

## OPTIMAL TREATMENT FOR A PHASE FIELD SYSTEM OF CAHN-HILLIARD TYPE MODELING TUMOR GROWTH BY ASYMPTOTIC SCHEME

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**ABSTRACT.** We consider a particular phase field system which physical context is that of tumor growth dynamics. The model we deal with consists of a Cahn-Hilliard equation governing the evolution of the phase variable which takes into account the tumor cells proliferation in the tissue coupled with a reaction-diffusion equation for the nutrient. This model has already been investigated from the viewpoint of well-posedness, long-time behavior, and asymptotic analyses as some parameters go to zero. Starting from these results, we aim to face a related optimal control problem by employing suitable asymptotic schemes. In this direction, we assume some quite general growth conditions for the involved potential and a smallness restriction for a parameter appearing in the system we are going to face. We provide the existence of optimal controls and a necessary condition for optimality is addressed.

**1. Introduction.** Over the last decades, there has been increasing attention by the mathematical community towards biological models for tumor growth (see [21]). Among them, the ones introduced by exploiting phase field approaches and continuum mixture theory cover an important role. The key idea consists in reading the physical evolution process like an interaction between two particular fluids which are designed to model the tumor cells and the healthy ones. In this regards, we especially point out two classes. The first one gives rise to the so-called Cahn-Hilliard-Darcy or Cahn-Hilliard-Brinkman systems which describe the tumor and healthy cells as inertia-less fluids including effects generated by fluid flow development. To this concerns, let us refer to [22, 29, 32, 33, 35, 53]. The second class neglects the velocity and consists of a Cahn-Hilliard equation (see, e.g., [43] and the huge references therein) for the phase variable coupled with a reaction-diffusion equation for the nutrient. Moreover, let us point out the papers [30, 31], where transport mechanisms such as chemotaxis and active transport are also taken into account. Further investigations and mathematical models related to biology can be found, e.g., in [27, 28].

Here, we try to describe the main purpose of the work postponing the technicalities and the investigation of the proper assumptions that will be specified in the

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forthcoming section. Before moving on, let us mention that with  $\Omega \subset \mathbb{R}^3$  we denote the set where the evolution takes place and, for a given final time  $T > 0$ , we define the standard parabolic cylinder and its boundary by

$$\begin{aligned} Q_t &:= \Omega \times (0, t), & \Sigma_t &:= \partial\Omega \times (0, t) \quad \text{for every } t \in (0, T], \\ Q &:= Q_T, & \Sigma &:= \Sigma_T. \end{aligned} \tag{1.1}$$

Hence, the model we are going to consider reads as follows

$$\alpha \partial_t \mu_\beta + \partial_t \varphi_\beta - \Delta \mu_\beta = P(\varphi_\beta)(\sigma_\beta - \mu_\beta) \quad \text{in } Q \tag{1.2}$$

$$\mu_\beta = \beta \partial_t \varphi_\beta - \Delta \varphi_\beta + F'(\varphi_\beta) \quad \text{in } Q \tag{1.3}$$

$$\partial_t \sigma_\beta - \Delta \sigma_\beta = -P(\varphi_\beta)(\sigma_\beta - \mu_\beta) + u_\beta \quad \text{in } Q \tag{1.4}$$

$$\partial_n \mu_\beta = \partial_n \varphi_\beta = \partial_n \sigma_\beta = 0 \quad \text{on } \Sigma \tag{1.5}$$

$$\mu_\beta(0) = \mu_0, \varphi_\beta(0) = \varphi_0, \sigma_\beta(0) = \sigma_0 \quad \text{in } \Omega, \tag{1.6}$$

for some positive constants  $\alpha$  and  $\beta$ . Let us emphasize that the notation  $\varphi_\beta$ , instead of the simplest  $\varphi$ , and the same goes for the other variables, is motivated by the fact that in the following we are going to let  $\beta \searrow 0$  and we will denote as  $\varphi$  the limit of  $\varphi_\beta$ . So, with the subscript  $\beta$ , we aim to stress the fact that such a variable corresponds to the system with  $\beta > 0$ .

The above system is a simplified version of the diffuse interface model originally introduced by Hawkins-Daruud et al. in [38], where the velocity and chemotaxis contributions are neglected (see also [20, 46, 37, 39, 54]) and it also includes some regularizing parameters. It is worth spending some words explaining the physical meaning of the model. The unknown  $\varphi_\beta$  is an order parameter and it is devoted to keeping track of the evolution of the tumor in the tissue. It is usually normalized between  $-1$  and  $+1$ , where these extreme values represent the pure phases, that is the tumor phase and the healthy cell phase, respectively. The second unknown  $\mu_\beta$ , as usual for Cahn-Hilliard equation, stands for the chemical potential for  $\varphi_\beta$ . Finally, the last unknown  $\sigma_\beta$  represents the nutrient-rich extracellular water volume fraction. It takes values between  $0$  and  $1$  with the following property: the closer to one, the richer of water the extracellular fraction is, while the closer to zero, the poorer it is. Furthermore,  $u_\beta$  is the so-called control variable which will allow us to interact in some sense with the above system. As usual in control theory, the system in which the control variable appears is referred to as state system. Likewise, we will refer to the solution  $(\mu_\beta, \varphi_\beta, \sigma_\beta)$  to as the associated state. As far as  $P$  and  $F$  are concerned, they are nonlinearities. The former is a proliferation function, while the latter is a double-well potential. Customary examples for  $F$  are the classical regular potential, the logarithmic potential, and the double-obstacle one. In this contribution, we focus the attention on the first one which is given by

$$F_{reg}(r) := \frac{1}{4}(r^2 - 1)^2 = \frac{1}{4}((r^2 - 1)^+)^2 + \frac{1}{4}((1 - r^2)^+)^2 \quad \text{for } r \in \mathbb{R}. \tag{1.7}$$

For different physically meaningful choices of the potentials we refer to [1], and to the references therein, where several numerical applications to tumor growth can be found as well.

We point out that the above model has been quite well-understood owing to the previous works [8, 11, 13] as far as well-posedness and long-time behavior are concerned. Moreover, in [26] the analyses of the same model without the relaxation terms  $\alpha \partial_t \mu_\beta$  and  $\beta \partial_t \varphi_\beta$ , has been performed considering regular potentials and allowing  $P$  to possess polynomial growth. Besides, as long-time behavior of

solutions are concerned, we also mention [40], where the author extends the well-posedness results proved in [11, 13], as  $\beta \searrow 0$ , to the case of unbounded domains. Moreover, we refer to [44], where the authors investigate the long-time behavior of the non-relaxed version of system 1.2–1.6, i.e. the case  $\alpha = \beta = 0$ , showing the existence of the global attractor in proper phase space. In view of such flourishing literature, a further natural aim is to investigate some corresponding optimal control problems in which the state system is governed by the evolution system 1.2–1.6. In this direction, we refer to the recent work [49], where, making extensive use of the terms  $\alpha \partial_t \mu_\beta$  and  $\beta \partial_t \varphi_\beta$ , an optimal control problem for the above system is tackled in a general framework for the potential, allowing both the classical and the logarithmic potential to be considered. Additionally, the same author proves in the subsequent work [48], via a proper asymptotic scheme known in the literature as to deep quench limit, that it is also possible to generalize the assumptions for the potentials in order to take into account also singular and nonregular potentials like the double-obstacle. Furthermore, let us refer to [12], where a similar optimal control problem is considered for the state system 1.2–1.6 without these relaxation terms. In addition, regarding some optimal control problem in which time is taken into account, we address the recent [5], where also long-time behavior of solutions has been investigated, and we also mention [34], where an optimal time therapeutic treatment has been investigated. Finally, we point out the contributions [23, 24] in which the authors investigate an optimal control problem for different tumor growth model based on the Cahn-Hilliard-Brinkman equation, which was previously investigated in [25], pointing out also some sufficient conditions for the optimality. Lastly, let us mention [51], where a distributed optimal control problem for the Cahn-Hilliard-Darcy system with mass source was studied.

Here, we actually aim to study an intermediate optimal control problem between [12] and [49]. In fact, we still consider the state system to be 1.2–1.6, but without considering the relaxation term  $\beta \partial_t \varphi_\beta$ . Let us emphasize the mathematical interest of this problem. On the one hand, the non-relaxed version of system 1.2–1.6 was investigated in [26] and the corresponding control problem was then tackled in [12]. On the other hand, the relaxed model was studied in [8]. There, due to the stronger regularity, the authors can take into account very general potentials. Then, by [11, 13], the asymptotic analysis as  $\alpha$  and  $\beta$  go to zero was performed. Moreover, the corresponding control problem was recently treated by the author in [49]. Hence, it will be interesting to understand how the optimal control problems associated with these similar models behave. Furthermore, we refer to [47], where the author, focus the attention on the case  $\alpha \searrow 0, \beta > 0$ .

As for the control problem, we are going to take into account the following tracking-type cost functional

$$\begin{aligned} \mathcal{J}(\varphi, \sigma, u) := & \frac{b_1}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{b_2}{2} \|\sigma - \sigma_Q\|_{L^2(Q)}^2 + \frac{b_3}{2} \|\sigma(T) - \sigma_\Omega\|_{L^2(\Omega)}^2 \\ & + \frac{b_0}{2} \|u\|_{L^2(Q)}^2, \end{aligned} \quad (1.8)$$

and the control-box constraints

$$\mathcal{U}_{ad} := \{u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q\}, \quad (1.9)$$

where  $u_*$  and  $u^*$  are functions that fix the admissible set in which the control variable  $u$  can be chosen. Furthermore,  $b_0, b_1, b_2, b_3$  stand for nonnegative constants, not all zero, while  $\varphi_Q, \sigma_Q, \sigma_\Omega$  denote some target functions defined in  $Q$  and  $\Omega$ , respectively.

Since our starting point is [49], we will refer several times to the results there proved. So, it is worth noting that the cost functional (1.8) is slightly less general with respect to the one there proposed. There, an additional term of the form  $\frac{k}{2}\|\varphi(T) - \varphi_\Omega\|_{L^2(\Omega)}^2$  also appears, for a nonnegative constant  $k$  and a prescribed function  $\varphi_\Omega$  which models the final configuration of the tumor colony. From a control viewpoint, this contribution allows us to force the final configuration of the tumor to be as close as possible, in the sense of  $L^2(\Omega)$ -norm, to the fixed target  $\varphi_\Omega$ . Here, we restrict the investigation to the case  $k = 0$ . This will be motivated by the analysis of the corresponding adjoint problem that leads to assume such a compatibility condition. To motivate this limitation, let us recall that (see [49, Syst. (2.22)-(2.26)]) the final conditions of the adjoint problem pointed out in [49] are the following (note that the constant  $k$  is called  $b_2$  in that paper)

$$p(T) - \beta q(T) = k(\bar{\varphi}(T) - \varphi_\Omega), \quad \alpha p(T) = 0.$$

However, if  $\alpha > 0$  and  $\beta = 0$ , we formally deduce that

$$\begin{cases} p(T) = k(\bar{\varphi}(T) - \varphi_\Omega) \\ \alpha p(T) = 0, \end{cases}$$

which yields  $p(T) = 0$  and therefore also that  $0 = k(\bar{\varphi}(T) - \varphi_\Omega)$ , which is not satisfied in general since  $\varphi_\Omega$  is arbitrary. Hence, to not lead to a contradiction we assume  $k = 0$ , so that the choice of the less general cost functional (1.8) is now justified.

Thus, the optimal control treated in [49] consists of solving the problem:

**(CP) $_\beta$**  Minimize  $\mathcal{J}(\varphi, \mu, u)$  subject to the control constraints (1.9) and under the requirement that the variables  $(\varphi, \sigma)$  yield a solution to 1.2–1.6.

There, the author confirmed the existence of, at least, one optimal control and also provide some first-order optimality conditions reading as variational inequalities.

