



Transfer of Energy from Flexural to Torsional Modes for the Fish-Bone Suspension Bridge Model

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Abstract. We consider a conservative coupled oscillators system which arises as a simplified model of the interaction of flexural and torsional modes of vibration along the deck of the so-called fish-bone (Berchio and Gazzola in *Nonlinear Anal* 121:54–72, 2015) model of suspension bridges. The elastic response of the cables is supposed to be asymptotically linear under traction, and asymptotically constant when compressed (a generalization of the slackening regime). We show that for vibrations of sufficiently large amplitude, transfer of energy from flexural modes to torsional modes may occur provided a certain condition on the parameters is satisfied. The main result is a non-trivial extension of a theorem in Marchionna and Panizzi (*Nonlinear Anal* 140:12–28, 2016) to the case when the frequencies of the normal modes are no more supposed to be the same. Several numerical computations of instability diagrams for various slackening models respecting our assumptions are presented.

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1. Introduction

In this paper, which is a completion of a previous work [15], we face the question of energy exchange between torsional and longitudinal modes in a generalized energy conserving suspension bridge model, the so-called *fish-bone* model, proposed by K.S. MOORE [21], in which the elastic response of the cables is asymptotically linear under traction, and asymptotically constant when compressed (a generalization of the slackening regime).

We follow the line of research carried out by F. GAZZOLA and coworkers in a series of papers [2, 4, 10] according to which internal nonlinear resonances may

occur even when the aeroelastic coupling is disregarded. To analyze the onset of resonances, the PDEs system (infinite dimensional system) is reduced to a coupled system of nonlinear ODEs, by projecting the infinite dimensional phase space on a two dimensional subspace through an approximate Galerkin technique. The ODEs system is then a simplified model of the interaction of flexural and torsional modes of vibration along the deck of the bridge.

The same problem was addressed in a previous paper [15], in which only the interaction between the modes corresponding to the fundamental flexural and torsional frequencies was considered. In the present work, we extend the study to the more general case in which modes of different frequencies can interact.

The main result is Theorem 2.1 in which we prove that for deflections of sufficiently large amplitude, pure flexural and periodic vibrations are unstable, provided a certain condition on the structural parameters of the bridge is satisfied, so that transfer of energy from flexural modes to torsional modes may occur even when no external forces are applied.

Our technique follows the lines of previous literature: we consider the differential of the Poincaré map relative to the torsional component of the system, around a periodic purely flexural solution. As is well-known this leads to a Hill equation; then, as the energy of the system tends to infinity, we compute the limit equation, which turns out to be simple enough to allow the analysis of its stability via the classical Floquet theory.

If the transverse and torsional modes are labeled by j and k respectively, an unexpected property holds when j is even: the instability condition at large deflections of Theorem 2.1 is never satisfied, since the limit system is an uncoupled system of linear, constant coefficients oscillators. Consequently, every solution of the limit system is periodic thus stable, so that purely flexural solutions are asymptotically linearly stable.

The extension of the result in [15] is non-trivial because 2 main steps are involved: the quite subtle derivation of the C^1 -regularity of the projected system under the relaxed regularity assumption (\mathbf{S}_0) on the slackening function; the computation of the limit Hill equation at large energies, which requires several technicalities.

In order to obtain a more general picture of the stability of the flexural component, in Sect. 3 we look at the Hill equation relative to the torsional component as dependent on two parameters. The first natural parameter is the maximum elongation of the purely flexural solution, which in Theorem 2.1 tends to ∞ ; we note that the dependence on such parameter is nonlinear. The choice of the second parameter is motivated by mathematical reasons, as the equation naturally presents a spectral parameter which, from the structural point of view, essentially corresponds to the torsional behavior of the bridge. In this way, we can compare the results related to our problem with the instability diagrams in the literature for classical two-parameters Hill equations. In particular, we consider a slackening model that has minimal regularity with respect to our requests, using both academic and real parameters corresponding to Tacoma narrow bridge; we numerically draw the corresponding instability diagrams, pointing out which properties are preserved and which diverge from the classical ones; some interesting mathematical aspects are

highlighted, such as the presence of resonance pockets typical of some two parameters Hill equations, *e.g.* the multi-step Meissner equation.

The remainder of this section is devoted to the presentation of the PDEs model, and its reduction to the ODEs system. In Sect. 2 we provide the proof of the main instability result. The work ends with three appendices: in ‘‘Appendix A’’ we prove the regularity of the ODEs system; the computation of the limit system at large energies is presented in ‘‘Appendix B’’; finally, ‘‘Appendix C’’ provides the explicit computation of the instability discriminant for a piecewise linear system leading to a multi-step Meissner equation.

1.1. The Bridge Model

We briefly recall the PDEs system. The dynamics of the midline of the deck, modeled as an Euler–Bernoulli beam of length L and width $2l$, is coupled with the elastic response of the suspension cables acting on the side ends of the deck. The cross section of the deck is assumed to be a rigid rod with mass density ρ , length $2l$ and thickness small with respect to l ; $Y(x, t)$ is the vertical downward deflection of the midline of the deck with respect to the unloaded state, $\Theta(x, t)$ is the angle of rotation of the deck with respect to the horizontal position. The corresponding PDEs system is given by,

$$\begin{cases} Y_{tt} + \frac{EI}{\rho S} Y_{xxxx} + f(Y + l \sin \Theta) + f(Y - l \sin \Theta) = 0, \\ \Theta_{tt} - \frac{GK}{\rho J} \Theta_{xx} + \frac{Sl}{J} \cos \Theta [f(Y + l \sin \Theta) - f(Y - l \sin \Theta)] = 0, \end{cases} \quad (1)$$

complemented with hinged boundary conditions:

$$Y(0, t) = Y(L, t) = Y_{xx}(0, t) = Y_{xx}(L, t) = 0, \quad \Theta(0, t) = \Theta(L, t) = 0. \quad (2)$$

The other constant parameters are: S the cross section area, I the planar second moment of area with respect to the plane $Y = 0$, J the polar second moment of area with respect to the x -axis and E and G respectively the Young modulus and the shear modulus, K the torsional constant.

The restoring force f exerted by the hangers is applied to both extremities of the deck whose displacements from the unloaded state are given by $Y \pm l \sin \Theta$. No external forces, except gravity, are taken in account.

In the classical slackening regime, the hangers behave as linear springs of elastic constant $k > 0$ if stretched and do not exert restoring force if compressed. A first model in which the system (1) acts as a linear, non coupled system for small displacements, was proposed by K.S MOORE and P.J. McKENNA [18, 21] (hereforth MMK model) assuming for f the following expression (g is the gravity),

$$f(r) = m [(r + r_0)^+ - r_0], \quad r_0 = \rho S g / 2k, \quad m = \frac{k}{\rho S}. \quad (3)$$

Subsequently many other forms for f have been proposed in [4, 15, 16, 19], some of these are nonlinear and smooth in a neighborhood of the origin, making instability feasible even at low energies. Two significant examples are (h is a positive constant):

$$f(r) = mr + \sqrt{(mr)^2 + h^2} - h, \quad (4)$$

$$f(r) = h(e^{mr/h} - 1), \quad (5)$$

Throughout the paper we assume that the function f satisfies the following mild regularity condition:

Assumption (S₀)

- (a) f is continuous, increasing, and $f(0) = 0$;
- (b) f is piecewise C^1 , that is its derivative is continuous with the exception of a finite (eventually empty) set of points $r_1 < r_2 < \dots < r_n$ **not including zero** in which there exist the finite limits:

$$\lim_{r \rightarrow r_i^\pm} f'(r);$$

- (c) $m := f'(0) > 0$.

In most cases the elastic response of the cables is supposed to be asymptotically linear under traction, and asymptotically constant when compressed, thus it is natural to assume at least one of the following conditions:

- (S₁) $\lim_{r \rightarrow -\infty} f'(r) = 0$
- (S₂) $M := \lim_{r \rightarrow +\infty} f'(r) > 0$.

We note that both (3) and (4) satisfy (S₀)–(S₁)–(S₂), unlike the function (5) which does not satisfies condition (S₂).

The problem (1)–(2) is well-posed in appropriate Sobolev spaces [4, 8], and enjoys two properties mostly relevant for our purposes: the total energy is conserved over time; it admits *pure flexural solutions*, that is motions in which the cross sections of the deck remain horizontal at all times so that no torsional vibrations occur.

1.2. The ODEs System

For small torsional angles, it is very convenient to replace the system (1) with a pre-linearized one, see [4, 15]. By the usual approximation: $\sin \Theta \sim \Theta$, $\cos \Theta \sim 1$, and by setting $Z = l\Theta$, the system (1) reduces to

$$\begin{cases} Y_{tt} + \frac{EI}{\rho S} Y_{xxxx} + f(Y + Z) + f(Y - Z) = 0 \\ Z_{tt} - \frac{GK}{\rho J} Z_{xx} + \frac{l^2 S}{J} [f(Y + Z) - f(Y - Z)] = 0, \end{cases} \quad (6)$$

and as far as the scope of this paper is concerned, nothing changes starting from the system (1) or from (6).

