Mediterr. J. Math. (2024) 21:72
https://doi.org/10.1007/s00009-024-02608-1
1660-5446/24/030001-15
published online March 18, 2024

Mediterranean Journal of Mathematics

Multiple Lines of Maximum Genus in $\mathbb{P}^3$

Enrico Schlesinger

Abstract. We introduce a notion of good cohomology for multiple lines in $\mathbb{P}^3$ and we classify multiple lines with good cohomology up to multiplicity 4. In particular, we show that the family of space curves of degree d, not lying on a surface of degree $< d$, and of maximal arithmetic genus is not irreducible already for $d = 4$ and $d = 5$.

Mathematics Subject Classification. 14H50, 14F06, 14F17.

Keywords. Space curves, arithmetic genus, quasiprimitive multiple lines, Hilbert schemes.

1. Introduction

By a *space curve* we mean a locally Cohen–Macaulay purely one dimensional subscheme of $\mathbb{P}^3$, the projective space over an algebraically closed field. Thus a curve is allowed to have several irreducible components and a nonreduced scheme structure, but it cannot have zero-dimensional components—neither isolated nor embedded. The most important invariants of a space curve C are its *arithmetic genus* $g(C) = 1 - \chi\mathcal{O}_C$, which does not depend on the embedding of C in $\mathbb{P}^3$; its *degree* $\deg(C)$, which is defined through the Hilbert polynomial $\chi(\mathcal{O}_C(n)) = n\deg(C) + 1 - g(C)$ and depends on the invertible sheaf $\mathcal{O}_C(1)$, but not on the sections of $\mathcal{O}_C(1)$ that define the embedding of C in $\mathbb{P}^3$; and the *minimum degree* $s(C)$ of a surface that contains C, which *does depend* on the embedding in $\mathbb{P}^3$. The maximum genus problem for space curves asks to determine the most basic relation between these invariants, that is, what is the maximum arithmetic genus $P(d, s)$ of a space curve of degree d in $\mathbb{P}^3$ that is not contained in a surface of degree $< s$—there is a huge literature on the maximum genus problem for *smooth* space curves, but we will not be concerned with smooth curves in this paper. The problem makes sense for pairs of integers (d, s) satisfying $1 \leq s \leq d$ because there exists a space curve of degree d that is not contained in a surface of degree $< s$ if and only if $d \geq s \geq 1$.

The author was partially supported by PRIN 2020 Squarefree Gröbner degenerations, special varieties and related topics, and is a member of GNSAGA of INDAM.

We now survey what is known about the maximum genus problem for (locally Cohen–Macaulay) space curves. Beorchia [2] proved a bound $B(d,s)$ for the maximum genus if the characteristic of the ground field is zero, and proved the bound is sharp if $s \le 4$; later the author [7] gave a different proof of this bound valid in any characteristic.

Theorem 1.1. ([2,7]) *Let C be a curve in $\mathbb{P}^3$ of degree d and genus g. Assume that C is not contained in any surface of degree $< s$. Then $d \ge s$ and*

$$g \le B(d,s) = \begin{cases} (s-1)d + 1 - \binom{s+2}{3}, & \text{if } s \le d \le 2s, \\ \binom{d-s}{2} - \binom{s-1}{3}, & \text{if } d \ge 2s+1. \end{cases} \qquad (\star)$$

To prove sharpness one needs to construct, for each pair $d \ge s$, a curve of genus $B(d,s)$ not lying on a surface of degree $< s$. The case $d = s$ is crucial because, if $P(s-1,s-1) = B(s-1,s-1)$, it follows that $P(d,s) = B(d,s)$ for every $d \ge 2s-1$—see [3]. The aim of this paper is to propose a framework for the classification of curves achieving the maximum genus $B(d,d)$ in the basic case $\deg(C) = s(C) = d$, together with the computation of the first few relevant examples; even for $d = 4$ this seems to be new.

One first observes that curves of degree d are always contained in surfaces of degree d, and that those that do not lie on a surface of degree $< d$ are supported on either one line or two disjoint lines—see Proposition 3.1 below. It is, therefore, necessary to study curves supported on a line L in $\mathbb{P}^3$. We denote by L_d the $(d-1)$th neighborhood of L in $\mathbb{P}^3$: the ideal sheaf of L_d is the dth power $\mathcal{I}_L^d$ of the ideal sheaf of L. Any degree d curve C supported on L, for short a d-uple line, is contained in L_d, and such a curve does not lie on a surface of degree $s < d$ if and only if $\mathrm{H}^0\big(\mathcal{I}_C(d-1)\big) = \mathrm{H}^0\big(\mathcal{I}_L^d(d-1)\big)$ as the latter vector space vanishes. It is clear how to strengthen this requirement to deal with the fact that the support of a curve of maximum genus in the case $d = s$ may consists of two disjoint lines, rather than only one.

Definition 1.2. Fix integers $d \ge 1$ and $\ell \ge 0$. We say that a degree d curve C supported on a line L is a $C_{d,\ell}$ if

- the genus of C is
$$g(C_{d,\ell}) = B(d,d) - \ell\binom{d}{2} = -(d-1) - \binom{d}{3} - \ell\binom{d}{2}$$
- the only surfaces of degree $\ell + d - 1$ containing C are those containing the entire neighborhood L_d of L as well:
$$\mathrm{H}^0\big(\mathcal{I}_C(\ell+d-1)\big) = \mathrm{H}^0\big(\mathcal{I}_L^d(\ell+d-1)\big).$$

In particular, a $C_{d,0}$ is a d-uple line of genus $B(d,d)$ that does not lie on a surface of degree $< d$, and the existence of a $C_{d,0}$ implies sharpness of Beorchia's bound $P(d,d) = B(d,d)$. But the definition is tailored so that, for each $1 \le k \le d-1$, if C and D are respectively a $C_{k,d-k}$ and a $C_{d-k,k}$ whose supports are disjoint lines, then the union of C and D is a also a curve of maximum genus, that is, a curve satisfying $\deg(C) = d$, $s(C) = d$ and $g(C) = B(d,d)$.

It was originally an idea of Beorchia, see [3], that one should construct curves of maximum genus as $C_{d,0}$ for $d \equiv 2$ modulo 3, and adding a line to a $C_{d-1,1}$ when $d \equiv 0$ modulo 3, or a suitable double line to a $C_{d-2,2}$ when $d \equiv 1$ modulo 3. Thus the problem of sharpness of the bound $B(d,d)$ is reduced to constructing d-lines with good cohomological properties when $d \equiv 2$ modulo 3, and this construction in [3] is reduced to an algebraic statement [3, Conjecture B on p. 142]. Sammartano and the author [6] are completing the proof of this Conjecture under the additional hypothesis the ground field has characteristic zero, thus showing the existence of curves $C_{d,\ell}$ for every $d \equiv 2$ modulo 3 and proving sharpness of Beorchia's bound in the case $s = d$.