Moreover, let us recall that the asymptotic analysis for system 1.2–1.6 has already been investigated in [8, 11, 13], where the authors carefully point out some sufficient conditions to let  $\alpha$  and  $\beta$  go to zero, both sequentially and separately. As a matter of fact, they proved that, as  $\beta \searrow 0$ , which is the case we are going to consider, providing to require additional assumptions, the unique solution to system 1.2–1.6 converges to some limit which yields a solution to the following problem

$$v^* \langle \partial_t(\alpha\mu + \varphi), v \rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega P(\varphi)(\sigma - \mu)v \quad \forall v \in V, \text{ a.e. in } (0, T) \quad (1.10)$$

$$\mu = -\Delta\varphi + F'(\varphi) \quad \text{in } Q \quad (1.11)$$

$$\partial_t\sigma - \Delta\sigma = -P(\varphi)(\sigma - \mu) + u \quad \text{in } Q \quad (1.12)$$

$$\partial_n\mu = \partial_n\varphi = \partial_n\sigma = 0 \quad \text{on } \Sigma \quad (1.13)$$

$$(\alpha\mu + \varphi)(0) = \alpha\mu_0 + \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \quad (1.14)$$

Furthermore, it was shown that, under a suitable smallness requirement on  $\alpha$ , that will be precised later on, the solution is indeed unique.

From a different perspective, one can take the above system as a starting point, trying to face the analysis of the corresponding optimal control problem. Namely, one can try to solve the following minimization problem: minimize  $\mathcal{J}(\varphi, \sigma, u)$  subject to the control constraints (1.9) and under the requirement that the variables

$(\varphi, \sigma)$  are solutions to 1.10–1.14. Actually, this is the optimal control problem we try to tackle by following a different strategy consisting of pass to the limit, as  $\beta \searrow 0$ , in the optimal control problem  $(CP)_\beta$ . This technique turns out to be particularly interesting since we still will obtain similar results with respect to [49]. At the same time, we will treat the optimal control problem avoiding the investigation of the linearized system, which is usually not so difficult and, mostly, we can avoid the discussion on the Fréchet differentiability of the associated control-to-state mapping, which is usually more challenging.

On the other hand, the first-order necessary condition of  $(CP)$  cannot be directly obtained by letting  $\beta \searrow 0$  in the optimality condition for the corresponding optimal control problem with  $\beta > 0$ . This would be the case if we ensure that every optimal control for  $(CP)$  can be recovered as limit of sequence of optimal controls for  $(CP)_\beta$ , which is quite a strong requirement. Unfortunately, we are unable to prove such a global result. However, a partial one can be stated localizing the problem by following the idea firstly introduced by Barbu in [2] (see also, e.g., [6, 7, 18, 48], where such a technique was applied). The idea consists in locally perturbing the problem  $(CP)_\beta$  in order to find the desired approximation result. For this purpose, the main ingredient is the so-called adapted cost functional that is defined as follows

$$\tilde{\mathcal{J}}(\varphi, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2, \quad (1.15)$$

where  $\bar{u}$  stands for an optimal control for  $(CP)$ . Thus, we are naturally lead to solving the following minimization problem:

$(\widetilde{CP})_\beta$  Minimize  $\tilde{\mathcal{J}}(\varphi, \mu, u)$  subject to the control constraints (1.9) and under the requirement that the variables  $(\varphi, \sigma)$  yield a solution to 1.2–1.6.

Next, instead of looking for approximating sequence of optima for  $(CP)_\beta$ , one take a sequence of controls which are optimal for  $(\widetilde{CP})_\beta$ . This technique will allow us to properly let  $\beta \searrow 0$  in the optimality condition for  $(\widetilde{CP})_\beta$  to recover the variational inequality which characterizes the necessary condition for optimality of  $(CP)$ .

Summing up, the current contribution has the purpose of showing, by means of asymptotic approaches, that the following control problem admits a solution:

$(CP)$  Minimize  $\mathcal{J}(\varphi, \mu, u)$  subject to the control constraints (1.9) and under the requirement that the variables  $(\varphi, \sigma)$  yield a solution to 1.10–1.14.

Moreover, we will also provide a necessary condition that an optimal control has to satisfy in terms of a variational inequality.

Lastly, let us sketch the physical background of the control problem we are dealing with. Roughly speaking, we are looking for the best choice  $u$  in such a way that, with the corresponding solution to 1.10–1.14, it minimizes the cost functional  $\mathcal{J}$ . Furthermore, the control  $u$  appears in equation (1.12), the one describing the evolution of the nutrient. Therefore, from the model viewpoint, it can be read as a supply of a nutrient or a drug in medical treatment. Moreover, for some given a priori targets  $\varphi_Q, \sigma_Q, \sigma_\Omega$ , minimizing the cost functional  $\mathcal{J}$  corresponds to force the system to approach a prescribed targets which should be taken as desirable configurations for clinical reasons, e.g., for surgery. In addition, the ratios among the constants  $b_0, b_1, b_2, b_3$  implicitly describe which targets hold the leading part in our application and the last term of the cost functional represents the cost we have to pay to take  $u$  into account. In fact, it should be read as the rate of risks to

afflict harm to the patient by following that strategy, namely the side-effect that may occur if too many drugs are dispensed to the patient.

The plan for the rest of the paper is as follows. In Section 2, we set the notation we are going to use and recollect the obtained results. From Section 3 onward, we start with the proofs of the stated results. Furthermore, Section 3 is devoted to investigating the well-posedness and the asymptotic behavior, as  $\beta \searrow 0$ , of the state system. Lastly, in Section 4, we discuss the control problem (CP) by invoking some asymptotic schemes. We check the existence of optimal control, study the well-posedness of the adjoint system, and provide first-order necessary condition for optimality.

**2. Assumptions and main results.** In this section, we aim to set the notation and collect our results. To begin with, we recall that  $\Omega$  stands for the space domain where the evolution takes place and we assume it to be a bounded, connected, smooth, and open set of  $\mathbb{R}^3$ , with boundary indicated by  $\Gamma$ . As the functional spaces are concerned, it turns out to be convenient to introduce the following

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\},$$

where  $\partial_n$  stands for the outward normal derivative. Furthermore, to work with Banach spaces, we endow them with their standard norms. For a general Banach space  $X$ , we use  $\|\cdot\|_X$  to designate its norm,  $X^*$  for its topological dual, and  ${}_{X^*}\langle \cdot, \cdot \rangle_X$  for the duality product between  $X^*$  and  $X$ . Likewise, we use the symbol  $\|\cdot\|_p$  for the usual norm in  $L^p(\Omega)$ . The above definitions, in turn, imply that  $(V, H, V^*)$  constitutes a Hilbert triplet, that is, the following injections  $V \subset H \equiv H^* \subset V^*$  are both continuous and dense and we have the standard identification between the dual product of  $V$  and the inner product of  $H$ . Namely, it holds that

$${}_{V^*}\langle u, v \rangle_V = \int_{\Omega} uv \quad \text{for every } u \in H \text{ and } v \in V.$$

As for the basic assumptions for the system 1.2–1.6 and for the cost functional (1.8), we postulate that

$$\alpha, \beta > 0. \tag{2.1}$$

$$b_0, b_1, b_2, b_3 \text{ are nonnegative constants, but not all zero.} \tag{2.2}$$

$$\varphi_Q, \sigma_Q \in L^2(Q), \sigma_{\Omega} \in H^1(\Omega), u_*, u^* \in L^{\infty}(Q) \text{ with } u_* \leq u^* \text{ a.e. in } Q. \tag{2.3}$$

$$P \in C^2(\mathbb{R}) \text{ is nonnegative, bounded and Lipschitz continuous.} \tag{2.4}$$

$$\varphi_0 \in H^3(\Omega) \cap W, \mu_0 \in H^1(\Omega), \sigma_0 \in H^1(\Omega). \tag{2.5}$$

Furthermore, we employ the following notation

$\mathcal{U}_R \subset L^2(Q)$  be a non-empty and bounded open set such that it contains  $\mathcal{U}_{ad}$  and  $\|u\|_2 \leq R$  for all  $u \in \mathcal{U}_R$ .

For the potential setting, we require that  $D(\widehat{B}) = \mathbb{R}$  and that

$$\widehat{B} : \mathbb{R} \rightarrow [0, +\infty) \text{ is convex and lower semicontinuous, with } 0 \in B(0). \tag{2.6}$$

$$\widehat{\pi} \in C^1(\mathbb{R}) \text{ is nonnegative, } \pi := \widehat{\pi}' \text{ is Lipschitz continuous with Lipschitz constant } L, \text{ i.e. } \|\pi'\|_{L^{\infty}(\mathbb{R})} \leq L. \tag{2.7}$$

Then, we define the potential  $F$ , and its derivative as the sum of these two contributions by

$$F : \mathbb{R} \rightarrow [0, +\infty], \quad F := \widehat{B} + \widehat{\pi} \quad \text{and} \quad F' := B + \pi, \quad (2.8)$$

where  $B$  is a maximal and monotone graph  $B \subset \mathbb{R} \times \mathbb{R}$  defined as the subdifferential of  $\widehat{B}$ , that is,  $B := \partial \widehat{B}$ . Unfortunately, we are not able to face the asymptotic analyses, as  $\beta$  goes to zero, without assuming proper growth restrictions for the potential  $F$ . Some sufficient conditions for our purposes are as follows

$$F = \widehat{B} + \widehat{\pi} \quad \text{is a } C^3 \text{ function which satisfies} \quad (2.9)$$

$$|B(r)| \leq C_B(1 + \widehat{B}(r)) \quad \text{for every } r \in \mathbb{R}, \quad (2.10)$$

for a positive constant  $C_B$ . Anyhow, we emphasize that, although we cannot work at the utmost generality for the potentials setting, all polynomially growing potentials, as well as exponential functions, comply with the requirements above; in particular, (1.7) is allowed. Furthermore, by combining the embedding  $W \subset L^\infty(\Omega)$  with the first of the initial conditions (2.5), it is straightforward to infer that  $F(\varphi_0)$  belongs to  $L^\infty(\Omega)$ . It also follows from the above framework that  $F''$  is bounded below in terms of the Lipschitz constant  $L$ . Indeed, we have that

$$F'' \geq -L. \quad (2.11)$$

It is worth noting that, in the case of (1.7), we can take  $L = 1$ , as can be easily checked by computing its second derivative.

Now, we first recall some results already presented in other contributions and then list our statements. The already mentioned optimal control problem  $(CP)_\beta$  has been tackled in [49]. On the other hand, since the above setting perfectly fits with the one of [49], all the results there proved are at our disposal. There, the author, after showing the existence of optimal controls, provides the first-order necessary condition for optimality making use of the so-called adjoint system to 1.2–1.6. For the sake of simplicity, we just recall here the adjoint system there founded. From now on,  $\bar{u}_\beta$  denotes an optimal control for  $(CP)_\beta$ , and  $(\bar{\mu}_\beta, \bar{\varphi}_\beta, \bar{\sigma}_\beta)$  the corresponding optimal state. Thus, the adjoint system reads as follows

$$\begin{aligned} \beta \partial_t q_\beta - \partial_t p_\beta + \Delta q_\beta - F''(\bar{\varphi}_\beta) q_\beta + P'(\bar{\varphi}_\beta)(\bar{\sigma}_\beta - \bar{\mu}_\beta)(r_\beta - p_\beta) \\ = b_1(\bar{\varphi}_\beta - \varphi_Q) \quad \text{in } Q \end{aligned} \quad (2.12)$$

$$q_\beta - \alpha \partial_t p_\beta - \Delta p_\beta + P(\bar{\varphi}_\beta)(p_\beta - r_\beta) = 0 \quad \text{in } Q \quad (2.13)$$

$$- \partial_t r_\beta - \Delta r_\beta + P(\bar{\varphi}_\beta)(r_\beta - p_\beta) = b_2(\bar{\sigma}_\beta - \sigma_Q) \quad \text{in } Q \quad (2.14)$$

$$\partial_n q_\beta = \partial_n p_\beta = \partial_n r_\beta = 0 \quad \text{on } \Sigma \quad (2.15)$$

$$p_\beta(T) - \beta q_\beta(T) = 0, \quad \alpha p_\beta(T) = 0, \quad r_\beta(T) = b_3(\bar{\sigma}_\beta(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (2.16)$$

The well-posedness of the above system has been already established in [49]. So, we just recall the obtained result.