Our *ansatz* is that, after a suitable rescaling of the space variable, the displacements can be reasonably well approximated by the $j - k$ mode of vibration, that is,

$$Y(x, t) \simeq y_j(t) \sin(jx), \quad Z(x, t) \simeq z_k(t) \sin(kx), \quad 0 \leq x \leq \pi.$$

Then, through a Galerkin projection, the PDEs system reduces to a coupled oscillators system which, dropping the indexes $j - k$, reads as follows:

$$\begin{cases} \ddot{y} + \alpha j^4 y + \psi_1(y, z) = 0 \\ \ddot{z} + \beta k^2 z + \gamma \psi_2(y, z) = 0, \end{cases} \quad (7)$$

with structural parameters, $\alpha = EI\pi^4/\rho SL^4$, $\beta = GK\pi^2/\rho JL^2$, $\gamma = l^2S/J$, and nonlinear coupling terms,

$$\psi_1(y, z) = \frac{2}{\pi} \int_0^\pi [f(y \sin(jx) + z \sin(kx)) + f(y \sin(jx) - z \sin(kx))] \sin(jx) dx, \quad (8)$$

$$\psi_2(y, z) = \frac{2}{\pi} \int_0^\pi [f(y \sin(jx) + z \sin(kx)) - f(y \sin(jx) - z \sin(kx))] \sin(kx) dx. \quad (9)$$

If we define

$$\Psi(y, z) = \frac{2}{\pi} \int_0^\pi [\mathcal{F}(y \sin(jx) + z \sin(kx)) + \mathcal{F}(y \sin(jx) - z \sin(kx))] dx,$$

where $\mathcal{F}(r) = \int_0^r f(s) ds$, we have $\psi_1 = \partial\Psi/\partial y$, $\psi_2 = \partial\Psi/\partial z$, so that the system (7) admits a conserved energy,

$$\mathcal{E}(y, \dot{y}, z, \dot{z}) = \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2\gamma} + \frac{\alpha j^4}{2} y^2 + \frac{\beta k^2}{2\gamma} z^2 + \Psi(y, z). \quad (10)$$

Note that, under Assumption **(S₀)**, $\Psi(y, z)$ is nonnegative. As a consequence all solutions of (7) are global and bounded.

Since $\psi_2(y, 0) \equiv 0$, the system (7) admits periodic pure flexural solutions, that is solutions of the form $y = u(t)$, $z \equiv 0$ with $u(t)$ periodic. We consider such solutions as parametrized by the initial displacement, and we define $u = u(t; q)$ as the solution of the initial value problem,

$$\ddot{u} + \alpha j^4 u + 2f_j(u) = 0 \quad u(0) = q, \quad \dot{u}(0) = 0. \quad (11)$$

where

$$f_j(r) := \frac{1}{2} \psi_1(r, 0) = \frac{2}{\pi} \int_0^\pi f(r \sin jx) \sin jx dx. \quad (12)$$

Assuming for the moment that $f \in C^1$, the linearization at a fixed energy level (iso-energetic linearization) of the system around the periodic orbit $(u(\cdot, q), 0)$ yields, for the torsional component, the Hill equation (see *e.g.* [6, 15] for details),

$$\ddot{v} + (\beta k^2 + 2\gamma g_{j,k}(u(t; q))) v = 0, \quad (13)$$

in which we have set,

$$g_{j,k}(r) = \frac{1}{2} \frac{\partial \psi_2}{\partial z}(r, 0) = \frac{2}{\pi} \int_0^\pi f'(r \sin jx) \sin^2 kx dx. \quad (14)$$

The problem we want to address is the stability of solutions of the Hill equation (13). It is worth noting that in the case $j = k$, we have $g_{j,j}(r) = f'_j(r)$, and the linearized system (11)–(13) is the same as the one studied in [15]. This is no longer true if $j \neq k$. In [15], under the assumptions **(S₀)**–**(S₁)**–**(S₂)**, we established a condition depending on a set of 3 parameters under which the flexural motions are unstable provided the energy parameter q is sufficiently large. The next section is devoted to the main result of the present paper which is a non-trivial extension of the result in [15] to the case $j \neq k$.

2. Instability of Pure Flexural $j - k$ Modes at High Energies

The stability analysis of (13) is carried out by means of Floquet's theorem, see [7, 14]. We recall the definition of the stability discriminant $\Delta = \Delta(q)$ of the Hill equation. Let $\tau(q)$ be the period of the solution u of the problem (11), and let $v_0(t)$, $v_1(t)$ be the solutions of (13) corresponding to initial data

$$v_0(0) = 1, \dot{v}_0(0) = 0, \quad v_1(0) = 0, \dot{v}_1(0) = 1;$$

then

$$\Delta(q) = v_0(\tau(q)) + v_1'(\tau(q)).$$

If $|\Delta| > 2$ the non-trivial solutions of the Hill equation are unbounded, if $|\Delta| < 2$ are all bounded. In the case when $\Delta = 2$ there exists at least a non trivial $\tau(q)$ -periodic solution, when $\Delta = -2$ there exists at least a non-trivial $2\tau(q)$ -periodic solution.

The main result of this section consists in the the computation of

$$\Delta_\infty := \lim_{q \rightarrow \infty} \Delta(q),$$

in the case when the elastic response of the cables is asymptotically linear, *i.e.* under assumptions (S₁)–(S₂). Referring to the characterization of instability $|\Delta_\infty| > 2$, we can establish a condition, depending on j , k , and the structural parameters, for which there is instability for sufficiently high energies. The proof mimics that of Theorem 1 in [15]. First we are able to compute the limit system of (11)–(13) as q goes to $+\infty$; it turns out that the Hill equation of the limit system is a two-step potential (Meissner equation) that can be integrated explicitly; the condition (15) expresses the instability condition for q sufficiently large through the discriminant Δ_∞ of the limit equation.

Theorem 2.1. *Assume that the function f satisfies the conditions (S₀)–(S₁)–(S₂).*

Assume that j is odd, and let the constants ω_\pm , A_\pm , ϕ_\pm , be defined as follows:

$$\begin{aligned} \omega_\pm^2 &= \alpha j^4 + M \left(1 \pm \frac{1}{j} \right), \\ A_\pm^2 &= \beta k^2 + \gamma M (1 \pm \epsilon_{j,k}), \quad \epsilon_{j,k} = \frac{1}{j} - \frac{\tan(\pi k/j)}{k\pi}, \\ \phi_\pm &= \frac{A_\pm}{\omega_\pm} \pi, \quad a = \frac{A_+}{A_-}. \end{aligned}$$

Then, if the following condition holds true,

$$\frac{|\Delta_\infty|}{2} := \left| \cos \phi_+ \cos \phi_- - \frac{a + a^{-1}}{2} \sin \phi_+ \sin \phi_- \right| > 1. \quad (15)$$

there exists q_0 such that, if $q > q_0$, the pure flexural periodic solution $(u(t; q), 0)$ of the (non-linear) system (7) is unstable.

From the above expression it is not obvious that A_\pm^2 are positive constants. Actually the two quantities $1 \pm \epsilon_{j,k}$ result as the two possible values (+ for $r > 0$, and – for $r < 0$) of the positive integral $\frac{4}{\pi} \int_0^\pi H(r \sin jx) \sin^2 kx \, dx$, see (18), (20) below, and “Appendix B”.

Remark 2.2. In Theorem 2.1 the case even j is not considered since the system decouples and solutions are bounded, hence stable. Indeed, thanks to (19), we can write explicitly the limit system as follows,

$$\begin{cases} \ddot{U}_\infty + (\alpha j^4 + M)U_\infty = 0 \\ \ddot{v}_\infty + (\beta k^2 + \gamma M)v_\infty = 0. \end{cases}$$

As consequence of the decoupling, condition (15) is never satisfied.

2.1. Linearized $j - k$ Mode: Technical Tools

First of all we need a regularity result which is crucial for the linearization process of the system (7) around a pure flexural solution.

Proposition 2.3. *If the function f satisfies assumption (\mathbf{S}_0) , then the functions ψ_1, ψ_2 , as defined in (8)–(9), are of class $C^1(\mathbb{R}^2)$.*

The proof of this Proposition is given in “Appendix A”.

