r)} = O(r^{\gamma + \frac{N}{2} + 1 - \delta})$ for some $\delta \in (0, 1)$ as $r \rightarrow 0$. In particular (2.9) holds.

Now let $\Phi \in \mathcal{C}$. We can rewrite

$$\varphi_0(\Phi^{-1}(\epsilon x)) = \varphi_0(\Phi_{\text{AE}}^{-1}(\epsilon G_\epsilon(x))), \quad \text{with } G_\epsilon(x) := \frac{(\Phi_{\text{AE}} \circ \Phi^{-1})(\epsilon x)}{\epsilon}.$$

Thanks to regularity properties of Φ_{AE} and Φ , we have that

$$G_\epsilon(x) = x + |x|^2 O(\epsilon), \quad \text{as } \epsilon \rightarrow 0.$$

As a consequence, one can prove that

$$\epsilon^{-\gamma} \varphi_0(\Phi_{\text{AE}}^{-1}(\epsilon G_\epsilon(x))) \rightarrow \psi_\gamma(x), \quad \text{in } H^1(B_R^+) \text{ as } \epsilon \rightarrow 0,$$

for all $R > 0$, thus concluding the proof of (2.9).

The proofs of (2.10) and (2.11) follow from (2.9) by making a change of variable in the integral and taking into account that, for all $R > 0$,

$$\chi_{\epsilon^{-1}\Phi(\Omega \cap B_{R\epsilon})} \rightarrow \chi_{B_R^+} \quad \text{a.e. in } \mathbb{R}^N,$$

with χ_A denoting as usual the characteristic function of a set $A \subset \mathbb{R}^N$, as one can easily deduce from the fact that $\Phi^{-1}(y) = y + O(|y|^2)$ as $y \rightarrow 0$. \square

In this section we consider a particular class of families of compact sets concentrating to the origin (which is assumed to belong to $\partial\Omega$), as described in Section 2.1. Let us fix $\Phi \in \mathcal{C}$ with

$$(6.5) \quad \Phi : \mathcal{U}_0 \rightarrow B_{R_0}$$

being \mathcal{U}_0 an open neighbourhood of 0 and $R_0 > 0$ such that $\Phi(\mathcal{U}_0 \cap \Omega) = B_{R_0}^+$ and $\Phi(\mathcal{U}_0 \cap \partial\Omega) = B'_{R_0}$. In the rest of this section, we will use the same notation as in the proof of Proposition 2.8 defining A and \hat{c} as in (6.2) (with Φ instead of Φ_{AE}).

Since $A \in C^{0,1}(B_{R_0}^+, \mathcal{M}_{N \times N})$ (with $\mathcal{M}_{N \times N}$ denoting the space of $N \times N$ real matrices) and $A(0) = I_N$, it is possible to choose $R_0 > 0$ small enough in order to have

$$\|A(x) - I_N\|_{\mathcal{M}_{N \times N}} \leq \frac{1}{2} \quad \text{and} \quad \hat{c}(x) \geq \frac{c_0}{2} \quad \text{for a.e. } x \in B_{R_0}^+$$

(with $\|\cdot\|_{\mathcal{M}_{N \times N}}$ denoting the operator norm on $\mathcal{M}_{N \times N}$). With this choice of R_0 , we have that

$$(6.6) \quad \begin{aligned} \int_{B_{R_0/\epsilon}^+} A(\epsilon x) \nabla u(x) \cdot \nabla u(x) \, dx &= \int_{B_{R_0/\epsilon}^+} (A(\epsilon x) - I_N) \nabla u(x) \cdot \nabla u(x) \, dx + \int_{B_{R_0/\epsilon}^+} |\nabla u(x)|^2 \, dx \\ &\geq \frac{1}{2} \int_{B_{R_0/\epsilon}^+} |\nabla u(x)|^2 \, dx \end{aligned}$$

and

$$(6.7) \quad \int_{B_{R_0/\epsilon}^+} \hat{c}(\epsilon x) u^2(x) \, dx \geq \frac{c_0}{2} \int_{B_{R_0/\epsilon}^+} u^2(x) \, dx$$

for all $u \in H^1(B_{R_0/\epsilon}^+)$.

Let $K_\epsilon \subseteq \bar{\Omega} \cap \mathcal{U}_0$ be a compact set for any $\epsilon \in (0, 1)$ such that (2.7) and (2.8) hold. In the following we denote

$$\tilde{K}_\epsilon := \Phi(K_\epsilon)/\epsilon.$$

For any compact set $H \subseteq \mathbb{R}^N$ we define the *radius* of H as follows

$$(6.8) \quad r(H) := \max_{x \in H} |x|.$$

Remark 6.1. Concerning hypothesis (2.8), one can prove that the convergence of $\mathbb{R}^N \setminus \tilde{K}_\epsilon$ to $\mathbb{R}^N \setminus K$ in the sense of Mosco, as $\epsilon \rightarrow 0$, introduced in Definition 2.6, is equivalent to the convergence of the space $H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ to the space $H_{0, K}^1(B_R^+)$ in the sense of Mosco for all $R > r(M)$. We recall that $H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ is said to *converge to $H_{0, K}^1(B_R^+)$ in the sense of Mosco* if the following holds:

- (1) the weak limit points (as $\epsilon \rightarrow 0$) in $H^1(B_R^+)$ of every family of functions $\{u_\epsilon\}_\epsilon \subseteq H^1(B_R^+)$, such that $u_\epsilon \in H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ for every ϵ , belong to $H_{0, K}^1(B_R^+)$;
- (2) for every $u \in H_{0, K}^1(B_R^+)$ there exists a family $\{u_\epsilon\}_\epsilon \subseteq H^1(B_R^+)$ such that $u_\epsilon \in H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ for every ϵ and $u_\epsilon \rightarrow u$ in $H^1(B_R^+)$, as $\epsilon \rightarrow 0$.

The proof of this equivalence is essentially based on the continuity of the extension operator for functions in $H^1(B_R^+)$ and of the restriction operator on B_R^+ for functions in $H^1(\mathbb{R}^N)$. Analogously, one can also prove that they are both also equivalent to the convergence of $H_{0, \tilde{K}_\epsilon \cup S_R^+}^1(B_R^+)$ to $H_{0, K \cup S_R^+}^1(B_R^+)$ in the sense of Mosco, as $\epsilon \rightarrow 0$. These equivalent hypotheses turn out to be more adequate for our needs in this final part of the section.

In view of Proposition 2.8 it is natural to consider the following rescalings of the limit eigenfunction φ_0 and of the φ_0 -capacitary potential of K_ϵ

$$(6.9) \quad \tilde{\varphi}_\epsilon(y) := \frac{\varphi_0(\Phi^{-1}(\epsilon y))}{\epsilon^\gamma} = \frac{\hat{\varphi}_0(\epsilon y)}{\epsilon^\gamma}, \quad \tilde{V}_\epsilon(y) := \frac{V_{K_\epsilon, \varphi_0}(\Phi^{-1}(\epsilon y))}{\epsilon^\gamma}, \quad y \in B_{R_0/\epsilon}^+,$$

where $\hat{\varphi}_0(z) = \varphi_0(\Phi^{-1}(z)) \in H^1(B_{R_0}^+)$. We have that $\tilde{\varphi}_\epsilon \in H^1(B_{R_0/\epsilon}^+)$, $\tilde{V}_\epsilon - \tilde{\varphi}_\epsilon \in H_{0, \tilde{K}_\epsilon}^1(B_{R_0/\epsilon}^+)$, and they satisfy

$$(6.10) \quad \int_{B_{R_0/\epsilon}^+} (A(\epsilon y) \nabla \tilde{\varphi}_\epsilon(y) \cdot \nabla \varphi(y) + \epsilon^2 \hat{c}(\epsilon y) \tilde{\varphi}_\epsilon(y) \varphi(y)) \, dy = \lambda_0 \epsilon^2 \int_{B_{R_0/\epsilon}^+} p(\epsilon y) \tilde{\varphi}_\epsilon(y) \varphi(y) \, dy,$$

for all $\varphi \in H_{0, S_{R_0/\epsilon}^+}^1(B_{R_0/\epsilon}^+)$ and

$$(6.11) \quad \int_{B_{R_0/\epsilon}^+} (A(\epsilon y) \nabla \tilde{V}_\epsilon(y) \cdot \nabla \varphi(y) + \epsilon^2 \hat{c}(\epsilon y) \tilde{V}_\epsilon(y) \varphi(y)) \, dy = 0,$$

for all $\varphi \in H_{0, \tilde{K}_\epsilon \cup S_{R_0/\epsilon}^+}^1(B_{R_0/\epsilon}^+)$.

The following Lemma provides a first, rough estimate of the boundary Sobolev capacity appearing in the asymptotic expansion in Theorem 2.5.

Lemma 6.2. *We have that*

$$\text{Cap}_{\bar{\Omega}, c}(K_\epsilon, \varphi_0) = O(\epsilon^{N+2\gamma-2}), \quad \text{as } \epsilon \rightarrow 0.$$

Proof. Recall the definition of M in (2.7) and that of $r(M)$ in (6.8). Since $K_\epsilon \subseteq \Phi^{-1}(\overline{\epsilon B_{r(M)}})$ and $\Phi^{-1}(y) = y + O(|y|^2)$ as $|y| \rightarrow 0$, there exists $R > 0$ such that $K_\epsilon \subset B_{R\epsilon}$ for ϵ sufficiently small. Now let $\eta_\epsilon \in C_c^\infty(\mathbb{R}^N)$ be such that

$$\begin{aligned} 0 \leq \eta_\epsilon(y) \leq 1, \\ |\nabla \eta_\epsilon| \leq \frac{2}{\epsilon R}, \end{aligned} \quad \eta_\epsilon(y) = \begin{cases} 0, & \text{for } y \in \mathbb{R}^N \setminus B_{2\epsilon R}, \\ 1, & \text{for } y \in B_{\epsilon R}. \end{cases}$$