**Proposition 1.** *Assume that 2.1–2.10 are in force. Then, system 2.12–2.16 admits a unique solution satisfying*

$$q_\beta, p_\beta, r_\beta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W). \quad (2.17)$$

Moreover, accounting for the solution of the adjoint system, the following necessary condition was pointed out.

**Theorem 2.1.** *Assume that 2.1–2.10 are fulfilled. Let  $\bar{u}_\beta \in \mathcal{U}_{ad}$  be an optimal control for  $(CP)_\beta$ ,  $(\bar{\mu}_\beta, \bar{\varphi}_\beta, \bar{\sigma}_\beta)$  be the corresponding optimal state and  $(p_\beta, q_\beta, r_\beta)$  the associated solution to the adjoint system 2.12–2.16. Then, the necessary condition for optimality is given by*

$$\int_Q (r_\beta + b_0 \bar{u}_\beta)(v - \bar{u}_\beta) \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \quad (2.18)$$

As sketched above, we would like to exploit the control problem  $(CP)_\beta$  in order to solve  $(CP)$  by employing some asymptotic schemes. In fact, in Section 4, we rigorously show that, as  $\beta \searrow 0$ , system 2.12–2.16 converge to

$$-\partial_t p + \Delta q - F''(\bar{\varphi})q + P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) = b_1(\bar{\varphi} - \varphi_Q) \quad \text{in } Q \quad (2.19)$$

$$q - \alpha \partial_t p - \Delta p + P(\bar{\varphi})(p - r) = 0 \quad \text{in } Q \quad (2.20)$$

$$-\partial_t r - \Delta r + P(\bar{\varphi})(r - p) = b_2(\bar{\sigma} - \sigma_Q) \quad \text{in } Q \quad (2.21)$$

$$\partial_n q = \partial_n p = \partial_n r = 0 \quad \text{on } \Sigma \quad (2.22)$$

$$\alpha p(T) = 0, \quad r(T) = b_3(\bar{\sigma}(T) - \sigma_\Omega) \quad \text{in } \Omega. \quad (2.23)$$

Below, you can find the precise meaning of the above sentence.

**Theorem 2.2.** *Assume that 2.1–2.10 are fulfilled and let  $(q_\beta, p_\beta, r_\beta)$  be the unique solution to system 2.12–2.16 satisfying (2.17). Then, there exists  $\alpha_0 \in (0, 1)$  such that, for every  $\alpha \in (0, \alpha_0)$ , as  $\beta \searrow 0$ , and up to a subsequence, we have the following convergences*

$$q_\beta \rightarrow q \quad \text{weakly in } L^2(0, T; W) \quad (2.24)$$

$$p_\beta \rightarrow p \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.25)$$

$$r_\beta \rightarrow r \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.26)$$

$$\beta q_\beta \rightarrow 0 \quad \text{strongly in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W). \quad (2.27)$$

Moreover, there exists a positive constant  $C_1$ , independent of  $\beta$ , such that

$$\begin{aligned} & \beta \|q_\beta\|_{H^1(0, T; H)} + \beta^{1/2} \|q_\beta\|_{L^\infty(0, T; V)} + \|q_\beta\|_{L^2(0, T; W)} \\ & + \|p_\beta\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} + \|r_\beta\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \leq C_1. \end{aligned} \quad (2.28)$$

In addition, the limit  $(q, p, r)$  is the unique solution to system 2.19–2.23 which possesses the following regularity

$$q \in L^2(0, T; W) \quad (2.29)$$

$$p, r \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W). \quad (2.30)$$

Next, we can address the results related to the control problem we are dealing with. We begin with the first fundamental result concerning the existence of optimal control.

**Theorem 2.3.** *Assume that 2.1–2.10 are in force. Then, the optimal control problem  $(CP)$  admits at least a solution  $\bar{u} \in \mathcal{U}_{ad}$ .*

Lastly, by employing a proper asymptotic scheme, we develop the first-order necessary condition for optimality.



**Theorem 2.4.** *Assume that 2.1–2.10 are satisfied. Let  $\bar{u} \in \mathcal{U}_{ad}$  be an optimal control for (CP) with its corresponding state  $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$  and let  $(p, q, r)$  be the solution to the associated adjoint system 2.19–2.23. Then, the following variational inequality*

$$\int_Q (r + b_0 \bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U}_{ad} \quad (2.31)$$

*is satisfied. Moreover, whenever  $b_0 \neq 0$ , the optimal control  $\bar{u}$  is the  $L^2(0, T; H)$ –projection of  $-r/b_0$  onto the subspace  $\mathcal{U}_{ad}$ .*

To conclude the section, we recall a well-known inequality and a general fact that is widely used in the sequel. First of all, let us remind the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \quad (2.32)$$

Furthermore, we recall the standard Sobolev continuous embedding

$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{which holds for every } q \in [1, 6]. \quad (2.33)$$

Throughout the paper, we convey to use the symbol small-case  $c$  for every constant which depend only on the final time  $T$ , on  $\Omega$ , on  $R$ , on the shape of the nonlinearities, on the norms of the involved functions, and possibly on  $\alpha$ . On the other hand, we will explicitly point out when an appearing constant may depend on  $\beta$ . For this reason, the meaning of  $c$  might change from line to line and even in the same chain of inequalities. Differently, we devote the capital letters to indicate particular constants which we eventually will refer later on.

**3. The state system.** From this section onward, we start with the proofs of the introduced statements. We begin with the task of investigating the well-posedness of system 1.2–1.6, and proving its asymptotic behavior as  $\beta \searrow 0$ . Again, we remark that the above system has already been discussed in [8, 11, 13], where the asymptotic analysis represents the core of the contributions. For this reason, some of the calculations below can also be found there. Anyhow, in order to handle the control problem (CP), it turns out that the results there obtained are insufficient. Therefore, we perform all the necessary estimates, having the care to emphasize when the appearing constants may depend on  $\beta$ .

**Theorem 3.1.** *Let the assumptions 2.1–2.10 be fulfilled and let  $\mu_0 := -\Delta\varphi_0 + F'(\varphi_0)$ . Then, there exists  $\alpha_0 \in (0, 1)$  such that, for every  $\alpha \in (0, \alpha_0)$ , system 1.2–1.6 admits a unique solution  $(\mu_\beta, \varphi_\beta, \sigma_\beta)$  that satisfies the following regularity*

$$\mu_\beta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (3.1)$$

$$\varphi_\beta \in W^{1, \infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad (3.2)$$

$$\sigma_\beta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W). \quad (3.3)$$

*Furthermore, there exists a positive constant  $C_2$  such that the following estimate is verified*

$$\begin{aligned} & \beta^{1/2} \|\partial_t \varphi_\beta\|_{L^\infty(0, T; H)} + \|\varphi_\beta\|_{H^1(0, T; V) \cap L^\infty(0, T; W) \cap L^\infty(Q)} \\ & \quad + \|\mu_\beta\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} + \|\sigma_\beta\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \\ & \leq C_2 (\|\mu_0\|_V + \|\varphi_0\|_W + \|\sigma_0\|_V + 1), \end{aligned} \quad (3.4)$$

*where  $C_2$  is a positive constant that depends on  $\Omega, T, R, \alpha$ , the shape of the functions  $P$  and  $\pi$ , but it is independent of  $\beta$ .*

Before moving to the proof, it is worth focusing the attention on the already obtained limit system 1.10–1.14 which was investigated in [13, Thm. 2.2, p. 41] (see also [11]). Forgetting the fact that 1.10–1.14 has founded as a result of a limit procedure, it can be considered as a starting point as a system of partial differential equations. In this regards, we only have at disposal the derivative of the linear combination  $\alpha\mu + \varphi$ , so that the initial condition, in general, reads as  $(\alpha\mu + \varphi)(0) = \eta_0$ , for a suitable element  $\eta_0$ . Furthermore, owing to what already proved for the system 1.10–1.14, we claim that, whenever  $\eta_0 \in V$ , the existence and the uniqueness of the solution follows, provide a smallness assumption of the constant  $\alpha$  is satisfied. Thus, we consider this problem under the additional assumptions

$$\begin{cases} \eta_0 = \alpha\mu_0 + \varphi_0 \\ \mu_0 = -\Delta\varphi_0 + F'(\varphi_0). \end{cases} \quad (3.5)$$

The first condition allows us to match the two approaches; that is, to see the above system as a direct problem and as the limit of system 1.2–1.6 as  $\beta \searrow 0$ . Conversely, the second condition states that  $\mu_0$  cannot be arbitrary chosen, but it has to be defined in terms of  $\varphi_0$  in a prescribed way. At this stage, that requirement could appear quite unnecessary and unnatural, but it will be motivated in view of a forthcoming estimate (see the Fourth estimate below) which is of crucial importance for the asymptotic analysis. Under this additional assumption, it follows that the second of (2.5) is rather a consequence of the first one combined with the strong regularity we postulated for  $\varphi_0$ , which can now be motivated by virtue of the regularity we want for  $\mu_0$ . Indeed, combining the growth assumption for the potential (2.10) with the continuous embedding  $W \subset L^\infty(\Omega)$ , we infer from the second of (3.5) that  $\mu_0 \in V$ . On the other hand, whenever  $\eta_0 \in V$  is given, let us claim that  $\mu_0$  and  $\varphi_0$  can be reconstructed providing to impose  $\partial_n\varphi_0 = 0$ . In fact, using (3.5), we realize that we are looking for a variable  $\varphi_0$  such that

$$\begin{cases} -\Delta\varphi_0 + \frac{\varphi_0 - \eta_0}{\alpha} + F'(\varphi_0) = 0 & \text{in } \Omega \\ \partial_n\varphi_0 = 0 & \text{on } \Gamma. \end{cases}$$

Again, the fact that  $\alpha$  has to be sufficiently small helps us and the existence and uniqueness of a solution to the above equation can be proved. Indeed, the nonlinear term  $F'(\varphi_0)$  can be split as  $F'(\varphi_0) = B(\varphi_0) + \pi(\varphi_0)$ , where the perturbation  $\pi(\varphi_0)$  does not bother since it is balanced by the term  $\frac{\varphi_0 - \eta_0}{\alpha}$  which dominates owing to the smallness of the denominator (we can indeed assume that  $\alpha L < 1$ , which gives  $\frac{1}{\alpha} > L$ ). Then, by combining  $\partial_n\varphi_0 = 0$  with the elliptic regularity theory, we are able to reconstruct  $\varphi_0$  which fulfills the first condition of (2.5). Finally, from the first equation of (3.5), we also recover  $\mu_0$  with the prescribed regularity. With these comments in mind, we can proceed with the proof of Theorem 3.1.

*Proof of Theorem 3.1.* For the uniqueness part, we refer the reader to [8, Sec. 3]. As the existence is concerned, we take into account an approximation scheme.