Then we list some general facts about the functions $f_j(r), g_{j,k}(r)$ defined in (12), (14). Given any function f , accordingly with [15], we define the integral transform,

$$\tilde{f}(r) := \frac{2}{\pi} \int_0^\pi f(r \sin x) \sin x \, dx. \quad (16)$$

We denote by $f_e(x) = \frac{1}{2}(f(x) + f(-x))$, $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ even and odd parts of a function respectively. We have the following,

Lemma 2.4. *If the function f satisfies the first two conditions of the assumption (\mathbf{S}_0) , then $f_j \in C^1(\mathbb{R})$, $f_j(0) = 0$ and $f'_j \geq 0$.*

Moreover we have,

$$f_j = \tilde{f}_o \quad (\text{even } j), \quad f_j = \tilde{f}_o + \frac{1}{j} \tilde{f}_e \quad (\text{odd } j). \quad (17)$$

Proof. The regularity and the sign of f'_j follow from the analogous properties of \tilde{f} proved in [15], condition $(\tilde{\mathbf{H}})$. To prove (17), we use the elementary identity $\sin(z + h\pi) = (-1)^h \sin z$, to compute

$$\begin{aligned} \int_0^\pi f(r \sin jx) \sin jx \, dx &= \frac{1}{j} \int_0^{j\pi} f(r \sin z) \sin z \, dz \\ &= \frac{1}{j} \sum_{h=0}^{j-1} (-1)^h \int_0^\pi f((-1)^h r \sin z) \sin z \, dz. \end{aligned}$$

By collecting the signs, and by distinguishing the two cases even or odd j , we obtain (17). \square

Lemma 2.5. *If j is even, then $g_{j,k}(r)$ is an even function for every k . More precisely, we have*

$$g_{j,k}(r) = \frac{2}{\pi} \int_0^\pi f'_o(r \sin jx) \sin^2 kx \, dx.$$

Proof. If j is even, $f'(r \sin jx)(\sin kx)^2$ has period π , then

$$g_{j,k}(r) = \frac{2}{\pi} \int_0^\pi f'(r \sin jx) \sin^2 kx \, dx = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f'(r \sin jx) \sin^2 kx \, dx.$$

It follows that

$$g_{j,k}(r) = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} (f'_e(r \sin jx) \sin^2 kx + f'_o(r \sin jx) \sin^2 kx) \, dx$$

and the integral of the first term is null since f'_e is odd. \square

We make an observation to highlight some differences that may exist between even and odd harmonics of the flexural component. If the function f is as in example (4), its odd part is linear, precisely we have $f_o(r) = mr$. Then, if j is even, the Eq. (13) reduces to the linear oscillator,

$$\ddot{v} + (\beta k^2 + 2\gamma m)v = 0.$$

Since $\beta k^2 + 2\gamma m > 0$, the $j - k$ modes are always linearly stable. This is no more true for odd j , see formula (17).

We end this section by introducing the functions which characterize the limit system as $q \rightarrow \infty$ ($(\cdot)^+$ = positive part, $H(\cdot)$ Heaviside function). We set

$$h_j(r) = \frac{2M}{\pi} \int_0^\pi (r \sin jx)^+ \sin jx \, dx, \quad s_{j,k}(r) = \frac{2M}{\pi} \int_0^\pi H(r \sin jx) \sin^2 kx \, dx, \quad (18)$$

then, by setting $r_0 = 0$ and $\theta = 0$ in Lemma B.2 in ‘‘Appendix B’’, we obtain their explicit expressions:

$$h_j(r) = \frac{M}{2}r, \quad s_{j,k}(r) = \frac{M}{2} \quad (\text{even } j); \quad (19)$$

$$h_j(r) = \frac{M}{2} \left[1 + \frac{\text{sign}(r)}{j} \right] r, \quad s_{j,k}(r) = \frac{M}{2} \left[1 + \text{sign}(r) \left[\frac{1}{j} - \frac{\tan(k\pi/j)}{k\pi} \right] \right] \quad (\text{odd } j). \quad (20)$$

2.2. Proof of Theorem 2.1

Proof. In [15] we proved the same facts for the simple mode with $j = k = 1$, therefore here we just outline the main points emphasizing a few differences.

Let u be the solution of the problem (11), and v be the solution of Eq. (13) with fixed initial data $v(0) = a$, $\dot{v}(0) = b$.

We rescale u by setting $U_q(t) = u(t)/q$, then the system becomes

$$\begin{cases} \ddot{U}_q + \alpha j^4 U_q + 2 \frac{f_j(qU_q)}{q} = 0 \\ \ddot{v}_q + (k^2 \beta + 2\gamma g_{j,k}(qU_q(t)))v_q = 0 \end{cases} \quad (21)$$

with initial data $U_q(0) = 1$, $\dot{U}_q(0) = 0$, $v_q(0) = a$, $\dot{v}_q(0) = b$.

Then we introduce the limit system of (21), as $q \rightarrow \infty$,

$$\begin{cases} \ddot{U}_\infty + \alpha j^4 U_\infty + 2h_j(U_\infty) = 0 \\ \ddot{v}_\infty + (\beta k^2 + 2\gamma s_{j,k}(U_\infty(t)))v_\infty = 0 \end{cases} \quad (22)$$

with initial data $U_\infty(0) = 1$, $\dot{U}_\infty(0) = 0$, $v_\infty(0) = a$, $\dot{v}_\infty(0) = b$.

We claim that, as $q \rightarrow +\infty$, the solutions of (21)–(22) satisfy the limit relations:

- (i) $U_q \rightarrow U_\infty$ in $C^0(\mathbb{R})$;
- (ii) $v_q \rightarrow v_\infty$ in $C^1([0, T])$, for every $T > 0$;
- (iii) $\tau_q \rightarrow \tau_\infty$, where τ_q, τ_∞ are the periods of U_q, U_∞ respectively.

Proof of (i): Let $\tilde{f}(r)$ be as in (16), then in [15] we proved that $\lim_{q \rightarrow +\infty} \tilde{f}(qr)/q = \tilde{h}(r)$, uniformly on compact sets. Thanks to (17), in the same way we obtain $\lim_{q \rightarrow +\infty} f_j(qr)/q = h_j(r)$, uniformly on compact sets. Owing to a classical continuous dependence theorem ([11], Th. 3, Ch.XV, p.297), we get that U_q converges uniformly on \mathbb{R} to U_∞ .

Proof of (ii): Let us consider the Hill equation in (22). By Lemma 5.1 in [15] (see also [22]), we have to prove that $g_{j,k}(qU_q(t)) \rightarrow s_{j,k}(U_\infty(t))$ in $L^1[0, T]$.

In the case when $\sin jx \neq 0$, we have $U_\infty(t) \neq 0$, thus by the uniform convergence of U_q , for q sufficiently large, the sign of $qU_q \sin jx$ is the same as that of $U_\infty \sin jx$. Thanks to the assumptions **(S₁)**–**(S₂)**, it follows that

$$\lim_{q \rightarrow +\infty} f'(qU_q(t) \sin jx) = MH(U_\infty(t) \sin jx).$$

Since $U_\infty(t) \neq 0$ almost everywhere in $[0, T]$, and $s_{j,k}$ is bounded, we get the required convergence of $g_{j,k}(qU_q)$ to $s_{j,k}(U_\infty)$.

The proof of (iii) is the same as in Proposition 5.2 of [15].

Finally, we come to the computation of the limit period τ_∞ and discriminant Δ_∞ . Recall that j is odd, then the limit system is a coupled (nonlinear) system with step coefficients $h_j(r)$, $s_{j,k}(r)$ defined by formula (20). It turns out that the limit Hill equation (22) is a two-step Meissner equation that can be integrated explicitly. First of all we note that the function $U_\infty(t)$ is even, so that we can fix our attention only on the half period. If we divide the interval $[0, \tau/2]$ in two subintervals $I^+ = [0, t_0]$, where $U_\infty \geq 0$, and $I^- = [t_0, t_1]$, where $U_\infty \leq 0$, we easily obtain

$$t_0 = \frac{\pi}{2\omega_+}, \quad t_1 = \frac{\pi}{2\omega_-}, \quad \tau = 2(t_0 + t_1).$$

It follows that the coefficient of the Eq. (22) is given by (here $\mathbb{1}_I(t)$ denotes the indicator function of I),

$$\beta k^2 + 2\gamma s_{j,k}(U_\infty(t)) = A_+^2 \mathbb{1}_{I^+}(t) + A_-^2 \mathbb{1}_{I^-}(t) \quad (0 \leq t \leq \tau/2).$$

A straightforward computation (see *e.g.* [15] or “Appendix C”), shows that

$$\frac{\Delta_\infty}{2} = v_\infty(\tau) = \cos \phi_+ \cos \phi_- - \frac{a + a^{-1}}{2} \sin \phi_+ \sin \phi_-.$$

By the instability characterization of Hill equations, if $|\Delta_\infty| > 2$, then the pure flexural solutions $u(t; q)$ are linearly unstable (thus unstable) for sufficiently large q . \square

3. The MMK Model: Instability Tongues and Some Numerical Results

A classical problem in the theory of Hill equations consists in studying the parametric resonance of equations of the type

$$\ddot{v} + (\beta + qp(t))v = 0. \quad (23)$$

Here $p(t)$ is a periodic function of period π (just for fixing one), and β, q are real parameters. The general picture is well known and can be briefly described as follows [14, ch. II, Th. 2.1] [7, ch. 2, Th. 2.3.1]. In the (q, β) -plane a sequence of separated regions of instability (instability tongues, Arnold's tongues) emerge from the points $(0, n^2)$, $n = 1, 2, \dots$ on the β -axis which are bounded by two curves $\beta = \beta_n^\pm(q)$. For values (q, β) in the interior of the tongues, i.e for $\beta_n^-(q) < \beta < \beta_n^+(q)$, all solutions of (23) are unstable, while outside are stable. At the boundaries, $\beta = \beta_n^\pm(q)$, we have at least a non-trivial π -periodic or π -anti-periodic solution. For some $p(t)$, and for some exceptional values of q , the curves $\beta^\pm(q)$ may intersect (co-existence of periodic solutions) and form the so-called resonance pockets, as in the case of the square wave case $p(t) = \text{sign } \cos(2t)$.

