Chapter 2

On the dynamics of coupled oscillators and its application to the stability of suspension bridges

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We describe and provide a computer assisted proof of the bifurcation graph for a system of coupled nonlinear oscillator described in a model of a bridge. We also prove the linear stability/instability of the branches of solutions.

1 Introduction and main results

The literature on the collapse of the Tacoma Narrows Bridge is vast, and numerous explanations of the collapse have been presented in the last 81 years. One point on which everybody agrees is that the collapse was caused by the unexpected appearance of torsional oscillations of the deck. In [4], a new mathematical model for the study of the dynamical behavior of suspension bridges was introduced, and a new explanation for the appearance of torsional oscillations during the Tacoma collapse was provided. The key point of such explanation is that, when the amplitude of the vertical oscillation is below some threshold, such an oscillation is stable. At the threshold a pitchfork bifurcation occurs. The main branch, with no torsional movement, becomes unstable, while the secondary branches, with torsional oscillations, are stable. This causes a transfer of energy from the vertical oscillation to the torsional oscillation. This explanation has been further expanded in [5] and [6].

Here we provide a rigorous proof of the dynamics described above. More precisely, we provide a rigorous bifurcation graph for the system introduced in Section 4 of [4], and we also prove the linear stability/instability of the branches of solutions. The full description and motivation of the model can be found in [4], see also [11]; here we provide a short outline.

In Figure 1 the rod represents a cross section of the deck of a bridge. We choose the mass $m = 1$ and the half length $\ell = 1$. The rod is subjected to a force exerted by the hangers C_1 and C_2, which is denoted, respectively, by $f(y + \ell \sin\theta)$ and $f(y - \ell \sin\theta)$; these terms take also into account gravity. The energy of the system is

Note 1: All labels of equations that have not been referenced were removed.

$$\mathcal{E}(\dot\theta, \dot y, \theta, y) = \mathcal{K}(\dot\theta, \dot y) + \mathcal{U}(\theta, y),$$

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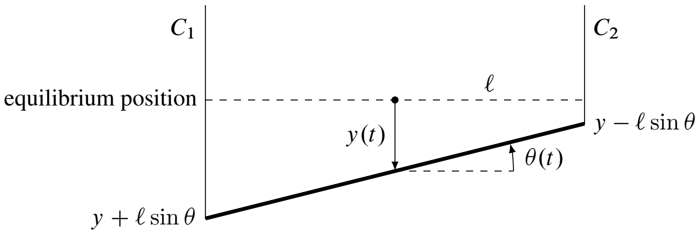


Figure 1. Vertical (y) and torsional (θ) displacements of the rod.

where $\mathcal{K}(\dot\theta, \dot y) = \dot y^2/2 + \dot\theta^2/6$ is the kinetic energy, and $\mathcal{U}(\theta, y) = F(y + \sin\theta) + F(y - \sin\theta)$ is the potential energy, with $F(s) = -\int_0^s f(\tau)\, d\tau$. The Euler–Lagrange equations corresponding to the Lagrangian $\mathcal{L} = \mathcal{K} - \mathcal{U}$ are

$$\begin{cases} \dfrac{\ddot\theta}{3} = \cos\theta\big(f(y + \sin\theta) - f(y - \sin\theta)\big), \\ \ddot y = f(y - \sin\theta) + f(y + \sin\theta). \end{cases} \tag{1}$$

For the nonlinear restoring force f, we follow the same choice as in [4], that is,

$$f(s) = -(s + s^2 + s^3) \implies F(s) = \frac{s^2}{2} + \frac{s^3}{3} + \frac{s^4}{4},$$

Note 2: The order of appearance was changed to match that of (1).

so that equation (1) becomes

$$\begin{cases} \ddot\theta = -\dfrac{3}{4}\big((6 + 8y + 12y^2)\sin(2\theta) - \sin(4\theta)\big), \\ \ddot y = -(2y + 2y^2 + 2y^3 + (6y + 2)\sin^2\theta). \end{cases} \tag{2}$$

We are interested in nontrivial periodic solutions of (2). Our results are displayed in Figure 2; we first describe the results, and refer to Theorems 1.1 and 1.2 for a precise statement.

The horizontal axis represents the period of the solution. The dotted line represents the trivial solution $(y, \theta) = (0, 0)$. At $T = \pi/\sqrt{2} = 2.221\ldots$, a pitchfork bifurcation occurs, and two branches of periodic solutions bifurcate out of the trivial solution, with increasing period (black line). The solutions on the secondary branches satisfy $\theta \equiv 0$. The two branches are related by the map $y(t) \mapsto y(-t)$. In fact, the pitchfork bifurcation is a symmetry breaking bifurcation, therefore we only study one of the secondary branches. At $T = 2.223\ldots$, a fold bifurcation occurs, and the branch continues with decreasing period (green line). The black and green branches of solutions are linearly stable. At $T = 1.952\ldots$, another pitchfork bifurcation occurs. Note that whenever we write a real number in truncated decimal notation followed by dots,

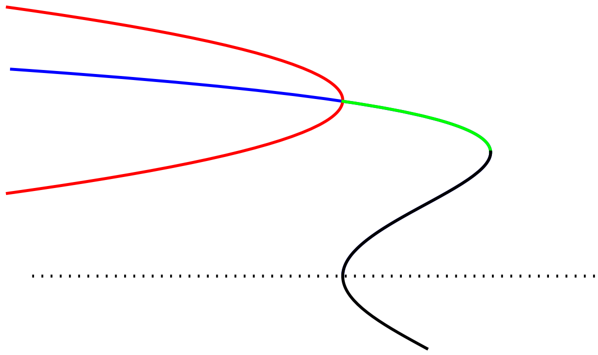


Figure 2. Bifurcation graph with respect of the period T of the solution (not in scale). The dotted line consists of trivial solutions, the black and green lines consist of stable solution with $\theta = 0$, the blue line consists of unstable solutions with $\theta = 0$, the red line consists of stable solutions with $\theta \neq 0$.

e.g., $2.223\ldots$, what we mean is that the exact value of the number (in case it is exactly representable by the IEEE standard) or a precise bound on such number (that is, an interval which contains the number, and whose center and radius are exactly representable by the IEEE standard) is available in the program or data files, and the first decimal digits are shown. The two new branches of the solution (red lines) have a torsional component, that is $\theta \neq 0$, they are linearly stable and they are related by the map $(y, \theta) \mapsto (y, -\theta)$. The branch consisting of purely horizontal oscillations (blue) becomes unstable.