**The approximating system** Let us take  $\varepsilon \in (0, 1)$  and consider the so-called Yosida approximation of the maximal and monotone operator  $B$ . Namely, for every  $r \in \mathbb{R}$ , we introduce

$$\widehat{B}_\varepsilon(r) := \min_{s \in \mathbb{R}} \left( \frac{1}{2\varepsilon} (s - r)^2 + \widehat{B}(s) \right), \quad B_\varepsilon(r) := \frac{d}{dr} \widehat{B}_\varepsilon(r), \quad \text{and} \quad F_\varepsilon := \widehat{B}_\varepsilon + \widehat{\pi}. \quad (3.6)$$

It turns out that  $\widehat{B}_\varepsilon$  is a well-defined  $C^1$  function,  $B_\varepsilon$  is Lipschitz continuous (see, e.g., [4, Prop. 2.11, p. 39]), and for every  $r \in \mathbb{R}$ , it holds that

$$0 \leq \widehat{B}_\varepsilon(r) \leq \widehat{B}(r) \quad \text{and} \quad \widehat{B}_\varepsilon(r) \nearrow \widehat{B}(r) \quad \text{as } \varepsilon \searrow 0. \quad (3.7)$$

Hence, in order to solve 1.2–1.6, we first solve the approximated system obtained by substituting  $F$  by  $F_\varepsilon$ , and then we let  $\varepsilon \searrow 0$  to prove the existence to the original problem. Therefore, we are going to face the following system

$$\alpha \partial_t \mu_{\beta,\varepsilon} + \partial_t \varphi_{\beta,\varepsilon} - \Delta \mu_{\beta,\varepsilon} = P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon}) \quad \text{in } Q \quad (3.8)$$

$$\mu_{\beta,\varepsilon} = \beta \partial_t \varphi_{\beta,\varepsilon} - \Delta \varphi_{\beta,\varepsilon} + F'_\varepsilon(\varphi_{\beta,\varepsilon}) \quad \text{in } Q \quad (3.9)$$

$$\partial_t \sigma_{\beta,\varepsilon} - \Delta \sigma_{\beta,\varepsilon} = -P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon}) + u_\beta \quad \text{in } Q \quad (3.10)$$

$$\partial_n \mu_{\beta,\varepsilon} = \partial_n \varphi_{\beta,\varepsilon} = \partial_n \sigma_{\beta,\varepsilon} = 0 \quad \text{on } \Sigma \quad (3.11)$$

$$\mu_{\beta,\varepsilon}(0) = \mu_0, \varphi_{\beta,\varepsilon}(0) = \varphi_0, \sigma_{\beta,\varepsilon}(0) = \sigma_0 \quad \text{in } \Omega. \quad (3.12)$$

Our starting point is the result below.

**Lemma 3.2.** *Assume that 2.1–2.10 are satisfied. Then, the approximating problem 3.8–3.12 admits a unique solution.*

As the uniqueness is concerned, it can be proved as a special case of [8, Sec. 3]. As regards the existence, let us only mention that, e.g., a Faedo-Galerkin scheme, along with some a priori estimates, will lead to proving the asserted result. For instance, as a basis of  $V$ , one could take into account the basis consisting of the eigenfunctions of the Laplacian operator with homogeneous Neumann boundary conditions. We decide to skip the details because the estimates we are going to perform below are very similar to the ones that could allow one to solve the approximating problem. In addition, let us point out that  $B_\varepsilon$  and the map which assigns  $(\mu_{\beta,\varepsilon}, \varphi_{\beta,\varepsilon}, \sigma_{\beta,\varepsilon}) \mapsto P(\varphi_{\beta,\varepsilon})(\mu_{\beta,\varepsilon} - \sigma_{\beta,\varepsilon}) =: R_{\beta,\varepsilon}$  are both smooth and Lipschitz continuous, for every  $\beta$  and every  $\varepsilon$ , and therefore the classical Picard-Lindelöf theorem directly yields the existence of a unique global solution to the system of ordinary differential equations given by that scheme.

**First estimate** We multiply (3.8) by  $\mu_{\beta,\varepsilon}$ , (3.9) by  $-\partial_t \varphi_{\beta,\varepsilon}$ , and (3.10) by  $\sigma_{\beta,\varepsilon}$ . Next, we integrate over  $Q_t$  and by parts, and add the resulting equalities to obtain that

$$\begin{aligned} & \frac{\alpha}{2} \int_\Omega |\mu_{\beta,\varepsilon}(t)|^2 + \int_{Q_t} |\nabla \mu_{\beta,\varepsilon}|^2 + \beta \int_{Q_t} |\partial_t \varphi_{\beta,\varepsilon}|^2 + \frac{1}{2} \int_\Omega |\nabla \varphi_{\beta,\varepsilon}(t)|^2 \\ & + \int_\Omega F_\varepsilon(\varphi_{\beta,\varepsilon}(t)) + \frac{1}{2} \int_\Omega |\sigma_{\beta,\varepsilon}(t)|^2 + \int_{Q_t} |\nabla \sigma_{\beta,\varepsilon}|^2 + \int_{Q_t} P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon})^2 \\ & = \frac{\alpha}{2} \int_\Omega |\mu_0|^2 + \frac{1}{2} \int_\Omega |\nabla \varphi_0|^2 + \int_\Omega F_\varepsilon(\varphi_0) + \frac{1}{2} \int_\Omega |\sigma_0|^2 + \int_{Q_t} u_\beta \sigma_{\beta,\varepsilon}, \end{aligned}$$

where we denote the integrals on the right-hand side by  $I_1, \dots, I_5$ , in this order. The terms on the left-hand side are nonnegative since they all are squares and  $P$  and  $F_\varepsilon$  are so by (2.4) and (2.7) along with 3.6–3.7, respectively. Moreover,  $I_1, I_2$  and  $I_4$  can be easily controlled owing to the assumptions on the initial conditions (2.5). As for  $I_3$ , we deduce that

$$|I_3| = \left| \int_\Omega F_\varepsilon(\varphi_0) \right| = \int_\Omega \widehat{B}_\varepsilon(\varphi_0) + \int_\Omega \widehat{\pi}(\varphi_0) \leq \int_\Omega \widehat{B}(\varphi_0) + \int_\Omega \widehat{\pi}(\varphi_0) \leq c,$$

by invoking the properties of  $\widehat{B}_\varepsilon$  pointed out by (3.7) and accounting for the features of the initial datum  $\varphi_0$  and on the function  $\widehat{\pi}$ . Then, by employing the Young inequality, we bound  $I_5$  as follows

$$|I_5| \leq \frac{1}{2} \int_{Q_t} |u_\beta|^2 + \frac{1}{2} \int_{Q_t} |\sigma_{\beta,\varepsilon}|^2.$$

Thus, a Gronwall argument yields that

$$\begin{aligned} & \|\mu_{\beta,\varepsilon}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \beta^{1/2} \|\partial_t \varphi_{\beta,\varepsilon}\|_{L^2(0,T;H)} + \|\nabla \varphi_{\beta,\varepsilon}\|_{L^2(0,T;H)} \\ & + \|F_\varepsilon(\varphi_{\beta,\varepsilon})\|_{L^\infty(0,T;L^1(\Omega))} + \|\sigma_{\beta,\varepsilon}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \end{aligned} \quad (3.13)$$

**Second estimate** Analyzing (3.13), we realize that it does not provide any information of  $\varphi_{\beta,\varepsilon}$  in  $L^2(0,T;H)$ . Hence, we try to reconstruct the full norm of  $\varphi_{\beta,\varepsilon}$  in  $L^2(0,T;V)$ . In this direction, we add equations (3.8) and (3.10) to get

$$\partial_t(\alpha\mu_{\beta,\varepsilon} + \varphi_{\beta,\varepsilon} + \sigma_{\beta,\varepsilon}) - \Delta(\mu_{\beta,\varepsilon} + \sigma_{\beta,\varepsilon}) = u_\beta.$$

Then, we test the above equation by  $\alpha\mu_{\beta,\varepsilon} + \varphi_{\beta,\varepsilon} + \sigma_{\beta,\varepsilon}$  and integrate over  $Q_t$  and by parts. Upon rearrange the terms, we obtain that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\alpha|\mu_{\beta,\varepsilon}(t)|^2 + |\varphi_{\beta,\varepsilon}(t)|^2 + |\sigma_{\beta,\varepsilon}(t)|^2) + \alpha \int_{Q_t} |\nabla \mu_{\beta,\varepsilon}|^2 + \int_{Q_t} |\nabla \sigma_{\beta,\varepsilon}|^2 \\ & = \frac{1}{2} \int_{\Omega} (\alpha|\mu_0|^2 + |\varphi_0|^2 + |\sigma_0|^2) - (\alpha + 1) \int_{Q_t} \nabla \mu_{\beta,\varepsilon} \cdot \nabla \sigma_{\beta,\varepsilon} \\ & \quad - \int_{Q_t} \nabla \mu_{\beta,\varepsilon} \cdot \nabla \varphi_{\beta,\varepsilon} - \int_{Q_t} \nabla \sigma_{\beta,\varepsilon} \cdot \nabla \varphi_{\beta,\varepsilon} + \int_{Q_t} u_\beta(\alpha\mu_{\beta,\varepsilon} + \varphi_{\beta,\varepsilon} + \sigma_{\beta,\varepsilon}), \end{aligned}$$

where the integrals on the right-hand side are denoted by  $I_1, \dots, I_5$ , in this order. Using (2.5), we immediately deduce that  $|I_1| \leq c$ . Furthermore, by combining the above estimate with the Young inequality, we find that the remaining terms can be estimated as

$$\begin{aligned} |I_2| + |I_3| + |I_4| + |I_5| & \leq c \int_{Q_t} (|\mu_{\beta,\varepsilon}|^2 + |\nabla \mu_{\beta,\varepsilon}|^2) + c \int_{Q_t} (|\varphi_{\beta,\varepsilon}|^2 + |\nabla \varphi_{\beta,\varepsilon}|^2) \\ & \quad + c \int_{Q_t} (|\sigma_{\beta,\varepsilon}|^2 + |\nabla \sigma_{\beta,\varepsilon}|^2) + \frac{1}{2} \int_{Q_t} |u_\beta|^2. \end{aligned}$$

Therefore, the Gronwall lemma entails that

$$\|\varphi_{\beta,\varepsilon}\|_{L^\infty(0,T;H)} \leq c. \quad (3.14)$$

Moreover, we also realize that

$$\|R_{\beta,\varepsilon}\|_{L^2(0,T;H)} \leq c. \quad (3.15)$$

**Third estimate** It is worth noting that (3.10) possesses a parabolic structure with respect to the variable  $\sigma_{\beta,\varepsilon}$  and, owing to the above results, it follows that its forcing term belongs to  $L^2(0,T;H)$ . Hence, the parabolic regularity theory for homogeneous Neumann problems with regular initial conditions, gives us

$$\|\sigma_{\beta,\varepsilon}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq c. \quad (3.16)$$

**Fourth estimate** Now, we present the key estimate for the forthcoming asymptotic analyses which motivates the unusual requirement (3.5). To begin with, let us formally differentiate (3.9) with respect to time to infer that

$$\partial_t \mu_{\beta,\varepsilon} = \beta \partial_{tt} \varphi_{\beta,\varepsilon} - \Delta \partial_t \varphi_{\beta,\varepsilon} + F_\varepsilon''(\varphi_{\beta,\varepsilon}) \partial_t \varphi_{\beta,\varepsilon}. \quad (3.17)$$

Next, we multiply it by  $\alpha$  and replace the first term of (3.8) with this new equation leading to obtain that

$$\begin{aligned} \alpha\beta\partial_{tt}\varphi_{\beta,\varepsilon} - \alpha\Delta\partial_t\varphi_{\beta,\varepsilon} + \alpha F''_\varepsilon(\varphi_{\beta,\varepsilon})\partial_t\varphi_{\beta,\varepsilon} + \partial_t\varphi_{\beta,\varepsilon} - \Delta\mu_{\beta,\varepsilon} \\ = P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon}). \end{aligned} \quad (3.18)$$

This formal procedure can be rigorously motivated. Indeed, by introducing the auxiliary variable  $z_\beta := \alpha\partial_t\varphi_{\beta,\varepsilon}$ , we can rewrite (3.18) as a parabolic equation as follows

$$\beta\partial_t z_\beta - \Delta z_\beta = f_\beta \quad \text{in } Q,$$

where  $f_\beta$  is defined by

$$f_\beta := \Delta\mu_{\beta,\varepsilon} - \partial_t\varphi_{\beta,\varepsilon} - \alpha F''_\varepsilon(\varphi_{\beta,\varepsilon})\partial_t\varphi_{\beta,\varepsilon} + P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon}).$$

Owing to the above estimates and to the growth conditions 2.9–2.10 for the potential, we easily realize that, for every  $\beta$ , the forcing term  $f_\beta \in L^2(0, T; V^*)$ . Therefore, the abstract theory for parabolic equations (see, e.g., [42]) guarantees the existence and the uniqueness of a solution  $z_\beta \in H^1(0, T; V^*) \cap L^2(0, T; V)$ , whenever the initial datum  $z_\beta(0)$  is sufficiently regular, that is, whenever  $z_\beta(0)$  belongs at least to  $H$ . As we will see, the particular choice of the initial datum  $\mu_0$  assumed by (3.5), entails, in the limit, that  $z_\beta(0) = 0$ , so that the required regularity is trivially fulfilled.