The main contribution of this paper is to show there are other components of curves of maximum genus $B(d,d)$ by giving examples of curves $C_{d,\ell}$ in cases $d = 3$ and $d = 4$. As a consequence, we show that the family of space curves satisfying $\deg(C) = d$, $s(C) = d$ and $g(C) = B(d,d)$ is not irreducible and contains curves that are scheme theoretically very different from the one constructed in [3]. We hope this will be useful for the problem of sharpness of Beorchia's bound in the intermediate range $s + 1 \leq d \leq 2s$, as curves of maximum genus in that range have to be constructed adding a plane curve to a curve satisfying $\deg(C) = s$, $s(C) = s$ and $g(C) = B(s,s)$ [7].

Our main theorem is the classification of the curves $C_{d,\ell}$ for $d \leq 4$. For this we need the notion of *quasiprimitive multiple structure* introduced in [1], a notion that we review in Sect. 2. A quasiprimitive d-line has an invariant, called type, that is a string of $d-1$ integers $(a; b_2, \ldots, b_{d-1})$. A quasiprimitive d-line is primitive if $b_2 = \ldots = b_{d-1} = 0$, so that for a primitive d-line the type is a single integer a. It is trivial to note that a line is a $C_{1,\ell}$ for any $\ell \geq 0$, and a double line is a $C_{2,\ell}$ if and only if it has genus $-1-\ell$, or, equivalently, it is a primitive double line of type $a = \ell$. In Sect. 5 we classify $C_{d,\ell}$'s for $d = 3$ and $d = 4$ proving

Theorem 1.3. *1. A triple line is a $C_{3,\ell}$ if and only if it is quasiprimitive of type $(\ell; 1)$. The family of $C_{3,\ell}$ curves is irreducible of dimension $5\ell + 12$.*
2. A quadruple line is a $C_{4,\ell}$ if and only if it is a general quasiprimitive quadruple line of type $(\ell; 2, 2)$. The family of $C_{4,\ell}$ curves is irreducible of dimension $9\ell + 21$.

Unfortunately, for $d \geq 5$ we don't have a classification. What we can say in general is that a $C_{d,\ell}$ is a quasiprimitive d-uple line of type $(a; b_2, \ldots, b_{d-1})$ where $\ell \leq a \leq \ell + \lfloor \frac{d-2}{3} \rfloor$. When $d \equiv 2$ modulo 3, a strategy for constructing a primitive $C_{d,\ell}$ of type $a = \ell + \frac{d-2}{3}$ is proposed in [3], and a proof that this works when the ground field has characteristic zero is being written up [6]. But we do not know even for $d = 5$ whether the quasiprimitive type is determined for a $C_{5,0}$ or whether the family of $C_{5,0}$'s is irreducible.

As an application of Theorem 1.3 we can show in the last section of the paper that the family of degree d curves of maximum genus $B(d,d)$ that do not lie on a surface of degree $< d$ is not irreducible already for $d = 4$ and 5. Specifically

Theorem 1.4. *1. The family of quadruple lines of maximum genus* $B(4,4) = -7$ *not lying on a cubic surface is not irreducible. It contains*
- *the* 22*-dimensional irreducible family whose general member is the disjoint union of two double lines of genus* -3*;*
- *the* 21*-dimensional family whose general member is the disjoint union of a line and a* $C_{3,1}$*;*

the closures of these two families are different components of the Hilbert scheme $H_{4,-7}$ *parametrizing space curve of degree* 4 *and genus* -7

2. The family of quintuple lines of maximum genus $B(5,5) = -14$ *not lying on a quartic surface is not irreducible. It contains*
- *the* 30*-dimensional irreducible family whose general member is a general primitive quintuple line of type* $a = 1$*;*
- *the* 34*-dimensional family whose general member is the disjoint union of a line and a general* $C_{4,1}$*;*
- *the* 35*-dimensional family whose general member is the disjoint union of a* $C_{3,2}$ *and a* $C_{2,3}$*;*

and there are no containment between the closures of these 3 *families in the Hilbert scheme* $H_{5,-14}$.

2. Quasiprimitive Multiple Lines in $\mathbb{P}^3$

By the term *d-uple line* we will mean a (locally Cohen–Macaulay) curve in $\mathbb{P}^3$ that has degree d and whose support is a line. The notion of *quasiprimitive* multiplicity structure on a smooth curve was introduced by Banica and Forster [1, § 3]; we recall what it means in our context.

Let C be a d-uple line with support L. Denote by C_j the subscheme of C obtained by removing the embedded points from $C \cap L_j$—as in the introduction, L_j is the infinitesimal neighborhood of L in $\mathbb{P}^3$ defined by $\mathcal{I}_L^j$. The *Cohen–Macaulay filtration* of C is:

$$L = C_1 \subset C_2 \subset \cdots \subset C_k = C \tag{2.1}$$

where k, $1 \leq k \leq d$, is the smallest integer such that $C \subset L_k$. The quotients $\mathcal{L}_j = \mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}}$ are vector bundles on L and $d = \deg(C) = 1 + \sum \operatorname{rank} \mathcal{L}_j$. The natural inclusions $\mathcal{I}_{C_i}\mathcal{I}_{C_j} \subset \mathcal{I}_{C_{i+j}}$ induce generically surjective multiplication maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$ and in particular we obtain generic surjections $\mathcal{L}_1^j \to \mathcal{L}_j$.

A multiple line C is *quasiprimitive* if it has generically embedding dimension two. This is the case if and only if $\operatorname{rank} \mathcal{L}_1 = 1$, or, equivalently, C does not contain the first infinitesimal neighborhood L_2 of its support L, so that the first filtrant C_2 has degree 2 (and C_j degree j for each j). If C is quasiprimitive, then the generic surjections $\mathcal{L}_1^j \to \mathcal{L}_j$ of invertible sheaves yield effective divisors D_j such that $\mathcal{L}_j \cong \mathcal{L}_1^j(D_j)$; the multiplication maps show that $D_i + D_j \leq D_{i+j}$.