The classical, and most studied, example is the Mathieu equation, $p(t) = \cos(2t)$, whose instability diagram was first drawn by Van Der Pol and Strutt [25] and can be found on many books [3, 20]. In this case we know the order of the instability tongues as $q \rightarrow 0$, the asymptotic behavior of their width as $n \rightarrow \infty$, and many other features, such as the fact that the width of the stability bands shrinks to zero as $q \rightarrow \infty$, so that the areas of instability invade the whole plane, see [13, 26].

Another case extensively studied is when $p(t)$ is a two-step function in the interval $[0, \pi]$ (Meissner equation). The papers [12, 24] provide very detailed results on the existence of resonance pockets and on the asymptotical behavior of the stability boundaries.

In [17] we studied a generalization of the Mathieu equation,

$$\ddot{v} + (\beta + g(u(t; q)))v = 0, \quad (24)$$

in which the periodic coefficient $g(u(t; q))$ depends on the solution $u = u(t; q)$ of an initial-value problem for a conservative second order differential equation,

$$\ddot{u} + 4u + f(u) = 0, \quad u(0) = q, \quad \dot{u}(0) = 0, \quad (25)$$

where $f(r) = O(r^2)$, $r \rightarrow 0$. Note that the Mathieu equation corresponds to the case $f = 0$ in (25), and $g(u) = u$ in (24). The Eq. (24) is again a Hill equation with two parameters that shares some aspects with the Mathieu equation. Indeed the main result in [17] concerns the order of tangency at $q = 0$ of the instability tongues:

$$\beta_n^+(q) - \beta_n^-(q) = O(q^n), \quad q \rightarrow 0, \quad (26)$$

which is the same as that of the Mathieu equation, at least if f and g are real analytic functions near the origin. However, two features make its analysis different (and more difficult): its period depends on the parameter q , as it is half the period of $u(t; q)$ if g is an even function, and is the same otherwise; the dependence on q in (24) is no more linear.

The system (11)–(13) has the same structure of (25)–(24), then for regular f, g the asymptotic formula (26) applies. Having at our disposal a result for high energies (Theorem 2.1) and another one for low energies (formula (26)), it would be interesting to have a general picture at least in the significant case of the MMK restoring force (3), which is a simple non-trivial model satisfying the Assumptions (\mathbf{S}_0) – (\mathbf{S}_1) – (\mathbf{S}_2) , being not smooth in a single point r_0 .

We now come to the presentation of the models adopted in our numerical simulations. We focus only on the system (11)–(13) in the case when function f is the MMK function (3), and on the system (29)–(30) below, which is a non smooth variation of it in the case $j = k$. We performed our simulations both with a simple set of parameters with the same order of magnitude (Fig. 1), and with the set as in [8,9] where the mechanical parameters of the Tacoma narrow bridge (TNB) and of other significant bridge models are listed, including the MMK model, and several numerical experiments are performed.

The analytic expression of the functions f_j , owing to (17) in Lemma 2.4, is known once we provide the projected form \tilde{f} of f in (3). This was already done in our previous paper [15], and its closed form is given by,

$$\tilde{f}(r) = mr, \quad r \geq -r_0, \quad \tilde{f}(r) = -\frac{2}{\pi}mr_0 \left[\frac{r}{r_0} \arcsin\left(\frac{r_0}{r}\right) + \sqrt{1 - \left(\frac{r_0}{r}\right)^2} \right], \quad r \leq -r_0. \quad (27)$$

As a consequence, for $j = k$, also the the functions $g_{j,j} = f'_j$ (see Lemma 2.4) have a simple closed form. Much more complicated is the derivation of the functions $g_{j,k}$ in (14) when $j \neq k$. Their complete computation is provided in “Appendix B”.

In order to compare different functions, modeling the same slackening regime, we introduce only for the case $j = k$ a non smooth variation of f_j , and f'_j which maintains the same shape as the original MMK model but which is adjusted in a way to have the same asymptotic behavior of f_j , that is $\lim_{r \rightarrow -\infty} \tilde{f}(r) = -\frac{4}{\pi}mr_0$. More precisely, we replace the function \tilde{f} in (27) with the function

$$\bar{f}(r) = m \left[(r + 4r_0/\pi)^+ - 4r_0/\pi \right] \quad (28)$$

Then f_j can be approximated, according to Lemma 2.4, with

$$\bar{f}_j(r) = \frac{1}{2} \bar{f}_o(r), \quad (\text{even } j), \quad \bar{f}_j(r) = \frac{1}{2}(\bar{f}_o(r) + \frac{1}{j} \bar{f}_e(r)), \quad (\text{odd } j),$$

and we introduce the system,

$$\ddot{u} + \alpha j^4 u + 2\bar{f}_j(u) = 0, \quad u(0) = q, \quad \dot{u}(0) = 0 \quad (29)$$

$$\ddot{v} + (\beta j^2 + 2\gamma \bar{f}'_j(u))v = 0, \quad (30)$$

Both u and v can be calculated explicitly and in “Appendix C” we provide the formula for the instability discriminant Δ for every (q, β) . The expression of $\Delta(q, \beta)$ is very complicated and is actually of little help from the analytical point of view, but it can be used to represent very quickly, using Matlab, the instability diagram in order to make a comparison with the numerical results regarding the system (11)–(13).

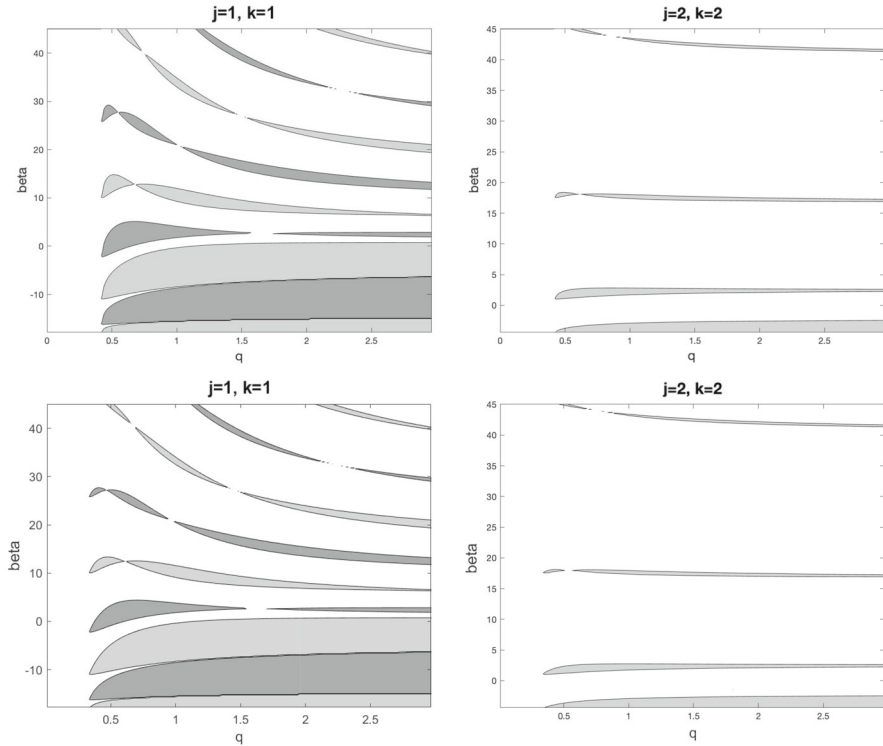


FIGURE 1 Instability diagrams of systems (29)–(30), first line, and (11, 13), second line. The fixed parameters are: $\alpha = 1$, $\gamma = 3$, $m = 3$, $r_0 = 1/3$. $\Delta(q, \beta) > 2$ in the light grey zones, and $\Delta(q, \beta) < -2$ in the dark grey zones. Note that on the right column the tongues corresponding to odd index vanish

Before proceeding with the discussion of the numerical results, let us fix the starting points $\beta_N^+(0) = \beta_N^-(0)$ of the instability tongues for our equations. Recall that we use β as spectral parameter.