Then, we multiply (3.18) by  $\partial_t\varphi_{\beta,\varepsilon}$  and integrate over  $Q_t$  and by parts to find that

$$\begin{aligned} \frac{\alpha\beta}{2} \int_\Omega |\partial_t\varphi_{\beta,\varepsilon}(t)|^2 + \alpha \int_{Q_t} |\nabla\partial_t\varphi_{\beta,\varepsilon}|^2 + \alpha \int_{Q_t} F''_\varepsilon(\varphi_{\beta,\varepsilon})|\partial_t\varphi_{\beta,\varepsilon}|^2 + \int_{Q_t} |\partial_t\varphi_{\beta,\varepsilon}|^2 \\ = \frac{\alpha\beta}{2} \int_\Omega |\partial_t\varphi_{\beta,\varepsilon}(0)|^2 + \int_{Q_t} P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon})\partial_t\varphi_{\beta,\varepsilon} \\ - \int_{Q_t} \nabla\mu_{\beta,\varepsilon} \cdot \nabla\partial_t\varphi_{\beta,\varepsilon}, \end{aligned} \quad (3.19)$$

where the integrals on the right-hand side are denoted by  $I_1, I_2, I_3$ , in this order. As the third integrals of the left-hand side is concerned, we remind that  $F_\varepsilon$  is defined by (3.6) and that  $F''$  is bounded below. Moreover, the same bound holds true for  $F_\varepsilon$  with the same constant  $L$ . Therefore, the third and fourth contributions on the left-hand side verify that

$$\alpha \int_{Q_t} F''_\varepsilon(\varphi_{\beta,\varepsilon})|\partial_t\varphi_{\beta,\varepsilon}|^2 + \int_{Q_t} |\partial_t\varphi_{\beta,\varepsilon}|^2 \geq (1 - \alpha L) \int_{Q_t} |\partial_t\varphi_{\beta,\varepsilon}|^2, \quad (3.20)$$

whereas the other terms on that side are nonnegative. As regards the right-hand side, let us emphasize that the definition (3.5) plays a fundamental role. Indeed, by taking  $t = 0$  in the equation (3.9), we get

$$\mu_{\beta,\varepsilon}(0) = \beta\partial_t\varphi_{\beta,\varepsilon}(0) - \Delta\varphi_{\beta,\varepsilon}(0) + F'_\varepsilon(\varphi_{\beta,\varepsilon}(0)) \quad \text{in } \Omega$$

Using (3.5), the initial conditions (3.12), and rearranging the terms lead to infer that

$$\beta\partial_t\varphi_{\beta,\varepsilon}(0) = \mu_0 + \Delta\varphi_0 - F'_\varepsilon(\varphi_0) = B(\varphi_0) - B_\varepsilon(\varphi_0) \quad \text{in } \Omega.$$

On the other hand, from well-known results (see, e.g., [4, Chapt. 2]), we also have the convergence, both *a.e.* in  $\Omega$  and in  $L^2(\Omega)$ , of the operator  $B_\varepsilon$  to element having minimum norm of the limit operator  $B$ . Actually, since  $\hat{B}$  is regular, it turns out

that its subdifferential  $B$  is single-valued, so that the element having minimum norm is  $B$  itself. Hence, we have

$$B_\varepsilon(r) \rightarrow B(r) \quad \text{strongly in } L^2(\Omega), \text{ a.e. in } \Omega, \text{ for every } r \in \mathbb{R}. \quad (3.21)$$

Therefore, we easily realize that  $\|B(\varphi_0) - B_\varepsilon(\varphi_0)\|_H^2 \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ . Meanwhile, the other integrals can be easily controlled thanks to the Young inequality. Recalling that  $P$  is bounded by (2.4), we control  $I_2$  by

$$|I_2| \leq \delta \int_{Q_t} |\partial_t \varphi_{\beta,\varepsilon}|^2 + c_\delta \int_{Q_t} (|\sigma_{\beta,\varepsilon}|^2 + |\mu_{\beta,\varepsilon}|^2), \quad (3.22)$$

for a positive constant  $\delta$ , yet to be determined. In a similar manner, we obtain

$$|I_3| \leq \frac{\alpha}{2} \int_{Q_t} |\nabla \partial_t \varphi_{\beta,\varepsilon}|^2 + c \int_{Q_t} |\nabla \mu_{\beta,\varepsilon}|^2. \quad (3.23)$$

Upon collecting 3.20–3.23, we rearrange the initial equation (3.19) to infer that

$$\begin{aligned} & \frac{\alpha\beta}{2} \int_{\Omega} |\partial_t \varphi_{\beta,\varepsilon}(t)|^2 + \frac{\alpha}{2} \int_{Q_t} |\nabla \partial_t \varphi_{\beta,\varepsilon}|^2 + (1 - \alpha L - \delta) \int_{Q_t} |\partial_t \varphi_{\beta,\varepsilon}|^2 \\ & \leq c(1 + \|B(\varphi_0) - B_\varepsilon(\varphi_0)\|_H^2) \end{aligned}$$

has been shown, where the right-hand side has been managed owing to the above estimates. On the other hand, we can assume  $\alpha$  to be sufficiently small in order to have that  $\alpha L < 1$ , so that it suffices to take  $\delta$  small enough to conclude  $(1 - \alpha L - \delta) > 0$ , which, in turn, implies that

$$\beta^{1/2} \|\partial_t \varphi_{\beta,\varepsilon}\|_{L^\infty(0,T;H)} + \|\partial_t \varphi_{\beta,\varepsilon}\|_{L^2(0,T;V)} \leq c(1 + \|B(\varphi_0) - B_\varepsilon(\varphi_0)\|_H). \quad (3.24)$$

**Fifth estimate** Moreover, (3.8) shows a parabolic structure with respect to the variable  $\mu_{\beta,\varepsilon}$ , so that

$$\alpha \partial_t \mu_{\beta,\varepsilon} - \Delta \mu_{\beta,\varepsilon} = f_\beta \quad \text{in } Q, \quad \text{where } f_\beta := -\partial_t \varphi_{\beta,\varepsilon} + P(\varphi_{\beta,\varepsilon})(\sigma_{\beta,\varepsilon} - \mu_{\beta,\varepsilon}).$$

On account of the above estimates, we easily realize that  $f_\beta \in L^2(0,T;H)$ . Thus, the regularity theory for homogeneous Neumann parabolic equations yields that

$$\|\mu_{\beta,\varepsilon}\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq c. \quad (3.25)$$

**Sixth estimate** Furthermore, let us read (3.9) as an elliptic equation with respect to the variable  $\varphi_{\beta,\varepsilon}$  as follows

$$-\Delta \varphi_{\beta,\varepsilon} + F'_\varepsilon(\varphi_{\beta,\varepsilon}) = \mu_{\beta,\varepsilon} - \beta \partial_t \varphi_{\beta,\varepsilon} \quad \text{in } Q.$$

Then, we consider the above equation written at time  $t$ , split  $F'_\varepsilon$  on account of (2.8), multiply it by  $-\Delta \varphi_{\beta,\varepsilon}(t)$  and integrate over  $\Omega$  and by parts to obtain that

$$\begin{aligned} & \int_{\Omega} |\Delta \varphi_{\beta,\varepsilon}(t)|^2 + \int_{\Omega} B'_\varepsilon(\varphi_{\beta,\varepsilon}(t)) |\nabla \varphi_{\beta,\varepsilon}(t)|^2 = - \int_{\Omega} \mu_{\beta,\varepsilon}(t) \Delta \varphi_{\beta,\varepsilon}(t) \\ & + \beta \int_{\Omega} \partial_t \varphi_{\beta,\varepsilon}(t) \Delta \varphi_{\beta,\varepsilon}(t) + \int_{\Omega} \pi(\varphi_{\beta,\varepsilon}(t)) \Delta \varphi_{\beta,\varepsilon}(t), \end{aligned}$$

where the terms on the right-hand side are denoted by  $I_1, I_2$  and  $I_3$ , in that order. Note that, at the first stage, the second term on the right-hand side can be neglected since it is nonnegative by the properties of  $B'_\varepsilon$ . On the other hand, Young's inequality, along with the above estimates, gives

$$|I_1| + |I_2| + |I_3| \leq \frac{3}{4} \int_{\Omega} |\Delta \varphi_{\beta,\varepsilon}(t)|^2 + c.$$

Hence, invoking first the elliptic theory, and then comparison in (3.9), lead to conclude that

$$\|\varphi_{\beta,\varepsilon}\|_{L^\infty(0,T;W)} + \|B_\varepsilon(\varphi_{\beta,\varepsilon})\|_{L^\infty(0,T;H)} \leq c. \quad (3.26)$$

Furthermore, the continuous embedding  $W \subset L^\infty(\Omega)$ , entails that

$$\|\varphi_{\beta,\varepsilon}\|_{L^\infty(Q)} \leq c. \quad (3.27)$$

**Passing to the limit** Lastly, we draw some consequences from the above a priori estimates, showing that we can let  $\varepsilon \searrow 0$  to complete the proof.

Owing to standard weak compactness arguments, we infer that, as  $\varepsilon \searrow 0$ , and up to a not relabeled subsequence, the following convergences

$$\mu_{\beta,\varepsilon} \rightarrow \mu_\beta \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad (3.28)$$

$$\varphi_{\beta,\varepsilon} \rightarrow \varphi_\beta \text{ weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \quad (3.29)$$

$$\sigma_{\beta,\varepsilon} \rightarrow \sigma_\beta \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad (3.30)$$

$$R_{\beta,\varepsilon} \rightarrow \zeta_\beta \text{ weakly in } L^2(0,T;H) \quad (3.31)$$

$$B_\varepsilon(\varphi_{\beta,\varepsilon}) \rightarrow \psi_\beta \text{ weakly star in } L^\infty(0,T;H) \quad (3.32)$$

are satisfied. Furthermore, compactness embedding results (see, e.g., [50, Sec. 8, Cor. 4]) easily imply that

$$\mu_{\beta,\varepsilon} \rightarrow \mu_\beta, \quad \varphi_{\beta,\varepsilon} \rightarrow \varphi_\beta, \quad \sigma_{\beta,\varepsilon} \rightarrow \sigma_\beta \text{ strongly in } C^0([0,T];H) \cap L^2(0,T;V).$$

These convergences give sense to initial conditions (1.6) and allow us to identify the limit of the nonlinear terms. Indeed, the boundedness and the regularity of  $P$ , along with the above strong convergences, yield that  $\zeta_\beta = R_\beta$ , where  $R_\beta := P(\varphi_\beta)(\sigma_\beta - \mu_\beta)$ . Arguing in a similar fashion, we infer that  $\pi(\varphi_{\beta,\varepsilon})$  strongly converges to  $\pi(\varphi_\beta)$  in  $L^2(0,T;H)$ . Lastly, from the monotonicity properties of the Yosida approximation introduced by (3.6), we get (see, e.g., [3, Lemma 1.3, p. 42]) that

$$\limsup_{\varepsilon \searrow 0} \int_Q B_\varepsilon(\varphi_{\beta,\varepsilon})\varphi_{\beta,\varepsilon} = \lim_{\varepsilon \searrow 0} \int_Q B_\varepsilon(\varphi_{\beta,\varepsilon})\varphi_{\beta,\varepsilon} = \int_Q B(\varphi_\beta)\varphi_\beta,$$

which implies  $\psi_\beta = B(\varphi_\beta)$ . In conclusion, the limit triplet  $(\mu_\beta, \varphi_\beta, \sigma_\beta)$  yields a solution to 1.2–1.6 and possesses the postulated regularity 3.1–3.3, so that Theorem 3.1 is completely proved.  $\square$

With the result below, we aim at improve the result on the asymptotic analysis of system 1.2–1.6, as  $\beta \searrow 0$  (compare with the regularity pointed out in [8, 13, 11]).