For a quasiprimitive d-uple line C in $\mathbb{P}^3$, we define the *type* $\sigma(C) = (a; b_2, \ldots, b_{d-1})$ of C setting $a = \deg(\mathcal{L}_1)$ and $b_j = \deg(D_j)$; it is convenient to set $b_1 = 0$ so that the inequalities

$$b_i + b_j \leq b_{i+j}$$

hold for every $i, j \geq 1$ such that $i + j \leq d - 1$.

Finally, a d-uple line C is called *primitive* if it has embedding dimension two everywhere. This is the case if and only if $\mathcal{L}_j \cong \mathcal{L}_1^j$ for every $1 \leq j \leq d-1$, so $b_2 = \ldots = b_{d-1} = 0$ and the type of C simplifies to the single integer $a = \deg(\mathcal{L}_1)$.

Proposition 2.1. (Genus of a quasiprimitive multiple line) *Let C be a quasiprimitive multiple line of type $(a; b_2, \ldots, b_{d-1})$ in $\mathbb{P}^3$. Then $a \geq -1$ and*

$$g(C) = -(d-1) - \frac{a}{2}d(d-1) - \sum_{j=2}^{d-1} b_j \tag{2.2}$$

Proof. Let L be the support of C. The inequality $a \geq -1$ follows from the fact that $\mathcal{I}_{L,C_2} \cong \mathcal{O}_L(a)$ is a quotient of the conormal bundle $\mathcal{I}_{L,L_2} \cong \mathcal{O}_L(-1) \bigoplus \mathcal{O}_L(-1)$. By definition of the type, $\mathcal{I}_{C_j,C_{j+1}} \cong \mathcal{O}_L(ja + b_j)$. The formula for the genus follows from the fact that $g(C) = \chi(\mathcal{I}_C)$. □

We next compute the dimension of the irreducible family of primitive d-uple lines of a given type $a \geq 0$. Let C be a primitive d-structure of type a on the line L in $\mathbb{P}^3$. Given a subscheme $X \subset \mathbb{P}^3$ we denote by the symbol $\mathscr{C}_X = \mathcal{I}_X/\mathcal{I}_X^2$ its conormal sheaf. Then [1] there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_L(da) \xrightarrow{\tau} \mathscr{C}_C \otimes \mathcal{O}_L \longrightarrow \mathscr{C}_L \longrightarrow \mathcal{O}_L \longrightarrow 0 \tag{2.3}$$

The morphism τ is induced by the inclusion $\mathcal{I}_L^d \hookrightarrow \mathcal{I}_C$ via the isomorphism $\mathcal{O}_L(de) \cong \mathcal{I}_L^d/\mathcal{I}_L^{d-1}\mathcal{I}_{C_2}$. If $a \geq 0$, it follows that

$$\mathscr{C}_C \otimes \mathcal{O}_L \cong \mathcal{O}_L(da) \oplus \mathcal{O}_L(-a-2). \tag{2.4}$$

By [1, Proposition 2.3] the set of primitive $d+1$-structures $\tilde{C}$ that contain C is parametrized by the set of retractions $\beta : \mathscr{C}_C \otimes \mathcal{O}_L \longrightarrow \mathcal{O}_L(da)$ of τ; the correspondence is given by $\mathcal{I}_{\tilde{C}}/\mathcal{I}_L\mathcal{I}_C = \mathrm{Ker}(\beta)$. Therefore, if $a \geq 0$, the set of such $\tilde{C}$'s is parametrized by the set of splittings of

$$0 \longrightarrow \mathcal{O}_L(da) \xrightarrow{\tau} \mathcal{O}_L(da) \oplus \mathcal{O}_L(-a-2) \longrightarrow \mathcal{O}_L(-a-2) \longrightarrow 0;$$

hence, by an affine space of dimension $(d+1)a + 3$. With a little extra effort one can check that the set $\mathcal{P}_L(d; a)$ of primitive d structures on L of type a is an algebraic affine bundle over $\mathcal{P}_L(d-1; a)$, hence inductively that $\mathcal{P}_L(d; a)$ is a smooth variety of dimension

$$(2a+3) + (3a+3) + \cdots + (da+3) = \frac{a}{2}(d^2 + d - 2) + 3(d-1)$$
$$= (d-1)(3 + \frac{a}{2}(d+2)).$$

If we let the line L vary as well, we obtain

$$\dim \mathcal{P}(d; a) = \frac{a}{2}(d-1)(d+2) + 3d + 1. \tag{2.5}$$

This is an interesting number as primitive d-uple line are usually the generic point of a component of the Hilbert scheme parametrizing curves of degree

d—see [4,5] for the first relevant examples. When $d = 2$ we recover the easy and well know fact that double lines of type $a \geq 0$, that is, of genus $-a-1 \leq -1$, form an irreducible family of dimension $2a+7$, which is a component of the Hilbert scheme if $a \geq 1$.

By a similar argument Nollet [4, Corollary 2.6] proves that the family $\mathcal{P}(3;a;b)$ of quasiprimitive triple lines of type $(a;b)$ with $a \geq 0$ is irreducible of dimension

$$\dim \mathcal{P}(3;a;b) = 5a + 2b + 10 \tag{2.6}$$

and one can prove, more generally, that the family $\mathcal{P}(d;a;0,\ldots,0,b)$ of quasiprimitive d-lines of type $(a;0,\ldots,0,b)$, with $a \geq 0$, is irreducible of dimension

$$\dim \mathcal{P}(d;a;0,\ldots,0,b) = \frac{a}{2}(d-1)(d+2) + 3d + 2b + 1. \tag{2.7}$$

With some extra effort Nollet and the author [5, Proposition 2.3] prove that the family $\mathcal{P}(d;a;b,c)$ of quasiprimitive 4-lines of type $(a;b,c)$, with $a \geq 0$, is irreducible of dimension

$$\dim \mathcal{P}(d;a;b,c) = 9a + 2b + 2c + 13 \tag{2.8}$$

The extra effort goes into proving [5, Lemma 2.2] that, for a quasiprimitive triple line C of type $(a;b)$ with $a \geq 0$ supported on the line L, the restriction of the conormal sheaf $\mathscr{C}_C \otimes \mathcal{O}_L$ has torsion, and modulo torsion is isomorphic to $\mathcal{O}_L(3a+b) \oplus \mathcal{O}_L(-a-b-2)$. A similar argument works for a quasiprimitive d-line C of type $(a;0,\ldots,0,b)$, with $a \geq 0$ and shows $\mathscr{C}_C \otimes \mathcal{O}_L$ modulo torsion is isomorphic to $\mathcal{O}_L(da+b) \oplus \mathcal{O}_L(-a-b-2)$. We can then prove the irreducibility of the family of quasiprimitive d-uple lines of type $(a;0,\ldots,0,b,c)$, with $a \geq 0$, and compute its dimension:

$$\dim \mathcal{P}(d;a;0,\ldots,0,b,c) = \dim \mathcal{P}(d-1;a;0,\ldots,0,b) + da + 2c + 3 \tag{2.9}$$