Since $\eta_\epsilon \varphi_0 \in H_{0, K_\epsilon}^1(\Omega)$, from (2.2) we have that

$$\begin{aligned} \text{Cap}_{\bar{\Omega}, c}(K_\epsilon, \varphi_0) &\leq q(\eta_\epsilon \varphi_0) \\ &\leq 2 \int_{\Omega \cap B_{2R\epsilon}} |\nabla \eta_\epsilon(x)|^2 \varphi_0(x) \, dx \\ &\quad + 2 \int_{\Omega \cap B_{2R\epsilon}} |\eta_\epsilon(x)|^2 |\nabla \varphi_0(x)|^2 \, dx + \|c\|_{L^\infty(\Omega)} \int_{\Omega \cap B_{2R\epsilon}} \varphi_0^2(x) \, dx \\ &\leq \left(\frac{8}{\epsilon^2 R^2} + \|c\|_{L^\infty(\Omega)} \right) \int_{\Omega \cap B_{2R\epsilon}} \varphi_0^2(x) \, dx + 2 \int_{\Omega \cap B_{2R\epsilon}} |\nabla \varphi_0(x)|^2 \, dx \end{aligned}$$

The conclusion then follows from (2.10) and (2.11). \square

The following lemma, whose proof is classical, is useful in order to pass from a global scale (functions in $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$) to a local one (meaning functions in $H^1(B_R^+)$) and vice versa.

Lemma 6.3. *Let $K \subseteq \overline{\mathbb{R}_+^N}$ be compact. If $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ is such that $f|_{B_R^+} \in H_{0,K}^1(B_R^+)$ for some $R > r(K)$, then $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$. Conversely, if $f \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$, then $f|_{B_R^+} \in H_{0,K}^1(B_R^+)$ for all $R > r(K)$.*

In the lemma below we compare the two notions of capacity arising in our work, namely Definition 2.1 and Definition 2.10.

Lemma 6.4 (Comparison of capacities). *Let $K \subseteq \overline{\mathbb{R}_+^N}$ be a compact set and let $R > r(K)$. Then there exists a constant $\alpha = \alpha(R) > 0$ such that*

$$\alpha^{-1} \text{cap}_{\overline{\mathbb{R}_+^N}}(K) \leq \text{Cap}_{B_R^+}(K) \leq \alpha \text{cap}_{\overline{\mathbb{R}_+^N}}(K).$$

Proof. Let $\eta_K \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta_K = 1$ in a neighbourhood of K .

In order to prove the left hand inequality, let $W_R \in H^1(B_R^+)$ be the potential achieving $\text{Cap}_{B_R^+}(K)$ and let $\hat{W}_R \in H^1(\mathbb{R}_+^N)$ be its extension to \mathbb{R}_+^N . Obviously $\hat{W}_R \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ and, by Lemma 6.3, we have that $\hat{W}_R - \eta_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$. Therefore \hat{W}_R is admissible for $\text{cap}_{\overline{\mathbb{R}_+^N}}(K)$ and hence

$$\begin{aligned} \text{cap}_{\overline{\mathbb{R}_+^N}}(K) &\leq \int_{\mathbb{R}_+^N} |\nabla \hat{W}_R|^2 dx \leq \int_{\mathbb{R}_+^N} (|\nabla \hat{W}_R|^2 + \hat{W}_R^2) dx \\ &\leq C_1(R) \int_{B_R^+} (|\nabla W_R|^2 + W_R^2) dx = C_1(R) \text{Cap}_{B_R^+}(K), \end{aligned}$$

where $C_1(R)$ is the constant related to the extension operator for Sobolev functions. In order to prove the other inequality, let $W_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ be the potential achieving $\text{cap}_{\overline{\mathbb{R}_+^N}}(K)$. Since $W_K - \eta_K \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$, then $W_K \in H^1(B_R^+)$ and, in view of Lemma 6.3, $W_K - 1 \in H_{0,K}^1(B_R^+)$. Hence W_K is admissible for $\text{Cap}_{B_R^+}(K)$. Moreover, by Hölder and Sobolev inequalities, we have that

$$\|W_K\|_{L^2(B_R^+)}^2 \leq \|W_K\|_{L^{2^*}(B_R^+)} \|B_R^+\|^{2/N} \leq C_2(R) \|\nabla W_K\|_{L^2(\mathbb{R}_+^N)}^2,$$

for some $C_2(R) > 0$ depending only on N and R . Then the following estimates hold

$$\text{Cap}_{B_R^+}(K) \leq \int_{B_R^+} (|\nabla W_K|^2 + W_K^2) dx \leq (1 + C_2(R)) \text{cap}_{\overline{\mathbb{R}_+^N}}(K),$$

thus concluding the proof. \square

In order to prove Theorem 2.12 the following Poincaré type inequality is needed.