We briefly recall the relation between this result and the dynamics of a suspension bridge, see [4–6] for a more detailed discussion. As long as the horizontal oscillation has small amplitude (black and green solutions), it is stable. But at a certain threshold, corresponding here to the bifurcation where the blue, the green and the red branches meet, the horizontal oscillation becomes unstable, while at the same time, a stable oscillation with a torsional component appears. Then, if the oscillation of the bridge passes that threshold, the torsional oscillation starts and eventually leads to a collapse.

A secondary purpose of this paper is the introduction of a computer assisted method for the study of the Poincaré map of a dynamical system. It is standard (and obvious) practice to represent periodic functions with Fourier series, and this choice would have made the study of the bifurcation graph easier. On the other hand, Fourier series are not suitable for the study of the variational equation associated with the system, that is, the linearisation around a solution, since one does not expect a periodic solution for such equation. Here we choose to solve equation (2) together with its variational equation, using a representation based on the Chebyshev expansion.

In order to study the stability of periodic solutions of (2), it is convenient to consider a Poincaré map. Since we are dealing with a system with a four-dimensional phase space and with one conserved quantity (the energy), the dynamics take place in a three-dimensional manifold. Then we can choose the subspace $y = 0$ crossed with $\dot{y} > 0$ as Poincaré section, so that the Poincaré map maps (a subset of) the plane $(\theta, \dot{\theta})$ into itself.

After rescaling time, we look for solutions of

$$\begin{cases} \ddot{y} = f(y, \theta, T) := -\dfrac{T^2}{2}(2y + 2y^2 + 2y^3 + (6y + 2)\sin^2\theta), \\ \ddot{\theta} = g(y, \theta, T) := -\dfrac{3T^2}{8}((6 + 8y + 12y^2)\sin(2\theta) - \sin(4\theta)), \\ y(-1) = y(1) = 0, \quad \dot{y}(-1) = \dot{y}(1) > 0, \\ \theta(-1) = \theta(1), \quad \dot{\theta}(-1) = \dot{\theta}(1), \end{cases} \tag{3}$$

where T is the period of the solution to be determined. Note that the condition $\dot{y}(-1) = \dot{y}(1)$ follows from the other boundary conditions because of the conservation of the energy. Clearly, solutions of (3) correspond to periodic solutions of (2) of period T. To compute the derivative of the Poincaré map, we need to solve the following initial value problem for the variation equations:

$$\begin{cases} \ddot{\theta}_1 = h(y, \theta, \theta_1, T) := -\dfrac{3T^2}{8}((2(6 + 8y + 12y^2)\cos(2\theta) - 4\cos(4\theta))\theta_1), \\ \theta_1(-1) = 1, \quad \dot{\theta}_1(-1) = 0, \\ \ddot{\theta}_2 = h(y, \theta, \theta_2, T), \quad \theta_2(-1) = 0, \quad \dot{\theta}_2(-1) = 1. \end{cases} \tag{4}$$

We look for solutions of (3) and (4) in a suitable space of analytic functions $\mathcal{A}$, see equation (5) for the definition. As described above, our first result is the existence branches of solutions connected with three pitchfork bifurcations and two fold bifurcations.

Theorem 1.1. (1) *There exists an analytic curve* $\alpha\colon [-1, 1] \mapsto \mathcal{A}^2 \times \mathbb{R}$ *and a real number* $T_F = 2.223\ldots$ *such that*:

- *for all* $\tau \in [-1, 1]$, $(y, \theta) = (\alpha_1(\tau), \alpha_2(\tau))$ *is a solution of* (3) *with* $\alpha_2(\tau) = 0$ *and* $T = \alpha_3(\tau)$.
- $(\alpha_1(-\tau)(-t), \alpha_3(-\tau)) = (\alpha_1(\tau)(t), \alpha_3(\tau))$ *for all* $\tau \in [0, 1]$ *and all* $t \in [-1, 1]$,
- $\alpha_3(1) = T_F$,
- $\alpha_3'(\tau) > 0$ *for all* $\tau \in (0, 1)$.

(2) *There exists an analytic curve* $\beta\colon [-1, 1] \mapsto \mathcal{A}^2 \times \mathbb{R}$ *such that*:

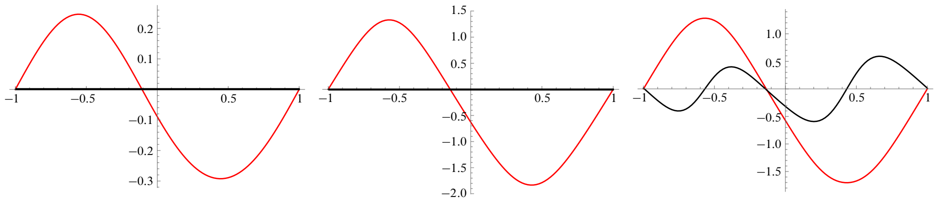


Figure 3. Approximate graph of three solutions (y is red and θ is black): branch β at $T = 2.221\dots$ (left), branch β at $T = 1.491\dots$ (center), branch γ at $T = 1.391\dots$ (right).

- *for all* $\tau \in [-1, 1]$, $(y, \theta) = (\beta_1(\tau), \beta_2(\tau))$ *is a solution of* (3) *with* $\beta_2(\tau) = 0$ *and* $T = \beta_3(\tau)$,
- *for all* $\tau \in [-1, 1)$, $\beta_3'(\tau) > 0$,
- $\beta_3(-1) = 1.40\dots$.