Proposition 3.1. *Let us suppose that the function f satisfies the assumption (\mathbf{S}_0) . Then instability tongues for the Hill equation (13) stem from the values of $\beta_N(0)$, $N = 1, 2, \dots$, in the β -axis, given by*

$$\beta_N(0) = \frac{\alpha j^4 + 2m}{4k^2} N^2 - \frac{2\gamma m}{k^2}.$$

If j is even the tongues corresponding to an odd index N disappear, since the actual period is half the period of $u(t; q)$.

We skip the proof which, after a translation of the spectral parameter β , follows from the fact that, if $\tau(q)$ is the period of $u(t; q)$, then its limit as $q \rightarrow 0$, is given by $\tau_0 = 2\pi/\sqrt{\alpha j^4 + 2m}$. The different behavior for even or odd j arises from the fact that for even j , $g_{j,k}(u(t))$ is an even function and the minimal period of the Eq. (13) is half the period $\tau(q)$ of $u(t; q)$.

In the first line of Fig. 1 we show the diagram of the instability tongues for the “irregular approximation” (29)–(30) when $j = 1$ (left), $j = 2$ (right). In the

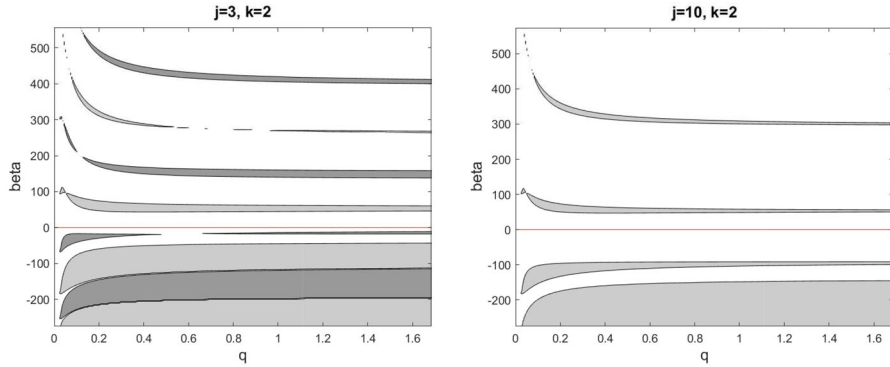


FIGURE 2 Instability diagrams of system (11)–(13) for the MMK function, and $j \neq k$. The fixed parameters (TNB) are: $\alpha = 8.0353 \cdot 10^{-4}$, $m = 185.1$, $r_0 = 0.0265$. The actual value $\beta = 8.1833 \cdot 10^{-5}$ of the TNB is highlighted

second line of Fig. 1 the diagrams of the corresponding instability tongues for the system (11)–(13), (\tilde{f} as in (27)), with $j = k$ and with the same parameters, in order to compare the diagrams with the first line. It is worth noting that both systems are linear decoupled systems for low values of q , then we have a trivial case of coexistence of periodic orbits, since all solutions are periodic.

Coming to the specific features of the system (29)–(30), the true instability tongues split from the points $(\frac{4}{\pi}r_0, \beta_N(0))$, where $\beta_N(0)$ is as in Proposition (3.1). We note the persistence of areas of stability even for large values of q , so that their widths do not shrink to 0, as $q \rightarrow \infty$. In this regard, it is worth comparing the instability diagram in Fig. 1, and the one when $p(t)$ is a step function in problem (23), see [24]. In both cases β has the same meaning, but in the first case the parameter q acts both on the period and the width of the steps of the Meissner equation, while in the second case, the parameters q in (23) acts only on the height of the steps, as the period and width of the steps are fixed. Then we note that the width at the splitting points $(\frac{4}{\pi}r_0, \beta_N(0))$, unlike (26), is $O(q)$, $q \rightarrow 0$. This is due to the singularity of \tilde{f}_j in $r = 4\pi/r_0$. Finally we observe the onset of the so called resonance pockets, that are typical of some Hill equations of Meissner type, see [12, 24].

The diagrams in the second line of Fig. 1 are obtained by the numerical solutions of the system (11)–(13). We used the Matlab solver `ode23t` with a quite high `reltol` and `abstol` setting. Even though we used the closed formulas for f_j , $g_{j,k}$ provided in “Appendix B”, a very long computation time was needed to obtain an accurate value of $\Delta(q, \beta)$, and a reasonably good quality diagram. Indeed, at every step in the grid value of q , a precise evaluation of the period of $u(t; q)$ was necessary.

Figure 2 shows two instability diagrams of the system (11)–(13) respectively for $j = 3$, $k = 2$ on the left, and $j = 10$, $k = 2$ on the right. In this case $j \neq k$, so that the approximated system (29)–(30) is no more available.

More precisely, we decided to use the set of structural parameters of the Tacoma Narrow bridge (TNB) reported in [8] that are derived from the Technical Report

(1941) of AMMANN et al. [1]. The mechanical parameters of the TNB are characterized by very different orders of magnitude. Nonetheless the general picture of the instability tongues (behavior at the splitting points and resonance pockets) remains the same and seems to be a constant feature of the MMK model.

The reports about the failure of the TNB show that the transfer from flexural to torsional energy regarded most probably the $9 - 2$ or the $10 - 2$ modes (see for example [10] for an extensive overview). We checked all the modes $j-2$, for $2 \leq j \leq 10$, and $0 < q \leq 1.6$ m (flexural oscillations of amplitude of about one and a half meters were reported by witness, see for example again [8], “Appendix A”). For our simplified model the tongues corresponding to an even N are relatively thick in the given interval for q , but stay far away from the significant value of β . Whereas the tongues related to an odd N get very thin as j grows, and also such modes do not present any instability near the significant value of β . We point out that the fish-bone model does not take into account the dynamics of the suspension cables and their mechanical parameters, then our simulations are interesting only from a mathematical point of view.

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Appendix A. Regularity of ψ_1 and ψ_2

Lemma A.1. *Let f be a real valued function such that $f \in C^0(\mathbb{R})$, $f \in C^1(\mathbb{R} \setminus \{r_0\})$, and $\sup_{\mathbb{R} \setminus \{r_0\}} |f'(x)| < \infty$, $r_0 \in \mathbb{R}$. Let I be the closed interval $[a, b]$, $u \in C^1(\mathbb{R}^{n+1})$, $w \in L^\infty(I)$, and define*

$$G(p) = \int_I f(u(x, p))w(x) dx, \quad (p \in \mathbb{R}^n).$$

If, for each $p \in \mathbb{R}^n$,

$$u(x, p) = r_0, \text{ has (at most) a finite number of solutions } x \in I, \quad (31)$$

then $G \in C^1(\mathbb{R}^n)$, and the following differentiation formula holds for every $p \in \mathbb{R}^n$,

$$\nabla G(p) = \int_I f'(u(x, p)) \nabla_p u(x, p) w(x) dx.$$

Proof. For simplicity we assume $n = 1$. The general case $n \geq 1$ follows exactly in the same way.

We set

$$\Gamma = \{(x, p) \in \mathbb{R}^2 : u(x, p) = r_0\}, \quad \Omega = \mathbb{R}^2 \setminus \Gamma,$$

and let $p_0 \in \mathbb{R}$ be a number such that there exists a finite number of points $(x_j, p_0) \in \Gamma$, $j = 0, 1, \dots, m$. Possibly by decomposing the interval I into a finite union of intervals I_j in which $u(x, p) = r_0$ has a single solution, we may assume that there exists a unique point $(x_0, p_0) \in \Gamma$.

Let us fix $\varepsilon > 0$ and define

$$K_\varepsilon^0 = \{(x, p_0) : x \in I, |x - x_0| \geq \varepsilon\}.$$

Since K_ε^0 is compact, its distance from the closed set Γ is strictly positive. Then there exists $\delta = \delta(\varepsilon)$, such that

$$K_\varepsilon^\delta = \{(x, p) : x \in I, |x - x_0| \geq \varepsilon, |p - p_0| \leq \delta\} \subseteq \Omega.$$

By the continuity of u , we have either $u < r_0$ or $u > r_0$ on each connected component of K_ε^δ , thus $f \circ u \in C^1(K_\varepsilon^\delta)$.