Lemma 6.5 (Poincaré Inequality). *Let $M, K \subseteq \overline{\mathbb{R}_+^N}$ and $\{K_\epsilon\}_{\epsilon \in (0,1)}$ satisfy (2.7)–(2.8) for some $\Phi \in \mathcal{C}$ and let $\tilde{K}_\epsilon := \Phi(K_\epsilon)/\epsilon$. Let us assume that $\text{cap}_{\overline{\mathbb{R}_+^N}}(K) > 0$. For any $R > r(M)$ there exist $\epsilon_0 \in (0, 1)$ and $C > 0$ (both depending on R and K) such that*

$$\int_{B_R^+} u^2 dx \leq C \int_{B_R^+} |\nabla u|^2 dx$$

for all $u \in H_{0,\tilde{K}_\epsilon}^1(B_R^+)$ and for all $\epsilon < \epsilon_0$.

Proof. By way of contradiction, suppose that, for a certain $R > r(M)$, there exist a sequence of real numbers $\epsilon_n \rightarrow 0^+$ and a sequence of functions $u_n \in H_{0,\tilde{K}_{\epsilon_n}}^1(B_R^+)$ such that

$$\int_{B_R^+} u_n^2 dx > n \int_{B_R^+} |\nabla u_n|^2 dx.$$

Now let us consider the sequence

$$v_n := \frac{u_n}{\|u_n\|_{L^2(B_R^+)}} ,$$

so that

$$\|v_n\|_{L^2(B_R^+)} = 1 \quad \text{and} \quad \int_{B_R^+} |\nabla v_n|^2 \, dx < \frac{1}{n} .$$

Therefore, since $\|v_n\|_{H^1(B_R^+)}$ is uniformly bounded with respect to n , there exists $v \in H^1(B_R^+)$ such that, up to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(B_R^+)$. By compactness $v_n \rightarrow v$ strongly in $L^2(B_R^+)$; this implies that $\|v\|_{L^2(B_R^+)} = 1$ and hence that $v \neq 0$. On the other hand, by weak lower semicontinuity,

$$\int_{B_R^+} |\nabla v|^2 \, dx \leq \liminf_{n \rightarrow \infty} \int_{B_R^+} |\nabla v_n|^2 \, dx = 0 ,$$

hence there exists a constant $\kappa \neq 0$ such that $v = \kappa$ a.e. in B_R^+ . Finally, since $\mathbb{R}^N \setminus \tilde{K}_{\epsilon_n}$ is converging to $\mathbb{R}^N \setminus K$ in the sense of Mosco, $v \in H_{0,K}^1(B_R^+)$ (see Remark 6.1) and this implies the existence of a sequence $\{w_n\}_n \subset C_c^\infty(\overline{B_R^+} \setminus K)$ such that $\|w_n - \kappa\|_{H^1(B_R^+)} \rightarrow 0$ as $n \rightarrow +\infty$. Letting $z_n = (\kappa - w_n)/\kappa$, we have that

$$z_n - 1 \in H_{0,K}^1(B_R^+) \quad \text{and} \quad \|z_n\|_{H^1(B_R^+)} \rightarrow 0$$

as $n \rightarrow +\infty$, thus implying $\text{Cap}_{B_R^+}(K) = 0$ and hence contradicting, in view of Lemma 6.4, the fact that $\text{cap}_{\mathbb{R}_+^N}(K) > 0$. \square

In the same spirit of Proposition 3.1, we have that the relative ψ_γ -capacity of the set K , denoted by $\text{cap}_{\mathbb{R}_+^N}(K, \psi_\gamma)$ (see Definition 2.10), is uniquely achieved, as asserted in the following lemma.

Lemma 6.6. *Let $\eta \in C_c^\infty(\overline{\mathbb{R}_+^N})$ be a cut-off function such that $\eta = 1$ in a neighbourhood of K . There exists a unique $\tilde{V} \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$ such that $\tilde{V} - \eta\psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$ and*

$$\int_{\mathbb{R}_+^N} \nabla \tilde{V} \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K),$$

i.e. weakly solving

$$\begin{cases} -\Delta \tilde{V} = 0, & \text{in } \mathbb{R}_+^N \setminus K, \\ \tilde{V} = \psi_\gamma, & \text{on } K, \\ \frac{\partial \tilde{V}}{\partial \nu} = 0, & \text{on } \partial \mathbb{R}_+^N \setminus K. \end{cases}$$

Moreover

$$\text{cap}_{\mathbb{R}_+^N}(K, \psi_\gamma) = \int_{\mathbb{R}_+^N} |\nabla \tilde{V}|^2 \, dx$$

and \tilde{V} does not depend on the choice of the cut-off function η .

Since we are in the case $N \geq 3$, the following Hardy-type inequality on half balls holds.