3. Rough Classification of Curves with $s(C) = \deg(C)$

The following proposition is easy and certainly well known [8], Remark 6.8, but we include a proof for the convenience of the reader:

Proposition 3.1. *For a space curve C, the inequality $s(C) \leq \deg(C)$ holds; if $s(C) = \deg(C)$, then*

1. *every subcurve $D \subseteq C$ also satisfies $s(D) = \deg(D)$;*
2. *the curve C_{red} is either a line or the disjoint union of two lines, and on each line in its support C has the structure of a quasiprimitive multiple line satisfying $a \geq 0$ (here $a = \deg(\mathcal{L}_1)$ is the first integer appearing in the type of C).*

Proof. The inequality $s(C) \leq \deg(C)$ is proven for example in [7]. Suppose from now on that $s(C) = \deg(C)$. If $D \subseteq C$ and S is a surface of degree $s(D)$ containing D, there is an exact sequence

$$0 \to \mathcal{I}_E(-s(D)) \to \mathcal{I}_C \to \mathcal{I}_{C\cap S,S} \to 0,$$

where E, the subscheme of C residual to $C \cap S$, is a locally Cohen–Macaulay curve of degree

$$\deg(E) = \deg(C) - \deg(C \cap S) \leq \deg(C) - \deg(D).$$

Hence, $s(E) \leq \deg(C) - \deg(D)$. On the other hand, by the above exact sequence $s(C) \leq s(E) + s(D)$, therefore, $s(C) \leq \deg(C) - \deg(D) + s(D)$. This together, with the hypothesis $s(C) = \deg(C)$, implies $\deg(D) \leq s(D)$, hence equality must hold.

For an irreducible and reduced curve D, the equality $s(D) = \deg(D)$ can hold only if D is a line—cf. [7, Proposition 3.2]. Note that C cannot contain the union D of two lines meeting at one point because such a D has $s(D) = 1 < \deg(D)$. Thus, the support of C is a union of disjoint lines. Since the union of 3 disjoint lines lies on a quadric surface, the support of C consists of at most two lines. Since the first infinitesimal neighborhood of a line in $\mathbb{P}^3$ has degree 3 and is contained in a quadric surface, it cannot be contained in C, so C has a quasiprimitive structure on each line in its support. Finally, since any degree 2 subcurve of C is not contained in a plane while a double line of type -1 is contained in a plane, we must have $a \geq 0$. □

4. Multiple Lines with Good Cohomological Properties

Fix homogeneous coordinates x, y, z, w on $\mathbb{P}^3$ so that L is the line of equations $x = y = 0$. The projection $\pi : L_d \to L$ from the line M of equations $z = w = 0$ corresponds to the inclusion of coordinate rings

$$H^0_*(\mathcal{O}_L) = k[z, w] \cong k[x, y, z, w]/(x, y) \hookrightarrow k[x, y, z, w]/(x, y)^d \cong H^0_*(\mathcal{O}_{L_d}).$$

From the isomorphism of $k[z, w]$-modules

$$H^0_*(\mathcal{O}_{L_d}) = \frac{k[x, y, z, w]}{(x^d, x^{d-1}y, \ldots, y^d)} \cong \bigoplus_{i=0}^{d-1} (k[z, w](-i))^{\oplus(i+1)} \tag{4.1}$$

it follows

$$\pi_*\mathcal{O}_{L_d} \cong \bigoplus_{i=0}^{d-1} (\mathcal{O}_L(-i))^{\oplus(i+1)}. \tag{4.2}$$

Proposition 4.1. *Let C be a curve of degree d supported on the line L, and let $\pi : L_d \to L$ denote the projection from a line M disjoint from L. Fix an integer $\ell \geq 0$. The following conditions are equivalent*

1. *the genus of C is*
$$g(C) = -(d-1) - \binom{d}{3} - \ell\binom{d}{2} = B(d, d) - \ell\binom{d}{2}$$
and
$$\mathrm{H}^0\big(\mathcal{I}_C(\ell + d - 1)\big) = \mathrm{H}^0\big(\mathcal{I}_L^d(\ell + d - 1)\big).$$
2. *the genus of C is $g(C) = B(d, d) - \ell\binom{d}{2}$ and $\mathrm{H}^1\big(\mathcal{I}_C(\ell + d - 1)\big) = 0$.*
3. *the sheaf $\pi_*\mathcal{I}_{C,L_d}$ is isomorphic to $(\mathcal{O}_L(-d-\ell))^{\oplus\frac{d(d-1)}{2}}$.*

Proof. If C has degree d and genus $B(d,d) - \ell\binom{d}{2} = g(C_{d,\ell})$, then the rank and the degree of the locally free sheaf $\mathcal{E} = \pi_* \mathcal{I}_{C,L_d}$ are the same as those of $\mathcal{O}_L(-d-\ell)^{\oplus \frac{d(d-1)}{2}}$. Thus $\mathcal{E} \cong \mathcal{O}_L(-d-\ell)^{\oplus \frac{d(d-1)}{2}}$ if and only if $h^0\mathcal{E}(\ell+d-1) = 0$, which is equivalent to $h^1\mathcal{E}(\ell+d-1) = 0$ because $\chi\mathcal{E}(\ell+d-1) = 0$. As $h^1\mathcal{I}_{L_d}(\ell+d-1) = h^2\mathcal{I}_{L_d}(\ell+d-1) = 0$, these vanishings are equivalent to those in the statement. □

Definition 4.2. Given a pair of integers $d \geq 1$ and $\ell \geq 0$, we say that a d-line is a $C_{d,\ell}$ if it satisfies the equivalent conditions of Proposition 4.1.

Note that a $C_{d,0}$ is a curve of degree d, not lying on a surface of degree $< d$, of maximum genus $B(d,d)$: if a $C_{d,0}$ exists for a given d, then Beorchia's bound $B(d,d)$ is sharp.