Let us set

$$g(p) = \int_I f'(u(x, p)) \frac{\partial u}{\partial p}(x, p) w(x) dx,$$

and split the function G in the following way:

$$G(p) = G_{1,\varepsilon}(p) + G_{2,\varepsilon}(p) \quad (|p - p_0| \leq \delta),$$

where

$$G_{1,\varepsilon}(p) = \int_{x \in I, |x - x_0| \geq \varepsilon} f(u(x, p)) w(x) dx, \quad G_{2,\varepsilon}(p) = \int_{x \in I, |x - x_0| < \varepsilon} f(u(x, p)) w(x) dx,$$

so that $G_{1,\varepsilon} \in C^1([p_0 - \delta, p_0 + \delta])$ by the standard rule for differentiation under the integral sign. As for the second term we have,

$$\begin{aligned} & \left| \frac{G_{2,\varepsilon}(p_0 + h) - G_{2,\varepsilon}(p_0)}{h} - \int_{x \in I, |x - x_0| < \varepsilon} f'(u(x, p_0)) \frac{\partial u}{\partial p}(x, p_0) w(x) dx \right| \\ & \leq \int_{x \in I, |x - x_0| < \varepsilon} \left| \frac{f(u(x, p_0 + h)) - f(u(x, p_0))}{h} w(x) \right| + \left| f'(u(x, p_0)) \frac{\partial u}{\partial p}(x, p_0) w(x) \right| dx \end{aligned}$$

Since f has bounded derivative (thus Lipschitz continuous), $w \in L^\infty$, and $u \in C^1$, the sum of both terms in the last integral is bounded above by some

positive constant M . In conclusion, by considering the incremental ratio of both $G_{1,\varepsilon}$, and $G_{2,\varepsilon}$ we get

$$\limsup_{h \rightarrow 0} \left| \frac{G(p_0 + h) - G(p_0)}{h} - g(p_0) \right| \leq 2M\varepsilon,$$

for every $\varepsilon > 0$, which proves that $G'(p_0) = g(p_0)$.

Now we prove the continuity of $g(p)$. With the same notations as before, for any fixed $\varepsilon > 0$, we have

$$g(p) = G'_{1,\varepsilon}(p) + \int_{x \in I, |x-x_0| < \varepsilon} f'(u(x,p)) \frac{\partial u}{\partial p}(x,p) w(x) dx \quad (|p - p_0| \leq \delta),$$

where $G'_{1,\varepsilon} \in C^0([p_0 - \delta, p_0 + \delta])$. As for the other term, as before the boundedness of f' , and the regularity of u , yield

$$\int_{x \in I, |x-x_0| < \varepsilon} \left| f'(u(x,p)) \frac{\partial u}{\partial p}(x,p) w(x) \right| dx \leq 2M\varepsilon \quad (|p - p_0| \leq \delta).$$

Therefore we infer that

$$|g(p) - g(p_0)| \leq |G'_{1,\varepsilon}(p) - G'_{1,\varepsilon}(p_0)| + 4M\varepsilon,$$

so that

$$\limsup_{p \rightarrow p_0} |g(p) - g(p_0)| \leq 4M\varepsilon,$$

which concludes the proof of the Lemma. \square

Proof of Proposition 2.3. We apply Lemma A.1 to f satisfying (\mathbf{S}_0) , $I = [0, \pi]$, $p = (y, z) \in \mathbb{R}^2$, $u(x, p) = y \sin(jx) \pm z \sin(kx)$, $w(x) = \sin(jx)$ or $w(x) = \sin(kx)$. The lemma has a local character and it is enough to verify the hypotheses in a neighborhood of each point $p_0 = (y_0, z_0) \in \mathbb{R}^2$, therefore we may assume that the function f is not differentiable at a single point r_0 , and that has bounded derivative elsewhere. The regularity of u , and w being obvious, we must verify the assumption (31) for $r_0 \neq 0$ (recall that this was required in (\mathbf{S}_0)). By contradiction, if for a fixed $(y, z) \in \mathbb{R}^2$, the equation $u(x, y, z) = r_0$ were satisfied for an infinite set of $x \in I$, since $x \mapsto u(x, y, z)$ then $u(x, y, z) = r_0$ for every $x \in \mathbb{R}$. This follows by the unique continuation principle applied to the analytic function $x \mapsto u(x, y, z)$. In particular we would have $u(0, y, z) = 0 = r_0$, which is a contradiction.

We conclude this Appendix with a few remarks on the differentiation Lemma A.1.

The lemma, in spite of its simplicity, cannot be derived from the traditional Lebesgue theorem of differentiation under integral, as one can easily verify. A simple example is provided by the function $G(p) = \int_0^1 |x - p| dx$, which is differentiable everywhere, and of class C^1 . As a matter of fact, Lemma A.1 is more a regularity result than a sufficient condition to differentiate under integral sign.

The condition (31) serves to our purposes but can be easily relaxed, for example by assuming that the set $\{x \in I : u(x, p) = r_0\}$ has (at most) a finite number of accumulation points for any fixed p . A trivial example in which this last condition is violated, for $p = 0$, is given by $G(p) = \int_0^1 |p| dx$, which of course is not differentiable at $p = 0$. In this regard, we point out the relevance in the application of the Lemma to the functions ψ_1 , and ψ_2 , in which $u(x, p) = y \sin(jx) \pm z \sin(kx)$, to the

assumption $r_0 \neq 0$. On the other hand, any relaxed version of condition (31) is far from being necessary, as the simple example $G(p) = \int_0^1 |p^3| dx$ shows. In literature there are other more or less classical differentiation results, see *e.g.* [23], but we have found that for our purposes the verification of the hypotheses would have been more difficult than the direct proof of the lemma. \square

Appendix B. Explicit Formulas for the MMK Function

The non-linear terms of the system (7) and the periodic coefficient of the Hill equation (13) are both defined as an integral. This fact becomes very time consuming when it comes to drawing the instability tongues with Matlab, as the solution of some ten thousands of systems is needed. So, in the case of the MMK slackening model (3), we decided to provide the explicit computation of $g_{j,k}(r)$ in (13), for every choice of j and k . The result in Lemma B.2 for $r_0 = 0$ is necessary for the computation of the limit system in Sect. 2.

We need a simple computational lemma.

Lemma B.1. *Let j, k be two positive integers, then*

$$q(j, k) := \sum_{n=1}^j \cos\left(\frac{2k}{j}n\pi\right) = \begin{cases} j & \text{if } k/j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

$$p(j, k) := \sum_{n=1}^j (-1)^n \sin\left(\frac{2k}{j}n\pi\right) = \begin{cases} 0 & \text{if } j \text{ even,} \\ -\tan\left(\frac{k}{j}\pi\right) & \text{otherwise.} \end{cases} \quad (33)$$

Proof. Let us prove (32). In the case when $k/j \in \mathbb{N}$, we obviously have $q(j, k) = j$; otherwise we get ($i = \sqrt{-1}$, $Re =$ real part),

$$q(j, k) = \sum_{n=0}^{j-1} \cos\left(\frac{2k}{j}n\pi\right) = Re \sum_{n=0}^{j-1} e^{\frac{2k\pi i}{j}n} = Re \frac{1 - e^{2k\pi i}}{1 - e^{2k\pi i/j}} = 0.$$

In order to prove (32), we observe that, if $\frac{2k}{j} \in \mathbb{N}$, then $p(j, k) = 0$. Otherwise, we have ($Im =$ imaginary part),

$$p(j, k) = \sum_{n=0}^{j-1} (-1)^n \sin\left(\frac{k}{j}2n\pi\right) = Im \sum_{n=0}^{j-1} \left(-e^{\frac{k}{j}2\pi i}\right)^n = Im \frac{1 - (-1)^j}{1 + e^{\frac{k}{j}2\pi i}},$$

thus $p(j, k) = 0$ for j even. When j is odd, we get

$$p(j, k) = -\frac{\sin\left(\frac{k}{j}2\pi\right)}{1 + \cos\left(\frac{k}{j}2\pi\right)} = -\tan\left(\frac{k}{j}\pi\right).$$

\square

The closed form of the function $g_{j,k}$, in the case of MMK function (3), is given by $2m/\pi$ times the function $H_{j,k}(r)$ defined here below.

Lemma B.2. *Let j, k be positive integers, $r_0 \geq 0$, and let $H_{j,k}(r)$ be the function defined as ($H[\cdot] = \text{Heaviside step function}$),*

$$H_{j,k}(r) := \int_0^\pi H(r \sin jx + r_0) \sin^2 kx \, dx \quad (r \in \mathbb{R}).$$

Let us set $\theta(r) = \text{asin}(r_0/|r|)$, for $|r| \geq r_0$.

Then, for every j, k , we have

$$H_{j,k}(r) = \frac{\pi}{2} \quad \text{for } |r| \leq r_0,$$

while for $|r| > r_0$ (note that for even j the second row vanishes),

$$\begin{aligned} H_{j,k}(r) = & \frac{\pi}{4} + \frac{\theta(r)}{2} - \frac{q(j,k)}{4k} \sin\left(\frac{2k\theta(r)}{j}\right) \\ & + \frac{1 - (-1)^j}{2} \frac{r}{|r|} \left[\frac{\pi}{4j} - \frac{\theta(r)}{2j} + \frac{p(j,k)}{4k} \cos\left(\frac{2k\theta(r)}{j}\right) + \frac{1}{4k} \sin\left(\frac{2k\theta(r)}{j}\right) \right]. \end{aligned} \quad (34)$$

Proof. If $|r| \leq r_0$, we have $r \sin jx + r_0 \geq 0$, thus $H_{j,k}(r) = \int_0^\pi \sin^2 kx \, dx = \pi/2$.