**Theorem 3.3.** *Suppose that 2.1–2.10 are satisfied and let  $\mu_0 := -\Delta\varphi_0 + F'(\varphi_0)$ . Then, there exists a sufficiently small  $\alpha_0 \in (0, 1)$  such that, for every  $\alpha \in (0, \alpha_0)$  and  $\beta \in (0, 1)$ , the unique solution  $(\mu_\beta, \varphi_\beta, \sigma_\beta)$  to problem 1.2–1.6, as  $\beta \searrow 0$ , satisfies*

$$\mu_\beta \rightarrow \mu \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad (3.33)$$

$$\varphi_\beta \rightarrow \varphi \text{ weakly star in } H^1(0,T;V) \cap L^\infty(0,T;W) \quad (3.34)$$

$$\sigma_\beta \rightarrow \sigma \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad (3.35)$$

$$\beta\varphi_\beta \rightarrow 0 \text{ strongly in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \quad (3.36)$$

at least for a subsequence. Moreover, the limit  $(\mu, \varphi, \sigma)$  turns out to be the unique solution to the limit system 1.10–1.14. Furthermore, there exists a subsequence for which we also have the strong convergences

$$\begin{aligned} \varphi_\beta &\rightarrow \varphi \text{ strongly in } C^0([0, T]; H^{2-\gamma}(\Omega)), \text{ for every } \gamma > 0, \\ &\text{which entails } \varphi_\beta \rightarrow \varphi \text{ strongly in } C^0(\overline{Q}) \end{aligned} \quad (3.37)$$

$$\mu_\beta \rightarrow \mu \text{ strongly in } C^0([0, T]; H) \cap L^2(0, T; V) \quad (3.38)$$

$$\sigma_\beta \rightarrow \sigma \text{ strongly in } C^0([0, T]; H) \cap L^2(0, T; V). \quad (3.39)$$

*Proof.* Since it immediately follows on account of standard techniques from estimate (3.4), we just sketch the proof and left the details to the reader.

The convergences 3.33–3.36 immediately follow by standard argument by estimate (3.4), while the strong convergences 3.37–3.39 can be obtained by invoking well-known compactness results (see, e.g., [50]). Hence, it remains to check that  $(\mu, \varphi, \sigma)$  yields a solution to 1.10–1.14. In principle, one should consider the variational formulation corresponding to system 1.2–1.6 and, using the above estimates, pass to the limit to conclude. Therefore, the only terms that deserve further comments are the nonlinear ones. Anyhow, the strong convergence (3.37) suffices since, along with (2.4), (2.8) and (3.4), yields that

$$F'(\varphi_\beta) \rightarrow F'(\varphi), \quad P(\varphi_\beta) \rightarrow P(\varphi), \quad \text{both strongly in } C^0(\overline{Q}).$$

Furthermore, we infer that

$$B(\varphi_\beta) \rightarrow B(\varphi) \text{ at least strongly in } L^2(0, T; H).$$

It is now a standard matter to complete the details. Finally, uniqueness follows as a consequence of [13, Thm. 2.3].  $\square$

Let us remark that by [13, Ex. 2.4] the authors pointed out a severe non-uniqueness result for the system 1.10–1.14 if  $\alpha L = 1$ . On the other hand, they pointed out a stability estimate whenever  $\alpha$  is sufficiently small ( $\alpha < \min\{\frac{1}{L}, \frac{1}{(1+L)^2}\}$ ), which directly implies uniqueness. The smallness condition is indeed motivated by the fact that we need the uniqueness for system 1.10–1.14.

**4. The control problem.** This last section is completely devoted to the investigation of the optimal control problem (CP). We prove the existence of an optimal control and point out a variational inequality which characterizes the first-order necessary condition for optimality.

**4.1. Existence of optimal controls.** To begin with, we check the existence of optimal controls, namely, we prove Theorem 2.3.

*Proof of Theorem 2.3.* For the proof, we employ the direct method of calculus of variations. In this direction, let us fix a sequence  $\{\beta_n\}_n$  which goes to zero as  $n \rightarrow \infty$ . Then, let  $\{u_n\}_n := \{u_{\beta_n}\}_n \subset \mathcal{U}_{ad}$  be a minimizing sequence for  $\mathcal{J}$  which, at every step, is made of optimal control for  $(CP)_{\beta_n}$ , and let  $(\mu_n, \varphi_n, \sigma_n)$  be the corresponding solution to system 1.10–1.14. From the bounds pointed out by estimate (3.4), we deduce that, as  $n \rightarrow \infty$ , there exist some  $\bar{u} \in \mathcal{U}_{ad}$ , a triple



$(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$ , such that, up to a not relabeled subsequence, the following convergences

$$\begin{aligned} u_n &\rightarrow \bar{u} \text{ weakly star in } L^\infty(Q) \\ \mu_n &\rightarrow \bar{\mu} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \\ \varphi_n &\rightarrow \bar{\varphi} \text{ weakly star in } H^1(0, T; V) \cap L^\infty(0, T; W) \\ \sigma_n &\rightarrow \bar{\sigma} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \end{aligned}$$

are satisfied. Moreover, as made in (3.37), standard compactness results (see, e.g., [50, Sec. 8, Cor. 4]) imply that

$$\varphi_n \rightarrow \bar{\varphi} \text{ strongly in } C^0(\bar{Q}),$$

which also gives sense to the initial condition  $\bar{\varphi}(0) = \varphi_0$ . Thus, this latter, along with (2.4), (2.8) and (3.4), allows us to identify the nonlinear terms in the limit. In fact, as  $n \rightarrow \infty$ , we realize that

$$F'(\varphi_n) \rightarrow F'(\bar{\varphi}), \quad P(\varphi_n) \rightarrow P(\bar{\varphi}), \quad \text{both strongly in } C^0(\bar{Q}).$$

Next, we take into account the variational formulation of system 1.10–1.14, written for  $(\mu_n, \varphi_n, \sigma_n)$ , and pass to the limit as  $n \rightarrow \infty$ . Therefore, we realize that  $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$  is the unique solution to 1.10–1.14 associated with  $\bar{u}$ . Lastly, invoking the weak sequential lower semicontinuity of the cost functional  $\mathcal{J}$ , it turns out that  $\bar{u}$  is the minimizer we are looking for.  $\square$

**4.2. Approximation of optimal controls.** Once the existence has been proved, we would like to characterize the optimality of  $(CP)$  on account of some asymptotic schemes. If we want to let  $\beta \searrow 0$  in the optimality condition for  $(CP)_\beta$  in order to obtain the one for  $(CP)$ , we should ensure that every optimal control  $\bar{u}$  for  $(CP)$  can be found as a limit of a sequence consisting of optimal controls for  $(CP)_\beta$ . As anticipated, this strong condition is out of reach, so that we follow a different way making use of the approximated optimal control problem  $(\widetilde{CP})_\beta$ . In fact, we first show that  $(\widetilde{CP})_\beta$  can be solved, and then we precise in which sense it can be useful to deduce the necessary condition for optimality of  $(CP)$ . Furthermore, as it complies with the framework of [49], it is straightforward to obtain the result below.

**Lemma 4.1.** *Under the assumptions 2.1–2.10, whenever  $\beta \in (0, 1)$  is given, the optimal control problem  $(\widetilde{CP})_\beta$  admits at least a solution.*

Moreover, as a consequence of [49], it also follows the first-order optimality condition for optimality (compare with Theorem 2.1).

**Theorem 4.2.** *Assume that 2.1–2.10 are satisfied and let  $\bar{u}_\beta \in \mathcal{U}_{ad}$  be an optimal control for  $(\widetilde{CP})_\beta$  with the corresponding optimal state  $(\bar{\mu}_\beta, \bar{\varphi}_\beta, \bar{\sigma}_\beta)$ . Moreover, let  $(p_\beta, q_\beta, r_\beta)$  be the solution to the adjoint system 2.12–2.16. Then, the first-order necessary conditions for optimality reads as follows*

$$\int_Q (r_\beta + b_0 \bar{u}_\beta + (\bar{u}_\beta - \bar{u}))(v - \bar{u}_\beta) \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \quad (4.1)$$

Now, all the ingredients are set and we are in a position to properly state the approximation result we mentioned above.

**Theorem 4.3.** *Assume that 2.1–2.10 are in force. Moreover, let  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$  be an optimal triple for  $(CP)$  and let  $\{\beta_n\}_n$  be a sequence which goes to zero as  $n \rightarrow \infty$ . Then, there exists an approximating optimal triple, namely a triple  $(\bar{\varphi}_{\beta_n}, \bar{\sigma}_{\beta_n}, \bar{u}_{\beta_n})$  which solves  $(\widetilde{CP})_{\beta_n}$  and a not relabeled subsequence such that, as  $n \rightarrow \infty$ , we have the following convergences*

$$\bar{u}_n := \bar{u}_{\beta_n} \rightarrow \bar{u} \text{ strongly in } L^2(Q) \quad (4.2)$$

$$\bar{\varphi}_n := \bar{\varphi}_{\beta_n} \rightarrow \bar{\varphi} \text{ weakly star in } H^1(0, T; V) \cap L^\infty(0, T; W) \quad (4.3)$$

$$\bar{\sigma}_n := \bar{\sigma}_{\beta_n} \rightarrow \bar{\sigma} \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (4.4)$$

$$\tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \rightarrow \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}). \quad (4.5)$$

This theorem is the best we can say as far as the approximation of optimal controls of  $(CP)$  by sequences of optimal controls for an approximating problem is concerned. The proof mainly relies on monotonicity and compactness arguments.