A line is a $C_{1,\ell}$ for every ℓ. A double line of genus $-\ell-1 < 0$ is a $C_{2,\ell}$, and conversely. Indeed, all non planar double lines arise as follows—see for example [4]: take a smooth surface S containing L of degree $\ell+2 \geq 2$ and let C be the divisor $2L$ on S. By adjunction

$$\mathcal{I}_{L,C} \cong \mathcal{O}_S(-L) \otimes \mathcal{O}_L \cong \mathcal{O}_L(\deg(S)-2)) = \mathcal{O}_L(\ell).$$

From the exact sequence

$$0 \to \mathcal{I}_{C,L_2} \to \mathcal{I}_{L,L_2} \to \mathcal{I}_{L,C} \cong \mathcal{O}_L(\ell) \to 0,$$

we see

$$\mathcal{I}_{C,L_2} \cong \mathcal{O}_L(-\ell-2),$$

that is, C is a $C_{2,\ell}$. Since $\mathcal{I}_{L,C} \cong \mathcal{O}_L(\ell)$, the double line C is primitive of type $a = \ell \geq 0$.

To analyze higher degree cases, we introduce an intermediate notion.

Definition 4.3. Given integers $d \geq 1$ and $\ell \geq 0$, we say that a d-uple line C satisfies condition $\mathscr{C}_{d,l}$ if

$$h^0(\mathcal{I}_{C,L_d}(\ell+d-1)) = 0 \qquad (\mathscr{C}_{d,l})$$

or, equivalently, $\mathrm{H}^0(\mathcal{I}_C(n)) = \mathrm{H}^0(\mathcal{I}_L^d(n))$ for every $n \leq \ell+d-1$.

For $\ell = 0$, the condition $\mathscr{C}_{d,0}$ means simply that C is not contained in any surface of degree $< d$, but for $\ell > 0$ the condition $\mathscr{C}_{d,\ell}$ is stronger as it says that the only surface containing C of degree up to $\ell+d-1$ are those containing the whole infinitesimal neighborhood L_d. By Proposition 4.6, a $C_{d,\ell}$ is a d-uple line of maximum genus among those satisfying condition $\mathscr{C}_{d,l}$. Note that a double line C of type a, that is, of genus $-a-1$, satisfies condition $\mathscr{C}_{2,l}$ if and only if $a \geq \ell$ because $\mathcal{I}_{C,L_2} \cong \mathcal{O}_L(-a-2)$.

Lemma 4.4. *If C satisfies condition $\mathscr{C}_{d,l}$ and $D \subset C$ is a locally Cohen–Macaulay subcurve of degree k, then D satisfies condition $\mathscr{C}_{k,l}$.*

Proof. This follows from $I_L^{d-k}\mathcal{I}_D \subseteq \mathcal{I}_C$. □

Remark 4.5. Unfortunately, it is not true that a degree k subcurve D of a $C_{d,\ell}$ is a $C_{k,\ell}$; the point of the previous lemma is that at least D satisfies condition $\mathscr{C}_{k,l}$.

A $C_{d,\ell}$, assuming it exists, has maximum genus among degree d multiple lines whose ideal agrees with that of L_d up to degree $\ell + d - 1$:

Proposition 4.6. *Suppose C is d-uple line with support L. If $\ell \geq 0$ and*

$$\mathrm{H}^0\big(\mathcal{I}_C(n)\big) = \mathrm{H}^0\big(\mathcal{I}_L^d(n)\big) \quad \text{for } n \leq \ell + d - 1,$$

then $g(C) \leq -(d-1) - \binom{d}{3} - \ell\binom{d}{2}$.

Proof. By hypothesis, $\mathrm{H}^0\big(\mathcal{I}_C(d-1)\big) = 0$, hence C is a curve of degree d that does not lie on a surface of degree $< d$. It follows $\mathrm{H}^1\big(\mathcal{O}_C(n)\big) = 0$ for $n \geq -1$ by [7, Proposition 3.2]. Hence,

$$\begin{aligned} d(\ell + d - 1) + 1 - g(C) &= h^0(\mathcal{O}_C(\ell + d - 1)) \\ &\geq h^0(\mathcal{O}_{\mathbb{P}^3}(\ell + d - 1)) - h^0(\mathcal{I}_L^d(\ell + d - 1)) \end{aligned}$$

which is equivalent to $g(C) \leq g(C_{d,\ell})$ because

$$d(\ell + d - 1) + 1 - g(C_{d,\ell}) = h^0(\mathcal{O}_{\mathbb{P}^3}(\ell + d - 1)) - h^0(\mathcal{I}_L^d(\ell + d - 1)).$$

□

In [3, p. 141], with a different terminology, it is noted that one could prove sharpness of Beorchia's bound in the case $d = s$ by constructing curves $C_{d,\ell}$ for all $d \equiv 2$ modulo 3 and $\ell = 0, 1, 2$: indeed for $d \equiv 2$ modulo 3, a $C_{d,0}$ has genus $B(d, d)$; when $d \equiv 0$ modulo 3 the disjoint union of a line and a $C_{d-1,1}$ has genus $B(d, d)$; finally, when $d \equiv 1$ modulo 3 the disjoint union of a $C_{d-2,2}$ and a double line of genus $1-d$ has genus $B(d, d)$. We introduced the notion of a $C_{d,\ell}$ to formalize and generalize this remark as follows:

Proposition 4.7. *Suppose $1 \leq k \leq d - 1$ and C and D are, respectively, a $C_{k,d-k}$ and $C_{d-k,k}$ whose supports are disjoint. Then the disjoint union of C and D is a curve of degree d and genus $B(d, d)$ that does not lie on a surface of degree $d - 1$.*

Proof. A direct calculation shows

$$g(C \cup D) = g(C_{k,d-k}) + g(C_{d-k,k}) - 1 = B(d, d).$$

Thus, we only need to show that $C \cup D$ is not contained in a surface of degree $< d$. By way of contradiction, suppose F is the equation of a degree $d-1$ surface S containing $C \cup D$. We can assume the support of C is the line of equations $x = y = 0$ and the support of D is the line $z = w = 0$. By assumption, the polynomial F must lie in $(x, y)^k$ because $C \subset S$ and in $(z, w)^{d-k}$ because $D \subset S$, but this contradicts $\deg(F) = d - 1$. □

A d-uple line C that is a $C_{d,\ell}$ is quasiprimitive, and there are some obvious numerical constraints on the type of C:

Proposition 4.8. *Suppose $d \geq 2$ and C is a $C_{d,\ell}$. Then C is quasiprimitive. If the type of C is $(a; b_2, \ldots, b_{d-1})$, then $\ell \leq a \leq \ell + \lfloor \frac{d-2}{3} \rfloor$ and*

$$\sum_{j=2}^{d-1} b_j + (a - \ell)\binom{d}{2} = \binom{d}{3}.$$

In particular, if C is primitive, then $d \equiv 2$ modulo 3 and $a = \ell + \frac{d-2}{3}$.