The other cases are much more involved. First of all, by changing the integration variable, we write

$$H_{j,k}(r) = \frac{1}{j} \int_0^{j\pi} H(r \sin z + r_0) \sin^2(kz/j) \, dz;$$

and observe that, for $z \in [0, 2\pi]$, we have

$$r \sin z + r_0 \geq 0 \quad \text{iff } z \in B_0^+(r) := [0, \pi + \theta(r)] \cup [2\pi - \theta(r), 2\pi] \quad (r > r_0),$$

$$r \sin z + r_0 \geq 0 \quad \text{iff } z \in B_0^-(r) := [0, \theta(r)] \cup [\pi - \theta(r), 2\pi] \quad (r < -r_0);$$

then we set $B_n^\pm(r) = B_0^\pm(r) + 2n\pi$ (the translated sets).

We begin with $r > r_0$, and j even. We get

$$H_{j,k}(r) = \frac{1}{j} \sum_{n=0}^{j/2-1} \int_{B_n^+(r)} \sin^2(kz/j) \, dz;$$

by direct computation of the integrals, we obtain

$$\begin{aligned} \frac{1}{j} \sum_{n=0}^{j/2-1} \int_{B_n^+(r)} \sin^2(kz/j) \, dz = & \frac{\pi}{4} + \frac{\theta}{2} + \frac{1}{4k} \sum_{n=0}^{j/2-1} (\sin(4kn\pi/j) - \sin(4k(n+1)\pi/j)) \\ & + \frac{1}{4k} \sum_{n=0}^{j/2-1} \left(\sin\left(\frac{2k}{j}((2n+2)\pi - \theta)\right) - \sin\left(\frac{2k}{j}((2n+1)\pi + \theta)\right) \right). \end{aligned}$$

The first summation on the right hand side is telescopic and cancels out. The last summation, by using the trigonometric addition formula, may be written as

$$\frac{1}{4k} \left(\sum_{n=1}^j (-1)^n \sin\left(\frac{2k}{j}n\pi\right) \right) \cos\left(\frac{2k}{j}\theta\right) - \frac{1}{4k} \left(\sum_{n=1}^j \cos\left(\frac{2k}{j}n\pi\right) \right) \sin\left(\frac{2k}{j}\theta\right).$$

Owing to (32), and (33), and putting together the various contributions, this concludes the proof in the case j even and $r > r_0$.

If j is even and $r < -r_0$, we replace $B_n^+(r)$ with $B_n^-(r)$ and follow the same procedure, to obtain

$$H_{j,k}(r) = \frac{\pi}{4} + \frac{\theta}{2} - \frac{p(j,k)}{4k} \cos\left(\frac{2k}{j}\theta\right) - \frac{q(j,k)}{4k} \sin\left(\frac{2k}{j}\theta\right) = \frac{\pi}{4} + \frac{\theta}{2} - \frac{q(j,k)}{4k} \sin\left(\frac{2k}{j}\theta\right).$$

Let us now consider the case when j is odd. In the simple case $j = 1$, the direct computation of $H_{j,k}(r)$ is trivial, and yields

$$H_{1,k}(r) = \frac{\pi}{2}, \quad \text{if } r > -r_0; \quad H_{1,k}(r) = \theta(r) - \frac{\sin(2k\theta(r))}{2k} \quad \text{if } r < -r_0,$$

which coincides with the formula (34) (although its verification is somehow hidden).

For j odd, and $j \geq 3$, and $r > r_0$, we get, again by following the same procedure as above,

$$\begin{aligned} H_{j,k}(r) &= \frac{1}{j} \sum_{n=0}^{(j-3)/2} \int_{B_n^+(r)} \sin^2\left(\frac{k}{j}z\right) dz + \frac{1}{j} \int_{(j-1)\pi}^{j\pi} \sin^2\left(\frac{k}{j}z\right) dz \\ &= \left(1 + \frac{1}{j}\right) \frac{\pi}{4} + \left(1 - \frac{1}{j}\right) \frac{\theta}{2} + \frac{p(j,k)}{4k} \cos\left(\frac{2k}{j}\theta\right) - \frac{1}{4k} (q(j,k) - 1) \sin\left(\frac{2k}{j}\theta\right); \end{aligned}$$

while, for $r < -r_0$,

$$\begin{aligned} H_{j,k}(r) &= \frac{1}{j} \sum_{n=0}^{(j-1)/2} \int_{B_n^-(r)} \sin^2\left(\frac{k}{j}z\right) dz - \frac{1}{j} \int_{j\pi}^{(j+1)\pi} \sin^2\left(\frac{k}{j}z\right) dz \\ &= \left(1 - \frac{1}{j}\right) \frac{\pi}{4} + \left(1 + \frac{1}{j}\right) \frac{\theta}{2} - \frac{p(j,k)}{4k} \cos\left(\frac{2k}{j}\theta\right) - \frac{1}{4k} (q(j,k) + 1) \sin\left(\frac{2k}{j}\theta\right). \end{aligned}$$

Again owing to (32), and (33), this concludes the proof of the lemma. \square

If $f(r)$ is the MMK function in (3), where $r_0 > 0$, then $g_{j,k}(r) = \frac{2m}{\pi} H_{j,k}(r)$. We can verify by direct inspection in (34) that this is a continuous function, as expected, because Proposition 2.3 says that $\frac{\partial \psi_2}{\partial z}(y, 0) = 2g_{j,k}(y)$ must be continuous. If we set instead $r_0 = 0$, (34) gives us the explicit formula for $s_{j,k}(r) = \frac{2M}{\pi} H_{j,k}(r)$, that is needed for defining the limit system in Theorem 2.1. This function is not continuous in $r = 0$ if j is odd.

This makes clear again that, if we weaken the condition b) in the assumption (\mathbf{S}_0) , substituting the MMK function with the similar one $f(r) = Mr^+$, the respective functions ψ_1, ψ_2 in (7) are no more smooth.

Appendix C. The Discriminant for the Approximation (28)

In this section we provide the computation of the instability discriminant Δ for each fixed value of β and q for the Hill equation (30). We find the explicit solution $u(t; q)$ of the Eq. (29), and its period $\tau = \tau(q)$. The periodic coefficient in (30) is affected only by the slope of the piecewise linear function $\bar{f}_j(r)$ whose values change at some transition points of $u(t; q)$. It turns out that, at any fixed value of q , (30) is a multi-step Hill equation or Meissner equation [5, 14].

We recall how to compute the discriminant of a Hill equation with a positive, multi-step potential. Let the interval $[0, \tau] = \cup_{i=0}^n I_i$ be the union of disjoint intervals,

each having length Δt_i . Let us consider a potential $Q(t)$ which is τ -periodic positive and constant on each subinterval I_i , that is,

$$Q(t) = \sum_{i=0}^n A_i^2 \mathbb{1}_{I_i}(t) \quad 0 \leq t \leq \tau.$$

The monodromy matrix, and the discriminant of the Hill equation $\ddot{v}(t) + Q(t)v(t) = 0$, are computed as follows, see [5] p. 12. If the L_i 's are the transition matrices

$$L_i = \begin{bmatrix} \cos(A_i \Delta t_i) & \frac{1}{A_i} \sin(A_i \Delta t_i) \\ -A_i \sin(A_i \Delta t_i) & \cos(A_i \Delta t_i) \end{bmatrix}, \quad i = 0, 1, \dots, n,$$

then we get

$$M = L_n L_{n-1} \dots L_1 L_0, \quad \Delta = \text{tr} M.$$

To simplify some calculations, it is important to note that the discriminant is invariant with respect to any cyclic permutation of the transition matrices, so for instance $M = L_2 L_1 L_0$, and $M' = L_1 L_0 L_2$ are in general different matrices having the same trace. In fact, they correspond to different translations in time of $Q(t)$, which of course leave the discriminant unchanged.

In applying the previous formulas to the Eq. (30), the A_i^2 's coefficients are known, being determined by the constant slopes of the function $\bar{f}_j(r)$. It remains to compute the length of the intervals I_i , and their ordering, modulo cyclic permutations.

First of all, we observe that, in both cases j even or odd, we have

$$\bar{f}_j(r) = mr, \quad \text{if } |r| \leq \bar{r} := \frac{4}{\pi} r_0,$$

therefore when the initial value $u(0) = q$ is less or equal to \bar{r} , the Eq. (29) is linear, with solution $u(t; q) = q \cos(\omega t)$, $\omega = \sqrt{\alpha j^4 + 2m}$. It follows that \bar{f}_j' equals to m , and the Hill equation (30) reduces to a linear oscillator with constant angular frequency $A = \sqrt{\beta j^2 + 2\gamma m}$. Thus it is stable and Δ is simply the trace of the matrix L with $\Delta t = 2\pi/\omega$, *i.e.* $\Delta = 2 \cos(2\pi A/\omega)$.