Lemma 6.7 (Hardy-type inequality). *For all $R > 0$ and $u \in H^1(B_R^+)$*

$$\frac{N-2}{2} \int_{B_R^+} \frac{u^2}{|x|^2} \, dx \leq \frac{N+1}{R^2} \int_{B_R^+} u^2 \, dx + \frac{N}{N-2} \int_{B_R^+} |\nabla u|^2 \, dx .$$

Proof. By integrating over B_R^+ the identity

$$\text{div} \left(u^2 \frac{x}{|x|^2} \right) = (N-2) \frac{u^2}{|x|^2} + 2u \nabla u \cdot \frac{x}{|x|^2} ,$$

we obtain that, for any $u \in C^\infty(\overline{B_R^+})$,

$$\begin{aligned} (N-2) \int_{B_R^+} \frac{u^2}{|x|^2} dx &= \int_{\partial B_R^+} u^2 \frac{x \cdot \nu}{|x|^2} dS - 2 \int_{B_R^+} u \nabla u \cdot \frac{x}{|x|^2} dx \\ &\leq \frac{1}{R} \int_{S_R^+} u^2 dS + \frac{N-2}{2} \int_{B_R^+} \frac{u^2}{|x|^2} dx + \frac{2}{N-2} \int_{B_R^+} |\nabla u|^2 dx, \end{aligned}$$

thanks to the fact that $x \cdot \nu = 0$ on $\{x_1 = 0\} \cap \partial B_R^+$ and $x = R\nu$ on S_R^+ .

On the other hand, integrating over B_R^+ the identity

$$\operatorname{div}(u^2 x) = 2u \nabla u \cdot x + Nu^2$$

and arguing in a similar way, we deduce that

$$\int_{S_R^+} u^2 dS \leq \frac{N+1}{R} \int_{B_R^+} u^2 dx + R \int_{B_R^+} |\nabla u|^2 dx.$$

Combining those two relations the lemma is proved. \square

The following proposition provides a blow-up analysis for scaled capacitary potentials, which will be the core of the proof of Theorem 2.12.

Proposition 6.8. *Let $\{K_\epsilon\}_{\epsilon>0} \subseteq \overline{\Omega}$ be a family of compact sets concentrating at $\{0\} \subseteq \partial\Omega$ as $\epsilon \rightarrow 0$ and satisfying (2.7)-(2.8) for some $\Phi \in \mathcal{C}$ and for some compact sets $M, K \subseteq \overline{\mathbb{R}_+^N}$ with $\operatorname{cap}_{\mathbb{R}_+^N}(K) > 0$. Let φ_0 be as in (2.4) and let γ, ψ_γ be as in (2.9)-(2.12). Let \tilde{V}_ϵ be as in (6.9) and \tilde{V} as in Lemma 6.6. Then*

$$\begin{aligned} \tilde{V}_\epsilon &\rightharpoonup \tilde{V} \quad \text{weakly in } H^1(B_R^+), \\ A(\epsilon x) \nabla \tilde{V}_\epsilon(x) &\rightharpoonup \nabla \tilde{V}(x) \quad \text{weakly in } L^2(B_R^+), \\ \epsilon^2 \hat{c}(\epsilon x) \tilde{V}_\epsilon(x) &\rightharpoonup 0 \quad \text{weakly in } L^2(B_R^+), \end{aligned}$$

as $\epsilon \rightarrow 0$ for all $R > r(M)$, where A and \hat{c} are as in (6.2) (with Φ instead of Φ_{AE}).

Proof. From Lemma 3.9 and Lemma 6.2 we have that

$$\int_{\Omega} |\nabla V_{K_\epsilon, \varphi_0}|^2 dx = \operatorname{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0)(1 + o(1)) = O(\epsilon^{N+2\gamma-2}),$$

as $\epsilon \rightarrow 0$. On the other hand, letting \mathcal{U}_0 and R_0 be as in (6.5), by a change of variables we have that

$$\int_{\Omega} |\nabla V_{K_\epsilon, \varphi_0}|^2 dx \geq \int_{\Omega \cap \mathcal{U}_0} |\nabla V_{K_\epsilon, \varphi_0}|^2 dx = \epsilon^{N+2\gamma-2} \int_{B_{R_0/\epsilon}^+} A(\epsilon x) \nabla \tilde{V}_\epsilon(x) \cdot \nabla \tilde{V}_\epsilon(x) dx$$

and so

$$(6.12) \quad \int_{B_{R_0/\epsilon}^+} A(\epsilon x) \nabla \tilde{V}_\epsilon(x) \cdot \nabla \tilde{V}_\epsilon(x) dx = O(1)$$

as $\epsilon \rightarrow 0$. From (6.12) and (6.6) it follows that

$$(6.13) \quad \int_{B_{R_0/\epsilon}^+} |\nabla \tilde{V}_\epsilon(x)|^2 dx \leq 2 \int_{B_{R_0/\epsilon}^+} A(\epsilon x) \nabla \tilde{V}_\epsilon(x) \cdot \nabla \tilde{V}_\epsilon(x) dx = O(1) \quad \text{as } \epsilon \rightarrow 0.$$

On the other hand, in view of (6.7),

$$\begin{aligned} \operatorname{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0) &\geq \int_{\Omega \cap \mathcal{U}_0} c(x) |V_{K_\epsilon, \varphi_0}|^2 dx \\ &= \epsilon^{N+2\gamma} \int_{B_{R_0/\epsilon}^+} \hat{c}(\epsilon x) |\tilde{V}_\epsilon(x)|^2 dx \geq \frac{c_0}{2} \epsilon^{N+2\gamma} \int_{B_{R_0/\epsilon}^+} |\tilde{V}_\epsilon(x)|^2 dx \end{aligned}$$

so that from Lemma 6.2 we deduce that

$$(6.14) \quad \epsilon^2 \int_{B_{R_0/\epsilon}^+} |\tilde{V}_\epsilon(x)|^2 dx = O(1) \quad \text{as } \epsilon \rightarrow 0.$$