*Proof of Theorem 4.3.* Let  $\beta \in (0, 1)$ ,  $(\bar{\varphi}_\beta, \bar{\sigma}_\beta, \bar{u}_\beta)$  be an optimal triple for  $(\widetilde{CP})_\beta$ , which exists by virtue of Lemma 4.1, and let  $\{\beta_n\}_n$  be a sequence which goes to zero as  $n \rightarrow \infty$ . For the sake of simplicity, with  $\bar{u}_n$  we denote the optimal control associated to  $\beta_n$ , that is,  $\bar{u}_n := \bar{u}_{\beta_n}$ . Likewise,  $\bar{\varphi}_n := \bar{\varphi}_{\beta_n}$  and  $\bar{\sigma}_n := \bar{\sigma}_{\beta_n}$ . In view of the boundedness of  $\mathcal{U}_{ad}$  and of estimates 3.33–3.35, there exist some  $\varphi, \sigma$ , and  $u$  such that, as  $n \rightarrow \infty$ , the convergences

$$\bar{u}_n \rightarrow u \text{ weakly star in } L^\infty(Q)$$

$$\bar{\varphi}_n \rightarrow \varphi \text{ weakly star in } H^1(0, T; V) \cap L^\infty(0, T; W)$$

$$\bar{\sigma}_n \rightarrow \sigma \text{ weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)$$

are verified. Moreover, we also realize that the limit  $(\varphi, \sigma, u)$  is an admissible triple for  $(CP)$ . Furthermore, we claim that  $(\varphi, \sigma, u)$  is nothing but  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$ , where  $\bar{u}$  is an optimal control for  $(CP)$ , whereas  $\bar{\varphi}$  and  $\bar{\sigma}$  are the corresponding states. Note that this would imply that the sequence  $(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n)$  approximates  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$  in the sense described above. The weak sequential lower semicontinuity of the adapted cost functional  $\tilde{\mathcal{J}}$  yields that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) &\geq \tilde{\mathcal{J}}(\varphi, \sigma, u) = \mathcal{J}(\varphi, \sigma, u) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 \\ &\geq \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) + \frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2, \end{aligned} \quad (4.6)$$

where we also take into account the optimality of  $(\bar{\varphi}, \bar{\sigma}, \bar{u})$  for  $(CP)$  and the definition of the adapted cost functional (1.15). On the other hand, the optimality of  $(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n)$  for  $(\widetilde{CP})_{\beta_n}$ , implies that

$$\tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) \quad \text{for every } n \in \mathbb{N}.$$

Hence, passing to the superior limit to both sides leads to deduce that

$$\limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}). \quad (4.7)$$

Finally, by combining (4.6) with (4.7), we infer that

$$\frac{1}{2} \|u - \bar{u}\|_{L^2(Q)}^2 = 0. \quad (4.8)$$

It is now straightforward to realize that also the corresponding states coincide, leading to conclude that  $(\varphi, \sigma, u) = (\bar{\varphi}, \bar{\sigma}, \bar{u})$ , as we claimed. Lastly, upon collecting the above information, we have the following chain of equality

$$\begin{aligned} \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) &= \tilde{\mathcal{J}}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \liminf_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \\ &= \lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \lim_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) + \frac{1}{2} \|\bar{u}_n - \bar{u}\|_{L^2(Q)}^2, \end{aligned} \quad (4.9)$$

Thus, we are reduced to show (4.2). Up to now, we have just proven that the weak limit of  $\bar{u}_n$  is  $\bar{u}$ . On the other hand, it easily follows from the above estimates, along with the lower semicontinuity of the cost functional, that

$$\begin{aligned} \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) &\leq \liminf_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \leq \limsup_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \\ &\leq \limsup_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \lim_{n \rightarrow \infty} \tilde{\mathcal{J}}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) = \mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}), \end{aligned}$$

so that

$$\mathcal{J}(\bar{\varphi}, \bar{\sigma}, \bar{u}) = \lim_{n \rightarrow \infty} \mathcal{J}(\bar{\varphi}_n, \bar{\sigma}_n, \bar{u}_n) \quad (4.10)$$

is verified. Therefore, by combining (4.9) with 4.8–4.10, we deduce that

$$\frac{1}{2} \|\bar{u}_n - \bar{u}\|_{L^2(Q)}^2 \rightarrow 0,$$

which conclude the proof.  $\square$

**4.3. The adjoint system.** Here, we are going to investigate the adjoint system proving Theorem 2.2. In order to avoid a heavy notation, we will omit writing the subscript  $\beta$  on the variables which occur in the calculations below, while we will reintroduce the correct notation at the end of each estimate. Before moving on, let us set

$$Q_t^T := \Omega \times [t, T], \quad \text{for every } t \in [0, T].$$

Below, we will proceed quite formally. The justification can be carried out rigorously, e.g., introducing a Galerkin scheme. Moreover, the adjoint system is linear and therefore, the uniqueness part easily follows by applying standard arguments from the existence part. On the other hand, system 2.12–2.16 has already been studied in [49, Sec. 4.4], and we refer there the interested reader for the details of the Galerkin technique.

*Proof of Theorem 2.2.* The a priori estimates we are going to point out will allow us to justify in a rigorous way the passage  $\beta \searrow 0$  in system 2.12–2.16.

**First estimate** To begin with, we add to both sides of (2.13) the term  $p$ . Then, we test (2.12) by  $-q$ , this new second equation by  $-\partial_t p$ , and (2.14) by  $r$ . Summing up and integrating over  $Q_t^T$  lead to

$$\begin{aligned} &\frac{\beta}{2} \int_{\Omega} |q(t)|^2 + \int_{Q_t^T} \partial_t p q + \int_{Q_t^T} |\nabla q|^2 + \int_{Q_t^T} F''(\bar{\varphi}) |q|^2 + \frac{1}{2} \int_{\Omega} |p(t)|^2 \\ &\quad - \int_{Q_t^T} \partial_t p q + \alpha \int_{Q_t^T} |\partial_t p|^2 + \frac{1}{2} \int_{\Omega} |\nabla p(t)|^2 + \frac{1}{2} \int_{\Omega} |r(t)|^2 + \int_{Q_t^T} |\nabla r|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{\beta}{2} \int_{\Omega} |q(T)|^2 + \frac{1}{2} \int_{\Omega} |p(T)|^2 + \frac{1}{2} \int_{\Omega} |\nabla p(T)|^2 + \frac{1}{2} \int_{\Omega} |r(T)|^2 \\
&\quad - b_1 \int_{Q_t^T} (\bar{\varphi} - \varphi_Q) q + b_2 \int_{Q_t^T} (\bar{\sigma} - \sigma_Q) r + \int_{Q_t^T} P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) q \\
&\quad + \int_{Q_t^T} P(\bar{\varphi})(p - r) \partial_t p - \int_{Q_t^T} p \partial_t p - \int_{Q_t^T} P(\bar{\varphi})(r - p) r. \tag{4.11}
\end{aligned}$$

On the left-hand side two integrals cancel out and, despite the fourth term, the others are nonnegative. As the fourth term is concerned, we remind that  $F''$  is bounded below in terms of the Lipschitz constant  $L$ , so that we have

$$\int_{Q_t^T} F''(\bar{\varphi}) |q|^2 \geq -L \int_{Q_t^T} |q|^2.$$

Next, we test (2.13) by  $Kq$ , for a positive constant  $K$  yet to be determined, and integrate over  $Q_t^T$  to get

$$K \int_{Q_t^T} |q|^2 = \alpha K \int_{Q_t^T} \partial_t p q - K \int_{Q_t^T} \nabla p \cdot \nabla q - K \int_{Q_t^T} P(\bar{\varphi})(p - r) q. \tag{4.12}$$

Then, after making use of the definition of the final conditions (2.16), we add 4.11–4.12 to obtain that

$$\begin{aligned}
&\frac{\beta}{2} \int_{\Omega} |q(t)|^2 + (K - L) \int_{Q_t^T} |q|^2 + \int_{Q_t^T} |\nabla q|^2 + \alpha \int_{Q_t^T} |\partial_t p|^2 \\
&\quad + \frac{1}{2} \int_{\Omega} |p(t)|^2 + \frac{1}{2} \int_{\Omega} |\nabla p(t)|^2 + \frac{1}{2} \int_{\Omega} |r(t)|^2 + \int_{Q_t^T} |\nabla r|^2 \\
&\leq \frac{b_3}{2} \int_{\Omega} |\bar{\sigma}(T) - \sigma_{\Omega}|^2 - b_1 \int_{Q_t^T} (\bar{\varphi} - \varphi_Q) q + b_2 \int_{Q_t^T} (\bar{\sigma} - \sigma_Q) r \\
&\quad + \int_{Q_t^T} P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) q + \int_{Q_t^T} P(\bar{\varphi})(p - r) \partial_t p \\
&\quad - \int_{Q_t^T} p \partial_t p - \int_{Q_t^T} P(\bar{\varphi})(r - p) r + \alpha K \int_{Q_t^T} \partial_t p q \\
&\quad - K \int_{Q_t^T} \nabla p \cdot \nabla q - K \int_{Q_t^T} P(\bar{\varphi})(p - r) q,
\end{aligned}$$

where we denote by  $I_1, \dots, I_{10}$  the integrals on the right-hand side, in that order. Now, we start estimating the terms on the right-hand side. Owing to assumptions 2.2–2.3, we easily realize that

$$|I_1| \leq c.$$

Meanwhile, the integrals  $I_2$  and  $I_3$  can be easily managed by applying the Young inequality and the fact that  $\bar{\varphi}$  and  $\bar{\sigma}$ , as solutions to 1.2–1.6, satisfy (3.4). In fact, we have that

$$|I_2| + |I_3| \leq \delta \int_{Q_t^T} |q|^2 + \frac{1}{2} \int_{Q_t^T} |r|^2 + c_{\delta},$$

for a small and positive  $\delta$  yet to be determined. Invoking the Hölder and Young inequalities, the continuous embeddings  $V \subset L^4(\Omega)$  and  $V \subset L^6(\Omega)$ , assumption

(2.4), and estimate (3.4), we infer that

$$\begin{aligned} |I_4| &\leq c \int_{Q_t^T} |\bar{\sigma} - \bar{\mu}| |r - p| |q| \leq c \int_t^T \|\bar{\sigma} - \bar{\mu}\|_6 \|r - p\|_2 \|q\|_3 \\ &\leq \delta \int_t^T \|q\|_V^2 + c_\delta \int_t^T (\|\bar{\sigma}\|_V^2 + \|\bar{\mu}\|_V^2) (\|r\|_H^2 + \|p\|_H^2) \\ &\leq \delta \int_{Q_t^T} (|q|^2 + |\nabla q|^2) + c_\delta \int_{Q_t^T} (|r|^2 + |p|^2). \end{aligned}$$

By the same token, using (2.32), we get

$$|I_5| \leq c \int_{Q_t^T} |p - r| |\partial_t p| \leq \delta \int_{Q_t^T} |\partial_t p|^2 + c_\delta \int_{Q_t^T} (|p|^2 + |r|^2).$$

Using the Young inequality once more, we realize that

$$|I_6| + |I_7| \leq \delta \int_{Q_t^T} |\partial_t p|^2 + c_\delta \int_{Q_t^T} |p|^2 + c \int_{Q_t^T} |r|^2,$$

and also that

$$|I_9| + |I_{10}| \leq \delta \int_{Q_t^T} |\nabla q|^2 + c_\delta \int_{Q_t^T} |\nabla p|^2 + \delta \int_{Q_t^T} |q|^2 + c_\delta \int_{Q_t^T} (|p|^2 + |r|^2).$$

Lastly, owing to (2.32),  $I_8$  can be dealt by

$$|I_8| = \left| \alpha K \int_{Q_t^T} \partial_t p q \right| \leq \frac{K}{2} \int_{Q_t^T} |q|^2 + \frac{\alpha^2 K}{2} \int_{Q_t^T} |\partial_t p|^2.$$

Collecting all the previous estimates, we realize that the backward-in-time Gronwall lemma yields the estimate we are looking for, provided we check that  $K$  and  $\delta$  can be chosen in such a way to satisfy the following condition

$$\min \left\{ K - \frac{K}{2} - L - 3\delta, 1 - 2\delta, \alpha - \frac{\alpha^2 K}{2} - 2\delta \right\} > 0.$$

Actually, considering that  $\delta$  can be taken arbitrarily small, we are reduced to show that there exists a positive constant  $K$  such that

$$\min \left\{ \frac{K}{2} - L, \alpha - \frac{\alpha^2 K}{2} \right\} > 0.$$

Let us claim that this is satisfied if  $\alpha$  is small enough. For instance, we take  $K = 3L$ , so that

$$\min \left\{ \frac{K}{2} - L, \alpha - \frac{\alpha^2 K}{2} \right\} = \min \left\{ \frac{L}{2}, \frac{\alpha}{2} (2 - \alpha 3L) \right\}.$$

Hence, we assume at once  $\alpha$  to be small in order that  $2 - \alpha 3L > 0$ . Lastly, we pick  $\delta$  small enough to conclude. Finally, a Gronwall argument implies that

$$\begin{aligned} &\beta^{1/2} \|q_\beta\|_{L^\infty(0,T;H)} + \|q_\beta\|_{L^2(0,T;V)} + \|p_\beta\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \\ &+ \|r_\beta\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \end{aligned} \quad (4.13)$$