Proof. If d-uple line C is a $C_{d,\ell}$, then in particular it does not lie on a surface of degree $< d$, hence it is quasiprimitive. By Lemma 4.4 the double line C_2 contained in C satisfies condition $\mathscr{C}_{2,\ell}$ and has type a, hence $a \geq \ell$.

Comparing the genus of a $C_{d,\ell}$ with the formula for the genus of a quasiprimitive multiple line, we obtain the equality:

$$a\binom{d}{2} + \sum_{j=2}^{d-1} b_j = \ell\binom{d}{2} + \binom{d}{3}.$$

If C is primitive, that is, all the b_j's are zero, it follows $d \equiv 2$ modulo 3 and $a = \ell + \frac{d-2}{3}$. For an arbitrary C, the integers b_j's are nonnegative, and the above equality implies $a \leq \ell + \lfloor \frac{d-2}{3} \rfloor$. □

5. Examples of Low Degree

Triple lines have been studied by Nollet [4]. In particular, he shows that the set of quasiprimitive triple lines of type $(a;b)$ with $a, b \geq 0$ is nonempty and irreducible of dimension $5a + 2b + 10$ [4, Corollary 2.6].

Proposition 5.1. (Triple lines) *Fix an integer $\ell \geq 0$. A triple line C satisfies condition $\mathscr{C}_{3,l}$ if and only if it is quasiprimitive of type $(a;b)$ and either $a = \ell$ and $b \geq 1$, or $a \geq \ell + 1$. Furthermore, C is a $C_{3,\ell}$ if and only if $a = \ell$ and $b = 1$. In particular, the family of $C_{3,\ell}$ curves is irreducible of dimension $5\ell + 12$.*

Proof. If C satisfies condition $\mathscr{C}_{3,l}$, then C is quasiprimitive because it does not lie on a surface of degree 2. Suppose the type of C is $(a;b)$. Then C contains a unique double line C_2, $\mathcal{I}_{L,C_2} \cong \mathcal{O}_L(a)$, $\mathcal{I}_{C_2,C} \cong \mathcal{O}_L(2a+b)$. By Lemma 4.4 the double line C_2 satisfies condition $\mathscr{C}_{2,l}$, therefore, $a \geq \ell$. Consider the exact sequence of $\mathcal{O}_L$-modules

$$0 \to \frac{\mathcal{I}_C}{\mathcal{I}_L\mathcal{I}_{C_2}} \to \frac{\mathcal{I}_{C_2}}{\mathcal{I}_L\mathcal{I}_{C_2}} \to \mathcal{O}_L(2a+b) \to 0.$$

As $\frac{\mathcal{I}_{C_2}}{\mathcal{I}_L\mathcal{I}_{C_2}} \cong \mathcal{O}_L(2a) \oplus \mathcal{O}_L(-2-a)$, we conclude $\frac{\mathcal{I}_C}{\mathcal{I}_L\mathcal{I}_{C_2}} \cong \mathcal{O}_L(-2-a-b)$. Then note that there is an obviously surjective map of $\mathcal{O}_L$-modules

$$\alpha : \mathcal{I}_{L,L_2} \otimes \mathcal{I}_{C_2,L_2} \to \frac{\mathcal{I}_L\mathcal{I}_{C_2}}{\mathcal{I}_L^3}$$

The sheaf on the left is isomorphic to $\mathcal{O}_L(-3-a)^{\oplus 2}$, while the sheaf on the right is locally free of rank two, thus α must be an isomorphism and $\frac{\mathcal{I}_L\mathcal{I}_{C_2}}{\mathcal{I}_L^3} \cong \mathcal{O}_L(-3-a)^{\oplus 2}$. Finally, from the exact sequence

$$0 \to \frac{\mathcal{I}_L\mathcal{I}_{C_2}}{\mathcal{I}_L^3} \to \mathcal{I}_{C,L_3} \to \frac{\mathcal{I}_C}{\mathcal{I}_L\mathcal{I}_{C_2}} \to 0$$

we conclude $\pi_*\mathcal{I}_{C,L_3} \cong \mathcal{O}_L(-3-a)^{\oplus 2} \oplus \mathcal{O}_L(-2-a-b)$ so that C satisfies condition $\mathscr{C}_{3,l}$ if and only if either $a = \ell$ and $b \geq 1$ or $a \geq \ell + 1$, and C is a $C_{3,\ell}$ if and only if it has type $(\ell;1)$. □

The case of quadruple lines is more difficult because the type of a quasiprimitive quadruple line C does not determine its postulation, that is, the sequence $n \mapsto h^0(\mathcal{I}_C(n))$. We show that a quadruple line is a $C_{4,\ell}$ if and only if it is a *sufficiently general* quasiprimitive quadruple line of type $(\ell; 2, 2)$.

Theorem 5.2. (Quadruple lines) *If a quadruple line C satisfies condition $\mathscr{C}_{4,l}$, then C is quasiprimitive of type $(a; b_2, b_3)$ and either $a = \ell$ and $b_2 \geq 2$, or $a \geq \ell + 1$.*

Furthermore, if a quadruple line C is a $C_{4,\ell}$, then C is quasiprimitive of type $(\ell; 2, 2)$; and a sufficiently general quasiprimitive quadruple line of type $(\ell; 2, 2)$ is a $C_{4,\ell}$. Such curves form a nonempty irreducible family of dimension $9\ell + 21$.

Proof. Quadruple lines have been studied in [5]. In particular, the dimension of the family of quasiprimitive quadruple lines of type $(a; b, c)$ is computed in [5, Proposition 2.3]) and we summarized the argument on page 6. Let C be a quasiprimitive 4-uple line of type $(a; b_2, b_3)$. Then C contains a unique double line C_2 and a unique triple line C_3, and

$$\mathcal{I}_{L,C_2} \cong \mathcal{O}_L(a), \quad \mathcal{I}_{C_2,C_3} \cong \mathcal{O}_L(2a + b_2), \quad \mathcal{I}_{C_3,C} \cong \mathcal{O}_L(3a + b_3).$$

If C satisfies condition $\mathscr{C}_{4,l}$, then C is quasiprimitive and C_3 satisfies condition $\mathscr{C}_{3,l}$, hence by 5.1 either $a = \ell$ and $b_2 \geq 1$ or $a \geq \ell + 1$.