From (6.13) we deduce that there exists $C_1 > 0$ (not depending on R) such that, for every $R > 0$,

$$(6.15) \quad \int_{B_R^+} |\nabla \tilde{V}_\epsilon|^2 dx \leq C_1 \quad \text{for all } \epsilon \in (0, R_0/R).$$

From the Poincaré inequality proved in Lemma 6.5, we deduce that, for every $R > r(M)$, there exist $C_2 = C_2(R) > 0$ and $\epsilon_{1,R} > 0$ (both depending on R) such that, if $\epsilon \in (0, \epsilon_{1,R})$,

$$\begin{aligned} \int_{B_R^+} |\tilde{V}_\epsilon|^2 dx &\leq 2 \int_{B_R^+} |\tilde{V}_\epsilon - \tilde{\varphi}_\epsilon|^2 dx + 2 \int_{B_R^+} |\tilde{\varphi}_\epsilon|^2 dx \\ &\leq 2C_2 \int_{B_R^+} |\nabla(\tilde{V}_\epsilon - \tilde{\varphi}_\epsilon)|^2 dx + 2 \int_{B_R^+} |\tilde{\varphi}_\epsilon|^2 dx \\ &\leq 4C_2 \int_{B_R^+} (|\nabla \tilde{V}_\epsilon|^2 + |\nabla \tilde{\varphi}_\epsilon|^2) dx + 2 \int_{B_R^+} |\tilde{\varphi}_\epsilon|^2 dx. \end{aligned}$$

Hence, from (2.9) and (6.15) we have that, for every $R > r(M)$, there exist $C_3 = C_3(R) > 0$ and $\epsilon_{2,R} > 0$ (both depending on R) such that, if $\epsilon \in (0, \epsilon_{2,R})$,

$$(6.16) \quad \int_{B_R^+} |\tilde{V}_\epsilon|^2 dx \leq C_3.$$

Combining (6.15) and (6.16) with a diagonal process, we deduce that there exists $W \in H_{\text{loc}}^1(\mathbb{R}_+^N)$ (not depending on R) such that, along a subsequence $\epsilon = \epsilon_n \rightarrow 0^+$,

$$(6.17) \quad \begin{aligned} \tilde{V}_\epsilon &\rightharpoonup W \quad \text{weakly in } H^1(B_R^+), \\ \tilde{V}_\epsilon &\rightarrow W \quad \text{strongly in } L^2(B_R^+), \\ \tilde{V}_\epsilon &\rightarrow W \quad \text{a.e. in } \mathbb{R}_+^N, \end{aligned}$$

for all $R > r(M)$. Since c is bounded and $\|A(\epsilon x) - I_N\|_{\mathcal{M}_{N \times N}} \leq C R \epsilon$ for all $x \in B_R^+$, from (6.17) it follows easily that

$$(6.18) \quad A(\epsilon x) \nabla \tilde{V}_\epsilon(x) \rightharpoonup \nabla W(x) \quad \text{weakly in } L^2(B_R^+),$$

$$(6.19) \quad \epsilon^2 \hat{c}(\epsilon x) \tilde{V}_\epsilon(x) \rightharpoonup 0 \quad \text{weakly in } L^2(B_R^+),$$

as $\epsilon = \epsilon_n \rightarrow 0$, for all $R > r(M)$.

Now let $\varphi \in C_c^\infty(\overline{\mathbb{R}_+^N} \setminus K)$. Then there exists $R > 0$ such that

$$\varphi \in C_c^\infty(B_R^+ \cup (B_R' \setminus K)) \subseteq H_{0, K \cup S_R^+}^1(B_R^+).$$

Since $H_{0, \tilde{K}_\epsilon \cup S_R^+}^1(B_R^+)$ is converging to $H_{0, K \cup S_R^+}^1(B_R^+)$ in the sense of Mosco (see hypothesis (2.8) and Remark 6.1), there exists a sequence $\psi_\epsilon \in H_{0, \tilde{K}_\epsilon \cup S_R^+}^1(B_R^+)$ such that

$$(6.20) \quad \psi_\epsilon \rightarrow \varphi \quad \text{strongly in } H^1(B_R^+), \quad \text{as } \epsilon \rightarrow 0.$$

From the equation (6.11) satisfied by \tilde{V}_ϵ we know that, for all $\epsilon \in (0, R_0/R)$ (so that $B_R^+ \subset B_{R_0/\epsilon}^+$)

$$\int_{B_R^+} (A(\epsilon x) \nabla \tilde{V}_\epsilon(x) \cdot \nabla \psi_\epsilon(x) + \epsilon^2 \hat{c}(\epsilon x) \tilde{V}_\epsilon(x) \psi_\epsilon(x)) dx = 0.$$