(3) *There exists an analytic curve* $\gamma\colon [-1, 1] \mapsto \mathcal{A}^2 \times \mathbb{R}$ *such that*:

- *for all* $\tau \in [-1, 1]$, $(y, \theta) = (\gamma_1(\tau), \gamma_2(\tau))$ *is a solution of* (3) *with* $T = \gamma_3(\tau)$,
- *for all* $\tau \in [-1, 0) \cup (0, 1]$, $\tau\gamma_3'(\tau) < 0$,
- $\gamma_2(\tau) = 0$ *if and only if* $\tau = 0$.

(4) $\alpha(0) = (0, 0, \pi/\sqrt{2})$, $\beta(1) = \alpha(1)$, *and* $\gamma(0) = \beta(0)$.

Our second result concerns the stability of the solutions.

Theorem 1.2. *All the branches described in Theorem 1.1 consist of linearly stable solutions, with the exception of the solutions* $\beta(\tau)$ *with* $\tau \in [-1, 0)$, *which are unstable.*

The remaining part of the paper contains the proofs of Theorems 1.1 and 1.2. In Section 2 we present the functional space where we look for solutions. In Section 3 we present the strategy we use for the proofs of the branches of solutions. In Section 4 we present the strategy we use for the proofs of the bifurcations. In Section 5 we link together the lemmas and propositions proved in the previous sections to obtain the proofs of the theorems. In Section 6 we discuss some details of the computer assisted proof.

2 Functional setting

Let $\rho = 5/4$ and

$$\mathcal{A} := \left\{ x\colon [-1, 1] \to \mathbb{R} \;\middle|\; x(t) = \sum_{j\geq 0} x_j T_j(t),\ \sum_{j\geq 0} |x_j| \rho^j < +\infty \right\}, \tag{5}$$

where $\{x_j\} \subset \mathbb{R}$ and T_j is the j-th Chebyshev polynomial. The space $\mathcal{A}$, equipped with the norm

$$\|x\| := \sum_{j\geq 0} |x_j|\rho^j,$$

is a Banach algebra, consisting of functions which admit an analytic extension to the interior of an ellipse in the complex plane with foci $\{-1, 1\}$ and semiaxes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$, see [10].

Note that, if $x(t)$ is represented as in (5), then

$$x(-1) = \sum_j (-1)^j x_j, \quad x(1) = \sum_j x_j,$$

and a primitive of a function expanded in Chebyshev series can be found, e.g., by using the following:

$$\int T_0(t)\,dt = T_1, \quad \int T_1(t)\,dt = \frac{T_0(t) + T_2(t)}{4}, \tag{6}$$

$$\int T_j(t)\,dt = \frac{T_{j+1}(t)}{2(k+1)} - \frac{T_{j-1}(t)}{2(k-1)}. \tag{7}$$

To solve the first equation in (3), we define D_D^{-2} as the inverse second derivative with homogeneous Dirichlet boundary conditions. More precisely, given $y \in \mathcal{A}$, with $y = \sum_{j\geq 0} y_j T_j$, $\hat{y} = D_D^{-2} y$ is the function in $\mathcal{A}$ obtained by applying twice (6)–(7) and choosing

$$\hat{y}_0 = -\sum_{j\geq 1} \hat{y}_{2j}, \quad \hat{y}_1 = -\sum_{j\geq 1} \hat{y}_{2j+1}.$$

Then the first equation in (3), together with the boundary conditions in y, is equivalent to

$$y = \mathbb{F}_1(y, \theta, T) := D_D^{-2} f(y, \theta, T).$$

To solve the equation $\ddot{\theta} = g(y, \theta)$, where periodic boundary conditions are required, we use the following lemma.

Lemma 2.1. *Let $\mathbb{F}_2 \colon \mathcal{A}^2 \times \mathbb{R} \to \mathcal{A}$ be defined as follows. Given $(y, \theta, T) \in \mathcal{A}^2 \times \mathbb{R}$, let*

$$s = \int_{-1}^{1} g(y(t), \theta(t), T)\,dt + \theta_0,$$

where θ_0 is the Chebyshev coefficient of order 0 of θ. Let $\hat{\theta}$ be the inverse second derivative of $g(y, \theta)$, where $\hat{\theta}_0 = s$ and $\hat{\theta}_1$ is chosen so that $\int_{-1}^{1} \hat{\theta}(t)\,dt = 0$. Set $\mathbb{F}_2(y, \theta, T) := \hat{\theta}$. Then θ is the solution of $\ddot{\theta} = g(y, \theta, T)$ with periodic boundary conditions if and only if $\mathbb{F}_2(y, \theta, T) = \theta$.

Proof. Assume that $\mathbb{F}_2(y,\theta,T)=\theta$. Clearly, $\ddot{\theta}=g(y,\theta,T)$. Since $\int_{-1}^{1}\theta(t)\,dt=0$, $\theta(-1)=\theta(1)$. Since θ is a fixed point of F, $s=\theta_0$, so that $\int_{-1}^{1}g(y(t),\theta(t))\,dt=\int_{-1}^{1}\ddot{\theta}(t)\,dt=0$, and therefore $\dot{\theta}(-1)=\dot{\theta}(1)$. ∎

Consider now the equations in (4).