**Second estimate** Then, we test (2.13) by  $-\Delta p$  and integrate over  $Q_t^T$  to obtain that

$$\frac{\alpha}{2} \int_\Omega |\nabla p(t)|^2 + \int_{Q_t^T} |\Delta p|^2 = \frac{\alpha}{2} \int_\Omega |\nabla p(T)|^2 + \int_{Q_t^T} q \Delta p + \int_{Q_t^T} P(\bar{\varphi})(p - r) \Delta p,$$

where we denote the terms on the right-hand side by  $I_1, I_2$ , and  $I_3$ , respectively. Owing to the final conditions (2.16), we easily conclude that  $I_1 = 0$ . Moreover, Young's inequality, combined with the boundedness of  $P$ , gives that

$$|I_2| + |I_3| \leq \frac{1}{2} \int_{Q_t^T} |\Delta p|^2 + \int_{Q_t^T} |q|^2 + c \int_{Q_t^T} (|p|^2 + |r|^2).$$

Hence, the previous estimate produces

$$\|\nabla p_\beta\|_{L^\infty(0,T;H)} + \|\Delta p_\beta\|_{L^2(0,T;H)} \leq c, \quad (4.14)$$

from which, applying standard elliptic regularity results for homogeneous Neumann boundary problems, we obtain that

$$\|p_\beta\|_{L^\infty(0,T;V) \cap L^2(0,T;W)} \leq c. \quad (4.15)$$

**Third estimate** Furthermore, we test (2.14) first by  $-\partial_t r$  and secondly by  $-\Delta r$  to get the following parabolic regularity

$$\|r_\beta\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq c. \quad (4.16)$$

**Fourth estimate** Next, by testing (2.12) by  $\Delta q$  and integrating over  $Q_t^T$ , we find that

$$\begin{aligned} \frac{\beta}{2} \int_{\Omega} |\nabla q(t)|^2 + \int_{Q_t^T} |\Delta q|^2 &= \frac{\beta}{2} \int_{\Omega} |\nabla q(T)|^2 + \int_{Q_t^T} \partial_t p \Delta q + \int_{Q_t^T} F''(\bar{\varphi}) q \Delta q \\ &\quad - \int_{Q_t^T} P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) \Delta q + \int_{Q_t^T} b_1(\bar{\varphi} - \varphi_Q) \Delta q, \end{aligned}$$

where we indicate the terms on the right-hand side by  $I_1, \dots, I_5$ , in that order. In a similar fashion as in the previous estimates, we first observe that  $I_1 = 0$ . Next, owing to the Young and Hölder inequalities, to the boundary of  $P$ , and to the continuous embedding (2.33), the remaining integrals can be dealt as

$$\begin{aligned} |I_2| + |I_3| + |I_4| + |I_5| &\leq \frac{4}{5} \int_{Q_t^T} |\Delta q|^2 + c \int_{Q_t^T} |\partial_t p|^2 + c \int_{Q_t^T} |q|^2 \\ &\quad + c \int_{Q_t^T} (|p|^2 + |r|^2) + c, \end{aligned}$$

where estimate (3.4) for the solutions  $\bar{\mu}$  and  $\bar{\sigma}$ , is also taken into account. Hence, we have that

$$\beta^{1/2} \|\nabla q_\beta\|_{L^\infty(0,T;H)} + \|\Delta q_\beta\|_{L^2(0,T;H)} \leq c, \quad (4.17)$$

and the regularity results for elliptic equations with homogeneous Neumann boundary conditions, entails that

$$\beta^{1/2} \|\nabla q_\beta\|_{L^\infty(0,T;H)} + \|q_\beta\|_{L^2(0,T;W)} \leq c. \quad (4.18)$$

**Fifth estimate** Lastly, we rearrange equation (2.12) in the following way

$$\beta \partial_t q = \partial_t p - \Delta q + F''(\bar{\varphi}) q - P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) + b_1(\bar{\varphi} - \varphi_Q).$$

Therefore, by comparison in the above equation, we also realize that

$$\beta \|\partial_t q_\beta\|_{L^2(0,T;H)} \leq c. \quad (4.19)$$

**Passing to the limit** Summing up, upon combining the above estimates we recover estimate (2.28). Moreover, we infer that there exist some variables  $q, p$  and  $r$  such that, up to a not relabeled subsequence, as  $\beta \searrow 0$ , the convergences mentioned by

2.24–2.27 hold. Furthermore, these uniform bounds, along with standard compactness embedding results, allow us to recover also the following strong convergences

$$p_\beta \rightarrow p \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V) \quad (4.20)$$

$$r_\beta \rightarrow r \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V). \quad (4.21)$$

Then, we try to draw some consequences from the aforementioned a priori bounds in order to pass to the limit, as  $\beta \searrow 0$ , in the adjoint system 2.12–2.16. For the sake of convenience, we rewrite its variational formulation which can be obtained by testing the system by an arbitrary  $v \in V$  and integrating over  $\Omega$ . It reads as follows

$$\begin{aligned} & \beta \int_{\Omega} \partial_t q_\beta(t) v - \int_{\Omega} \partial_t p_\beta(t) v - \int_{\Omega} \nabla q_\beta(t) \cdot \nabla v - \int_{\Omega} F''(\bar{\varphi}_\beta(t)) q_\beta(t) v \\ & \quad + \int_{\Omega} P'(\bar{\varphi}_\beta(t)) (\bar{\sigma}_\beta(t) - \bar{\mu}_\beta(t)) (r_\beta(t) - p_\beta(t)) v = \int_{\Omega} b_1(\bar{\varphi}_\beta(t) - \varphi_Q(t)) v \\ & \hspace{15em} \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & \int_{\Omega} q_\beta(t) v - \alpha \int_{\Omega} \partial_t p_\beta(t) v + \int_{\Omega} \nabla p_\beta(t) \cdot \nabla v \\ & \quad + \int_{\Omega} P(\bar{\varphi}_\beta(t)) (p_\beta(t) - r_\beta(t)) v = 0 \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & - \int_{\Omega} \partial_t r_\beta(t) v + \int_{\Omega} \nabla r_\beta(t) \cdot \nabla v + \int_{\Omega} P(\bar{\varphi}_\beta(t)) (r_\beta(t) - p_\beta(t)) v \\ & \quad = \int_{\Omega} b_2(\bar{\sigma}_\beta(t) - \sigma_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T), \end{aligned}$$

and the final condition

$$\int_{\Omega} r_\beta(T) v = \int_{\Omega} b_3(\bar{\sigma}_\beta(T) - \sigma_\Omega) v \quad \text{for every } v \in V.$$

At this point, we would invoke the above convergences 2.24–2.27 to show that in the limit, as  $\beta \searrow 0$ , we find

$$\begin{aligned} & - \int_{\Omega} \partial_t p(t) v - \int_{\Omega} \nabla q(t) \cdot \nabla v - \int_{\Omega} F''(\bar{\varphi}(t)) q(t) v \\ & \quad + \int_{\Omega} P'(\bar{\varphi}(t)) (\bar{\sigma}(t) - \bar{\mu}(t)) (r(t) - p(t)) v = \int_{\Omega} b_1(\bar{\varphi}(t) - \varphi_Q(t)) v \\ & \hspace{15em} \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & \int_{\Omega} q(t) v - \alpha \int_{\Omega} \partial_t p(t) v + \int_{\Omega} \nabla p(t) \cdot \nabla v \\ & \quad + \int_{\Omega} P(\bar{\varphi}(t)) (p(t) - r(t)) v = 0 \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T) \\ & - \int_{\Omega} \partial_t r(t) v + \int_{\Omega} \nabla r(t) \cdot \nabla v + \int_{\Omega} P(\bar{\varphi}_\beta(t)) (r_\beta(t) - p_\beta(t)) v \\ & \quad = \int_{\Omega} b_2(\bar{\sigma}(t) - \sigma_Q(t)) v \quad \text{for every } v \in V, \text{ for a.a. } t \in (0, T), \end{aligned}$$

and the final condition

$$\int_{\Omega} r(T) v = \int_{\Omega} b_3(\bar{\sigma}(T) - \sigma_\Omega) v \quad \text{for every } v \in V,$$

which corresponds to the variational formulation associated to system 2.19–2.23. Nonetheless, since there appear some nonlinear terms, some care is in order. First, let us recall that both  $P$  and  $F$  are regular due to (2.4) and (2.9) and that (3.27) holds. Hence, exploiting the strong convergence (3.37), we claim that, as  $\beta \searrow 0$ , we have

$$F''(\bar{\varphi}_\beta) \rightarrow F''(\bar{\varphi}) \quad \text{strongly in } C^0(\bar{Q}) \quad (4.22)$$

$$P(\bar{\varphi}_\beta) \rightarrow P(\bar{\varphi}) \quad \text{strongly in } C^0(\bar{Q}). \quad (4.23)$$

To prove the former, it suffices to combine (2.10) and (3.37) with the estimate (3.27), while for the latter we simply account for (3.37) and for the boundedness of  $P$ . Moreover, having in mind the weak convergence (2.24) and the strong ones 4.20–4.21, we can prove that the nonlinear terms can be identified in the limit. In fact, from (2.24), we have that

$$q_\beta \rightarrow q \quad \text{at least weakly in } L^2(0, T; H),$$

which, along with (4.22), leads to infer that

$$F''(\bar{\varphi}_\beta)q_\beta \rightarrow F''(\bar{\varphi})q \quad \text{at least weakly in } L^2(0, T; H).$$

Similarly, combining 4.20–4.21 with (4.23), we also deduce that

$$P(\bar{\varphi}_\beta)(p_\beta - r_\beta) \rightarrow P(\bar{\varphi})(p - r) \quad \text{strongly in } L^2(Q)$$

$$P'(\bar{\varphi}_\beta)(\bar{\sigma}_\beta - \bar{\mu}_\beta)(r_\beta - p_\beta) \rightarrow P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) \quad \text{strongly in } L^1(Q),$$

where the first one requires the help of the strong convergence pointed out by (3.37) and the boundedness of  $P$ , whereas in the second we again owe to (3.37), along with the strong convergences 3.38–3.39. To completely recover system 2.19–2.23 it suffices to check that the regularity is enough to rewrite the system in a strong form. So, this is the sense in which we can say that system 2.12–2.16 converges, as  $\beta \searrow 0$ , to 2.19–2.23.  $\square$

**4.4. First-order necessary condition.** We conclude the paper providing the first-order necessary condition that an optimal control, which exists in view of Theorem 2.3, has to verify.

*Proof of Theorem 2.4.* As previously mentioned, in order to get inequality (2.31), we cannot pass to the limit as  $\beta \searrow 0$  in the variational inequality (2.18) since nothing ensures that in such a passage, the control  $\bar{u}_\beta$  will converge to a limit that is also optimal for (CP). Therefore, the investigation made in the subsection 4.2 helps to rigorously handle this issue. Indeed, we are going to consider a sequence  $\{\beta_n\}$  which goes to zero as  $n \rightarrow \infty$  and then we take into account  $\bar{u}_n := \bar{u}_{\beta_n}$  instead of  $\bar{u}_\beta$ . Therefore, after extraction of a subsequence  $\{\beta_{n_k}\}$ , the asymptotics pointed out by 2.24–2.27 and 4.2–4.5 allow us to pass to the limit as  $k \rightarrow \infty$  in (4.1) to obtain (2.31).

Furthermore, the last sentence immediately follows as a straightforward application of the Hilbert projection theorem, since  $\mathcal{U}_{ad}$  is a non-empty, closed and convex subset of  $L^2(0, T; H)$ . Moreover, let us note that (2.31) implies that, whenever  $b_0 > 0$ , the optimal control  $\bar{u}$  can be implicitly characterized as follows (see, e.g., [52])

$$\bar{u}(x, t) = \max\left\{u_*(x, t), \min\left\{u^*(x, t), -\frac{1}{b_0}r(x, t)\right\}\right\} \quad \text{for a.a. } (x, t) \in Q.$$

$\square$



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