Therefore, combining (6.18), (6.19) and (6.20) we obtain that

$$\int_{B_R^+} \nabla W \cdot \nabla \varphi dx = 0.$$

Summing up we have that

$$(6.21) \quad \int_{\mathbb{R}_+^N} \nabla W \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\overline{\mathbb{R}_+^N} \setminus K).$$

By weak lower semicontinuity and (6.15) we have that

$$(6.22) \quad \int_{\mathbb{R}_+^N} |\nabla W|^2 \, dx < \infty.$$

Now let $R > r(M)$ and $\epsilon < R_0/R$. Thanks to Lemma 6.7 and estimates (6.13) and (6.14) we have that

$$\begin{aligned} \int_{B_R^+} \frac{|\tilde{V}_\epsilon|^2}{|x|^2} \, dx &\leq \int_{B_{R_0/\epsilon}^+} \frac{|\tilde{V}_\epsilon|^2}{|x|^2} \, dx \\ &\leq \frac{2(N+1)}{(N-2)R_0^2} \epsilon^2 \int_{B_{R_0/\epsilon}^+} \tilde{V}_\epsilon^2 \, dx + \frac{2N}{(N-2)^2} \int_{B_{R_0/\epsilon}^+} |\nabla \tilde{V}_\epsilon|^2 \, dx \leq C_4, \end{aligned}$$

for a certain $C_4 > 0$ not depending on ϵ and R . By weak lower semicontinuity we deduce that $\int_{B_R^+} \frac{|W|^2}{|x|^2} \, dx \leq C_4$ for all $R > r(M)$, hence

$$\int_{\mathbb{R}_+^N} \frac{|W|^2}{|x|^2} \, dx < \infty.$$

Thanks to this and to (6.22), we have that $W \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N})$; moreover, by density of $C_c^\infty(\overline{\mathbb{R}_+^N} \setminus K)$ in $\mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$ we have that (6.21) holds for any $\varphi \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$.

Let $\eta \in C_c^\infty(\overline{\mathbb{R}_+^N})$ be such that $\eta = 1$ in a neighbourhood of M : since, for every $R > r(M)$, $\tilde{V}_\epsilon - \eta \tilde{\varphi}_\epsilon \in H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ and since $H_{0, \tilde{K}_\epsilon}^1(B_R^+)$ is converging to $H_{0, K}^1(B_R^+)$ in the sense of Mosco, then, passing to the weak limit, there holds $W - \eta \psi_\gamma \in H_{0, K}^1(B_R^+)$ (see (2.9)). Hence, in view of Lemma 6.3, we have that $W - \eta \psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$. Combining this fact with (6.21), by uniqueness (stated in Lemma 6.6) we have that $W = \tilde{V}$ and, by Urysohn's Subsequence Principle, we conclude that the convergences (6.17), (6.18), and (6.19) hold as $\epsilon \rightarrow 0$ (not only along a sequence $\epsilon_n \rightarrow 0^+$). \square

Now we are ready for the proof of our second main result.

Proof of Theorem 2.12. Let $R > r(M)$ and let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta = 1$ in B_R . Also let $\tilde{R} > 0$ be such that $\text{supp } \eta \subseteq B_{\tilde{R}}$. For $\epsilon > 0$ small we define

$$\eta_\epsilon(x) = \begin{cases} \eta\left(\frac{1}{\epsilon}\Phi(x)\right), & \text{if } x \in \mathcal{U}_0, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \mathcal{U}_0, \end{cases}$$

and observe that, if ϵ is sufficiently small, $\eta_\epsilon \in C_c^1(\mathbb{R}^N)$ and $\eta_\epsilon \equiv 1$ in a neighbourhood of K_ϵ . Testing equation (3.1) with $V_{K_\epsilon, \varphi_0} - \varphi_0 \eta_\epsilon$ leads to

$$\begin{aligned} \text{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0) &= \int_{\Omega} (|\nabla V_{K_\epsilon, \varphi_0}|^2 + c|V_{K_\epsilon, \varphi_0}|^2) \, dx \\ &= \int_{\Omega} (\nabla V_{K_\epsilon, \varphi_0} \cdot \nabla(\varphi_0 \eta_\epsilon) + cV_{K_\epsilon, \varphi_0} \varphi_0 \eta_\epsilon) \, dx \\ &= \int_{\Omega \cap \mathcal{U}_0} (\nabla V_{K_\epsilon, \varphi_0} \cdot \nabla(\varphi_0 \eta_\epsilon) + cV_{K_\epsilon, \varphi_0} \varphi_0 \eta_\epsilon) \, dx. \end{aligned}$$