Lemma 2.2. *For $i=1,2$, define $\mathbb{G}_i\colon \mathcal{A}^3\times\mathbb{R}\to\mathcal{A}$ as follows. Let*

$$\hat{\theta}_i=\int h(y(t),\theta(t),\theta_i(t),\phi(t),T)\,dt,$$

with $(\hat{\theta}_i)_0=\sum_{j\geq 1}(-1)^j(\hat{\theta}_i)_j+(2-i)$. Then let

$$\tilde{\theta}_i=\int\hat{\theta}_i(t)\,dt,$$

with $(\tilde{\theta}_i)_0=\sum_{j\geq 1}(-1)^j(\hat{\theta}_i)_j+i-1$. Set $\mathbb{G}_i(y,\theta,\theta_i,T)=\tilde{\theta}_i$. Then $(\theta_1,\theta_2)\in\mathcal{A}^2$ is a solution of (4) if and only if

$$\mathbb{G}_i(y,\theta,\theta_i)=\theta_i,\quad i=1,2.$$

Proof. If θ_i is a fixed point of $\mathbb{G}_i$, then $\ddot{\theta}_i=h(y,\theta,\theta_i,T)$. Furthermore, $\hat{\theta}_i=\dot{\theta}_i$, and with our choice of $(\hat{\theta}_i)_0$, we have $\hat{\theta}_i(-1)=\dot{\theta}_i(-1)=(2-i)$. Finally, $\theta_i(-1)=i-1$ because of our choice of $(\tilde{\theta}_i)_0$. ∎

To find a solution of both (3) and (4), we equip $\mathcal{A}^4$ with the norm

$$\|(y,\theta,\theta_1,\theta_2)\|:=\|y\|+\|\theta\|+\|\theta_1\|+\|\theta_2\|,$$

and set $\mathbb{F}_T\colon\mathcal{A}^4\to\mathcal{A}^4$ as follows:

$$\mathbb{F}_T(y,\theta,\theta_1,\theta_2)=\big(\mathbb{F}_1(y,\theta,T),\mathbb{F}_2(y,\theta,T),\mathbb{G}_1(y,\theta,\theta_1,T),\mathbb{G}_2(y,\theta,\theta_2,T)\big).$$

The proof of the following lemma is straightforward.

Lemma 2.3. *The map $\mathbb{F}_T$ is compact, and $(y,\theta,\theta_1,\theta_2)\in\mathcal{A}^4$ is a solution of (3) and (4) if and only if it is a fixed point of $\mathbb{F}_T$.*

3 Branches of solutions

Consider the subset of the bifurcation graph displayed in Figure 4, and let $I_A=[1.34\ldots,1.952\ldots]$, $I_B=[1.40\ldots,1.952\ldots]$, $I_C=[1.954\ldots,2.22\ldots]$. We want to prove the existence of three branches of solutions of (3) and (4) when T varies in the intervals I_A,I_B,I_C. Note that the exact values of the endpoints of these intervals are

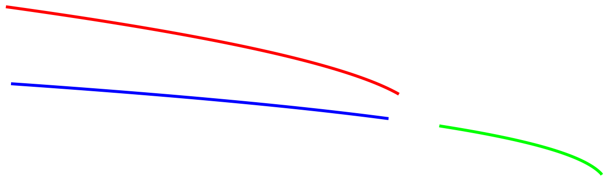


Figure 4. Three branches of solutions. T varies in I_A (branch A, red), I_B (branch B, green), I_C (branch C, blue).

provided in the program, see below for details, and in the next section, we will show that the branches are joint by a bifurcation. To prove the existence of the branches we use a technique introduced in [7].

We write all the coefficients in the Chebyshev expansion of $(y, \theta, \theta_1, \theta_2)$ as Taylor polynomials in T:

$$(y, \theta, \theta_1, \theta_2) = \sum_{j \geq 0} (y_j(T), \theta_j(T), \theta_{1j}(T), \theta_{2j}(T)) T_j(t), \tag{8}$$

$$(y_j(T), \theta_j(T), \theta_{1j}(T), \theta_{2j}(T)) = \sum_{l=0}^{L} (y_{jl}, \theta_{jl}, \theta_{1jl}, \theta_{2jl}) \left(\frac{T - T_0}{T_1}\right)^l, \tag{9}$$

where T_0, T_1 and L are defined below. We call (8)–(9) a Taylor–Chebyshev expansion, and its truncation a Taylor–Chebyshev polynomial. We choose some Taylor–Chebyshev polynomial $\bar{X} = (\bar{y}, \bar{\theta}, \bar{\theta}_1, \bar{\theta}_2) \in \mathcal{A}^4$, that is, an approximate fixed point of $\mathbb{F}_T$, and some finite rank operator $M_T : \mathcal{A}^4 \to \mathcal{A}^4$ such that $\mathbb{I} - M_T$ is an approximate inverse of $\mathbb{I} - D\mathbb{F}_T(\bar{X})$. Then for $h \in \mathcal{A}^4$, we define

$$\mathcal{M}_T(h) = \mathbb{F}_T(\bar{X} + \Lambda_T h) - \bar{X} + M_T h, \quad \Lambda_T = \mathbb{I} - M_T. \tag{10}$$

Clearly, if h is a fixed point of $\mathcal{M}_T$, then $X = \bar{X} + \Lambda_T h$ is a fixed point of $\mathbb{F}_T$ and, hence, X solves (3) and (4). Given $r > 0$ and $w \in \mathcal{A}^4$, let $B_r(w) = \{v \in \mathcal{A}^4 : \|v - w\| < r\}$. We partition the interval I_A in 13 subintervals, the interval I_B in 8 subintervals and the interval I_C in 10 subintervals. These subintervals have various widths, depending on the distance from the bifurcation points. The squares of the exact values of the center T_0 and width T_1 of each subinterval, together with the degree L of the Taylor expansion used for each subinterval, are available in the file `params.ads`, and they are denoted by `T2C` and `T2W`. Then we prove the following lemma with the aid of a computer, see Section 6.

* **Lemma 3.1.** *The following holds for each value of* T_0, T_1 *and* L *listed in the file* `params.ads`. *There exist a Taylor–Chebyshev polynomial* $\bar{X}(T)$ *of degree* L*, as described in* (9)*, a bounded linear operator* M_T *on* $\mathcal{A}$*, and positive real numbers* ε, r, K*,*

satisfying $\varepsilon + Kr < r$, *such that*

$$\|\mathcal{M}_T(0)\| \le \varepsilon, \quad \|D\mathcal{M}_T(Y)\| \le K \quad \textit{for all } Y \in B_r(0),$$

and all $T \in \mathbb{C}$, *with* $|T - T_0| < T_1$. *Furthermore, for each adjacent pair of subintervals, either*

$$B_{r^1}(\bar{X}(T_0^1 + T_1^1)) \subset B_{r^2}(\bar{X}(T_0^2 - T_1^2))$$

or

$$B_{r^2}(\bar{X}(T_0^2 + T_1^2)) \subset B_{r^1}(\bar{X}(T_0^1 - T_1^1)),$$

where the superscripts 1, 2 *refer to the values of* T_0, T_1, r *relative to each subinterval, and the* θ *component of each solution is identically* 0.