It remains to exclude the case $a = \ell$ and $b_2 = 1$. By [5, Lemma 2.2], if one defines $\mathcal{J} = \mathcal{I}_L\mathcal{I}_{C_3} + \mathcal{I}_{C_2}^2$, then $\mathcal{I}_{C_3}/\mathcal{J} \cong \mathcal{O}_L(3a+b_2) \oplus \mathcal{O}_L(-a-b_2-2)$. When $b_2 = 1$, it follows $h^0((\mathcal{I}_{C_3}/\mathcal{J})(a+3)) = 4a+6$. On the other hand, by the proof of 5.1, $\pi_*\mathcal{I}_{C_3,L_3} \cong \mathcal{O}_L(-3-a)^{\oplus 3}$ hence

$$h^0(\mathcal{I}_{C_3}(a+3)) = h^0(\mathcal{I}_{L_4}(a+3)) + 4(a+1) + 3.$$

Now suppose $a = \ell$ and let $p = d + \ell - 1 = a + 3$: then

$$h^0(\mathcal{I}_C(p)) \geq h^0(\mathcal{J}(p)) \geq h^0(\mathcal{I}_{C_3}(p)) - h^0((\mathcal{I}_{C_3}/\mathcal{J})(p)) = h^0(\mathcal{I}_{L_4}(p)) + 1.$$

We conclude that C does not satisfy condition $\mathscr{C}_{4,l}$ when $a = \ell$ and $b_2 = 1$.

Now suppose C is a $C_{4,\ell}$, then looking at the genus C we see C must be quasiprimitive of type $(\ell; 2, 2)$. Let us show that a sufficiently general quasiprimitive quadruple line C of type $(\ell; 2, 2)$ is a $C_{4,\ell}$.

Using the same notation as above, in this case $\mathcal{I}_{C_3,C} \cong \mathcal{O}_L(3a+2)$ and $\mathcal{I}_{C_3}/\mathcal{J} \cong \mathcal{O}_L(3a+2) \oplus \mathcal{O}_L(-a-4)$, hence $\mathcal{I}_C/\mathcal{J} \cong \mathcal{O}_L(-a-4)$, thus it is enough to show that, for a general choice of C_3, the sheaf $\mathcal{F} = \pi_*\mathcal{J}/\mathcal{I}_L^4$ is isomorphic to $\mathcal{O}_L(-a-4)^{\oplus 5}$. To prove this, we recall Nollet's description [4] of the ideal of a quasiprimitive 3-line C_3 of type $(a; 2)$: there exist forms

- $f, g \in k[z, w]$ of degree $a+1$ with no common zero;
- $p, q \in k[z, w]$ of degree 2 and $3a+4$ respectively, with no common zero;
- $r, s, t \in k[z, w]$ of degree $a+2$ such that $q = rf^2 + sfg + tg^2$;

such that

(i) in the exact sequence

$$0 \to \frac{\mathcal{I}_{C_3}}{\mathcal{I}_L\mathcal{I}_{C_2}} \to \frac{\mathcal{I}_{C_2}}{\mathcal{I}_L\mathcal{I}_{C_2}} \cong \mathcal{O}_L(2a) \oplus \mathcal{O}_L(-a-2) \to \mathcal{I}_{C_2,C_3} \cong \mathcal{O}_L(2a+2) \to 0$$

the last map is given by $[p, q]$;

(ii) the homogeneous ideal of C_3 is

$$I_{C_3} = I_L^3 + < xF, yF, G >$$

where $F = xg - yf$ and $G = pF - rx^2 - sxy - ty^2$.

Let $H_1 = xG$ and $H_2 = yG$, and consider the map β of $\mathcal{O}_L$-modules

$$\mathcal{E} = \mathcal{O}_L(-2a-4) \oplus \mathcal{O}_L(-a-5)^{\oplus 2} \oplus \mathcal{O}_L(-a-4)^{\oplus 3} \xrightarrow{\beta} \mathcal{G} = \pi_* \frac{\mathcal{I}_{C_3}}{\mathcal{I}_L^4}$$

that sends the generators of $\mathcal{E}$ to the classes of F^2, H_1, H_2, x^2F,xyF, y^2F; as all these polynomials are in $I_C^2 + I_L I_{C_3}$, the map β factors through the inclusion $\mathcal{F} = \pi_* \mathcal{J}/\mathcal{I}_L^4 \hookrightarrow \mathcal{G}$. By the proof of Proposition 5.1 $\pi_* \frac{\mathcal{I}_{C_3}}{\mathcal{I}_L^3}$ is isomorphic to $\mathcal{O}_L(-a-3)^{\oplus 2} \oplus \mathcal{O}_L(-a-4)$, and so

$$\mathcal{G} = \pi_* \frac{\mathcal{I}_{C_3}}{\mathcal{I}_L^4} \cong \mathcal{O}_L(-3)^{\oplus 4} \oplus \mathcal{O}_L(-a-3)^{\oplus 2} \oplus \mathcal{O}_L(-a-4)$$

with generators corresponding to x^3, x^2y, xy^2, y^3, xF, yF and G. With respect to the chosen basis, the matrix of $\beta : \mathcal{E} \to \mathcal{G}$ is

$$\begin{bmatrix} 0 & -r & 0 & g & 0 & 0 \\ 0 & -s & -r & -f & g & 0 \\ 0 & -t & -s & 0 & -f & g \\ 0 & 0 & -t & 0 & 0 & -f \\ g & p & 0 & 0 & 0 & 0 \\ -f & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Generically, the map β has rank 5 as one can see for example by computing the 5×5 minors of its matrix. On the other hand, the section $[p, -g, f, -r, -s, -t]^T$ of $H^0(\mathcal{E}(2a+6))$ is in the kernel of $H^0(\beta(2a+6))$ by a direct check or because

$$pF^2 - gH_1 + fH_2 - rx^2F - sxyF - ty^2F = 0.$$

As f and g have no common zeros on L, we conclude that the kernel of β is isomorphic to $\mathcal{O}_L(-2a-6)$. As we have already observed, the image of β is contained in $\mathcal{F} = \pi_* \mathcal{J}/\mathcal{I}_L^4$ hence we have an exact sequence

$$0 \to \mathcal{O}_L(-2a-6) \to \mathcal{E} \xrightarrow{\beta} \mathcal{F}.$$

Finally, from the exact sequence $0 \to \frac{\mathcal{J}}{\mathcal{I}_L^4} \to \frac{\mathcal{I}_{C_3}}{\mathcal{I}_L^4} \to \frac{\mathcal{I}_{C_3}}{\mathcal{J}} \to 0$ we compute that $\mathcal{F} = \pi_* \mathcal{J}/\mathcal{I}_L^4$ has the same rank 5 and the same degree $-5a - 20$ as the image of β, hence $\mathcal{E} \xrightarrow{\beta} \mathcal{F}$ is surjective.