Then, by the change of variable $x = \Phi^{-1}(\epsilon y)$, we obtain

$$(6.23) \quad \text{Cap}_{\overline{\Omega}, c}(K_\epsilon, \varphi_0) = \epsilon^{2\gamma+N-2} \int_{B_{\tilde{R}}^+} (A(\epsilon y) \nabla \tilde{V}_\epsilon(y) \cdot \nabla(\eta \tilde{\varphi}_\epsilon)(y) + \epsilon^2 \hat{c}(\epsilon y) \tilde{V}_\epsilon(y) \eta(y) \tilde{\varphi}_\epsilon(y)) \, dy$$

where \tilde{V}_ϵ and $\tilde{\varphi}_\epsilon$ are defined in (6.9). From Proposition 6.8 and Proposition 2.8 it follows that

$$(6.24) \quad \lim_{\epsilon \rightarrow 0} \int_{B_R^+} (A(\epsilon y) \nabla \tilde{V}_\epsilon(y) \cdot \nabla(\eta \tilde{\varphi}_\epsilon)(y) + \epsilon^2 \hat{c}(\epsilon y) \tilde{V}_\epsilon(y) \eta(y) \tilde{\varphi}_\epsilon(y)) \, dy = \int_{\mathbb{R}_+^N} \nabla \tilde{V} \cdot \nabla(\eta \psi_\gamma) \, dy.$$

Finally, testing the equation satisfied by \tilde{V} (see Lemma 6.6) with $\tilde{V} - \eta \psi_\gamma \in \mathcal{D}^{1,2}(\overline{\mathbb{R}_+^N} \setminus K)$ we have that

$$\text{cap}_{\mathbb{R}_+^N}(K, \psi_\gamma) = \int_{\mathbb{R}_+^N} |\nabla \tilde{V}|^2 \, dx = \int_{\mathbb{R}_+^N} \nabla \tilde{V} \cdot \nabla(\eta \psi_\gamma) \, dx.$$

This, combined with (6.23) and (6.24), concludes the proof. \square

7. SET SCALING TO AN INTERIOR POINT

In this last section we consider the case in which the perturbing sets K_ϵ are concentrating to an interior point in a way that resembles (and comprehends) the scaling of a fixed compact set and we sketch the steps that lead to the proof of Theorem 2.14 (counterpart of Theorem 2.12). Always in the case $N \geq 3$, we assume that $0 \in \Omega$ and that the family of compact sets $K_\epsilon \subseteq \Omega$ satisfy (2.13) and (2.14). Heuristically speaking, in the previous section the rescaled domain Ω/ϵ was “approaching” the half space \mathbb{R}_+^N , due to the fact that $0 \in \partial\Omega$ and that $\partial\Omega$ was smooth in a neighbourhood of the origin. In this section, since $0 \in \Omega$ the “limit” domain of Ω/ϵ turns out to be the whole space \mathbb{R}^N . For the same reason the role of half balls B_R^+ is played, in this section, by balls B_R .

Let κ and ζ_κ be as in (2.15). As in Lemma 6.2, by testing $\text{Cap}_{\overline{\Omega},c}(K_\epsilon, \varphi_0)$ with φ_0 suitably cutted off, it is possible to prove that

$$\text{Cap}_{\overline{\Omega},c}(K_\epsilon, \varphi_0) = O(\epsilon^{N+2\kappa-2}), \quad \text{as } \epsilon \rightarrow 0.$$

Also in this framework, a Poincaré type inequality holds and the proof follows the same steps as Lemma 6.5.

Lemma 7.1 (Poincaré Inequality). *Let $M, K \subseteq \mathbb{R}^N$ and $\{K_\epsilon\}_{\epsilon \in (0,1)}$ satisfy (2.13) and (2.14) and let $\tilde{K}_\epsilon := K_\epsilon/\epsilon$. Let us assume that $\text{cap}_{\mathbb{R}^N}(K) > 0$. For any $R > r(M)$ there exist $\epsilon_0 \in (0, 1)$ and $C > 0$ (both depending on R and K) such that*

$$\int_{B_R} u^2 \, dx \leq C \int_{B_R} |\nabla u|^2 \, dx$$

for all $u \in H_{0, \tilde{K}_\epsilon}^1(B_R)$ and for all $\epsilon < \epsilon_0$.

Furthermore, the capacity $\text{cap}_{\mathbb{R}^N}(K, \zeta_\kappa)$, whose definition is recalled in Definition 3.10, is attained by a potential $\hat{V} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, analogously to what is stated in Lemma 6.6. Also in this context it is possible to prove an Hardy type inequality, which reads as follows.

Lemma 7.2 (Hardy-type inequality). *We have that*

$$\frac{N-2}{2} \int_{B_R} \frac{u^2}{|x|^2} \, dx \leq \frac{N+1}{R^2} \int_{B_R} u^2 \, dx + \frac{N}{N-2} \int_{B_R} |\nabla u|^2 \, dx$$

for all $u \in H^1(B_R)$ and for all $R > 0$.

Following the same steps as in the proof of Proposition 6.8 and Theorem 2.12 and adapting the ideas and the computations to the current framework, it is possible to prove Theorem 2.14.

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VERONICA FELLI

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO–BICOCCA
 VIA COZZI 55, 20125 MILANO, ITALY
 Email address: veronica.felli@unimib.it

BENEDETTA NORIS

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO
 PIAZZA LEONARDO DA VINCI 32, 20133 MILANO, ITALY
 Email address: benedetta.noris@polimi.it

ROBERTO OGNIBENE

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI MILANO–BICOCCA
 VIA COZZI 55, 20125 MILANO, ITALY
 Email address: roberto.ognibene@unimib.it