This lemma, together with the contraction mapping theorem and the implicit function theorem, implies the following.

Proposition 3.2. *Equations* (3) *and* (4) *admit a solution for all* $T \in I_A$, *all* $T \in I_B$, *and all* $T \in I_C$. *The solutions depend analytically on* T.

Since Lemma 3.1 provides also rigorous bounds on the solution of the variation equation, the proof of the linear stability or instability of the solutions follows easily, see also Appendix A for the computation of the derivative of the Poincaré map.

Lemma 3.3. *Let* $\lambda_{1T}^X, \lambda_{2T}^X$ *be the eigenvalues of the derivative of the Poincaré map corresponding to solutions belonging to the branches* $X = A, B, C$ *and period* T. *For all values of* $T \in I_A$, *the eigenvalues* $\lambda_{1T}^A, \lambda_{2T}^A$ *are real, and* $|\lambda_{1T}^A| < 1 < |\lambda_{2T}^C|$. *For all values of* $T \in I_B$ *(resp.* $T \in I_C$*),* $\lambda_{1T}^B = \bar{\lambda}_{2T}^B$ *(resp.* $\lambda_{1T}^C = \bar{\lambda}_{2T}^C$*), and, since the determinant of the derivative of the Poincaré map is equal to* 1, $|\lambda_{1T}^B| = |\lambda_{2T}^B| = |\lambda_{1T}^C| = |\lambda_{2T}^C| = 1$.

4 Bifurcations

To prove Theorem 1.1, we need to prove the existence of pitchfork and fold bifurcations, and show that the branches of solutions proved in Section 3 are connected to such bifurcations. Let $u = (y, \theta)$, define $\mathbb{F}: \mathbb{R} \times \mathcal{A}^2 \to \mathcal{A}^2$ by

$$\mathbb{F}(T, u) = (\mathbb{F}_1(u, T), \mathbb{F}_2(u, T)) - u,$$

and note that u is a solution of (3) if and only if $\mathbb{F}(T, u) = 0$. A simple computation shows that $D\mathbb{F}(T, \cdot)$ has a (simple) eigenvalue zero at $T = \pi/\sqrt{2}$. Also, we found numerical evidence that, along the branch of solutions, $D\mathbb{F}(T, \cdot)$ has two other (simple) eigenvalues zero at $T = 1.953\ldots$ and $T = 2.222\ldots$. This suggests the

i	T_0	T_1	λ_1	M
1	$1.953\ldots$	2^{-8}	1/32	9
2	$2.222\ldots$	$21 \cdot 2^{-12}$	9/32	35

Table 1. Parameters for the bifurcation.

possibility of bifurcations which take place in a two-dimensional submanifold. We parametrize this surface by using the parameter T and a coordinate λ for the range of a suitable one-dimensional projection ℓ. Then we define a two-parameter family of functions $u(T,\lambda)$ by solving

$$(\mathbb{I}-\ell)\mathbb{F}(T,u(T,\lambda)) = 0, \quad \ell u(T,\lambda) = \lambda\hat{u}, \tag{11}$$

where $\hat{u}$ is a fixed nonzero function in the range of ℓ. For $\hat{u}$, we choose a pair of Chebyshev polynomials that approximate the eigenvector of $D\mathbb{F}(T,\cdot)$ corresponding to the eigenvalue closest to zero. The projection ℓ is defined by

$$\ell u = \ell_0(u)\hat{u}, \quad \ell_0(u) = \sum_{i=1,2}\sum_{j=0}^{N} u_{ij}\hat{u}_{ij},$$

where u_{ij} and $\hat{u}_{ij}$ are the Chebyshev coefficients of u and $\hat{u}$, respectively. Our goal is to show that for a rectangle $I \times J$ in the parameter space, equation (11) has a smooth and locally unique solution $u\colon I \times J \to \mathcal{A}^2$. Then, locally, the solutions of $\mathbb{F}(T,u) = 0$ are determined by the zeros of the function g, defined by

$$g(T,\lambda) = \ell_0\mathbb{F}(T,u(T,\lambda)).$$

We write all the coefficients in the Chebyshev expansion of u as Taylor polynomials in T,λ:

$$u(T,\lambda) = \sum_j (u_{1j},u_{2j})T_j, \quad u_{ij} = \sum_{0\le l+m\le M} u_{ijlm}\Big(\frac{T-T_0}{T_1}\Big)^l\Big(\frac{\lambda}{\lambda_1}\Big)^m, \tag{12}$$

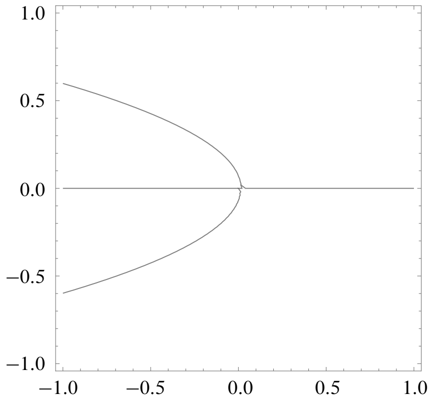
where T_0, T_1, λ_1, M are given in Table 1.