In the case when $q > \bar{r}$, we must identify the intervals I_i at whose end points $u(t; q) = \pm \bar{r}$. We start when j is even. Owing to (17) we have in the Eq. (29),

$$\bar{f}_j(r) = \frac{1}{2}(mr + m\bar{r}), \quad \text{if } r > \bar{r}, \quad \bar{f}_j(r) = \frac{1}{2}(mr - m\bar{r}), \quad \text{if } r < -\bar{r},$$

therefore the two angular frequencies of the Hill equation (30) are

$$A_0 = \sqrt{\beta j^2 + \gamma m}, \quad \text{if } |u| \geq \bar{r}, \quad A_1 = \sqrt{\beta j^2 + 2\gamma m}, \quad \text{if } |u| < \bar{r}.$$

To determine the intervals in which the two cases occur, we note that $\bar{f}_j(r)$ is an odd function, so that the potential for the Eq. (29) is even. Therefore it is enough to compute $u(t; q)$ for a quarter of a period, starting from $t = 0$ to the first positive time $t_1 = t_1(q)$ when $u(t_1; q) = 0$, so that $\tau = 4t_1$.

If we call $t_0 = t_0(q) < t_1(q)$ the first positive time such that $u(t_0; q) = \bar{r}$, then the cycle $[-t_0, \tau - t_0]$ is the union of disjoint intervals $\cup_{i=0}^3 I_i$, such that

$$|u| \geq \bar{r}, \quad t \in I_0 \cup I_2; \quad |u| \leq \bar{r} \quad t \in I_1 \cup I_3.$$

Their length are $\Delta t_i = 2t_0$, for $i = 0, 2$, and $\Delta t_i = 2(t_1 - t_0)$, for $i = 1, 3$. As a consequence, the potential of the Eq. (30) is given by

$$Q(t) = A_0^2 \mathbb{1}_{I_0}(t) + A_1^2 \mathbb{1}_{I_1}(t) + A_2^2 \mathbb{1}_{I_2}(t) + A_3^2 \mathbb{1}_{I_3}(t), \quad t \in [-t_0, \tau - t_0],$$

and we have 4 transition matrices with $L_0 = L_2$, $L_1 = L_3$, so that

$$M = (L_1 L_0)^2, \quad \Delta(q) = \text{tr}M.$$

As for the computation of t_0 , t_1 , we define the 2 angular frequencies of the Eq. (29),

$$\omega_0 = \sqrt{\alpha j^4 + m}, \quad \omega_1 = \sqrt{\alpha j^4 + 2m},$$

so that u satisfies the equation $\ddot{u} + \omega_0^2 u = -m\bar{r}$ on I_0 , and $\ddot{u} + \omega_1^2 u = 0$ on I_1 . On the first interval $I_0 = [-t_0, t_0]$, we obtain

$$u(t; q) = \left(q + \frac{m\bar{r}}{\omega_0^2} \right) \cos \omega_0 t - \frac{m\bar{r}}{\omega_0^2}.$$

By solving the equation $u(t; q) = \bar{r}$, we get

$$t_0 = \frac{1}{\omega_0} \text{acos} \left(\frac{\omega_0^2 \bar{r} + m\bar{r}}{\omega_0^2 q + m\bar{r}} \right).$$

On the second interval I_1 , u solves the equation $\ddot{u} + \omega_1^2 u = 0$, with initial data $u(t_0; q)$ and $\dot{u}(t_0; q)$. Thus

$$u(t; q) = \bar{r} \cos(\omega_1(t - t_0)) + \dot{u}(t_0; q) \sin(\omega_1(t - t_0))/\omega_1.$$

Since we have

$$\dot{u}(t_0; q) = -\omega_0 \left(q + \frac{m\bar{r}}{\omega_0^2} \right) \sqrt{1 - \cos^2 \omega_0 t_0} = -\sqrt{(q - \bar{r})[\omega_0^2(q + \bar{r}) + 2m\bar{r}]},$$

by solving the equation $\dot{u}(t; q) = 0$, we get

$$t_1 - t_0 = \frac{1}{\omega_1} \text{atan} \left(\frac{\omega_1 \bar{r}}{B(q)} \right), \quad B(q) = \sqrt{(q - \bar{r})[\omega_0^2(q + \bar{r}) + 2m\bar{r}]}.$$

Now we come to the calculations in the case when j is odd. The function $\bar{f}_j(r)$ in (29), for $|r| > \bar{r}$, is given by

$$\begin{aligned} \bar{f}_j(r) &= \frac{m}{2}(1 + 1/j)r + \frac{m}{2}(1 - i/j)\bar{r}, \quad r > \bar{r}, \\ \bar{f}_j(r) &= \frac{m}{2}(1 - 1/j)r - \frac{m}{2}(1 + 1/j)\bar{r}, \quad r < -\bar{r}. \end{aligned}$$

The procedure is similar to the previous case, only a bit longer, because $\bar{f}_j(r)$ is no more odd, so that we need to compute $u(t; q)$ on a half period. We define t_0 the first positive time such that $u(t_0; q) = \bar{r}$, and again we consider the cycle $[-t_0, \tau - t_0]$ which is the union $\cup_{i=0}^3 I_i$ of disjoint intervals, such that

$$u \geq \bar{r}, \quad t \in I_0; \quad |u| \leq \bar{r}, \quad t \in I_1 \cup I_3; \quad u \leq -\bar{r}, \quad t \in I_2.$$

The Hill equation (30) is now a four steps Meissner equation with potential

$$Q(t) = A_0^2 \mathbb{1}_{I_0}(t) + A_1^2 \mathbb{1}_{I_1}(t) + A_2^2 \mathbb{1}_{I_2}(t) + A_3^2 \mathbb{1}_{I_3}(t), \quad t \in [-t_0, \tau - t_0],$$

where

$$A_0 = \sqrt{\beta j^2 + \gamma m(1 + 1/j)}, \quad A_1 = A_3 = \sqrt{\beta j^2 + 2\gamma m}, \quad A_2 = \sqrt{\beta j^2 + \gamma m(1 - 1/j)},$$

then the matrix M becomes $M = L_1 L_2 L_1 L_0$.

We define the 3 angular frequencies for the Eq. (29),

$$\omega_0 = \sqrt{\alpha j^4 + m(1 + 1/j)}, \quad \omega_1 = \sqrt{\alpha j^4 + 2m}, \quad \omega_2 = \sqrt{\alpha j^4 + m(1 - 1/j)},$$

and the constants

$$D = m(1 - 1/j)\bar{r}, \quad E = m(1 + 1/j)\bar{r},$$

so that u satisfies the equation $\ddot{u} + \omega_0^2 u = -D$ on I_0 , the equation $\ddot{u} + \omega_1^2 u = 0$ on $I_1 \cup I_3$, and $\ddot{u} + \omega_2^2 u = E$ on I_2 .

Proceeding as above, for $t \in I_0$ we have,

$$u(t; q) = \left(q + \frac{D}{\omega_0^2} \right) \cos \omega_0 t - \frac{D}{\omega_0^2},$$

thus, by finding t_0 , we obtain half the length of I_0 , that is

$$\frac{\Delta t_0}{2} = \frac{1}{\omega_0} \operatorname{acos} \left(\frac{\omega_0^2 \bar{r} + D}{\omega_0^2 q + D} \right).$$

For $t \in I_1$, starting with the initial conditions at time t_0 , we get the expression

$$\begin{aligned} u(t; q) &= \bar{r} \cos(\omega_1(t - t_0)) - \frac{B}{\omega_1} \sin(\omega_1(t - t_0)), \quad B = -u(t_0; q) \\ &= \sqrt{(q - \bar{r})[\omega_0^2(q + \bar{r}) + 2D]}. \end{aligned}$$

Thus, by finding the first time greater than t_0 such that $u(t; q) = 0$, we obtain half the length of I_1 :

$$\frac{\Delta t_1}{2} = \frac{1}{\omega_1} \operatorname{atan} \left(\frac{\omega_1 \bar{r}}{B} \right).$$

Finally, for $t \in I_2$, we need to solve the third equation $\ddot{u} + \omega_2^2 u = E$ with initial data at $t^* = t_0 + \Delta t_1$, that is $u(t^*; q) = -\bar{r}$, $\dot{u}(t^*; q) = \dot{u}(t_0; q) = -B$. Therefore, we have

$$u(t; q) = - \left(\bar{r} + \frac{E}{\omega_2^2} \right) \cos \omega_2(t - t^*) - \frac{B}{\omega_2} \sin \omega_2(t - t^*) + \frac{E}{\omega_2^2}.$$

By finding the first time greater than t^* such that $\dot{u}(t; q) = 0$, we obtain half the length of I_2 :

$$\frac{\Delta t_2}{2} = \frac{1}{\omega_2} \operatorname{atan} \left(\frac{\omega_2 B}{\omega_2^2 \bar{r} + E} \right).$$

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