We can now show that $\mathcal{F} \cong \mathcal{O}_L(-a-4)^{\oplus 5}$ if C_3 is general. As $\mathcal{F}$ is locally free of the same rank and degree as $\mathcal{O}_L(-a-4)^{\oplus 5}$, it is enough to prove $H^0\mathcal{F}^\vee(-a-5) = 0$. Dualizing and twisting the above exact sequence we obtain

$$0 \to \mathcal{F}^\vee(-a-5) \to \mathcal{O}_L(-1)^{\oplus 3} \oplus \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(a-1) \to \mathcal{O}_L(a+1) \to 0.$$

Thus, what we need is that the map

$$H^0\mathcal{O}_L^{\oplus 2} \oplus H^0\mathcal{O}_L(a-1) \overset{[f,-g,p]}{\rightarrow} H^0\mathcal{O}_L(a+1)$$

be injective. Now this is certainly the case is f, g and p are chosen general, as it is injective if we choose $f = z^{a+1}$, $g = w^{a+1}$ and $p = zw$. □

6. Families of Maximum Genus Curves of Low Degree

To summarize, the curves $C_{d,\ell}$ of which we know existence are:

1. when $d = 3$, quasiprimitive multiple lines of type $(\ell; 1)$—this paper, Proposition 5.1;
2. when $d = 4$, quasiprimitive multiple lines of type $(\ell; 2, 2)$—this paper, Theorem 5.2;
3. when $d = 3m - 1 \leq 119$, primitive multiple lines of type $a = \ell + \frac{d-2}{3}$—these are constructed in [3], with the aid of *Macaulay2* for $m \geq 4$; at least in characteristic zero, in [6] we will show how to extend this result to all degrees $d \equiv 2$ modulo 3.

One would be tempted to guess from these examples that the family of $C_{d,\ell}$'s supported on a line L is irreducible, but there might be counterexamples already for $d = 5$: I do not know if there are quasiprimitive quintuple lines of type $(0; 2, 2, 6)$ or of type $(0; 2, 3, 5)$ that do not lie on a quartic surface, or whether, if they exist, they lie in the closure of the family of primitive 5 lines of type $a = 1$.

We close the paper enumerating the known examples of degree d curves of maximum genus $B(d, d)$ not lying on surfaces of degree $< d$ for small d, thereby proving Theorem 1.4 in the introduction.

For $d = 2$, the maximum genus $B(2, 2)$ is -1 and every curve of degree 2 and $g = -1$ is not contained in a plane; the 7 dimensional irreducible family of double lines of genus -1, that is of $C_{2,0}$'s, is contained in the closure of the family of two disjoint lines, which is the general member of the 8 dimensional Hilbert scheme $H_{2,-1}$.

For $d = 3$, the maximum genus $B(3, 3)$ is -3; Proposition 4.7 provides two irreducible families of degree 3 curves of maximum genus -3 not lying on a quadric: the 12 dimensional family of quasiprimitive triple lines of type $(0; 1)$, and the 13 dimensional family whose general member is the disjoint union of a line and a double line of genus -2; the first family is in the closure of the second by [4, Proposition 3.3].

For $d = 4$, the family of quadruple lines of maximum genus $B(4, 4) = -7$ not lying on a cubic surface is not irreducible. It contains by Proposition 4.7

- the 22-dimensional irreducible family F_1 whose general member is the disjoint union of two double lines of genus -3;
- the 21-dimensional family F_2 whose general member is the disjoint union of a line and a quasiprimitive triple line of type $(1; 1)$;
- the 21-dimensional family F_3 whose general member is a general quasiprimitive quadruple line of type $(0; 2, 2)$.

It is clear the second family F_2 cannot be in the closure of family F_1, and the closure of these two families are in fact a component of the Hilbert scheme $H_{4,-7}$ by [5, Theorem 6.2], while the family F_3 is in the closure of F_1 by [5, Proposition 3.3].

For $d = 5$, the family of quintuple lines of maximum genus $B(5, 5) = -14$ not lying on a quartic surface is not irreducible. It contains

- the 35-dimensional irreducible family G_1 whose general member is the disjoint union of a $C_{3,2}$ and a $C_{2,3}$;
- the 34-dimensional irreducible family G_2 whose general member is the disjoint union of a line and a general $C_{4,1}$;
- the 30-dimensional irreducible family G_3 whose general member is a general primitive quintuple line of type $a = 1$.

and there are no containment between the closures of these 3 families in the Hilbert scheme $H_{5,-14}$: one reason for which G_3 is not in the closure of G_1 or G_2 is that its general member is a curve that has embedding dimension two at each of its points, a property that does not hold for any curve in G_1 and G_2.

Acknowledgements

The Author would like to thank warmly Scott Nollet to whom he is indebted for several discussions on the topic of multiple lines.

Author contributions ES is the only author of the paper.

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare no competing interests.

References

[1] Bănică, C.(R-NISTC)., Forster, O.(D-MUTU): Multiplicity structures on space curves. In: The Lefschetz centennial conference, Part I (Mexico City, 1984), pp. 47–64. Contemp. Math., vol. 58. American Mathematical Society, Providence, RI (1986)

[2] Beorchia, V.: Bounds for the genus of space curves. Math. Nachr. **184**, 59–71 (1997)

[3] Beorchia, V., Lella, P., Schlesinger, E.: The maximum genus problem for locally Cohen-Macaulay space curves. Milan J. Math. **86**, 137–155 (2018)

[4] Nollet, S.: The Hilbert schemes of degree three curves. Ann. Scient. E.N.S., 4^e Série **30**, 367–384 (1997)

[5] Nollet, S., Schlesinger, E.: Hilbert schemes of degree four curves. Compos. Math. **139**(2), 169–196 (2003)

[6] Sammartano, A., Schlesinger, E.: Maximum genus curves and generic forms **(in preparation)**

[7] Schlesinger, E.: A new proof of a theorem of Beorchia on the genus of space curves. Math. Nachr. **194**, 197–203 (1998)

[8] Schlesinger, E.: A speciality theorem for Cohen-Macaulay space curves. Trans. Am. Math. Soc. **351**, 2731–2743 (1999)

Enrico Schlesinger
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci 32
20133 Milan
Italy
e-mail: enrico.schlesinger@polimi.it

Received: October 11, 2023.
Revised: October 11, 2023.
Accepted: February 5, 2024.