Equation (11) for $u = u(T,\lambda)$ is equivalent to the fixed point equation for the map $\mathbb{F}_{T,\lambda}$, defined by

$$\mathbb{F}_{T,\lambda}(u) = (\mathbb{I}-\ell)\mathbb{F}_T(u) + \lambda\hat{u}.$$

As in the last subsection, we use the contraction mapping principle to solve this fixed point problem. In place of the map defined in (10), we use the map $\mathcal{M}_{T,\lambda}$ defined by

$$\mathcal{M}_{T,\lambda}(h) = \mathbb{F}_{T,\lambda}(\bar{u} + \Lambda_{T,\lambda}h) - \bar{u} + M_{T,\lambda}h, \quad \Lambda_{T,\lambda} = \mathbb{I} - M_{T,\lambda}.$$



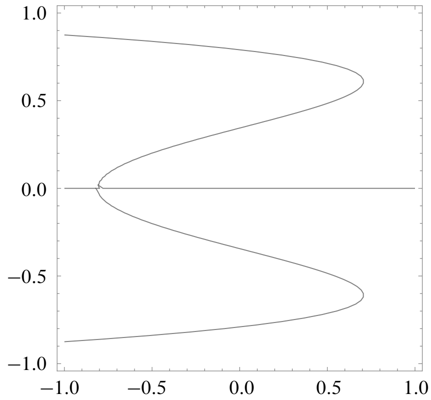


Figure 5. Approximate zeros of the functions $g(T,\lambda)$ for $i = 1$ (left) and $i = 2$ (right).

Here, $\bar{u}$ is an approximate fixed point of $\mathbb{F}_{T_0,0}$, and $M_{T,\lambda}$ is a finite rank operator such that $\Lambda_{T,\lambda} = \mathbb{I} - M_{T,\lambda}$ is an approximate inverse of $\mathbb{I} - D\mathbb{F}_{T_0,0}(\bar{u})$. By the implicit function theorem, the solution depends analytically on the two parameters T and λ. Denote by $D_r(z)$ a disk in the complex plane of radius r and center z. The following lemma is proved with the aid of a computer, see Section 6.

Lemma 4.1. *For $i = 1,2$, let T_0, T_1, λ_1, M be as in Table 1, $I = D_{T_1}(T_0)$ and $J = D_{\lambda_1}(0)$. There exists a Taylor–Chebyshev polynomial $\hat{u}$, a Chebyshev polynomial $\bar{u}(T,\lambda)$ as in (12), and positive constants ε, r, K satisfying $\varepsilon + Kr < r$, such that*

$$\|\mathcal{M}_{T,\lambda}(0)\| \leq \varepsilon, \quad \|D\mathcal{M}_{T,\lambda}(v)\| \leq K.$$

for all $v \in B_r(0)$ and for all $T \in I$ and all $\lambda \in J$.

As we had in the previous subsection, this lemma, together with the Contraction Mapping Theorem and the Implicit Function Theorem, implies the following:

Proposition 4.2. *For $i = 1,2$, let T_0, T_1, λ_1, M be as in Table 1, and let $I = D_{T_1}(T_0)$ and $J = D_{\lambda_1}(0)$. For every (T,λ) in $I \times J$, equation (11) has a unique solution $u(T,\lambda)$ in $B_r(\bar{u}(T,\lambda))$, and the map $(T,\lambda) \mapsto u = u(T,\lambda)$ is analytic. For any given real $T \in I$, a function u in $B \cap \ell^{-1}(J\hat{u})$ is a fixed point of F_T if and only if $u = u(T,\lambda)$ for some real $\lambda \in J$, and $g(T,\lambda) = \ell_0 u(T,\lambda) = 0$.*

This leaves the problem of verifying that the zeros of g correspond to a bifurcation. Figure 5 represents the approximate zeros of the functions g obtained in Proposition 4.2. Figure 5 clearly suggests the existence of a pitchfork bifurcation (left), and a pitchfork bifurcation together with two fold bifurcations (right). Note that the straight line in the graph on the right represents the trivial solution $(y,\theta) = 0$, and since we know from a direct computation that the $D\mathbb{F}(T,\cdot)$ is not invertible at $T = \pi/\sqrt{2}$, the apparent pitchfork bifurcation occurs exactly at that particular value of T. A sufficient set of conditions for the existence of a fold or a pitchfork bifurcation is the following, see [8] for the proofs.

Let g be a differentiable function of two variables, and denote by $\dot{g}$ and g' the partial derivatives of g with respect to the first and second argument. Let $I = [T_1, T_2]$ and $J = [-b, b]$.

Lemma 4.3 (Fold bifurcation). *Let g be a real-valued C^3 function on an open neighborhood of $I \times J$ such that $g(T,0) = 0$ for all $T \in I$, and*

(1) $g'' > 0$ *on* $I \times J$,

(2) $\dot{g} < 0$ *on* $I \times J$,

(3) $g(T_1, 0) \pm \frac{1}{2} b g'(T_1, 0) > 0$,

(4) $g(T_2, \pm b) > 0$,

(5) $g(T_2, 0) < 0$.

Then the solution set of $g(T,\lambda) = 0$ in $I \times J$ is the graph of a C^2 function $T = a(\lambda)$, defined on a proper subinterval J_0 of J. This function takes the value T_2 at the endpoints of J_0, and satisfies $T_1 < a(z_2) < T_2$ at all interior points of J_0, which includes the origin.

Lemma 4.4 (Pitchfork bifurcation). *Let g be a real-valued C^3 function on an open neighborhood of $I \times J$ such that $g(T,0) = 0$ for all $T \in I$, and*

(1) $g''' > 0$ *on* $I \times J$,

(2) $\dot{g}' < 0$ *on* $I \times J$,

(3) $g'(T_1, 0) \pm \frac{1}{2} b g''(T_1, 0) > 0$,

(4) $\pm g(T_2, \pm b) > 0$,

(5) $g'(T_2, 0) < 0$.

Then $g(T,\lambda) = \lambda G(T,\lambda)$ for some C^2 function G, and the solution set of $G(T,\lambda) = 0$ in $I \times J$ is the graph of a C^2 function $T = a(\lambda)$, defined on a proper subinterval J_0 of J. This function takes the value T_2 at the endpoints of J_0, and satisfies $T_1 < a(z_2) < T_2$ at all interior points of J_0, which includes the origin.

We also need the following lemma, whose proof is straightforward, in order to prove that the pitchfork and the fold bifurcation are connected with a continuous branch of solutions.

Lemma 4.5. *Let g be a real-valued C^3 function on an open neighborhood of $I \times J$, and*

(1) $\dot{g} > 0$ *on* $I \times J$,

(2) *$g' < 0$ on $I \times J$,*

(3) *$\pm g'(T_1, \pm b) < 0$,*

(4) *$\pm g(T_2, \pm b) > 0$,*

Then the solution set of $g(T,\lambda) = 0$ in $I \times J$ is the graph of a C^2 function $\lambda = a(T)$, defined on J, with $a'(T) > 0$ for all T.

The following lemmas are proved with the aid of a computer. Essentially, they imply that the zeros of the functions $\ell_0 u(T,\lambda)$ look as they are represented in Figure 5. The first lemma concerns the pitchfork bifurcation at $T = 1.953\ldots$, which connects the three branches of the solution mentioned in Proposition 3.2.

Lemma 4.6. *Consider the solution $u(T,\lambda)$ obtained in Proposition 4.2 in the case $i = 1$, and let $I = [-1.95367\ldots, -1.95233\ldots]$ (the exact values are available in the program files) and $J = [-2^{-5}, 2^{-5}]$. For any $u(T,\lambda) \in B_r(\bar{u}(T,\lambda))$, the function $g(T,\lambda) = \ell_0 u(-T,\lambda)$ satisfies the assumptions of Lemma 4.4.*

The second lemma concerns both a pitchfork and a fold bifurcation, both at T close to $2.222\ldots$. The main branch of the pitchfork bifurcation is the trivial solution. The secondary branches reach a fold. Let $T_1 = 2.22133\ldots$, $T_2 = 2.22185\ldots$, $T_3 = 2.22239\ldots$, $T_4 = 2.22303\ldots$, $T_5 = 2.22343\ldots$, $\lambda_1 = -0.070\ldots$, $\lambda_2 = 0.0615\ldots$, $\lambda_3 = 0.1889\ldots$, $\lambda_4 = 0.0834\ldots$, $\lambda_5 = 0.149\ldots$, $\lambda_6 = 0.228\ldots$, (the exact values are available in the program files).

Lemma 4.7. *Consider the solution $u(T,\lambda)$ obtained in Proposition 4.2 in the case $i = 2$.*

- Pitchfork: *Let $I = [T_1, T_2]$ and $J = [-\lambda_1, \lambda_1]$. For any $u(T,\lambda) \in B_r(\bar{u}(T,\lambda))$, the function $g(T,\lambda) = \ell_0 u(T,\lambda)$ satisfies the assumptions of Lemma 4.4.*
- Connecting branch: *For any $u(T,\lambda) \in B_r(\bar{u}(T,\lambda))$, the function $g(T,\lambda) = \ell_0 u(T,\lambda)$ satisfies the assumptions of Lemma 4.5 both in $I \times J = [T_2, T_3] \times [\lambda_2, \lambda_3]$ and in $I \times J = [T_3, T_4] \times [\lambda_4, \lambda_3]$.*
- Fold: *Let $I = [-T_5, -T_4]$ and $J = [\lambda_5, \lambda_6]$. For any $u(T,\lambda) \in B_r(\bar{u}(T,\lambda))$, the function $g(T,\lambda) = \ell_0 u(-T,\lambda)$ satisfies the assumptions of Lemma 4.3.*

To conclude the proof, we need to check that the branches of solutions obtained in Section 3 are connected to the bifurcation points. This is the result of the following lemmas, also proved with computer assistance.

Lemma 4.8. *Call u_A, u_B the (y,θ) components of the solutions obtained in Lemma 3.1 at $T_{A,B} = 1.952\ldots$ for the branches A and B, call u_C the (y,θ) components of the solution at $T_C = 1.954\ldots$ for the branch C. Let $B_r(\bar{u}(T,\lambda))$ the enclosure of the solution obtained in the case $i = 1$ in Lemma 4.1. Then $u_A \in B_r(\bar{u}(T_A,\lambda))$, $u_B \in B_r(\bar{u}(T_B,\lambda))$, $u_C \in B_r(\bar{u}(T_C,\lambda))$.*

Lemma 4.9. *Call u_C the (y,θ) components of the solution at $T_C = 2.22\ldots$ for the branch C. Let $B_r(\bar{u}(T,\lambda))$ be the enclosure of the solution obtained in the case $i = 2$ in Lemma 4.1. Then $u_C \in B_r(\bar{u}(T_C,\lambda))$.*

5 Proofs of Theorems 1.1 and 1.2

By Lemma 4.7, there exists a pitchfork bifurcation from the trivial solution at $T = \pi/\sqrt{2}$, and the secondary branch is defined for $T \in [\pi/\sqrt{2}, \tilde{T}]$, with $\tilde{T} = 2.223\ldots$, and at $T = \tilde{T}$, a fold bifurcation occurs, so that the branch continues for decreasing values of T. Because of Lemmas 3.1 and 4.9, this branch exists for $T \in [1.954\ldots, \tilde{T}]$. By Lemma 4.8, the branch is connected to the pitchfork bifurcation proved in Lemma 4.6. The remaining branches A and B, whose existence is proved in Lemma 3.1, are also connected to the primary and secondary branches of the pitchfork bifurcation because of Lemma 4.8. The remaining branches shown in Figure 2 are obtained by symmetry, $(y,\theta) \mapsto (y,-\theta)$.

6 The computer assisted part of the proof

The methods used here can be considered in perturbation theory: given an approximate solution, prove bounds that guarantee the existence of a true solution nearby. But the approximate solutions needed here are too complex to be described without the aid of a computer, and the number of estimates involved is far too large. The first part (finding approximate solutions) is a strictly numerical computation. The rigorous part is still numerical, but instead of truncating series and ignoring rounding errors, it produces guaranteed enclosures at every step along the computation. This part of the proof is written in the programming language Ada.[1] The following is meant to be a rough guide for the reader who wishes to check the correctness of the programs. The complete details can be found in [1].

Note 3: New footnote and the corresponding bibliographic entry removed.

In the present context, a "bound" on a map $f\colon \mathcal{X} \to \mathcal{Y}$ is a function F that assigns to a set $X \subset \mathcal{X}$ of a given type (`Xtype`) a set $Y \subset \mathcal{Y}$ of a given type (`Ytype`), in such a way that $y = f(x)$ belongs to Y for all $x \in X$. In Ada, such a bound F can be implemented by defining a `procedure F(X: in Xtype; Y: out Ytype)`.

To represent balls in a real Banach algebra $\mathcal{X}$ with unit $\mathbf{1}$, we use the pair `S` = (`S.C`,`S.R`), where `S.C` is a representable number (`Rep`) and `S.R` a nonnegative representable number (`Radius`). The corresponding ball in $\mathcal{X}$ is $\langle \texttt{S}, \mathcal{X}\rangle = \{x \in \mathcal{X} : \|x - (\texttt{S.C})\mathbf{1}\| \le \texttt{S.R}\}$.

[1] See the Ada reference manual, ISO/IEC 8652:2012 (E).

* In the case $\mathcal{X} = \mathbb{R}$, the data type described above is called `Ball`. Our bounds on some standard functions involving the type `Ball` are defined in the packages `Flts_Std_Balls`. Other basic functions are covered in the packages `Vectors` and `Matrices`. Bounds of this type have been used in many computer-assisted proofs; so we focus here on the more problem-specific aspects of our programs.

The computation and validation of branches involves Taylor series in one variable, which are represented by the type `Taylor1` with coefficients of type `Ball`. The definition of the type and its basic procedures are in the package `Taylors1`. A description of the type and its implementation can be found in [2]. The computation and validation of the bifurcations involves Taylor series in two variables, which are represented by the type `Taylor2` with coefficients of type `Ball`. The definition of the type and its basic procedures are in the package `Taylors2`. The structure of `Taylors2` is similar to `Taylors2`.

Consider now the space $\mathcal{A}$. Functions in $\mathcal{A}$ are represented by the type `Cheb` defined in the package `Chebs`, which accepts coefficients in some Banach algebra with unit $\mathcal{X}$. In our application the coefficients of `Cheb` are `Taylor1` or `Taylor2`. The type `Cheb` consists of a triple $\texttt{F} = (\texttt{F.C},\texttt{F.E},\texttt{F.R})$, where `F.C` is an `array(0..K) of Ball`, `F.E` is an `array(0..2*K) of Radius`, and `F.R` is type `Radius`, which represents the domain of analyticity of functions in $\mathcal{A}$, that is, $\texttt{F.R} = 5/4$. The corresponding set $\langle\texttt{F},\mathcal{A}\rangle$ is the set of all function $u = p + h \in \mathcal{A}$, where

$$p(t) = \sum_{j=0}^{K} \langle \texttt{F.C(J)}, \mathcal{X}\rangle T_j(t), \quad h = \sum_{j=0}^{2K} h^j, \quad h^j(t) = \sum_{m\ge j} h^j_m T_m(t),$$

with $\|h^J\| \le \texttt{F.E(J)}$, for all `J`. For the operations that we need in our proof, this type of enclosure allows for simple and efficient bounds. A bound on the map $\mathbb{F}_T$ is implemented by the procedure `GMap` in the package `Taylors1.Cheb.Fix`. Defining and estimating a contraction like $\mathcal{M}_T$ is a common task in many of our computer-assisted proofs. An implementation is done via two generic packages, `Linear` and `Linear.Contr`. For a description of this process we refer to [9].

A The Poincaré map and its derivative

Consider the n-dimensional system

$$\dot{x} = f(x), \quad x = (x_1, \ldots, x_n), \quad f = (f_1, \ldots, f_n), \tag{13}$$

and let $x(t)$ be a periodic solution such that $x(0) = x(T) = 0$. Let $y(t)$ be a solution of the variational equation

$$\dot{y} = \nabla f(x) y, \quad \nabla f = \left(\frac{\partial f_i}{\partial x_j}\right), \quad y(0) = \mathrm{Id}.$$

Let $\varphi(t,x)$ be the flow of (13), so that $\nabla_x \varphi(t,x) = y(t)$. Consider the Poincaré section $x_1 = 0$, and let $x_{-1} = \{x_2, \ldots, x_n\}$. The time of first return to the Poincaré section $t(x_{-1})$ satisfies

$$\varphi_1(t(x_{-1}), (0, x_{-1})) = 0.$$

Then, for all $j = 2, \ldots, n$, we have

$$\begin{aligned} 0 &= \partial_j \varphi_1(t(x_{-1}), (0, x_{-1})) \\ &= \partial_t \varphi_1(t(x_{-1}), (0, x_{-1}))\, \partial_j t(x_{-1}) + \partial_j \varphi_1(t(x_{-1}), (0, x_{-1})), \end{aligned}$$

where $\partial_j = \partial / \partial x_j$, so that

$$\partial_j t(x_{-1})|_{x_{-1}=0} = -\frac{\partial_j \varphi_1(0,T)}{\partial_t \varphi_1(T,0)} = -\frac{y_{1j}(T)}{f_1(0)}.$$

The Poincaré map is defined as

$$P(x_{-1}) = \big(\varphi_2(t(x_{-1}), (0, x_{-1})), \ldots, \varphi_n(t(x_{-1}), (0, x_{-1}))\big),$$

therefore

$$\partial_j P_i(x_{-1})|_{x_{-1}=0} = \partial_t \varphi_i(T,0)\, \partial_j t(x_{-1}) + y_{ij}(T) = y_{ij}(T) - y_{1j}(T) \frac{f_i(0)}{f_1(0)}.$$

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Note 4: The bibliography has been crosschecked and updated with MathSciNet. Since entries may be mismatched, please carefully double-check each one.

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